A COMBINATORIAL CONSTRAINT SATISFACTION PROBLEM DICHOTOMY CLASSIFICATION CONJECTURE

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Abstract. We further generalise a construction – the *fibre construction* – that was developed in an earlier paper of the first two authors. The extension in this paper gives a polynomial-time reduction of CSP(H) for any relational system H to CSP(P) for any relational system P that meets a certain technical partition condition, that of being *K_3-partitionable*.

Moreover, we define an equivalent condition on P, that of being *block projective*, and using this show that our construction proves NP-completeness for exactly those CSPs that are conjectured to be NP-complete by the CSP dichotomy classification conjecture made by Bulatov, Jeavons and Krohkin, and by Larose and Zádori. We thus provide two new combinatorial versions of the CSP dichotomy classification conjecture.

As with our previous version of the fibre construction, we are able to address restricted versions of the dichotomy conjecture. In particular, we reduce the Feder-Hell-Huang conjecture to the CSP dichotomy classification conjecture, and we prove the Kostochka-Nešetřil-Smolíková conjecture. Although these results were proved independently by Jonsson et. al. and Kun respectively, we give different, shorter, proofs.

1. Introduction

Many combinatorial problems can be expressed as Constraint Satisfaction Problems (CSPs). This concept originated in the context of Artificial Intelligence (see e.g. [33]) and is very active in several areas of Computer Science. CSPs include standard satisfiability problems and many combinatorial optimization problems, thus are also a very interesting class of problems from the theoretical point of view. The whole area was revitalized by Feder and Vardi [11], who reformulated CSPs as homomorphism problems (or H-colouring problems) for relational structures. Motivated by the results of [40] and [16], they formulated the following.

Conjecture 1.1. (Dichotomy) *Every Constraint Satisfaction Problem is either in P or NP-complete.*

Throughout the paper we will assume that $P \neq NP$, and we call a relational structure NP-complete if the associated CSP is NP-complete. Schaefer [40] established the dichotomy for CSPs with boolean domains, and Hell Nešetřil [16] established the dichotomy for undirected graphs; it follows from [11] that the dichotomy for CSPs can be reduced to the dichotomy problem for H-colouring for

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oriented graphs. This setting, and related problems, have motivated intensive research in descriptive complexity theory. This is surveyed, for example, in [8], [17], [14], and [18].

Recently the whole area was put into yet another context by Peter Jeavons and his collaborators, in [21] and [6], when they recast the complexity of CSPs as properties of algebras and polymorphisms of relational structures. In particular, they related the complexity of CSPs to a Galois correspondence between polymorphisms and definable relations (obtained by Bodnarčuk et al. [2] and by Geiger [13]; see [38] and [39]). This greatly simplified elaborate and tedious reductions of particular problems and led to the solution of the dichotomy problem for CSPs with three element domains [3] and other results which are surveyed, for example, in [6] and [15]. This approach to studying CSPs via certain algebraic objects yields, in particular, that every *projective* structure H is NP-complete [22], [21]. It also led to Conjecture 4.11, (first formulated in [6]), which strengthens Dichotomy Conjecture 1.1 by actually conjecturing what the dichotomy is.

The success of these general algebraic methods gave motivation for some older results to be restated in this new context. For example, [4] treats H-colouring problems for undirected graphs in such a way that the dichotomy between the tractable and NP-complete cases of H-colouring problem agrees with Conjecture [6]. A substantial generalization of this dichotomy result was obtained in [1] for the H-colouring problems over digraphs with no sources and no sinks.

Since [6], other algebraic interpretations of Conjecture 4.11 have been found. In particular, equivalent versions are implicit in the papers [27] and [30].

In [36], the first two authors proposed a new combinatorial approach to the dichotomy problem, generalising a construction of the second author from [42] (and [41]) that gave the first combinatorial proof that any projective relational structure is NP-complete. We were able to show that subprojective relational structures are NP-complete. It was then that we applied the name *fibre construction* to the construction. An example provided by Ralph McKenzie [31] gave us strong motivation to extend our results: he showed that there are structures that are NP-complete by the results of [6], that are not subprojective. We mentioned this extension in [36] and give the details here.

In this paper, we present an incarnation of the fibre construction that is general enough to provide a combinatorial version of Conjecture 4.11. In Section 3, we extend the fibre construction to all structures which are what we call *K_3-partitionable*, thus showing that all K_3-partitionable structures are NP-complete (Theorem 3.2, Corollary 3.3).

In Section 4 we recall the reduction that allows us to assume that a structure is idempotent for questions of its complexity. We then define what we call *block projective structures* and show that they are K_3-partitionable. As well as being a cleaner quantification than K_3-partitionable, block projectivity is useful in showing that the structures that are NP-complete by Conjecture 4.11 are also K_3-partitionable.

In [36] we suggested that the set of K_3-partitionable structures may be greater than the set of structures that conjectured to be NP-complete in 4.11. Here we prove that the two sets of structures are in fact the same. This is included in Section 4.4 of this paper.

Thus we get the following equivalences for idempotent structures.

$$\begin{array}{rl} & NP\text{-complete by Conjecture 4.11} \\ \iff & \text{block projective} \\ \iff & K_3\text{-partitionable} \end{array}$$

This allows us to rephrase the CSP Dichotomy Classification Conjecture in terms of either K_3-partitionable, or block projective structures, see Conjecture 4.16.

In [10], Feder, Hell, and Huang conjecture that any CSP that is NP-complete, is NP-complete for instances of bounded degree. In [36], we extended results of [42] to show that for any subprojective structure H, CSP(H) is NP-complete for instances of maximum degree at most $4 \cdot \Delta(H)^6$, where $\Delta(H)$ is the maximum degree of H. In Section 5 we get similar bounds for block projective structures, in particular, this gives far better bounds for the graph case than were achieved in [42]. Independently, Jonsson, Krokhin, and Kuivinen [23], reduced the Feder-Hell-Huang conjecture to the CSP Dichotomy Classification Conjecture. Such a reduction is an immediate corollary of our main results.

It was conjectured by Kostochka, Nešetřil, and Smolíková [24] that for any integer $\ell \geq 3$, and any graph H that is NP-complete, CSP(H) is NP-complete for instances of girth at least ℓ. Although a recent result of Kun [25] for general relational structures settles this conjecture in the positive, we give a simpler proof in Sections 6 and 7.

It is interesting to note how flexible the notion of the fibre construction is, and as this is a culmination of several earlier papers [41, 42, 36], we include, in Section 8 the general setting. We generalise the notion of K_3-partitionable, to $\mathcal{G}$-partitionable for arbitrary relational structures $\mathcal{G}$, and point out why this will be important in future applications.

2. Standard Definitions

We work with finite relational structures of a given finite type. A *type* is a (finite) vector $K = (k_i)_{i \in I}$ of positive integers, called *arities.* A *relational structure* $\mathcal{H}$ of type K, consists of a vertex set $V = V(\mathcal{H})$, and a k_i-ary relation $R_i = R_i(\mathcal{H}) \subset V^{k_i}$ on V, for each $i \in I$. An element of R_i is called a k_i-tuple.

Thus a digraph is just a relational structure of type $K = (2)$. A graph is the same, but in which the single 2-ary relation is symmetric and irreflexive.

Throughout the paper, we will use script letters, such as $\mathcal{G}, \mathcal{H}$ and $\mathcal{P}$, to represent relational structures except in the case that we are talking specifically of graphs.

Given two relational structures $\mathcal{G}$ and $\mathcal{H}$ of the same type, an $\mathcal{H}$*-colouring of* $\mathcal{G}$, or a *homomorphism from* $\mathcal{G}$ *to* $\mathcal{H}$, is a map $\phi : V(\mathcal{G}) \to V(\mathcal{H})$ such that for all $i \in I$ and every k_i-tuple $(v_1, \dots, v_{k_i}) \in R_i(\mathcal{G})$, $(\phi(v_1), \dots, \phi(v_{k_i}))$ is in $R_i(\mathcal{H})$. For a fixed relational structure $\mathcal{H}$, CSP($\mathcal{H}$) is the following decision problem:

Problem CSP($\mathcal{H}$)
Instance: A relational structure $\mathcal{G}$;
Question: Does there exist an $\mathcal{H}$-colouring of $\mathcal{G}$?

We write $\mathcal{G} \to \mathcal{H}$ to mean that $\mathcal{G}$ has an $\mathcal{H}$-colouring. A relational structure $\mathcal{H}$ is a *core* if its only $\mathcal{H}$-colourings are automorphisms. It is well known, (see, for example, [17]) that $\mathcal{G} \to \mathcal{H}$ if and only if $\mathcal{G}' \to \mathcal{H}'$, where $\mathcal{G}'$ and $\mathcal{H}'$ are the cores of $\mathcal{G}$ and $\mathcal{H}$ respectively. This allows us to restrict our attention to core relational structures in particular problems related to CSP.

All relational structures of a given type form a category with nice properties. In particular, this category has products and powers which are defined explicitly as follows:

Given a relational structure $\mathcal{H}$, and a positive integer d, the d-ary *power* $\mathcal{H}^d$ of $\mathcal{H}$ is the relational structure of the same type as $\mathcal{H}$, defined as follows.

- $V(\mathcal{H}^d) = \{(v_1, \dots, v_d) \mid v_1, \dots, v_d \in V(\mathcal{H})\}$.
- For $i \in I$, $((v_{1,1}, v_{1,2}, \dots, v_{1,d}), \dots, (v_{k_i,1}, \dots, v_{k_i,d}))$ is in $R_i(\mathcal{H}^d)$ if and only if all of $(v_{1,1}, v_{2,1}, \dots, v_{k_i,1}), \dots, (v_{1,d}, \dots, v_{k_i,d})$ are in $R_i(\mathcal{H})$.

We will write $f(v_1, \dots, v_d)$ in place of $f((v_1, \dots, v_d))$ when f is a function on $V(\mathcal{H}^d)$. An $\mathcal{H}$-colouring of $\mathcal{H}^d$ (i.e. a homomorphism $\mathcal{H}^d \to \mathcal{H}$) is called a *$d$-ary polymorphism* of $\mathcal{H}$. A d-ary polymorphism ϕ is called a *projection* if there exists some $i \in \{1, \dots, d\}$ such that $\phi(v_1, \dots, v_d) = v_i$ for any $v_1, \dots, v_d \in V(\mathcal{H})$. A d-ary polymorphism ϕ is called *idempotent* if $\phi(v, \dots, v) = v$ for all $v \in V(\mathcal{H})$. A relational structure is called *projective* if all of its idempotent polymorphisms are projections. One of our main definitions in this paper, Definition 4.3, generalises these structures.

3. The Fibre Construction

In this section we define the notion of K_3-partitionable structures, one of the two main combinatorial concepts that we introduce in the present paper. We shall prove a theorem that connects the CSP over these structures with the K_3-colouring problem of graphs. Before we get to the definition of K_3-partitionable structures, we mention some notational conventions that we will use.

We will often define sets of indexed vertices such as $W^* = \{w_1^*, \dots, w_d^*\}$. A copy W^a of the set W^* will mean the set $W^a = \{w_1^a, \dots, w_d^a\}$. Given two copies W^a and W^b of the same set W^* we say that we *identify W^a and W^b index-wise* to mean we identify the vertices w_i^a and w_i^b for $i = 1, \dots, d$. When we define a function f on W^*, we will assume it to be defined on any copy W^a of W^* by $f(w_\alpha^a) = f(w_\alpha^*)$ for all $\alpha = 1, \dots, d$. We refer to a function f on an set W^* as a *pattern* of W^*. In the case that the image of f is contained in the vertex set of some structure $\mathcal{H}$ we speak about $\mathcal{H}$-*pattern* of W^*. We will often describe $\mathcal{H}$-patterns of W^* explicitly as vectors of elements of $\mathcal{H}$. For example:

$$f(\{w_1^*, w_2^*, w_3^*, w_4^*\}) = [h, h, h', h],$$

for $h, h' \in V(\mathcal{H})$.

Definition 3.1. (K_3-partition) Let $\mathcal{H}$ be a relational structure. An instance $\mathcal{M}$ of CSP($\mathcal{H}$) is called a *K_3-partition* if $V(\mathcal{M})$ contains two disjoint copies W^1 and W^2 of some set W^* of indexed vertices, and there are three disjoint sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ of $\mathcal{H}$-patterns of W^* such that the following properties are met.

(i) Under every $\mathcal{H}$-colouring ϕ of $\mathcal{M}$, $\phi|_{W^1}$ and $\phi|_{W^2}$ are in different sets in $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$.

(ii) There are representative $\mathcal{H}$-patterns P_1, P_2, and P_3 of $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$ respectively such that for every choice of $i \neq j \in \{1, 2, 3\}$ there is an $\mathcal{H}$-colouring ϕ_{ij} of $\mathcal{M}$ for which $\phi_{ij}|_{W^1} = P_i$ and $\phi_{ij}|_{W^2} = P_j$.

If $\mathcal{H}$ has a K_3-partition, it is called *K_3-partitionable.*

The following theorem sheds light on the relationship between CSPs over K_3-partitionable structures and CSP(K_3). We view K_3 as a relational structure with one binary relation containing six ordered 2-tuples, thus an instance of CSP(K_3) also has ordered 2-tuples. In this way the classical problem of 3-colouring undirected graphs is just the subproblem of CSP(K_3) in which the instances have a symmetric and irreflexive relation.

Theorem 3.2. *(Fibre construction) If a relational structure $\mathcal{H}$ has a K_3-partition $\mathcal{M}$, then there is a polynomial time construction that provides, for any instance G of* CSP(K_3) *an instance $\mathcal{M}(G)$ of* CSP($\mathcal{H}$) *such that*

$$G \to K_3 \iff \mathcal{M}(G) \to \mathcal{H}.$$

Proof. Let $\mathcal{M}$ be a K_3-partition of $\mathcal{H}$, and let G be any instance of CSP(K_3) (i.e., a digraph). We construct the necessary instance $\mathcal{M}(G)$ of CSP($\mathcal{H}$) as follows.

(i) For each vertex v of G let W^v be a copy of W^*.
(ii) For each edge $e = (u, v)$ of G let $\mathcal{M}^e$ be a copy of $\mathcal{M}$. Identify, index-wise, W^u and W^v with W^1 and W^2 of $\mathcal{M}^e$ respectively.

Thus $\mathcal{M}(G)$ consists of $|V(G)|$ copies of W^* and $|E(G)|$ copies of $\mathcal{M}$. All vertices are distinct unless identified above [1]. We now show that $G \to K_3 \iff \mathcal{M}(G) \to \mathcal{H}$. Since $|V(\mathcal{M}(G))|$ is polynomial in $|V(G)|$, this will prove the theorem.

Let ϕ be an $\mathcal{H}$-colouring of $\mathcal{M}(G)$. For each $v \in V(G)$, W^v is a copy of W^1 or W^2 in some copy of $\mathcal{M}$, so by (i) of Definition 3.1, ϕ restricts on it to some $\mathcal{H}$-colouring in $\mathcal{P}_i$ for $i = 1, 2$, or 3. Thus $\phi' : v \to 1, 2, 3$ is well defined by

$$\phi'(v) = i \text{ if } \phi \text{ restricts on } W^v \text{ to a pattern in } \mathcal{P}_i.$$

Moreover, since for any edge $e = (u, v)$ of G, W^u and W^v are identified with W^1 and W^2 in the copy $\mathcal{M}^e$ of $\mathcal{M}$, we have that $\phi'(u) \neq \phi'(v)$, again by (i) of Definition 3.1. Thus ϕ' is a K_3-colouring of G.

On the other hand, let ϕ be a K_3-colouring of G. We define an $\mathcal{H}$-colouring ϕ' of $\mathcal{M}(G)$ as follows.

- For each $v \in V(G)$, let ϕ' restricted to W^v be the $\mathcal{H}$-pattern $P_{\phi(v)}$ from (ii) of Definition 3.1.
- For each edge $e = (u, v)$ of G, the copies of W^1 and W^2 are already coloured with the $\mathcal{H}$-patterns $P_{\phi(u)}$ and $P_{\phi(v)}$, where $\phi(u) \neq \phi(v) \in \{1, 2, 3\}$, so by (ii) of Definition 3.1 we can extend ϕ' to an $\mathcal{H}$-colouring of $\mathcal{M}^e$.

□

Corollary 3.3. *If a relational structure $\mathcal{H}$ has a K_3-partition $\mathcal{M}$, then* CSP($\mathcal{H}$) *is NP-complete.*

Proof. This follows immediately from the theorem by the fact, shown in [12], that CSP(K_3) is NP-complete (even for undirected instances).

□

[1] Observe that this construction is not the amalgamation of $\mathcal{M}$ and G which is often used in indicator constructions. In fact, for a given copy W^v of W^* in $\mathcal{M}(G)$ that has been identified with copies of W^1 and W^2 from different copies of $\mathcal{M}$, W^v induces the union of the edges induced by W^1 and W^2.

4. The CSP Dichotomy Classification Conjecture

In this section we consider how the notion of K_3-partitionable structures relates to the CSP Dichotomy Classification Conjecture. In Subsection 4.1 we recall a common reduction used for CSP problems. While the definition of K_3-partitionable structures may be quite intuitive to those familiar with indicator constructions it is somewhat technical when compared to earlier versions of the fibre construction, and less concrete. In Subsection 4.2 we give a more manageable definition, our second main combinatorial concept in the the paper: *block projective structure*. In Subsection 4.3 we introduce the necessary algebra, and then state the CSP Dichotomy Classification Conjecture. In Subsection 4.4 we give an alternate statement of the CSP Dichotomy Classification Conjecture in terms of K_3-partitionable and block projective structures.

4.1. **A Preliminary Reduction.** Recall that a d-ary polymorphism ϕ is called *idempotent* if $\phi(v, \dots, v) = v$ for all $v \in V(\mathcal{H})$. A relational structure is *idempotent* if its only polymorphisms are idempotent. There is a simple construction one can use to make a relational system idempotent.

Definition 4.1. Given a relational structure $\mathcal{H}$, let $\mathrm{Id}(\mathcal{H})$ be the idempotent structure constructed from $\mathcal{H}$ by adding, for each vertex v of $\mathcal{H}$, the unary relation R_v containing the single 1-tuple (v).

For CSPs, it is generally much easier to deal with idempotent structures, so it is common to reduce a problem to the CSP of the corresponding idempotent structure. The following, cf. [7], allows us to do this.

Proposition 4.2. *Let $\mathcal{H}$ be a core relational structure. Then* $\mathrm{CSP}(\mathcal{H})$ *is polynomial-time equivalent to* $\mathrm{CSP}(\mathrm{Id}(\mathcal{H}))$. *In particular,* $\mathrm{CSP}(\mathcal{H})$ *is NP-complete if and only if* $\mathrm{CSP}(\mathrm{Id}(\mathcal{H}))$ *is.*

In Section 5 we prove a bounded degree version of the second part of this theorem.

4.2. **Block Projective Structures.** The following definition is new, and is an extension of the idea of *subprojective* which was introduced in [36].

Definition 4.3. (Block Projective.) A pair $\{0, 1\}$ of vertices of $\mathcal{H}$ is *block projective* if there are associated disjoint sets $H_a, H_b \subset V(\mathcal{H})$ (called blocks) such that the following is true. For any polymorphism $\phi : \mathcal{H}^d \to \mathcal{H}$ of $\mathcal{H}$, there is some $i \leq d$ such that for any $(s_1, \dots s_d) \in \{0, 1\}^d$,

$$\phi(s_1, \dots, s_d) \in H_{s_i}.$$

A relational system is *block projective* if it contains a block projective pair.

Proposition 4.4. *Let $\mathcal{H}$ be a block projective relational structure. Then $\mathcal{H}$ is K_3-partitionable, and so by Corollary 3.3, NP-complete.*

(A very similar proof that subprojective structures are K_3-partitionable is embedded in the proof of the fibre construction in [36].)

Proof. Let $\{0, 1\}$ be a block projective pair in $\mathcal{H}$. Let $\mathcal{G} = \mathcal{H}^6$ and define copies W^1 and W^2 of W^* in $V(\mathcal{G})$ by [2]

$$W^1 = \{(001111), (110011), (111100)\} \text{ and } W^2 = \{(110101), (011110), (101011)\}.$$

[2] We drop the commas in the 6-tuples, as it helps readability.

Since the sets H_0 and H_1 from Definition 4.3 are disjoint, the following sets of $\mathcal{H}$-patterns of W^* are also disjoint.

$$\begin{aligned}
\mathcal{P}_1 &= \{[x, y_1, y_2] \mid x \in H_0;\ y_1, y_2 \in H_1\} \\
\mathcal{P}_2 &= \{[y_1, x, y_2] \mid x \in H_0;\ y_1, y_2 \in H_1)\} \\
\mathcal{P}_3 &= \{[y_1, y_2, x] \mid x \in H_0;\ y_1, y_2 \in H_1)\}.
\end{aligned}$$

We now observe that the properties (i) - (ii) of Definition 3.1 are satisfied.

Property (i): Let ϕ be an $\mathcal{H}$-colouring of $\mathcal{G} = \mathcal{H}^6$. Since $\{0,1\}$ is block projective, there is some projection $\pi : \{0,1\}^6 \to \{0,1\}$ such that for any $\vec{v} \in \{0,1\}^6$, $\phi(\vec{v}) \in H_{\pi(\vec{v})}$. So we have

$$\phi(W^1) = [\phi(001111), \phi(110011), \phi(111100)] = [x, y, z],$$

where $x \in H_{\pi(001111)}, y \in H_{\pi(110011)}, z \in H_{\pi(111100)}$, and

$$\phi(W^2) = [x', y', z'],$$

where $x' \in H_{\pi(110101)}, y' \in H_{\pi(011110)}, z' \in H_{\pi(101011)}$. Whichever slot π projects onto, we get that $\phi(W^1)$ and $\phi(W^2)$ are in different sets of $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$.

Property (ii): This is easy. Observe that any projection $\pi : \mathcal{G} = \mathcal{H}^6 \to \mathcal{H}$ is an $\mathcal{H}$-colouring, $0 = \pi(000000) \in H_0$ and $1 \in H_1$. Take the patterns $P_1 = [0,1,1], P_2 = [1,0,1]$, and $P_3 = [1,1,0]$. The six projections are thus the $\mathcal{H}$-colourings that we need.

□

4.3. **Algebraic Approach.** An algebra $\mathbf{A} = (A, F)$ consists of a non-empty set A, called the *base set* of $\mathbf{A}$, and a set F of finitary operations on A. It is *finite* if A is finite and is *trivial* if $|A| = 1$. Given a relational structure $\mathcal{H}$, recall that $\mathrm{Pol}(\mathcal{H})$ is the set of polymorphisms of $\mathcal{H}$. This defines an algebra $\mathbf{A}_{\mathcal{H}} = (V(\mathcal{H}), \mathrm{Pol}(\mathcal{H}))$. We say that an algebra of the form $\mathbf{A}_{\mathcal{H}}$ is NP-complete (in P) if CSP($\mathcal{H}$) is NP-complete (in P). This definition is well defined, as it is shown in [6] that the computational complexity of a relational structure over $V(\mathcal{H})$ depends only on $\mathrm{Pol}(\mathcal{H})$, that is, if we add finitely many of the relations over $V(\mathcal{H})$ that are preserved by $\mathrm{Pol}(\mathcal{H})$, we do not change the complexity of the structure. Further, it is known from [2] and [13] that the set of relations preserved by $\mathrm{Pol}(\mathcal{H})$ are exactly those that are primitive positive in the relations of $\mathcal{H}$ (that is, can be described using the relations of $\mathcal{H}$, the equality relation, conjunction, and first order existential quantification).

We require the following basic algebraic definitions.

Definition 4.5. Let $\mathbf{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f \in F$ and for any $b_1, \ldots, b_d \in B$, where d is the arity of f, we have $f(b_1, \ldots, b_d) \in B$. Then the algebra $\mathbf{B} = (B, F|_B)$ is called a *subalgebra of* $\mathbf{A}$, where $F|_B$ consists of the restrictions of all operations in F to B.

Definition 4.6. Let $\mathbf{B} = (B, F^{\mathbf{B}})$ and $\mathbf{C} = (C, F^{\mathbf{C}})$ be algebras such that $F^B = \{f_i^{\mathbf{B}} \mid i \in I\}$ and $F^{\mathbf{C}} = \{f_i^{\mathbf{C}} \mid i \in I\}$, where both $f_i^{\mathbf{B}}$ and $f_i^{\mathbf{C}}$ are d_i-ary, for all

$i \in I$. Then $\mathbf{C}$ is a *homomorphic image of* $\mathbf{B}$ if there exists a surjection $\psi : B \to C$ such that the following holds for all $i \in I$, and all $b_1, \ldots, b_{d_i} \in B$.

$$\psi \circ f_i^{\mathbf{B}}(b_1, \ldots, b_{d_i}) = f_i^{\mathbf{C}}(\psi(b_1), \ldots, \psi(b_{d_i})).$$

(Notationally, we will say that $\psi \circ f_i^{\mathbf{B}} = f_i^{\mathbf{C}} \circ \psi$ on B^{d_i}.)

Definition 4.7. Given an algebra $\mathbf{A} = (A, F^{\mathbf{A}})$, any homomorphic image $\mathbf{C} = (C, F^{\mathbf{C}})$ of a subalgebra $\mathbf{B} = (B, F^{\mathbf{B}})$ of $\mathbf{A}$ is called a *factor* of $\mathbf{A}$.

Definition 4.8. The *product* $\mathbf{A} \times \mathbf{B}$ of two algebras $\mathbf{A} = (A, F^{\mathbf{A}})$ and $\mathbf{B} = (B, F^{\mathbf{B}})$ is the algebra $(A \times B, F^{\mathbf{A}\times\mathbf{B}})$ such that each operation $f^{\mathbf{A}\times\mathbf{B}} \in F^{\mathbf{A}\times\mathbf{B}}$ acts componentwise on $A \times B$ via $f^{\mathbf{A}}$ and $f^{\mathbf{B}}$. For a set of algebras, the product is defined in a similar manner.

The subalgebras of finite powers of an algebra are sometimes called the *invariant relations* of the algebra. By the remark at the end of the first paragraph of Subsection 4.3, for any relational structure $\mathcal{H}$ the invariant relations of the algebra $\mathbf{A}_{\mathcal{H}}$ are exactly the relations definable by primitive positive formulas over the relations of $\mathcal{H}$.

Definition 4.9. A class of algebras is called a *variety* if it is closed under taking subalgebras, homomorphic images and products. A variety generated by an algebra is the smallest variety containing the algebra.

Given an algebra $\mathbf{C} = (C, F)$, the *term operations* of $\mathbf{C}$ refer to the set of finitary operations of C that can be built from F and the projections via superposition of functions, or equivalently, that preserve the same relations on C as F does.

The following is a consequence of Corollary 7.3 in [6].

Theorem 4.10. *For an idempotent relational structure $\mathcal{H}$, the algebra $\mathbf{A}_{\mathcal{H}}$ is NP-complete if it has a non-trivial factor, all of whose term operations are projections.*

In fact, it is conjectured in [6] that this is the only situation in which an idempotent structure is NP-complete, provided that $P \neq NP$. That is, the following CSP dichotomy classification conjecture is formulated in [6, Conjecture 7.5].

Conjecture 4.11. *For an idempotent relational structure $\mathcal{H}$,* CSP($\mathcal{H}$) *is NP-complete if $\mathbf{A}_{\mathcal{H}}$ has a non-trivial factor, all of whose term operations are projections. Otherwise, it is polynomial time solvable.*

By Proposition 4.2, this conjecture implies that a relational structure $\mathcal{H}$ is NP-complete if the algebra $\mathbf{A}_{\mathrm{Id}(\mathcal{H}')}$ associated with its core $\mathcal{H}'$ has a non-trivial factor all of whose term operations are projections, and is otherwise polynomial time solvable. As such, it is a strengthening of Conjecture 1.1.

4.4. **Comparison.**

Proposition 4.12. *Let $\mathcal{H}$ be a relational structure such that $\mathbf{A}_{\mathcal{H}}$ has a subalgebra $\mathbf{B} = (B, F^{\mathbf{B}})$ with a non-trivial homomorphic image $\mathbf{C} = (C, F^{\mathbf{C}})$, all of whose term operations are projections. Then $\mathcal{H}$ is block projective.*

Proof. Let ψ be a surjective homomorphism from $\mathbf{B}$ to $\mathbf{C}$. Since ψ is surjective, $\psi^{-1}(c)$ is non-empty for every $c \in C$. Let 0 and 1 be vertices in C, and let $\overline{0}$ and $\overline{1}$ be arbitrary elements in $\psi^{-1}(0)$ and $\psi^{-1}(1)$ respectively. We will show that $\{\overline{0}, \overline{1}\}$ is a block projective pair in $\mathcal{H}$.

Let $H_{\overline{0}} = \psi^{-1}(0)$ and $H_{\overline{1}} = \psi^{-1}(1)$. These subsets of $B \subset V(\mathcal{H})$ are clearly disjoint. Let $\phi : \mathcal{H}^d \to \mathcal{H}$ be an idempotent polymorphism of $\mathcal{H}$. We must show that for some $i \in [d]$ and all $\vec{v} = (s_1, \ldots, s_d) \in \{\overline{0}, \overline{1}\}^d$, $\phi(\vec{v}) \in H_{s_i}$.

Let $f^{\mathbf{B}}$ be the restriction of ϕ to B^d, and $f^{\mathbf{C}}$ be the member of $F^{\mathbf{C}}$ such that $\psi \circ f^{\mathbf{B}} = f^{\mathbf{C}} \circ \psi$. Since $f^{\mathbf{C}}$ is a projection, assume, wlog, that it projects onto the i^{th} slot. Then we have that for any $\vec{v} = (s_1, \ldots, s_d) \in \{\overline{0}, \overline{1}\}^d$,

$$\psi \circ \phi(\vec{v}) = \psi \circ f^{\mathbf{B}}(\vec{v}) = f^{\mathbf{C}} \circ \psi(\vec{v}) = \psi(s_i).$$

So $\phi(\vec{v}) \in \psi^{-1}(s_i) = H_{s_i}$. This completes the proof of the proposition.

$\square$

Thus by putting together Propositions 4.12 and 4.4 we get a combinatorial proof of Theorem 4.10.

It turns out that one can use a K_3-partition of $\mathcal{H}$ to build a graph relation G out of primitive positive formulas over the relations of a power of $\mathcal{H}$, such that G retracts to a triangle. This yields an alternate proof of the fact that $\mathcal{H}$ is NP-complete. The construction of such a graph G will be described in detail in the proof of Proposition 4.14. The proof requires a few more results that we recall now.

In [5], Bulatov and Jeavons show that an idempotent algebra $\mathbf{B}$ has a non-trivial factor, all of whose term operations are projections if and only if, in the language of tame congruence theory, the variety generated by $\mathbf{B}$ admits type 1. In [30], it is shown that this is true if and only if $\mathbf{B}$ admits no weak near unanimity term operation.

Definition 4.13. A operation ϕ of arity at least 3 of an algebra $\mathbf{A} = (A, F)$ is called a *weak near unanimity operation* (or *weak nu operation*) if it is idempotent and satisfies the following identity for all $x, y \in A$.

$$\phi(y, x, x, \ldots, x, x) = \phi(x, y, x, \ldots, x, x) = \cdots = \phi(x, x, x, \ldots, x, y)$$

In order to prove that a variety admits type 1 it suffices to show that some of the algebras in the variety generates a subvariety which admits type 1. With this in mind, we complete the picture which shows that for an idempotent K_3-partitionable structure $\mathcal{H}$, the algebra $\mathbf{A}_{\mathcal{H}}$ has a non-trivial factor, all of whose term operations are projections.

Proposition 4.14. *For an idempotent K_3-partitionable structure $\mathcal{H}$, the variety generated by $\mathbf{A}_{\mathcal{H}}$ contains an algebra $\mathbf{B}$ that admits no weak near unanimity term operation. Thus $\mathbf{A}_{\mathcal{H}}$ has a non-trivial factor all of whose term operations are projections.*

Proof. Let $\mathcal{M}$ be a K_3-partition of $\mathcal{H}$, and let W^*, W^1, and W^2 be as in Definition 3.1. Let $w = |W^*|$ and $\mathbf{B} = \mathbf{A}_{\mathcal{H}}^w$. So $\mathbf{B}$ is an algebra in the variety generated by $\mathbf{A}_{\mathcal{H}}$. Further, elements in the base set of $\mathbf{B}$ can be viewed as maps from W^* to $V(\mathcal{H})$. We show that $\mathbf{B}$ has no weak nu term operation. Towards contradiction let us assume that t is a d-ary weak nu term operation of $\mathbf{B}$, $d \geq 2$.

Now we define a graph G on $\mathbf{B}$. The set E of edges of G is defined as follows: $(f, g) \in E$ if and only if there exist $h_1, h_2 : \mathcal{M} \to \mathcal{H}$ such that $h_1|_{W^1} = f, h_1|_{W^2} = g$, and $h_2|_{W^1} = g, h_2|_{W^2} = f$. Since the definition of E is primitive positive in terms of invariant relations of $\mathbf{A}_{\mathcal{H}}$, t preserves E. This fact can also be seen directly as follows. By restricting the set of homomorphisms from $\mathcal{M}$ to $\mathcal{H}$ to $W_1 \cup W_2$ we get a $2w$-ary invariant relation R_1 of $\mathbf{A}_{\mathcal{H}}$. By switching the variables of R_1 that correspond to the elements of W_1 with those that correspond to the elements of W_2 we get another $2w$-ary invariant relation, say R_2 of $\mathbf{A}_{\mathcal{H}}$. Clearly, the relation $Q = R_1 \cap R_2$ is also a $2w$-ary invariant relation of $\mathbf{A}_{\mathcal{H}}$. Now, observe that $(f, g) \in E$ if and only if the $2w$-tuple determined by the two w-tuples f and g is in Q. Since Q is an invariant relation of $\mathbf{A}_{\mathcal{H}}$ and the operations of $\mathbf{A}_{\mathcal{H}}^w$ act componentwise, E is an invariant binary relation of $\mathbf{B} = \mathbf{A}_{\mathcal{H}}^w$.

Clearly, E is symmetric and by Definition 3.1 (i), E is irreflexive, too. Observe that by Definition 3.1(ii), there is a triangle T in G through the vertices P_1, P_2, P_3 and by Definition 3.1 (i) the graph G retracts onto T, say under the retraction r. But then the operation $rt|_T$ is a weak nu operation admitted by the triangle T. This contradicts the fact that the triangle is projective.

□

From Propositions 4.12, 4.4 and 4.14, we immediately get the following.

Theorem 4.15. *For an idempotent relational structure $\mathcal{H}$, the following are equivalent.*

(i) $\mathbf{A}_{\mathcal{H}}$ *has a factor all of whose term operations are projections.*
(ii) $\mathcal{H}$ *is block projective.*
(iii) $\mathcal{H}$ *is K_3-partitionable.*

Thus we have the following equivalent version of Conjecture 4.11

Conjecture 4.16. *An idempotent relational structure $\mathcal{H}$ is NP-complete if it satisfies one of the equivalent conditions of Theorem 4.15. Otherwise, it is polynomial time solvable.*

5. Bounded Degree CSP Dichotomy

For a relational structure $\mathcal{H}$, the *degree* $\deg(V)$ of a vertex v of $\mathcal{H}$ is the number of tuples it occurs in in all relations of $\mathcal{H}$. The *maximum degree* $\Delta(\mathcal{H})$ of $\mathcal{H}$ is the maximum of $\deg(V)$ over all vertices v of $\mathcal{H}$.

In the introduction of the paper we mentioned the following conjecture of Feder, Hell, and Huang.

Conjecture 5.1 ([10]). *For any relational structure $\mathcal{H}$ that is NP-complete, there exists an integer $b = b(\mathcal{H})$ such that the problem* CSP$(\mathcal{H})$ *is NP-complete even when restricted to instances of degree at most $b(\mathcal{H})$.*

As our proof that a K_3-partitionable structure $\mathcal{H}$ is NP-complete is an explicit construction, it takes very little work to show the stronger statement that there is some constant b, depending on $\mathcal{H}$, for which the problem CSP$(\mathcal{H})$ is NP-complete even when restricted to instances of degree at most b.

Indeed, a finer reading of the proof of Theorem 3.2 yields the following version of Corollary 3.3.

Corollary 5.2. *If a core relational structure $\mathcal{H}$ has a K_3-partition $\mathcal{M}$, then $b(\mathcal{H})$ exists and is at most $4 \cdot \Delta(\mathcal{M})$.*

Proof. It follows from a result of [20] that CSP(K_3) is NP-complete for instances of maximum degree at most 4. (The result in [20] is for undirected graphs, but arbitrarily directing edges, it holds for directed graphs.) Where $\mathcal{M}$ is the K_3-partition of $\mathcal{H}$ from Theorem 3.2, and G is an instance of K_3 with maximum degree at most 4, the instance $\mathcal{M}(G)$ from Theorem 3.2 has maximum degree at most $4 \cdot \Delta(\mathcal{M})$. Thus CSP($\mathcal{H}$) is NP-complete for instances with maximum degree at most $4 \cdot \Delta(\mathcal{M})$. □

If a structure $\mathcal{H}$ is block projective, we explicitly find the K_3-partition $\mathcal{M} = \mathcal{H}^6$, and so get that

$$b(\mathcal{H}) < 4 \cdot \Delta(\mathcal{H})^6. \tag{1}$$

This bound was observed for subprojective structures in [36]. It holds for all idempotent relational structures satisfying the conditions of Conjecture 4.16. In the next subsection, we prove a bounded degree version of the second part of Proposition 4.2 which allows us to extend this bound, slightly weakened, to all relational structures $\mathcal{H}$ with cores $\mathcal{H}'$ for which Id($\mathcal{H}'$) is block projective.

Thus we show that the truth Conjecture 5.1 is a consequence of the truth of Conjecture 4.16. In [23], Jonsson, Krokhin, and Kuivinen independently reduced Conjecture 5.1 to Conjecture 4.11.

5.1. Bounded degree version of Proposition 4.2.

We start with a definition.

Definition 5.3. Given a structure $\mathcal{H}$ containing a vertex v, the structure $v\mathcal{H}v'$, read 'v-cloned $\mathcal{H}$', is defined as follows.

- $V(v\mathcal{H}v') = V(\mathcal{H}) \cup \{v'\}$, where $v' \notin V(\mathcal{H})$.
- For any relation $R_i(\mathcal{H})$, $R_i(v\mathcal{H}v')$ is $R_i(\mathcal{H})$ with a second copy of any k_i-tuple containing v in which all occurrences of v are replaced with v'.

If $\mathcal{H}$ is a core, then in an $\mathcal{H}$-colouring of $v\mathcal{H}v'$, v and v' have the same image. We will use $v\mathcal{H}v'$ in the proof of the following proposition, which is a bounded degree version of the second part of Proposition 4.2.

Proposition 5.4. *Let $\mathcal{H}$ be a core relational structure. If* CSP(Id($\mathcal{H}$)) *is NP-complete for instances of degree at most b, then* CSP($\mathcal{H}$) *is NP-complete for instances of degree at most*

$$b(\mathcal{H}) = \max(b + \Delta(\mathcal{H}), 3\Delta(\mathcal{H})).$$

Proof of Proposition 5.4. It will be enough to give a polynomial time construction that provides, for any instance $\mathcal{G}$ of CSP(Id($\mathcal{H}$)), an instance $\mathcal{G}'$ of CSP($\mathcal{H}$), such that

(i) $\mathcal{G} \to \mathrm{Id}(\mathcal{H}) \iff \mathcal{G}' \to \mathcal{H}$, and
(ii) $\Delta(\mathcal{G}') = \max(\Delta(\mathcal{G}) + \Delta(\mathcal{H}), 3\Delta(\mathcal{H}))$.

Let $\mathcal{G}$ be an instance of CSP(Id($\mathcal{H}$)). For each $v \in V(\mathcal{H})$ let $R_v(\mathcal{G})$ denote the relation of $\mathcal{G}$ that corresponds to the unary relation introduced for v in the definition of Id($\mathcal{H}$) (Definition 4.1). Construct the instance $\mathcal{G}'$ of CSP($\mathcal{H}$) from $\mathcal{G}$ by doing the following for each $v \in V(\mathcal{H})$.

- Remove $R_v(\mathcal{G})$ for any $v \in V(\mathcal{H})$.
- String together $|R_v(\mathcal{G})|$ v-clones $v\mathcal{H}v'$ of $\mathcal{H}$ by identifying the copy of v' in the i^{th} with the copy of v in the $i + 1^{th}$ copy, for $i = 1, \ldots, |R_v(\mathcal{G})| - 1$.
- For the j^{th} 1-tuple (g) in $R_v(\mathcal{G})$, take a new copy of $v\mathcal{H}v'$. Identify the v with the vertex g of $\mathcal{G}$, and identify the v' with the copy of v in the j^{th} copy of $v\mathcal{H}v'$ from the second step.

Let H be a new copy of $\mathcal{H}$ and for each $v \in V(\mathcal{H})$ identify the last copy of v' in the string of v-clones from the second step with the copy of v in H. Call this whole structure $\mathcal{G}'$.

It is not hard to verify that $\Delta(\mathcal{G}') = \max(\Delta(\mathcal{G}) + \Delta(\mathcal{H}), 3\Delta(\mathcal{H}))$. We conclude the proof by verifying that $\mathcal{G} \to \mathrm{Id}(\mathcal{H}) \iff \mathcal{G}' \to \mathcal{H}$.

Let ϕ be an $\mathcal{H}$-colouring of $\mathcal{G}'$. This clearly restricts to a mapping $\phi' : V(\mathcal{G}) \to V(\mathcal{H}) = V(\mathrm{Id}(\mathcal{H}))$ which preserves all the relations of $\mathcal{G}$. We show that ϕ' composed with some automorphism of $\mathcal{H}$ also preserves the relations R_v for $v \in V(\mathcal{H})$, thus is an $\mathrm{Id}(\mathcal{H})$-colouring of $\mathcal{G}$.

Indeed, since $\mathcal{H}$ is a core, ϕ restricts on the copy H of $\mathcal{H}$ in $\mathcal{G}'$ to an automorphism α of $\mathcal{H}$. The map $\alpha^{-1} \circ \phi' : V(\mathcal{G}) \to V(\mathrm{Id}(\mathcal{H}))$ still preserves the relations preserved by ϕ'. Furthermore, ϕ maps the vertices v and v' in all copies of $v\mathcal{H}v'$ to the same vertex $\alpha(v)$ of $\mathcal{H}$, so $\alpha^{-1} \circ \phi$ maps them to v. In particular, it maps any vertex of $\mathcal{G}'$ in the relation $R_v(\mathcal{G}')$, to v. Since $(v) \in R_v(\mathrm{Id}(\mathcal{H}))$, this preserves R_v as needed.

On the other hand, let ϕ be an $\mathrm{Id}(\mathcal{H})$-colouring of $\mathcal{G}$. Then ϕ can be extended to an $\mathcal{H}$-colouring of $\mathcal{G}'$, and in a unique way.

□

Applying this to the bound given in Equation (1) we get the following.

Corollary 5.5. *For any core relational structure* $\mathcal{H}$ *such that* $\mathrm{Id}(\mathcal{H})$ *is block projective,* $\mathrm{CSP}(\mathcal{H})$ *is* NP*-complete for instances of degree at most*

$$b(\mathcal{H}) = 4 \cdot \Delta(\mathrm{Id}(\mathcal{H}))^6 + \Delta(\mathcal{H}) \leq 4 \cdot (\Delta(\mathcal{H}) + 1)^6 + \Delta(\mathcal{H}).$$

Corollary 5.6. *If Conjecture 4.16 is true, then so is Conjecture 5.1.*

Proof. It suffices to prove the claim for core relational structures. Assume that Conjecture 4.16 is true, if $\mathcal{H}$ is a core relational structure for which $\mathrm{CSP}(\mathcal{H})$ is NP-complete, then $\mathrm{Id}(\mathcal{H})$ is block projective, and so by the previous corollary, there exists a finite $b(\mathcal{H})$ such that $\mathrm{CSP}(\mathcal{H})$ is NP-complete for instances with maximum degree at most $b(\mathcal{H})$. □

6. The Large Girth CSP Dichotomy

The *girth* of a graph H is the length of its shortest cycle. The problem $\mathrm{CSP}(H)$ restricted to instances of large girth was considered for graphs H in [24] where the following conjecture was made:

Conjecture 6.1. *Let* H *be a non-bipartite graph, and* $\ell \geq 3$ *be an integer. Then* $\mathrm{CSP}(H)$ *is* NP*-complete for instances of girth at least* ℓ*.*

The conjecture was shown in [24] to be true if $\ell \leq 7$ or if H is *symmetric*. A generalisation of this conjecture to all relational structures that are NP-complete has recently been proved by Gabor Kun in [25], independent of the Dichotomy Classification Conjecture.

In this section we give a simpler proof of Conjecture 6.1. Our proof is independent of Kun's proof that relies on the heavy machinery of expander graph constructions. The argument we give uses our main results, requiring only two main steps.

We have to construct K_3-partitions with a given girth, and we have to prove a girth version of the second part of Proposition 4.2.

To construct K_3-partitions of a given girth, we must apply what is known as a 'girth pumping' theorem to an ordinary K_3-partition. The concept of K_3-partitionable does not allow us to prove such a theorem, but the concept of block projective does. Thus we apply girth pumping to a block projective graph, and from there, build the large girth K_3-partition. The actual girth pumping theorem, Theorem 7.2, is quite technical, and of independent interest, so we put it off until Section 7. For now, we give just a couple corollaries that follow from Theorem 7.2.

As we intend on applying the girth pumping to $\mathrm{Id}(G)$ of a graph G, we must prove it for relational structures with one symmetric irreflexive binary relation, and any number of unary relations, (note that these are not loops). We refer to such structures as *u-graphs* and observe that graphs are indeed u-graphs. We define the *girth* of a u-graph as the girth of the binary relation. This agrees with the common definition of girth of a relational structure in terms of the girth of its incidence structure. An *edge* of a u-graph is a tuple in the binary relation. The tuples of the unary relations are called 1-*tuples*.

Corollary 6.2. *Let H be a core graph such that $\mathrm{Id}(H)$ is block projective with block projective pair $\{0,1\}$, and associated disjoint sets H_0 and H_1. Let $\ell \geq 3$ be an integer. There exists a u-graph M and an injection $\alpha : \{0,1\}^6 \to V(M)$ such that the following are true.*

(i) *For any $\mathrm{Id}(H)$-colouring ϕ of M there is some $i \leq 6$ such that for any $(s_1, \dots, s_6) \in \{0,1\}^6$,*
$$\phi(\alpha(s_1, \dots, s_6)) \in H_{s_i}.$$
(ii) *M has girth at least ℓ.*
(iii) *Vertices of the range of α are distance at least ℓ apart in M.*

This is a corollary of Theorem 7.2, and is proved in Section 7. Now letting $W^1 = \{\alpha(001111), \alpha(110011), \alpha(111100)\}$ and $W_2 = \{\alpha(110101), \alpha(011110), \alpha(101011)\}$ much like we did in the proof of Proposition 4.4, and analogously defining the sets $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ as in that proof, we get that M is a K_3-partition of $\mathrm{Id}(H)$ having girth at least ℓ and such that vertices of $W^1 \cup W^2$ are distance at least ℓ apart. This ensures, when we use M to build an instance $M(G)$ of $\mathrm{CSP}(\mathrm{Id}(H))$ from any instance G of $\mathrm{CSP}(K_3)$, that $M(G)$ has girth at least ℓ.

The following is also a consequence of Theorem 7.2, and proved in Section 7.

Corollary 6.3. *Let H be a graph core, v be a vertex of H, and $\ell \geq 3$ be an integer. Then there exists a graph $vH_\ell v'$ with the following properties.*

(i) *$vH_\ell v'$ has girth at least ℓ.*
(ii) *The vertices v and v' of $vH_\ell v'$ are distance at least ℓ apart.*
(iii) *Under any H-colouring ϕ of $vH_\ell v'$, $\phi(v) = \phi(v')$.*
(iv) *For any automorphism α of H, there is an H-colouring ϕ_α of $vH_\ell v'$ such that $\phi_\alpha(v) = \alpha(v)$.*

6.1. **Girth version of Proposition 4.2.**

Proposition 6.4. *Let H be a core graph. If $\mathrm{CSP}(\mathrm{Id}(H))$ is NP-complete for instances of minimum girth ℓ, then $\mathrm{CSP}(H)$ is NP-complete for instances of minimum girth ℓ.*

Proof. We proceed as in the proof of Proposition 5.4 except that we use a more complicated construction in place of the clones vHv'. We use graphs $vH_\ell v'$ whose existence is guaranteed by Corollary 6.3. Assuming the existence of $vH_\ell v'$ the graph $\mathcal{G}'$ constructed as in the proof of Proposition 5.4 from a graph $\mathcal{G}$ of girth at least ℓ will still have girth at least ℓ. So the proposition follows. □

6.2. **Proof of Conjecture 6.1.** Without loss of generality we may assume that H is a core graph. It was shown in [4] that for any non-bipartite core graph H, the algebra $\mathbf{A}_{\mathrm{Id}(H)}$, has a non-trivial factor, all of whose term operations are projections. Hence, by Proposition 4.12, $\mathrm{Id}(H)$ is block projective. Applying Corollary 6.2 (and the discussion following it), we get a K_3-partition $\mathcal{M}$ of $\mathrm{Id}(H)$ such that the instances $\mathcal{M}(G)$ of $\mathrm{CSP}(\mathrm{Id}(H))$ constructed in Theorem 3.2 all have girth at least ℓ. Thus $\mathrm{CSP}(\mathrm{Id}(H))$ is NP-complete for instances of girth at least ℓ. Proposition 6.4 then shows that $\mathrm{CSP}(H)$ is NP-complete for instances of girth at least ℓ. We therefore get the following theorem.

Theorem 6.5. *Let H be a non-bipartite graph, and $\ell \geq 3$ an integer. Then the problem $\mathrm{CSP}(H)$ is NP-complete when restricted to instances with girth $\geq \ell$.*

7. Block Projectivity and Girth

Answering a question of Erdős, Müller [34] proved that for any integers $k, \ell \geq 3$, and any set $\mathscr{S}$ of k-colourings of a set of vertices W, there is a graph G of girth ℓ containing the vertices W such that the restrictions of the k-colourings of G to W, up to permutation of colours, are exactly those in $\mathscr{S}$. The girth condition aside, this result underlies our fibre construction. However, the difficult part is maintaining control of the colourings while applying the girth condition. Müller did this for k-colouring, i.e. for mapping to cliques. In [37], this result was extended to projective graphs, and in [36] to subprojective graphs. In this section we extend the result to block projective graphs, and show how it implies Corollary 6.2.

First, we need a definition, which generalises a concept introduced in [37], that of a graph being H-pointed. (Recall, u-graphs are defined before Corollary 6.2.)

Definition 7.1. Let M, H be u-graphs. Let S_M be a subset of $V(M)$, and $\mathscr{S}_H$ be a family of disjoint subsets of $V(H)$. Then S_M and $\mathscr{S}_H$ are said to be a *system of (M,H)-block pointed subsets* if the following is true. For any two homomorphisms $g, g' : M \to H$, if $g(x)$ and $g'(x)$ are in the same set in $\mathscr{S}_H$ for all $x \in S_M$, $x \neq x_0$ (for some fixed vertex $x_0 \in S_M$), then $g(x_0)$ and $g'(x_0)$ are also in the same set in $\mathscr{S}_H$.

Theorem 7.2. *For every u-graph M and every choice of positive integers k and ℓ there exists a u-graph M' with the following properties.*

(i) *$g(M') > \ell$;*
(ii) *For every u-graph H with at most k vertices, there exists a homomorphism $g : M' \to H$ if and only if there exists a homomorphism $f : M \to H$.*

Furthermore, there exists a surjective homomorphism $c : M' \to M$ such that

(iii) *There exists a set $\{s' \in c^{-1}(s) \mid s \in S_M\}$ of representatives of the sets $c^{-1}(s)$ that are pairwise distance at least ℓ apart.*
(iv) *For every u-graph H with at most k vertices, every system of (M,H)-block pointed subsets S_M and $\mathscr{S}_H$, and every homomorphism $g : M' \to H$ there exists an H-colouring f of M such that for every $v \in c^{-1}(S_M)$, $g(v)$ and $f \circ c(v)$ are in the same element of $\mathscr{S}_H$.*

Before we prove the theorem, we show how it implies Corollaries 6.2 and 6.3.

Proof of Corollary 6.2. Let H be a core graph such that $\mathrm{Id}(H)$ has a block projective set $\{0,1\}$, with associated disjoint sets H_0 and H_1. Let $M = \mathrm{Id}(H)^6$. Then M has property (i) of Corollary 6.2, where α is taken as the identity on $\{0,1\}^6 \subset V(M)$.

Apply Theorem 7.2 to M for the given ℓ and $k = |V(\mathrm{Id}(H))|$ to get a u-graph M'. We show that is the graph promised by the corollary.

To get property (i) of the corollary, we need to define $\alpha : \{0,1\}^6 \to V(M')$. Theorem 7.2 gives us a surjective homomorphism $c : M' \to M$. For $v \in \{0,1\}^6 \subset V(M)$, let $\alpha(v)$ be the particular representative of $c^{-1}(v)$ singled out by item (iii) of the theorem. Further, let $S_M = \{0,1\}^6 \subset V(M)$, and $\mathscr{S}_H = \{H_0, H_1\}$. This is a system of $(M, \mathrm{Id}(H))$-block pointed subsets. Because M satisfied property (i) of Corollary 6.2, and M' satisfies property (iv) of Theorem 7.2, M' also satisfies property (i) of Corollary 6.2.

That M' satisfies properties (ii) and (iii) of Corollary 6.2 is immediate from properties (i) and (iii) of the theorem. □

Proof of Corollary 6.3. Let H be a core, v be a vertex of H, and $\ell \geq 3$ be an integer. Let vHv' be v-cloned H, as in Definition 5.3. Already vHv' satisfies properties (iii) and (iv) of the Corollary. Let $S = V(vHv')$ and $\mathscr{S}_H = \{\{x\} \mid x \in V(H)\}$. This is a system of (vHv', H)-pointed subsets. Applying Theorem 7.2 to $M = vHv'$ we get a graph M' (with vertices $c^{-1}(v)$ and $c^{-1}(v')$) that satisfies all properties (i) - (iv). □

Now we prove Theorem 7.2.

Proof. We follow the proof of Theorem 1.2 from [37] very closely, and we refer to this paper for many of the details.

Where M has a vertices $\{1, \ldots, a\}$ and q edges, let $V_1, \ldots, V_a$ be disjoint sets of n vertices each. Let M_0 be the u-graph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_a$, and edge set

$$\{xy \mid x \in V_i, y \in V_j, ij \in E(M)\},$$

and in which each vertex of V_i occurs in all the same unary relations as the vertex i of M does.

Thus M_0, which is often referred to as the n-blowup of M, has qn^2 edges. Let $\mathcal{M}$ be the set of all subgraphs of M_0 with all 1-tuples, and $m = \lfloor qn^{1+\varepsilon} \rfloor$ edges, where $0 < \varepsilon < 1/\ell$. Let $\delta = \min\{\varepsilon\ell, 1/k\}$.

Asymptotically, almost all graphs G of $\mathcal{M}$ satisfy the following properties.

(a) G has at most n^δ cycles of length $\leq \ell$, moreover, these cycles are vertex disjoint.

(b) For any two non-empty sets $A \subset V_i$ and $B \subset V_j$ of $V(G)$ (with ij in M) such that $|A| + |B| \geq \delta n$, the subgraph of G induced by $A \cup B$ is not a matching (set of mutually disjoint edges,) with fewer than n^δ edges.

(c) There is a choice of vertices $\{v_1, \ldots, v_a\}$, such that $v_i \in V_i$, and for any $1 \leq i \neq j \leq a$, the distance in G between v_i and v_j is at least ℓ.

It was shown in [37], using standard calculations, that asymptotically, almost all graphs G of $\mathcal{M}$ satisfy properties (a) and (b), thus we prove that almost all graphs of $\mathcal{M}$ satisfy properties (a) - (c), by proving the following claim.

Claim 7.3. *Almost all graphs G of $\mathcal{M}$ satisfy property (c) above.*

Proof. For a graph G chosen uniformly at random from $\mathcal{M}$, the probability that a given vertex u is distance ℓ or less from a vertex v is less than $n^{\ell\varepsilon - 1}$. Thus the probability that a given set of a vertices $\{v_1, \ldots, v_a\}$, with $v_i \in V_i$ for all i, fail to satisfy property (c) is less than $a^2 n^{\ell\varepsilon - 1}$. As $\varepsilon < 1/\ell$, this goes to zero as n goes to infinity, so not only do almost all graphs G of $\mathcal{M}$ satisfy property (c), almost all choices of the set $\{v_1, \ldots, v_a\}$ in almost all G satisfy (c). □

We now continue with the proof of Theorem 7.2. Let G be any graph of $\mathcal{M}$ that satisfies the properties (a), (b) and (c). It is clear that we can remove a matching of size at most n^δ from G and end up with a graph M' having the following corresponding properties.

(a') $g(M') > \ell$
(b') For any two non-empty sets $A \subset V_i$ and $B \subset V_j$ of $V(M')$ (with ij in M) such that $|A| + |B| \geq \delta n$, there is at least one edge of M' with both endpoints in $A \cup B$.
(c') Same as (c).

We now verify that M' satisfies properties (i) - (iv) of the theorem. Property (i) is given by property (a').

Letting $c : M' \to M$ be the M-colouring defined by

$$c(v) = i \text{ where } v \in V_i,$$

it is clear that for every graph H, and every H-colouring f of M, $g = f \circ c$ is an H-colouring of M'. To finish the proof that M' satisfies property (ii), it suffices to show that for any graph H with at most k vertices, and any H-colouring g of M' there is an H-colouring f of M.

Let such an H-colouring g of M' be given, and define $f : M \to H$ as follows. For each vertex i of M, there exists, by the pigeonhole principle, a vertex h of H such that $|V_i \cap g^{-1}(h)| \geq n/k > \delta n$. Let $f(i) = h$ for any such h. We now show that f is an H-colouring of M. Let ij be an edge of M. There is an edge of M' whose endpoints map to $f(i)$ and $f(j)$ (under g), and so f maps ij to an edge of H. Indeed, the sets $A = V_i \cap g^{-1}(f(i))$ and $B = V_j \cap g^{-1}(f(j))$ both have size at least n^δ and so by property (b'), there is an edge of M' with one endpoint in A and one in B. This edge clearly maps to $f(i)f(j)$, and so property (ii) is proved.

Property (iii) of the theorem follows directly from property (c').

To show property (iv) of the theorem, assume that S_M and $\mathscr{S}_H$ are (M, H)-block pointed subsets, where H has at most k vertices, and assume that g is an H-colouring of M'.

The main point is that for any vertex s in S_M, g takes everything from V_s to the same set in $\mathscr{S}_H$. Indeed let v be any vertex of V_s and define $f_v : V(M) \to V(H)$ by letting $f_v(s) = g(v)$, and otherwise letting f_v be defined as f in the proof of property (ii). That is, for $i \neq s$, set $f_v(i) = h$ for some vertex h of H such that $|V_i \cap g^{-1}(h)| \geq n/k > \delta n$. By almost the same argument as before, we get that f_v is an H-colouring of M. Thus if g takes some v in V_s to some where other than the set it takes s to, then we get different H-colourings of M that differ only on $s \in S_M$. This contradicts the fact that S_M and $\mathscr{S}_H$ are (M, H)-pointed.

The statement that g and $f \circ c$ restrict on S_M to the same function, uniquely determines the function f on S_M, We have to show that there exists an f such that g and $f \circ c$ take every thing in S_M to the same set in $\mathscr{S}_H$. Because g is constant on V_s for all $s \in S_M$, the function f defined as in the proof of property (ii) is such that g and $f \circ c$ restrict on $c^{-1}(S_M)$ to the same function. Thus M' has property (iv).

□

8. The General Fibre Constructions

In showing that a structure $\mathcal{H}$ has a K_3-partition, we reduce CSP(K_3) to CSP($\mathcal{H}$), and so show that CSP($\mathcal{H}$) is NP-complete. Naturally we could replace K_3 with any other structure that we know is NP-complete, and arrive at the same conclusion. In practise it will often be easier to show that a structure is $\mathcal{G}$-partitionable for some other NP-complete structure $\mathcal{G}$, than it will be to show that it is K_3-partitionable. Such a situation is seen in [43]. In this section we will define $\mathcal{G}$-partitions, and outline the proof that any structure with such a $\mathcal{G}$-partition is NP-complete, provided that $\mathcal{G}$ is NP-complete.

Definition 8.1. Let $\mathcal{H}$ and $\mathcal{G}$ be relational structures. Then $\mathcal{H}$ has a $\mathcal{G}$-partition if there exists some set W^* of indexed vertices and a family $\{\mathcal{P}_v \mid v \in V(\mathcal{G})\}$ of disjoint sets of $\mathcal{H}$-patterns of W^*, with $\mathcal{P}_v$ containing a representative pattern P_v for each $v \in V(\mathcal{G})$, such that the following condition holds.

For any integer k and any k-ary relation R of $\mathcal{G}$, there is an instance $\mathcal{M} = \mathcal{M}^{\mathcal{H}}_{\mathcal{G}(R)}$ of CSP($\mathcal{H}$), containing copies $W^1, \dots, W^k$ of W^*, for which the following properties are met.

(i) Under every $\mathcal{H}$-colouring ϕ of $\mathcal{M}$, there is some $(v_1, \dots v_k) \in R$ such that for $i = 1, \dots, k$, $\phi|_{W^i} \in \mathcal{P}_{v_i}$.
(ii) For every $\vec{v} = (v_1, \dots, v_k) \in R$, there is an $\mathcal{H}$-colouring $\phi_{\vec{v}}$ of $\mathcal{M}$ for which $\phi_{\vec{v}}|_{W^i} = P_{v_i}$ for $i = 1, \dots, k$.

Furthermore, we can generalise Theorem 3.2 to the following.

Theorem 8.2. *If a relational structure $\mathcal{H}$ has a $\mathcal{G}$-partition then there is a polynomial time construction that provides, for any instance $\mathcal{I}$ of* CSP($\mathcal{G}$) *an instance $\mathcal{M}^{\mathcal{H}}_{\mathcal{G}}(\mathcal{I})$ of* CSP($\mathcal{H}$) *such that*

$$\mathcal{I} \to \mathcal{G} \iff \mathcal{M}^{\mathcal{H}}_{\mathcal{G}}(\mathcal{I}) \to \mathcal{H}.$$

We do not prove this in detail, but we do provide the construction in detail.

Construction 8.3. *(General Fibre Construction $\mathcal{I} \to \mathcal{M}^{\mathcal{H}}_{\mathcal{G}}(\mathcal{I})$.) Let $\mathcal{H}$ and $\mathcal{G}$ be relational structures, such that $\mathcal{H}$ has a $\mathcal{G}$-partition. Let $\mathcal{I}$ be an instance of* CSP($\mathcal{G}$), *we define the instance $\mathcal{M}^{\mathcal{H}}_{\mathcal{G}}(\mathcal{I})$ of* CSP($\mathcal{H}$) *as follows.*

(i) *For each vertex $v \in \mathcal{I}$, let W^v be a new copy of the set W^*.*
(ii) *For each relation R of $\mathcal{G}$ and each tuple $\vec{v} = (v_1, \dots, v_k) \in R$, let $\mathcal{M}^{\vec{v}}$ be a new copy of $\mathcal{M}^{\mathcal{H}}_{\mathcal{G}(R)}$ from Definition 8.1, and for $i = 1, \dots, k$ identify the copy of W^i in $\mathcal{M}^{\vec{v}}$ with W^{v_i} index-wise.*

The proof of Theorem 8.2 would involve showing two things. The first is that for every $\mathcal{H}$-colouring ϕ of $\mathcal{M}^{\mathcal{H}}_{\mathcal{G}}(\mathcal{I})$, the mapping $\phi' : V(\mathcal{I}) \to V(\mathcal{G})$ defined by

$$\phi'(v) = g \text{ if } \phi|_{W^v} \in \mathcal{P}_g,$$

is a $\mathcal{G}$-colouring of $\mathcal{I}$. The second is that for every $\mathcal{G}$-colouring of $\mathcal{I}$, there is an $\mathcal{H}$-colouring ϕ of $\mathcal{M}^{\mathcal{H}}_{\mathcal{G}}(\mathcal{I})$ such that for every $v \in V(\mathcal{I})$,

$$\phi|_{W^v} = P_{\phi'(v)}.$$

The details of this just follow the proof of Theorem 3.2.

Corollary 8.4. *If a relational structure $\mathcal{H}$ has a $\mathcal{G}$-partition where $\mathcal{G}$ is an NP-complete relational structure then $\mathcal{H}$ is NP-complete.*

The following proposition, along with [4] and Proposition 4.12, gives us in particular, that if a structure is $\mathrm{Id}(G)$-partitionable for any non-bipartite core graph G, then it is K_3-partitionable.

Proposition 8.5. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be relational structures. If $\mathcal{C}$ is $\mathcal{B}$-partitionable and $\mathcal{B}$ is $\mathcal{A}$-partitionable, then $\mathcal{C}$ is $\mathcal{A}$-partitionable.*

Proof. The proof is broken into three parts for readability.
Notation. In this proof, we will be referring to Definition 8.1 for three different partitions. Thus to simplify notation, we will relabel the elements W^*, $\mathcal{P}_v$ and P_v for each of these partitions. We will denote vertices of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by a, b, and c respectively.

Let $\mathcal{C}$ be $\mathcal{B}$-partitionable. Let S^* denote the corresponding copy of W^*. Let $\{\mathcal{Q}_b \mid b \in V(\mathcal{B})\}$, where Q_b is the representative element of $\mathcal{Q}_b$, denote the corresponding family of disjoint $\mathcal{C}$-patterns of S^*. Let $s = |S^*|$. Thus for any instance $\mathcal{G}$ of $\mathrm{CSP}(\mathcal{B})$ we get the instance $\mathcal{M}^{\mathcal{C}}_{\mathcal{B}}(\mathcal{G})$ of $\mathrm{CSP}(\mathcal{C})$ containing $|V(\mathcal{G})|$ copies of S^*, one copy S^v for each vertex v of $\mathcal{G}$.

Let $\mathcal{B}$ be $\mathcal{A}$-partitionable. Let T^* denote the corresponding copy of W^*. Let $\{\mathcal{X}_a \mid a \in V(\mathcal{A})\}$ (with representative X_a of $\mathcal{X}_a$) be the corresponding family of disjoint $\mathcal{B}$-patterns of T^*. Let $t = |T^*|$. Thus for any k-ary relation R of $\mathcal{A}$ we have the instance $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$ containing k copies $T^1, \dots, T^k$, of T^*.

Definition of the $\mathcal{A}$-partition of $\mathcal{C}$. We will now define the W^* and $\{\mathcal{P}_a \mid a \in V(\mathcal{A})\}$ that are necessary to exhibit that $\mathcal{C}$ is $\mathcal{A}$-partitionable. Let $W^* = S^* \times T^* = \{(s^*, t^*) \mid s^* \in S^*, t^* \in T^*\}$. For any fixed $t^*_0 \in T^*$ the subset

$$W^*(t^*_0) = \{(s^*_i, t^*_0) \mid s^*_i \in S^*)\}$$

is an copy of the set S^*. Thus for any function $\phi : W^* \to V(\mathcal{C})$, and any $t^*_0 \in T^*$, $\phi|_{W^*(t^*_0)}$ is an $\mathcal{C}$-pattern that may be in one of the $\mathcal{Q}_b$. Define $\phi' : T^* \to V(\mathcal{B}) \cup \{0\}$ by

$$\phi'(t^*_i) = b \text{ if } \phi|_{W^*(t^*_i)} \in \mathcal{Q}_b, \text{ and } \phi'(t^*_i) = 0 \text{ otherwise.}$$

Define $\phi'' : T^* \to V(\mathcal{B}) \cup \{0\}$ by

$$\phi''(t^*_i) = b \text{ if } \phi|_{W^*(t^*_i)} = Q_b, \text{ and } \phi''(t^*_i) = 0 \text{ otherwise.}$$

For $a \in V(\mathcal{A})$, let $\mathcal{P}_a$ be the set of functions $P : W^* \to V(\mathcal{C})$ such that $P' \in \mathcal{X}_a$. Let P_a be the function such that $P''_a = X_a$.

For any k-ary relation R of $\mathcal{A}$, $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$ is an instance of $\mathrm{CSP}(\mathcal{B})$. Let the needed instance $\mathcal{M}^{\mathcal{C}}_{\mathcal{A}(R)}$ of $\mathrm{CSP}(\mathcal{C})$ be $\mathcal{M}^{\mathcal{C}}_{\mathcal{B}}(\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)})$ which is constructed from $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$ by Construction 8.3. For $\alpha = 1, \dots, k$, $W^\alpha \subset V(\mathcal{M}^{\mathcal{C}}_{\mathcal{A}(R)})$ is the copy of W^* defined by

$$W^\alpha = \{(s_i, t^\alpha_j) \mid s_i \in S^{t^\alpha_j}, t^\alpha_j \in T^\alpha\}.$$

Verification that this is an $\mathcal{A}$-partition. We must show that $\mathcal{M} = \mathcal{M}^{\mathcal{C}}_{\mathcal{A}(R)} = \mathcal{M}^{\mathcal{C}}_{\mathcal{B}}(\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)})$ satisfies properties (i) and (ii) of Definition 8.1.
Property (i). Let ϕ be an $\mathcal{C}$-colouring of $\mathcal{M}$. For each vertex $v \in \mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$, $\mathcal{M}$ contains a copy S^v of S^*. Let $\phi' : V(\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}) \to V(\mathcal{B})$ be defined by

$$\phi'(v) = b \text{ if } \phi|_{S^v} \in \mathcal{Q}_b.$$

By the proof of Theorem 8.2, (see remarks following Construction 8.3), ϕ' is a $\mathcal{B}$-colouring of $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$. In particular, this means that there is some k-tuple

$(a_1,\dots,a_k) \in R(\mathcal{A})$ such that $\phi'|_{T^i} \in \mathcal{X}_{a_i}$ for $i = 1,\dots,k$. Thus by the definition of $\mathcal{P}_i$, $\phi|_{W^i} \in \mathcal{P}_{a_i}$ for $i = 1,\dots,k$. This is what we needed to show.
Property (ii). Let $\vec{a} = (a_1,\dots a_k) \in R$. Because $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$ is an $\mathcal{A}(R)$-partition of $\mathcal{B}$, there a is $\mathcal{B}$-colouring ϕ' of $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)}$ such that for $i = 1,\dots,k$, $\phi'|_{T^i} = X_{a_i}$.

Define $\phi : \cup_{i=1,\dots k} W^i \to \mathcal{C}$ as follows. For each $t^i_j \in T^i$, let ϕ restricted to the copy $W^i(t^i_j) = \{(s_\alpha, t^i_j) \mid s_\alpha \in S^{t^i_j}\}$ of S^* be the $\mathcal{C}$-pattern $Q_{X_{a_i}(t^i_j)} \in \mathcal{Q}_{X_{a_i}(t^i_j)}$. Thus ϕ restricts to the $\mathcal{C}$-pattern P_{a_i} on W^i. If ϕ can be extended to an $\mathcal{C}$-colouring of $\mathcal{M}$, we are done.

For any k_i-ary relation R_i of $\mathcal{B}$, and any k_i-tuple $\vec{u} \in R_i(\mathcal{M}^{\mathcal{B}}_{\mathcal{A}(R)})$, we have by Theorem 8.2 that for $j = 1,\dots,k_i$, ϕ restricted to the copy of S^j in the copy $\mathcal{M}^{\vec{u}}$ of $\mathcal{M}^{\mathcal{C}}_{\mathcal{B}(R_i)}$ from Construction 8.3 is $Q_{u_i} \in \mathcal{Q}_{u_i}$. Thus by ϕ can be extended to an $\mathcal{C}$-colouring of $\mathcal{M}^{\vec{u}}$.

Thus ϕ is the necessary $\mathcal{C}$-colouring $\phi_{\vec{a}}$.

□

9. Additional Comments

9.1. **Idempotence.** Observe that while $\mathrm{Id}(K_3)$ is block projective, K_3 is not. So to use Proposition 4.4 to show that K_3 is NP-complete we must also use the reduction to idempotence (Proposition 4.2). However, with a slight variation, we could build this reduction right into the fibre construction (Theorem 3.2) thus showing that a core $\mathcal{H}$ is NP-complete if $\mathrm{Id}(\mathcal{H})$ is block projective. (This would be done by integrating the proof of Proposition 4.2 into the fibre construction. Specifically, we could use clones of $\mathcal{H}$ to identify corresponding vertices in the diagonal copies of $\mathcal{H}$ in the edge-gadgets $\mathcal{M}$. All of this could be integrated into the definition of block projective, so that $\mathcal{H}$ would be block projective if and only if $\mathrm{Id}(\mathcal{H})$ is.) This is a semantic difference.

At the same time, K_3 is K_3-partitionable. It would be nice to show that if $\mathrm{Id}(\mathcal{H})$ is block projective then $\mathcal{H}$ is K_3-partitionable. Then Conjecture 4.16 would be that $\mathcal{H}$ is NP-complete if $\mathcal{H}$ is K_3-partitionable, and is otherwise polynomial time solvable. This would be more than a semantic difference.

9.2. **Theorem 4.15 and decidability.** Based on the definitions, it is not clear that the conditions of Theorem 4.15 are decidable. It is shown in [5] that they are. But it also follows from our results. If a structure does meet these conditions, then in particular it is block projective, and so by the proof of Proposition 4.4 it has a K_3-partition defined on $\mathcal{H}^6$. Thus to decide if $\mathcal{H}$ is K_3-partitionable, we only have to check if $\mathcal{H}^6$ is a K_3-partition of $\mathcal{H}$. This is decidable. In fact, it follows from [5], and [43] (which uses a variation of the fibre construction) that deciding if a structure satisfies these conditions is $coNP$-complete.

9.3. **Bounded Degree Dichotomy.** It is known that the directed triangle with one extra edge added in the other direction, is NP-complete. If we develop the fibre construction in terms of this graph, which has four tuples, instead of K_3, which has six (directed) tuples, then we can replace the power of 6 in the bound in Corollary 5.5 with a power of 4.

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