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Algebraic properties and dismantlability of finite posets

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This paper is dedicated to Simon David Larose, whose birth coincided with the completion of the proof of the main result.

Abstract

We show that every finite connected poset which admits certain operations such as Gumm or Jonsson operations, or a near unanimity function is dismantlable. This result is used to prove that a finite poset admits Gumm operations if and only if it admits a near unanimity function. Finite connected posets satisfying these equivalent conditions are characterized by the property that their idempotent subalgebras are dismantlable. As a consequence of these results we obtain that the proslem of determining if a finite poset admits a near unanimity function is decidable.

1. Introduction

Since the classification of maximal clones by Rosenberg [11], finite bounded posets have attracted much attention in clone theory and universal algebra, due to the special nature of their clone of isotone operations. Some straightforward questions about maximal clones, such as when they are finitely generated, remain unanswered for the class determined by finite bounded posets. In this paper we shall be investigating algebraic properties of finite posets in general.

We say that a poset P admits an n-ary operation g, if g is an isotone operation on P. An algebra is order-primal with respect to a poset P if its base set is equal to the base set of P and its clone of term operations is equal to the clone of all isotone operations on P.

Certain Maltsev conditions have been studied for varieties generated by order primal algebras, conditions such as congruence modularity, congruence distributivity,

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and existence of a near unanimity term operation, and some important results have been obtained. By the well-known results of Gumm [4] and Jónsson [5], the variety generated by an order-primal algebra will be congruence modular (distributive) exactly when the corresponding poset admits Gumm (Jónsson) operations. In [1] Davey showed that the existence of Jónsson operations is equivalent to the existence of Gumm operations for directed posets. In the finite case directed posets reduce to bounded ones and the result becomes obvious. It is not so trivial to extend Davey's result to all finite posets. This was done by McKenzie in [8] where he gave some additional characterizations for the existence of Jónsson operations in the clone of isotone operations under some boundedness conditions on the poset.

In general the presence of a near unanimity function in the clone of term operations of an algebra implies the presence of some Jónsson operations, see [10], but the converse fails to be true. It is not hard to come up with finite algebras which have Jónsson operations as term operations but do not have a near unanimity term operation, see [9]. In [1, 8] the question was raised whether the existence of Jónsson operations implies the existence of a near unanimity operation in the clones of isotone operations of finite posets. For finite bounded posets this was answered affirmatively in [14], but it was not clear how to extend the result to arbitrary finite posets.

In the present paper we generalize some results given in [8] and [14] to arbitrary finite posets. We show that for every finite poset the existence of Gumm or Jónsson operations is equivalent to the existence of a near unanimity function in the clone of isotone operations, see Theorem 4.3 and Corollary 4.4. It follows that there exists a finite procedure to determine if a finite poset admits a near unanimity function or not. The crucial observation that leads to the proof of our main result is that the presence of certain operations such as Gumm or Jónsson operations, or a near unanimity operation in the clone of any finite connected poset ensures the dismantlability of the poset, see Lemma 4.1 and Theorem 4.2. In general, one would like to see an 'order theoretical' description which reflects the existence of certain identities in the clone of isotone operations, and the above observation is a step in this direction. The proofs given here are simpler and are more revealing of the structure of finite posets admitting Gumm operations than the ones given in [14]. Nevertheless, at some point we have to refer to a result on zigzags whose proof is the most technical in [14].

2. Finite posets

We now gather the basic results about finite posets and isotone maps we shall need later. We use boldface capital letters to denote a poset throughout the paper. A poset is called *bounded*, if it has a largest and a smallest element. A *subposet* **Q** of a poset **P** is a subset Q of P together with the restriction of the order relation on **P** to Q. An element of a poset **P** is *irreducible* if it possesses either a unique upper cover or a unique lower cover in **P**. A finite poset **P** is *dismantlable* if P is one element or $P = \{x_1, \ldots, x_n\}$ such that for all $i = 1, \ldots, n - 1$, x_i is an irreducible element in the subposet of **P** induced by $\{x_1, \ldots, x_n\}$. A finite poset **P** is *ramified* if |P| > 1 and it contains no irreducible element.

For our purposes, a sequence $\{x_0, ..., x_n\}$ of elements in poset P is a *path of length n from* x_0 to x_n if x_i and x_{i+1} are comparable for all i = 0, ..., n-1. For $a, b \in P$, the *distance from a to b in* P is the least integer *n* such that there exists a path of length *n* from *a* to *b*. A poset P is a *fence* if $P = \{x_0, ..., x_n\}$ and $x_0 > x_1 < x_2$... or $x_0 < x_1 > x_2$... are the only comparabilities in P. Notice that in any poset, if x and y are at distance d then every path of length d from x to y is isomorphic to a fence.

Let P and Q be posets. The poset \mathbb{P}^{Q} consists of all isotone maps from Q to P ordered pointwise, i.e., $f \leq g$ if $f(g) \leq g(g)$ for all $q \in Q$. For $p \in P$ let \bar{p} denote the map in \mathbb{P}^{Q} with constant value p. If P is connected, then clearly the constant maps from Q to P lie in the same connected component of \mathbb{P}^{Q} . We shall denote this component by $C_{Q,P}$. In the following id_v shall denote the identity map on P. A subposet R of P is a *retract* of P if there exists an isotone map $r \in \mathbb{P}^{P}$ such that r(P) = R and $r^{2} = r$. Such a map is called a *retraction* from P to R.

Lemma 2.1 (Stong [12], Duffus and Rival [3]). Let P be a finite poset. Then the following conditions are equivalent:

- (1) P is dismantlable.
- (2) No retract of P is ramified.
- (3) $\mathbf{P}^{\mathbf{Q}}$ is connected for every finite poset \mathbf{Q} .

Lemma 2.2. Let P be a finite poset. Then the following conditions are equivalent:

- (1) P is ramified.
- (2) Every automorphism of \mathbf{P} is alone in its connected component of $\mathbf{P}^{\mathbf{P}}$.

Proof. It is proved in [12] that a poset is ramified if and only if the identity map is alone in its connected component of $\mathbf{P}^{\mathbf{P}}$. Now if (1) holds and σ is an automorphism of \mathbf{P} such that $\sigma \leq \tau$ for some map τ , then composing on each side by σ^{-1} gives that $\mathrm{id}_{\mathbf{P}} \leq \tau \sigma^{-1}$. Thus, $\tau \sigma^{-1}$ is the identity and consequently $\tau = \sigma$. Applying the same reasoning to the dual case shows that σ is comparable only to itself in $\mathbf{P}^{\mathbf{P}}$.

The following simple result can be found in [6], where it is used in the study of the fixed point property and general algebraic properties of finite posets (see also [7]).

Lemma 2.3. Let P and Q be finite posets, P connected and let $f \in P^{Q}$. If there exists a dismantlable subposet X of P which contains the image of f then $f \in C_{Q,P}$.

Proof. By Lemma 2.1, X^Q is connected, so there exists a path from f to some constant map in X^Q . This poset embeds naturally into P^Q and thus we obtain a path from f to a constant map in P^Q .

3. Zigzags

The basic notions and properties concerning zigzags are given in [14, 15]. We review here those we shall need later in proofs.

We say that a poset Q is contained in P if $Q \subseteq P$ and $\leq_Q \subseteq \leq_P Q$. If Q is contained in P we write $Q \subseteq P$. We say that Q is properly contained in P if $Q \subseteq P$ and $Q \neq P$.

Let P and Q be posets. A pair (Q, f) is called a P-colored poset if f is a partially defined map from Q to P. We shall refer to the elements in the domain of f as colored elements. If f can be extended to a fully defined isotone map $f: Q \rightarrow P$ on Q then f and (Q, f) are called P-extendible; otherwise f and (Q, f) are called P-nonextendible. A P-zigzag is a P-nonextendible, P-colored poset (H, f), where H is finite and for every K properly contained in H, the P-colored poset (K, $f|_K$) is P-extendible. Roughly speaking, the P-zigzags are the finite, minimal, nonextendible P-colored posets. When it is clear what P is we omit it in terms such as P-zigzags, P-extendible, etc.

For two P-colored posets (H, f) and (Q, g) we say that (H, f) is contained in (Q, g) and we write (H, f) \subseteq (Q, g) if H \subseteq Q and $f = g|_{H}$. Observe that every finite nonextendible colored poset contains a zigzag.

For $n \ge 3$ an n-ary operation f on a set A is called a *near unanimity function*, *nuf* for short, it it obeys the identities

 $f(x, \ldots, x, y) \approx f(x, \ldots, x, y, x) \approx \ldots \approx f(y, x, \ldots, x) \approx x.$

The following fact, first observed by Tardos in [13] (see also [14, Remark 2.4]), describes via zigzags the finite posets admitting an *n*-ary near unanimity function.

Fact 3.1. Let $n \ge 3$. A finite poset **P** admits an n-ary near unanimity function if and only if in every **P**-zigzag the number of colored elements is at most n - 1.

The next result describes the zigzags of an ordinal sum of posets in terms of the summands when one of these satisfies special conditions.

Theorem 3.2 (Zádori [15]). Let **P** be a finite poset and let **A** be a finite poset such that every **A**-zigzag has at most one noncolored element. Let $\mathbf{P}' = \mathbf{P} \oplus \mathbf{A}$. Then every **P**'-zigzag with at least one noncolored element is of the following form: a **P**-zigzag in which every maximal element is colored, or an **A**-zigzag in which every minimal element is colored, or it can be obtained from an isotone **P**-zigzag (**H**, f), such that above each noncolored maximal element of (**H**, f) we place the colored elements of some **A**-zigzag having a noncolored minimal element. Moreover, every **P**'-colored poset of this form will be a **P**'-zigzag.

4. Main result

Our first task in this section will be to prove that finite connected posets admitting Gumm operations are dismantlable. This will be achieved by showing that an isotone operation on a poset which is 'minimally ramified' must be a projection if it satisfies identities of a certain form. This statement will be made more precise in the following lemma.

Lemma 4.1. Let **P** be a finite, ramified, connected poset whose proper retracts are dismantlable. For $n \ge 1$ let d be an n-ary isotone operation on **P**. Assume that there exist maps $a, b: \{1, ..., n\} \rightarrow \{1, ..., n\}$ and an integer $1 \le i \le n$ such that d satisfies the following identities:

$$d(x_{a(1)},\ldots,x_{a(n)})\approx x_i,\tag{1}$$

$$d(x_{b(1)},\ldots,x_{b(n)})\approx x_i.$$
(2)

Let $A = \{s: a(s) = i\}$ and $B = \{s: b(s) = i\}$. If $A \cap B \neq \emptyset$, then d satisfies

$$d(y_1,\ldots,y_n)\approx x_i,\tag{3}$$

where $y_s = x_i$ for all $s \in A \cap B$.

Proof. Let P be a finite, ramified, connected poset whose proper retracts are dismantlable. Let d be an *n*-ary isotone operation obeying (1) and (2) on P. We show first that d obeys the following identities as well:

$$d(y_1,\ldots,y_n)\approx x,\tag{4}$$

$$d(z_1,\ldots,z_n)\approx w,\tag{5}$$

where $y_s = x$ for all $s \in A$ and $z_t = w$ for all $t \in B$. We will show that d satisfies identity (4), the proof of the other identity is similar. Let l = |A|. By permuting variables, we may assume that $A = \{1, ..., l\}$. Consider the map

$$\phi: \mathbb{P}^{n-1} \to \mathbb{P}^{\mathbb{P}}$$

where for all $(p_{l+1}, \ldots, p_n) \in \mathbf{P}^{n-l}$ and all $x \in \mathbf{P}$

$$\Phi(p_{l+1},\ldots,p_n)(x)=d(x,\ldots,x,p_{l+1},\ldots,p_n)$$

Clearly, ϕ is isotone. Since *d* satisfies (1) we have that $\phi(p, p, ..., p)$ is the identity map for all $p \in P$. Since \mathbf{P}^{n-1} is connected, so must $\phi(\mathbf{P}^{n-1})$, thus by Lemma 2.2 we have that $\phi(\mathbf{P}^{n-1}) = \{id_p\}$. This means that

 $d(x,\ldots,x,p_{l+1},\ldots,p_n)=x$

for all $(p_{l+1}, \ldots, p_n) \in \mathbf{P}^{n-l}$ and all $x \in \mathbf{P}$, i.e., d satisfies (4).

Now by permuting variables, we may assume that $A \cap B = \{1, ..., k\}$ for some $1 \leq k \leq n$. Let $C = C_{P,P}$, and consider the isotone map

$$\Psi: C^{n-k} \rightarrow \mathbb{P}^{\mathbb{P}},$$

where for all $(f_{k+1}, \ldots, f_n) \in C^{n-k}$ and all $x \in \mathbf{P}$

$$\Psi(f_{k+1},\ldots,f_n)(x) = d(x,\ldots,x,f_{k+1}(x),\ldots,f_n(x)).$$

Let **R** be a proper retract of **P** which is maximal with respect to inclusion, i.e., if $R \subseteq S \subseteq P$ and if **S** is a retract of **P** then R = S or R = P. Let *r* be a retraction from **P** to **R**. Since **R** is dismantlable, by Lemma 2.3 *r* is in *C*. Let *q* be an element of *P* not in *R*. Now consider the map $g = \Psi(f_{k+1}, \ldots, f_n)$ where $f_j = r$ if $j \in A \setminus B$ and $f_j = \bar{q}$ otherwise. We show that *g* fixes every element of *R* and also fixes *q*. Indeed, for $x \in R$ we have

$$g(x) = d(x, ..., x, f_{k+1}(x), ..., f_n(x)),$$

where

$$f_j(x) = \begin{cases} x & \text{if } j \in A \setminus B, \\ q & \text{otherwise.} \end{cases}$$

Since d satisfies (1) we must have g(x) = x. A similar argument using (5) shows that g(q) = q. There exists an integer m such that g^m is a retraction to its image $\mathbf{S} = g^n(\mathbf{P})$, and since g fixes every element of R and fixes q we have that $R \cup \{q\} \subseteq S$. By maximality of R this implies that S = P and thus $g^m = id_P$, i.e., g is an automorphism of P. By Lemma 2.2 we conclude that $\Psi(C^{n-k}) = \{g\}$. By (1), we then have that

$$g(p) = \Psi(\bar{p}, \ldots, \bar{p})(p) = d(p, \ldots, p, p, \ldots, p) = p$$

for all $p \in P$ so $g = id_P$. Thus for all $p_{k+1}, \ldots, p_n, x \in P$ we have

$$d(x,\ldots,x,p_{k+1},\ldots,p_n)=\Psi(\overline{p_{k+1}},\ldots,\overline{p_n})(x)=x,$$

i.e., d satisfies (3).

Examples. Suppose d is isotone on a poset P satisfying the hypotheses of $t_{\rm bt}$ last lemma. If d satisfies

 $d(x, x, u, u, v, x, y, y, x) \approx x$

and

 $d(x, x, y, y, y, u, u, u) \approx x,$

then the lemma asserts that d must satisfy

 $d(x, x, y_3, y_4, y_5, y_6, y_7, y_8, y_9) \approx x.$

Notice that we can deduce something similar if d satisfies only one identity: if we have that $d(x, y, y) \approx x$ then we can conclude that d is the projection in the first variable.

Theorem 4.2. Let **P** be a finite connected poset. If **P** admits Gumm operations then **P** is dismantlable.

Proof. A poset P admits Gumm operations if there exist 3-ary isotone operations d_0, \ldots, d_n, p on P that satisfy:

$$d_0(x, y, z) \approx x, \tag{6}$$

$$d_i(x, y, x) \approx x$$
 for all *i*, (7)

$$d_i(x, x, y) \approx d_{i+1}(x, x, y) \text{ for } i \text{ even},$$
(8)

$$d_i(x, y, y) \approx d_{i+1}(x, y, y) \text{ for } i \text{ odd},$$
(9)

$$d_n(x, y, y) \approx p(x, y, y), \tag{10}$$

$$p(x, x, y) \approx y. \tag{11}$$

Suppose for a contradiction that there exists a non-dismantlable connected poset which admits Gumm operations. Let \mathbf{Q} be a poset of minimum cardinality with these properties. It is easy to verify that every retract of this poset will also admit Gumm operations, and thus all proper retracts of \mathbf{Q} are dismantlable. Furthermore, \mathbf{Q} is ramified by Lemma 2.1. Applying Lemma 4.1 to p and equation (11), we see that p must be the projecton in the third variable and from (10) we conclude that $d_n(x, y, y) \approx y$. Let j be the largest index such that d_j is the projection in the first variable. Clearly j < n. If j is even, we conclude from (8) and (7) that

$$d_{i+1}(x, y, x) \approx d_{i+1}(x, x, y) \approx x.$$

Applying Lemma 4.1 to d_{j+1} and these equations we find that d_{j+1} is the projection in the first variable, a contradiction. If j is odd, a similar argument using (9) shows that d_{j+1} must be the projection in the first variable, a contradiction. \Box

Let P be a finite poset and for all $n \ge 1$ let $I^{(n)}(\mathbf{P})$ denote the set of *n*-ary idempotent isotone operations on P (an operation is idempotent if it satisfies $f(x, ..., x) \approx x$). When we view this set as a subposet of \mathbf{P}^{p^n} we shall denote it by $\mathbf{I}^{(n)}(\mathbf{P})$. We call a subset of a finite power of P an *idempotent* P-subalgebra if it is preserved by all idempotent isotone maps on P. An idempotent P-subalgebra $X \subseteq P^k$ shall be denoted by X when viewed as a subposet of \mathbf{P}^k .

We now have all the tools at our disposal to state and prove our main result. We characterize finite connected posets admitting a near unanimity operations in terms of other algebraic properties, the poset structure of idempotent **P**-subalgebra and in terms of **P**-zigzags.

Theorem 4.3. For a finite connected poset **P** the following conditions are equivalent:

- (1) **P** admits a near unanimity function.
- (2) P admits Jónsson operations.
- (3) P admits Gumm operations.
- (4) Every idempotent P-subalgebra is dismantlable.
- (5) The two projections are connected via a path of idempotent operations in $\mathbf{P}^{\mathbf{P}^2}$.
- (6) There exists a finite upper bound for the diameters of P-zigzags.
- (7) The number of P-zigzags is finite.

Proof. (1) \Rightarrow (2): It is well-known and easy to see that Jónsson operations can be built using a near-unanimity function, see for example [8].

(2) \Rightarrow (3): Immediate.

(3) \Rightarrow (4): Let **Q** be an idempotent **P**-subalgebra, say $Q \subseteq P^{l}$. Since **P** admits Gumm operations d_{0}, \ldots, d_{n} , p and these are idempotent, **Q** also admits Gumm operations. Thus by Theorem 4.2 it suffices to show that **Q** is connected. Let a and b be two elements of Q; we show that there is a path in **Q** from a to b. By Theorem 4.2, **P** is dismantlable and by Lemma 2.1, $P^{P^{l}}$ is connected. For $i = 0, \ldots, n$ define maps

$$\Phi_i: \mathbb{P}^{\mathbb{P}^2} \to \mathbb{Q}$$

where

$$\Phi_i(f) = d_i(a, f(a, b), b)$$

for all $f \in \mathbf{P}^{p^2}$. Clearly these maps are isotone, and thus the image of Φ_i is connected for all i = 0, ..., n. In fact, this image is contained in Q: indeed, for any $f, \Phi_i(f)$ is equal to the map $d_i(\pi_1, f, \pi_2)$ evaluated at (a, b), where π_1 and π_2 are the two projections. By (7) this map is idempotent, and since a and b are in Q, $\Phi_i(f)$ must also be in Q. Consequently, the images of the Φ_i are connected subsets of Q. Now a is in the image of Φ_0 by (6). For i even we have by (8) that

$$\Phi_i(\pi_1) = d_i(a, a, b) = d_{i+1}(a, a, b) = \Phi_{i+1}(\pi_1)$$

and for i odd we have by (9) that

$$\Phi_i(\pi_2) = d_i(a, b, b) = d_{i+1}(a, b, b) = \Phi_{i+1}(\pi_2)$$

Thus the images of Φ_i and Φ_{i+1} intersect for all i = 0, ..., n. We conclude from (10) that a and p(a, b, b) are in the same connected component of **Q**.

Let K denote the connected component of Q that contains a and p(a, b, b). Since P^i is connected, there exists a shortest path in P^i from b to some member of K, say of length m. Let $b = c_0, \ldots, c_{2k}$ be a shortest possible path amongst the paths of even length from b to some member of K. Here $c_{2k} \in K$, and clearly $2k \leq m + 1$. By applying the argument of the preceding paragraph to c_{2k} and b we have that $p(c_{2k}, b, b)$ is in K. Now consider the sequence $p(c_{2k-j}, c_j, b), j = 0, \ldots, k$. Because the path $b = c_0, \ldots, c_{2k}$ is of even length, this is a path in P^i of length k from $b = p(c_k, c_k, b)$ to $p(c_{2k}, c_0, b) \in K$. Consequently we have that $m \leq k$. On the other hand, we saw earlier that $2k \leq m + 1$ and thus $m \leq 1$. We conclude that b is comparable to some member of K, i.e., b is in K.

(4) \Rightarrow (5): This is clear since $I^{(2)}(\mathbf{P})$ is an idempotent **P**-subalgebra.

(5) \Rightarrow (6) Let $\mathbf{K} = \mathbf{P}^{\mathbf{p}}$ and let π_1 and π_2 be the two projections in \mathbf{K} .

Suppose that π_1 and π_2 are connected by a path of idempotent operations in K. Let $\pi_1 = g_1, \ldots, g_m = \pi_2$ be such a path of least possible length. The elements of this path form a fence in K. Let this fence be denoted by F.

Suppose for a contradiction that there exists a P-zigzag (H, f) with diameter $n \ge m + 1$. Let h_1 and h_2 be two elements in H which have distance n in H. Since

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 $n \ge m + 1$, it is easy to see that there exists an isotone map $g: \mathbb{H} \to \mathbb{F}$ such that $g(h_1) = \pi_1$ and $s(h_2) = \pi_2$ (see [15] or [3] for details). By deleting h_i in (\mathbb{H}, f) the remaining colored poset is extendible. Let f_i be the extension obtained by deleting h_i for each i = 1, 2.

We define a map $f': H \rightarrow P$ by

$$f'(h) = \begin{cases} f_2(h_1) & \text{if } h = h_1, \\ f_1(h_2) & \text{if } h = h_2, \\ g(h)(f_2(h), f_1(h)) & \text{otherwise.} \end{cases}$$

We claim that f' is isotone. When restricted to $H \setminus \{h_1, h_2\}$, f' is a composition of isotone maps. Thus it remains to verify the claim for comparable pairs $\{h_i, h\}$ where i = 1 or i = 2. Without loss of generality we consider the case where $h_1 < h$. Note that h and h_2 are distinct since $n \ge 2$. By using the facts that g and f_2 are isotone on their domains we have that

$$\begin{aligned} f'(h_1) &= f_2(h_1) \leq f_2(h) = \pi_1(f_2(h), f_1(h)) \\ &= g(h_1)(f_2(h), f_1(h)) \leq g(h)(f_2(h), f_1(h)) = f'(h). \end{aligned}$$

Finally, observe that f' extends f which contradicts the nonextendibility of (H, f).

(6) \Rightarrow (7): Let *m* be a finite upper bound for the diameters of P-zigzags. Let $\mathbf{Q} = 1 \oplus 2 \oplus \mathbf{P} \oplus 2 \oplus 1$. It is easy to describe the $2 \oplus 1$ -zigzags, see [15]. It turns out that they have at most one noncolored element. Similarly, the same claim holds for the $1 \oplus 2$ -zigzags. We apply Theorem 3.2 to posets \mathbf{P} and $2 \oplus 1$ and then its dual to posets $1 \oplus 2$ and $\mathbf{P} \oplus 2 \oplus 1$ to conclude that the diameter of every \mathbf{Q} -zigzag is at most *m*. Since \mathbf{Q} is a finite bounded poset, by (5) \Rightarrow (6) of Theorem 4.1 in [14] the number of \mathbf{Q} -zigzag is finite. Another application of Theorem 3.2 now gives us that the number of \mathbf{P} -zigzags is finite.

(7) \Rightarrow (1): Apply Fact 3.1 \Box

Corollary 4.4. For a finite poset P the following conditions are equivalent:

- (1) **P** admits a near unanimity function.
- (2) P admits Jónsson operations.
- (3) P admits Gumm operations.

Proof. It suffices to prove $(3) \Rightarrow (1)$. Since Gumm operations are idempotent, it is easy to see that they must preserve each connected component of **P**. Thus by Theorem 4.3 each of these components admits an nuf. Notice that **P** lies in the order variety generated by its connected components and the two-element antichain: in fact, it is a retract of a finite product of copies of these posets. Since existence of an nuf is preserved under finite product and retraction (see for example [2]) we conclude that **P** admits an nuf. \Box

Corollary 4.5. The problem of determining whether a finite poset admits a near unanimity function is decidable.

Remarks. (i) There is an interesting interpretation of Theorem 4.3 in purely topological terms. Indeed, it is well known that the category of finite posets with isotone maps is equivalent to that of finite T_0 topological spaces with continuous maps. Under this equivalence conditions (3) and (4) can be restated as follows:

(3') Every idempotent P-subalgebra is a contractible space,

(4') The two binary projections are homotopic relative to the diagonal (in symbols, $\pi_1 \simeq \pi_2$ (rel 4)).

(ii) As we pointed out in the introduction, the equivalence of (2) and (3) in Theorem 4.3 had been shown previously by McKenzie for all finite posets using commutator theory [8]. He had also proved these conditions to be equivalent to (5) in the bounded case. The equivalence of (1)-(7) except (4) was first shown by Zádori for bounded posets [14]. From the proof of (5) \Rightarrow (6) of Theorem 4.1 in [14] we infer that, if a finite connected poset P admits an nuf then it admits one of arity at most k^{kd} , where k is the size of P and d is the distance between the two projections in $I^{(2)}(P)$. Unfortunately, in the general case we do not know any significantly better upper bound for d than k^{k^2} . The existence of an upper bound polynomial in k for d would imply that the probem mentioned in Corollary 4.5 is in NP. What we know is that there is a close relationship between d and diameters of P-zigzags given by the following proposition.

Proposition 4.6. Let **P** be a finite connected poset admitting a near unanimity function. Let d denote the distance between the two projections in $I^{(2)}(\mathbf{P})$ and let u denote the maximum value of the diameters of **P**-zigzags. Then $0 \le d - u \le 1$.

Proof. For $u \le d$ see the proof of $(5) \Rightarrow (6)$ of Theorem 4.3. We show that $d \le u + 1$. Suppose that d > u + 1. Let **F** be a fence of length u + 1. Consider the poset $\mathbf{P}^2 \times \mathbf{F}$. Let a_0 and a_{u+1} be the two endpoints of **F**. We define a partial map f from the poset $\mathbf{P}^2 \times \mathbf{F}$ to **P** by $f(p, q, a_0) = p$, $f(p, q, a_{u+1}) = q$ and f(p, p, a) = p for all $p, q \in P$ and $a \in \mathbb{F}$. Observe that the **P**-colored poset ($\mathbf{P}^2 \times \mathbf{F}$, f) is nonextendible; otherwise we would get a path of length less than d between the two projections in $\lambda^{(2)}(\mathbf{P})$. So ($\mathbf{P}^2 \times \mathbf{F}$, f) contains a zigzag (**H**, f). Notice that (**H**, f) must have elements in both $P^2 \times \{a_0\}$ and $P^2 \times \{a_{u+1}\}$. Hence (**H**, f) is a zigzag with diameter at least u + 1, which contradicts the definition of u.

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