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Order Varieties Generated by Finite Posets

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Abstract. In a 1981 paper, Duffus and Rival define an order variety as a class of posets that is closed under the formation of products and retracts. They also introduce the notion of an irreducible poset. In the present paper we define nonextendible colored posets and certain minimal nonextendible colored posets that we call zigzags. We characterize via nonextendible colored posets the order varieties generated by a set of posets. If the generating set contains only finite posets our characterization is via zigzags. By using these theorems we give a characterization of finite irreducible posets.

As an application we show that two different finite irreducible posets generate two different order varieties. We also show that there is a poset which has two different representations by irreducible posets. We thereby settle two open problems listed in the Duffus and Rival paper.

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Definitions and Notation

We use the same boldface and slanted capital letters to denote a poset and its base set, respectively. If $\mathbf{Q}$ and $\mathbf{P}$ are posets then we write $f\colon \mathbf{Q}\to\mathbf{P}$ if f is a monotone map from $\mathbf{Q}$ to $\mathbf{P}$. A poset $\mathbf{Q}=(Q, \leqslant_{\mathbf{Q}})$ is a *subposet* of $\mathbf{P}=(P, \leqslant_{\mathbf{P}})$ if $Q\subseteq P$ and $\leqslant_{\mathbf{Q}} = \leqslant_{\mathbf{P}}|_Q$. We say that a poset $\mathbf{Q}$ is *contained* in $\mathbf{P}$ if $Q\subseteq P$ and $\leqslant_{\mathbf{Q}} \subseteq \leqslant_{\mathbf{P}}|_Q$. If $\mathbf{Q}$ is contained in $\mathbf{P}$ we write $\mathbf{Q}\subseteq\mathbf{P}$. We say that $\mathbf{Q}$ is *properly contained* in $\mathbf{P}$ if $\mathbf{Q}\subseteq\mathbf{P}$ and $\mathbf{Q}\neq\mathbf{P}$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A pair $(\mathbf{Q},f)$ is called a $\mathbf{P}$-*colored poset* if f is a partially defined map from Q to P. If f can be extended to a fully defined monotone map $f'\colon \mathbf{Q}\to\mathbf{P}$ on Q then f and $(\mathbf{Q},f)$ are called $\mathbf{P}$-*extendible*, otherwise f and $(\mathbf{Q},f)$ are called $\mathbf{P}$-*nonextendible*. A $\mathbf{P}$-*zigzag* is a $\mathbf{P}$-nonextendible, $\mathbf{P}$-colored poset $(\mathbf{H},f)$, where H is finite and for every $\mathbf{K}$, properly contained in $\mathbf{H}$, the $\mathbf{P}$-colored poset $(\mathbf{K},f|_K)$ is $\mathbf{P}$-extendible. Roughly speaking, the $\mathbf{P}$-zigzags are the finite, minimial $\mathbf{P}$-nonextendible $\mathbf{P}$-colored posets. The term zigzag was introduced by Tardos in [6] when he found all $\mathbf{P}$-zigzags for a particular eight-element poset. The notion of a zigzag is related to that of the gap [1], hole [5] and obstruction [4]. For more on these relationships see [7]. For two $\mathbf{P}$-colored posets $(\mathbf{H},f)$ and $(\mathbf{Q},g)$ we write $(\mathbf{H},f)\subseteq(\mathbf{Q},g)$ if $\mathbf{H}\subseteq\mathbf{Q}$ and $f=g|_H$. When it is clear what $\mathbf{P}$ is we omit it in the terms $\mathbf{P}$-zigzags, $\mathbf{P}$-extendible, etc.

We define two important constructions of posets: product and retract. Let I be an index set and let $\mathbf{P}_i$, $i \in I$, be posets. Then the *product* $\Pi_{i \in I} \mathbf{P}_i$ is a poset with the base set $\Pi_{i \in I} P_i$ and the ordering $(a_i)_{i \in I} \leqslant (b_i)_{i \in I}$ iff $a_i \leqslant_{\mathbf{P}_i} b_i$ for every $i \in I$. Let $\mathbf{P}$ and $\mathbf{R}$ be two posets. We say $\mathbf{R}$ is a *retract* of $\mathbf{P}$ if there are monotone functions $r: \mathbf{P} \to \mathbf{R}$ and $e: \mathbf{R} \to \mathbf{P}$ such that $r \circ e$ is equal to the identity function of R. The maps r and e are called *retraction* and *coretraction*, respectively.

Let K be a class of posets. The *order variety* generated by K is the smallest class of posets containing K and closed under the retract and product constructions. In [1] it is shown that the order variety generated by K exists and is equal to $\mathscr{R}\mathscr{P}(K)$, where $\mathscr{R}$ is the operator of taking retracts of posets and $\mathscr{P}$ is the operator of taking products of posets.

The Order Variety of Posets of Finite Type

First we want to introduce an order variety that turns out to be useful when working with finite posets.

We define a poset $\mathbf{P}$ to be of *finite type* if every $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ is extendible whenever every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is extendible. Since every finite nonextendible colored poset contains a zigzag we have the following simple proposition.

PROPOSITION 1. *Let* $\mathbf{P}$ *be a poset of finite type. A* $\mathbf{P}$*-colored poset is* $\mathbf{P}$*-extendible if and only if it does not contain a* $\mathbf{P}$*-zigzag.*

One expects the following result.

PROPOSITION 2. *Every finite poset* $\mathbf{P}$ *is of finite type.*

Proof. Let $\mathbf{P}$ be a finite poset and let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored poset. Suppose that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\mathbf{P}$-extendible. We want to show that f is $\mathbf{P}$-extendible to $\mathbf{H}$. We give a compactness argument using Tikhonov's theorem, which states that a product of compact topological spaces is compact. For every $h \in H$ we define a compact topological space T_h, namely, if f is defined on h then T_h is the one element set $\{f(h)\}$ otherwise T_h is P with the discrete topology. Then $\Pi_{h \in H} T_h$ can be considered to be the set of all functions from H to P which extend f to H. For $h_1 < h_2 \in H$ $a \nleqslant b$, where $a, b \in P$, we define an open set of $\Pi_{h \in H} T_h$ in the product topology consisting of those elements of $\Pi_{h \in H} T_h$ whose h_1-component is a and whose h_2-component is b. Let S denote the set of all open sets obtained in this way. Observe that every nonmonotone $\mathbf{P}$-extension of f is in one of the open sets of S. Let us suppose that f has no monotone extension to $\mathbf{H}$. Then the open sets of S cover $\Pi_{h \in H} T_h$. But since $\Pi_{h \in H} T_h$ is compact there are only finitely many members of S which cover $\Pi_{h \in H} T_h$. Now, we can find a finite set $M \subseteq H$ which contains all the elements of H which occur in the definition of this finite cover. Then we have that every element of $\Pi_{h \in H} T_h$ is nonmonotone on the finite subposet of $\mathbf{H}$ having base set M. But this contradicts the assumption.

PROPOSITION 3. *The posets of finite type form an order variety.*

Proof. First we show that a product $\Pi_{i \in I} \mathbf{P}_i$ of posets $\mathbf{P}_i$, $i \in I$, of finite type is of finite type. Let $(\mathbf{H}, f)$ be a $\Pi_{i \in I} \mathbf{P}_i$-colored poset such that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\Pi_{i \in I} \mathbf{P}_i$-extendible. Looking at the i-th component of f as a coloring on $\mathbf{H}$ the so obtained $\mathbf{P}_i$-colored poset is $\mathbf{P}_i$-extendible since $\mathbf{P}_i$ is of finite type. Now, the i-th component of a $\Pi_{i \in I} \mathbf{P}_i$-extension of f is defined by the above extension.

Second we show that a retract $\mathbf{R}$ of a poset $\mathbf{P}$ of finite type is of finite type. Let $r: \mathbf{P} \to \mathbf{R}$ be an onto retraction and let $e: \mathbf{R} \to \mathbf{P}$ be a corresponding coretraction. Let $(\mathbf{H}, f)$ be an $\mathbf{R}$-colored poset such that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\mathbf{R}$-extendible. The $\mathbf{P}$ colored poset $(\mathbf{H}, e \circ f)$ is $\mathbf{P}$-extendible since $\mathbf{P}$ is of finite type. So there exists a monotone map $g: \mathbf{H} \to \mathbf{P}$ which extends $e \circ f$ to $\mathbf{H}$. But then $r \circ g: \mathbf{H} \to \mathbf{R}$ is an $\mathbf{R}$-extension of f to $\mathbf{H}$ since $r \circ e \circ f = f$.

COROLLARY 4. *The order variety generated by all finite posets is a subvariety of the order variety of all posets of finite type.*

Proof. Apply Proposition 2 and 3.

We do not know if these two varieties are different.

Order Varieties and Nonextendible Colored Posets

Next we shall prove some general theorems concerning order varieties and, in particular, order varieties generated by finite posets.

For a poset $\mathbf{P}$ let $E(\mathbf{P})$, $N(\mathbf{P})$, and $Z(\mathbf{P})$ denote the class of all $\mathbf{P}$-extendible $\mathbf{P}$-colored posets, the class of all nonextendible $\mathbf{P}$-colored posets and the class of all $\mathbf{P}$-zigzags, respectively.

Throughout the following proofs we frequently use the fact that, if $(\mathbf{H}, f) \in Z(\mathbf{P})$ and $g: \mathbf{P} \to \mathbf{Q}$ is a monotone map then $(\mathbf{H}, g \circ f) \in Z(\mathbf{Q}) \cup E(\mathbf{Q})$.

PROPOSITION 5. *Let* $\mathbf{P}$ *and* $\mathbf{R}$ *be two posets. Then there exists a retraction r from* $\mathbf{P}$ *onto* $\mathbf{R}$ *if and only if there is a monotone map e from* $\mathbf{R}$ *to* $\mathbf{P}$ *such that for every* $\mathbf{R}$*-colored poset* $(\mathbf{H}, f)$ *in* $N(\mathbf{R})$ *the* $\mathbf{P}$*-colored poset* $(\mathbf{H}, e \circ f)$ *is in* $N(\mathbf{P})$.

Proof. Let r be a retraction from $\mathbf{P}$ onto $\mathbf{R}$ with a corresponding coretraction e. Let $(\mathbf{H}, f) \in N(\mathbf{R})$. Now, let us suppose $(\mathbf{H}, e \circ f)$ is $\mathbf{P}$-extendible. By applying r to this $\mathbf{P}$-extension of $(\mathbf{H}, e \circ f)$ we get an $\mathbf{R}$-extension of $(\mathbf{H}, r \circ e \circ f) = (\mathbf{H}, f)$ contradicting $(\mathbf{H}, f) \in N(\mathbf{R})$.

Conversely, let us suppose that there is a monotone map e from $\mathbf{R}$ to $\mathbf{P}$ such that for every $\mathbf{R}$-colored poset $(\mathbf{H}, f) \in N(\mathbf{R})$ we have $(\mathbf{H}, e \circ f) \in N(\mathbf{P})$. For every $a \not\leqslant b$, $a, b \in R$, the two element chain colored by a at the bottom and colored by b at the top, is in $N(\mathbf{R})$. Hence the two element chain colored by $e(a)$ at the bottom and by $e(b)$ at the top, is in $N(\mathbf{P})$ and so $e(a) \not\leqslant e(b)$. Thus e preserves the relations $\leqslant$ and $\not\leqslant$. So e is an order embedding of $\mathbf{R}$ into $\mathbf{P}$. Let us look at the $\mathbf{R}$-colored poset $(\mathbf{P}, e^{-1})$. This is $\mathbf{R}$-extendible by a monotone function r, for otherwise, $(\mathbf{P}, e^{-1}) \in N(\mathbf{R})$ and then by the hypothesis, $(\mathbf{P}, e \circ e^{-1}) = (\mathbf{P}, id_{e(R)}) \in N(\mathbf{P})$. This is impossible since id_p extends $id_{e(R)}$ to $\mathbf{P}$. Clearly, $r \circ e = id_R$ showing that r is a retraction of $\mathbf{P}$ onto $\mathbf{R}$.

COROLLARY 6. *Let* $\mathbf{P}$ *be a poset and let* $\mathbf{R}$ *be a poset of finite type. There exists a retraction* r *from* $\mathbf{P}$ *onto* $\mathbf{R}$ *if and only if there is a monotone map* e *from* $\mathbf{R}$ *to* $\mathbf{P}$ *such that* $(\mathbf{H}, e \circ f) \in Z(\mathbf{P})$ *whenever* $(\mathbf{H}, f) \in Z(\mathbf{R})$.

Proof. Use Proposition 1 and Proposition 5.

PROPOSITION 7. *Let* $\mathbf{P}_i$, $i \in I$, *be a set of posets. Then a* $\Pi_{i \in I} \mathbf{P}_i$*-colored poset* $(\mathbf{H}, f) \in N(\Pi_{i \in I} \mathbf{P}_i)$ *if and only if there exist two subsets* A *and* B *of* I, *with* $A \neq \emptyset$ *and* $A \cup B = I$, *such that* $(\mathbf{H}, f_i) \in N(\mathbf{P}_i)$ *for every* $i \in A$, $(\mathbf{H}, f_i) \in E(\mathbf{P}_i)$ *for every* $i \in B$, *where* f_i *is the i-th component of* f.

Proof. The claim is obvious.

COROLLARY 8. *Let* $\mathbf{P}_i$, $i \in I$, *be a set of posets. Then a* $\Pi_{i \in I} \mathbf{P}_i$*-colored poset* $(\mathbf{H}, f) \in Z(\Pi_{i \in I} \mathbf{P}_i)$ *if and only if there exist two subsets* A *and* B *of* I, *with* $A \neq \emptyset$ *and* $A \cup B = I$, *such that* $(\mathbf{H}, f_i) \in Z(\mathbf{P}_i)$ *for every* $i \in A$, $(\mathbf{H}, f_i) \in E(\mathbf{P}_i)$ *for every* $i \in B$, *where* f_i *is the i-th component of* f.

Proof. Use Proposition 7.

PROPOSITION 9. *Let* $\mathbf{P}$ *be a poset and let* K *be a set of posets. Then* $\mathbf{P} \in \mathscr{RP}(K)$ *if and only if for every* $\mathbf{P}$-colored poset $(\mathbf{H}, f) \in N(\mathbf{P})$ *there is a* $\mathbf{Q} \in K$ *and a monotone map* $g: \mathbf{P} \to \mathbf{Q}$ such that $(\mathbf{H}, g \circ f) \in N(\mathbf{Q})$.

Proof. Let $\mathbf{P}$ be a retract of the product of $\Pi_{i \in I} \mathbf{P}_i$, where $\mathbf{P}_i \in K$ for every $i \in I$ and let $(\mathbf{H}, f) \in N(\mathbf{P})$. Then by Proposition 5 there exists a monotone map $e: \mathbf{P} \to \Pi_{i \in I} \mathbf{P}_i$ which sends $(\mathbf{H}, f) \in N(\mathbf{P})$ to $(\mathbf{H}, e \circ f) \in N(\Pi_{i \in I} \mathbf{P}_i)$. Then by Proposition 7 there exists an i such that $(\mathbf{H}, \pi_i \circ e \circ f) \in N(\mathbf{P}_i)$, where π_i is the i-th projection map. So taking $\mathbf{P}_i \in K$ as $\mathbf{Q}$ and $\pi_i \circ e$ as g we get one direction of the claim.

To prove the other direction, by Proposition 5 it suffices to show that there exist a set I, posets $\mathbf{P}_i \in K$, $i \in I$, and a monotone map $e: \mathbf{P} \to \Pi_{i \in I} \mathbf{P}_i$ such that for every $(\mathbf{H}, f) \in N(\mathbf{P})$ we have $(\mathbf{H}, e \circ f) \in N(\Pi_{i \in I} \mathbf{P}_i)$. Since K is a set we may define a set I which contains all pairs $(g, \mathbf{Q})$, where $\mathbf{Q} \in K$ and g is a monotone map from $\mathbf{P}$ to $\mathbf{Q}$. Now we define a map e from $\mathbf{P}$ to $\Pi_{(g, \mathbf{Q}) \in I} \mathbf{Q}$ by $e_{(g, \mathbf{Q})}(a) = g(a)$, $a \in P$. Now, e is trivially monotone. Moreover, if $(\mathbf{H}, f) \in N(\mathbf{P})$ then $(\mathbf{H}, e \circ f) \in N(\Pi_{(g, \mathbf{Q}) \in I} \mathbf{Q})$ since by the hypothesis there exists some $(g, \mathbf{Q}) \in I$ such that $(\mathbf{H}, g \circ f) \in N(\mathbf{Q})$, and we can apply Proposition 7.

We have an unpleasant hypothesis in the previous proposition, namely that K must be a set rather than a class of posets. This could be avoided if we knew that for any poset $\mathbf{P}$, there exists a cardinal $\kappa(\mathbf{P})$ such that any $\mathbf{P}$-nonextendible colored poset

$(\mathbf{H}, f)$ contains a $\mathbf{P}$-nonextendible one of size less than $\kappa(\mathbf{P})$. Alan Mekler has shown, [3], if for every cardinal λ there exists a strongly compact cardinal greater than λ, then $\kappa(\mathbf{P})$ exists. It is known that the existence of a strongly compact cardinal cannot be proven from ZFC, see Theorem 80 in [2], and the assumption used by Mekler is considered to be a very strong one in set theory. Observe, for every poset $\mathbf{P}$ of finite type we can let $\kappa(\mathbf{P}) = \omega$. So we state the following proposition.

PROPOSITION 10. *Let* $\mathbf{P}$ *be a poset of finite type and* K *be a class of posets. Then* $\mathbf{P} \in \mathcal{RP}(K)$ if and only if *for every* $(\mathbf{H}, f) \in Z(\mathbf{P})$ *there is a* $\mathbf{Q} \in K$ *and a monotone map* $g: \mathbf{P} \to \mathbf{Q}$ *such that* $(\mathbf{H}, g \circ f) \in Z(\mathbf{Q})$.

Proof. We get the proof copying the proof of Proposition 9, replacing $N(\ldots)$ by $Z(\ldots)$, Proposition 5 and 7 by Corollary 6 and 8, and changing the definition of I to a set of pairs $(g, \mathbf{Q})$ such that for every $(\mathbf{H}, f) \in Z(\mathbf{P})$ there exists a monotone $g: \mathbf{P} \to \mathbf{Q}$ with $(\mathbf{H}, g \circ f) \in Z(\mathbf{Q})$.

By a similar argument we can derive Theorem 8 in [4]. From Proposition 9 we easily get a well known result of [1].

COROLLARY 11. *Let a finite poset* $\mathbf{P} \in \mathcal{RP}(K)$, *where* K *is a finite set of finite posets. Then* $\mathbf{P}$ *is a retract of a finite product of some members of* K.

Proof. It follows from the only if part of Proposition 9 and from the proof of the if part of Proposition 9 since I will be finite.

Finite Irreducible Posets

In the final section of the paper we study the finite irreducible posets. The following definitions can be found in [1]. A *representation* of a poset $\mathbf{P}$ is a family $(\mathbf{P}_i \mid i \in I)$ of posets such that $\mathbf{P}_i$ is a retract of $\mathbf{P}$ for each $i \in I$, and $\mathbf{P}$ is a retract of $\Pi_{i \in I} \mathbf{P}_i$. A poset $\mathbf{P}$ is *irreducible* if for every representation $(\mathbf{P}_i \mid i \in I)$ of $\mathbf{P}$, $\mathbf{P}$ is a retract of $\mathbf{P}_i$ for some $i \in I$. If $\mathbf{P}$ is not irreducible then it is called *reducible*.

For example, the two element antichain, fences and crowns are known to be irreducible posets, see [1]. The following problem on the unique representation of posets is mentioned as Problem 2 in [1].

PROBLEM 12. *Let us suppose that we have two representations* $(\mathbf{P}_i \mid i \in I)$ *and* $(\mathbf{Q}_j \mid j \in J)$ *of* $\mathbf{P}$ *such that* $\mathbf{P}_i$ *and* $\mathbf{Q}_j$ *are irreducible for every* $i \in I$ *and* $j \in J$. *For every* $i \in I$ *is there a* $j \in J$ *such that* $\mathbf{P}_i$ *is a retract of* $\mathbf{Q}_j$?

We will show that this problem is equivalent to the following.

PROBLEM 13. *Does there exist an irreducible poset* $\mathbf{Q}$ *which is a retract of* $\Pi_{j \in J} \mathbf{Q}_j$, *where for all* $j \in J$, $\mathbf{Q}_j$ *is an irreducible poset and* $\mathbf{Q}$ *is not a retract of* $\mathbf{Q}_j$?

CLAIM 14. *If the answer is no for Problem* 12 *then the answer is yes for Problem* 13 *and conversely.*

Proof. Let us suppose that the answer is no for Problem 12. Then there exists $\mathbf{P}$ which has two representations $(\mathbf{P}_i \mid i \in I)$ and $(\mathbf{Q}_j \mid j \in J)$ and there exists an $i \in I$ such that $\mathbf{P}_i$ is not a retract of $\mathbf{Q}_j$ for any $j \in J$. Now, $\mathbf{P}_i$ is a retract of $\mathbf{P}$ and hence of $\Pi_{j \in J} \mathbf{Q}_j$. So, in Problem 13 taking $\mathbf{Q} = \mathbf{P}_i$ we get the answer yes.

For the converse, let us suppose that we answered Problem 13 affirmativcly, i.e., there exists an irreducible poset $\mathbf{Q}$ which is a retract of $\Pi_{j \in J} \mathbf{Q}_j$, where $\mathbf{Q}_j, j \in J$, are irreducible posets, such that $\mathbf{Q}$ is not a retract of $\mathbf{Q}_j$ for any $j \in J$. Then let $\mathbf{P} = \Pi_{j \in J} \mathbf{Q}_j$. Now, $(\mathbf{Q}_j \mid j \in J)$ is a representation of $\mathbf{P}$ by irreducibles since the j-th projection map on $\mathbf{P}$ is a retraction. Another representation of $\mathbf{P}$ by irreducible posets is given by $\mathbf{Q}$ and $\mathbf{Q}_j$, $j \in J$, since we can define a retraction from $\mathbf{Q} \times \Pi_{j \in J} \mathbf{Q}_j$ onto $\mathbf{P}$ by $f(q, q_1, q_2, \ldots) = (q_1, q_2, \ldots)$, where $q \in Q$ and $q_j \in Q_j$, $j \in J$. Thus, we have the above two representations of $\mathbf{P}$ by irreducibles, and because $\mathbf{Q}$ is not a retract of $\mathbf{Q}_j$ for any $j \in J$, the answer is no for Problem 12.

The following example, see footnote 4, p. 85 in [1], shows that Problem 13 has the answer yes and so by the previous claim Problem 12 has the answer no.

EXAMPLE 15. *Let* $\mathbf{Q}$ *be the two element antichain and let* $\mathbf{Q}_j$ *be a j-element fence* $j \in J = \omega$. *Then* $\mathbf{Q}$ *is a retract of the product* $\Pi_{j \in J} \mathbf{Q}_j$, *but* $\mathbf{Q}$ *is not a retract of any* $\mathbf{Q}_j$, $j \in J$.

Proof. Since $J = \omega$, $\Pi_{j \in J} \mathbf{Q}_j$ is not connected so $\mathbf{Q}$ is a retract of $\Pi_{j \in J} \mathbf{Q}_j$. On the other hand, a retraction preserves the connectedness of posets. So the nonconnected poset $\mathbf{Q}$ cannot be a retract of a connected poset $\mathbf{Q}_j$, $j \in J$.

We note that Problem 13 turns into Problem 5 in [1] if we assume that $\{\mathbf{Q}_j \mid j \in J\}$ is a finite set of finite posets. That problem remains unsolved.

Now we characterize finite irreducible posets in terms of their zigzags. We begin with a consequence of Proposition 10.

PROPOSITION 16. *A finite poset* $\mathbf{P}$ *is irreducible if and only if there exists a* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *such that for every monotone map* $g : \mathbf{P} \to \mathbf{P}$ *either the range of* g *is not a subset of a proper retract of* $\mathbf{P}$ *or* $(\mathbf{H}, g \circ f)$ *is* $\mathbf{P}$*-extendible.*

Proof. Let us suppose that $\mathbf{P}$ is irreducible and for every $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ there is a monotone map $g : \mathbf{P} \to \mathbf{P}$ such that the range of g is a subset of a proper retract $\mathbf{R}$ of $\mathbf{P}$ and $(\mathbf{H}, g \circ f)$ is not $\mathbf{P}$-extendible. Since $\mathbf{R}$ is a subposet of $\mathbf{P}$, $(\mathbf{H}, g \circ f)$ is also not $\mathbf{R}$-extendible. Hence $(\mathbf{H}, g \circ f)$ is an $\mathbf{R}$-zigzag. Then by Proposition 10, $\mathbf{P}$ is a retract of a product of its proper retracts, which contradicts the fact that $\mathbf{P}$ is irreducible.

Now, let $\mathbf{P}$ be reducible. Then there exists a representation of $\mathbf{P}$ by $(\mathbf{P}_i \mid i \in I)$ where each $\mathbf{P}_i$ is a retract of $\mathbf{P}$ with $|P_i| < |P|$. We can assume that each $\mathbf{P}_i$, $i \in I$, is the image of a monotone, idempotent map $r_i : \mathbf{P} \to \mathbf{P}$. Then by Proposition 10, for every **P**-zigzag $(\mathbf{H},f)$ there is an $i \in I$ and a monotone $g:\mathbf{P}\to\mathbf{P}_i$ such that $(\mathbf{H}, g\circ f)$ is a $\mathbf{P}_i$-zigzag. Observe that g maps **P** to **P** and $(\mathbf{H}, g\circ f)$ is not **P**-extendible otherwise $(\mathbf{H}, r_i \circ g\circ f) = (\mathbf{H}, g\circ f)$ would be $\mathbf{P}_i$-extendible. So the range of g is in a proper retract of **P** and $(\mathbf{H}, g\circ f)$ is not **P**-extendible, which proves the claim.

For a **P**-zigzag $(\mathbf{H},f)$ and a monotone map $g:\mathbf{P}\to\mathbf{P}$, if the **P**-colored poset $(\mathbf{H}, g\circ f)$ is **P**-extendible we say that g *collapses* $(\mathbf{H},f)$.

We can strengthen the result of Proposition 16.

PROPOSITION 17. *Let* **P** *be a finite poset. Then* **P** *is irreducible if and only if there exists a* **P**-*zigzag* $(\mathbf{H},f)$ *such that every monotone, non-onto map* $g:\mathbf{P}\to\mathbf{P}$ *collapses* $(\mathbf{H},f)$.

Proof. Let **P** be a finite, irreducible poset. For every finite poset **H** let $S_{\mathbf{H}}$ be the set of all the **P**-zigzags $(\mathbf{H},f)$ such that for every monotone map $g:\mathbf{P}\to\mathbf{P}$ the range of g is not a subset of a proper retract of **P** or g collapses $(\mathbf{H},f)$. By Proposition 16 we can select a poset **H** for which $S_{\mathbf{H}}$ is not empty. We define a relation $<$ on $S_{\mathbf{H}}$ by $(\mathbf{H},f_1) < (\mathbf{H},f_2)$ if and only if there is a monotone, non-onto map $g:\mathbf{P}\to\mathbf{P}$ such that $f_1 = g\circ f_2$. Clearly, $<$ is transitive. We claim that $<$ is irreflexive. Let us suppose instead that $(\mathbf{H},f) < (\mathbf{H},f)$ in $S_{\mathbf{H}}$. Then there exists a monotone, non-onto $g:\mathbf{P}\to\mathbf{P}$ such that $(\mathbf{H},f) = (\mathbf{H}, g\circ f)$. This equality implies $(\mathbf{H},f) = (\mathbf{H}, g^n\circ f)$ for any finite n. Let us select an n for which g^n is an idempotent map. Since g is not onto and g^n is idempotent, the range of g^n is in a proper retract of **P**. Because $(\mathbf{H},f) = (\mathbf{H}, g^n\circ f)$ the map g^n does not collapse $(\mathbf{H},f)$. This contradicts $(\mathbf{H},f) \in S_{\mathbf{H}}$.

Let $(\mathbf{H},f)$ be a minimal element of $(S_{\mathbf{H}}, <)$. Let $g:\mathbf{P}\to\mathbf{P}$ be an arbitrary monotone, non-onto map. We show that g collapses $(\mathbf{H},f)$. Let us suppose this is not true. Then we claim that the **P**-zigzag $(\mathbf{H}, g\circ f) \in S_{\mathbf{H}}$. Let $g':\mathbf{P}\to\mathbf{P}$ be a monotone map. Since $(\mathbf{H},f) \in S_{\mathbf{H}}$ the range of $g'\circ g$ is not in a proper retract of **P** or $g'\circ g$ collapses $(\mathbf{H},f)$. Hence either the range of g' is not in a proper retract of **P** or g' collapses $(\mathbf{H}, g\circ f)$. So $(\mathbf{H}, g\circ f) \in S_{\mathbf{H}}$, which contradicts the minimality of $(\mathbf{H},f)$ in $(S_{\mathbf{H}}, <)$. The other direction of the proof obviously follows from Proposition 16.

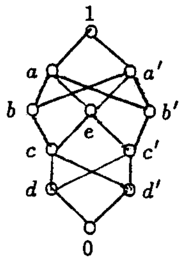
Proposition 17 is a useful tool of proving irreducibility for particular posets.

EXAMPLE 18. *The poset* **P** *in the figure below is irreducible.*

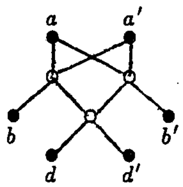
Proof. It is easily seen that the **P**-colored poset $(\mathbf{H},f)$ in the figure below is a **P**-zigzag. We show that $(\mathbf{H},f)$ is collapsed by every monotone, non-onto map on **P**. Let us suppose this is not true. Then there exists a monotone non-onto map $g:\mathbf{P}\to\mathbf{P}$ such that $(\mathbf{H}, g\circ f)$ is a **P**-zigzag. First, observe that $g(a) \parallel g(a')$, $g(b) \parallel g(b')$ and $g(d) \parallel g(d')$, otherwise $(\mathbf{H}, g\circ f)$ would be **P**-extendible. Now, $g(a) < a', a$ is impossible. Otherwise $g(a') < a', a$ as well and we cannot find the six element subposet determined by $g(a)$, $g(a')$, $g(b)$, $g(b')$, $g(d)$ and $g(d')$ in **P**. Thus $g(\{a, a'\}) = \{a, a'\}$. Dually, $g(\{d, d'\}) = \{d, d'\}$. Hence $g(\{b, b'\}) = \{b, b'\}$, otherwise $(\mathbf{H}, g \circ f)$ would be extendible. Let r be an idempotent power of g. Then r is monotone and fixes a, a', b, b', d and d'. These properties of r imply that r is the identity map on **P**. But this contradicts the fact that g is non-onto. So by Proposition 17, **P** is irreducible.

By the same argument one can prove that the posets of the form $1 + 2 + \mathbf{F} + 2 + 1$, where **F** is a fence with at least four elements and $+$ denotes the ordinal sum, are irreducible.

P=



$(\mathbf{H}, f)$=



With the help of Proposition 17 we can answer Problem 4 in [1].

THEOREM 19. *Let* **P** *and* **Q** *be two nonisomorphic, finite, irreducible posets. Then the order varieties generated by them are different.*

Proof. Let us suppose the claim is not true. By Proposition 17 there exists a **P**-zigzag $(\mathbf{H}, f)$ such that every monotone, non-onto map on **P** collapses $(\mathbf{H}, f)$. Since $\mathbf{P} \in \mathscr{RP}(\{\mathbf{Q}\})$, by Proposition 10 there exists a monotone map $g: \mathbf{P} \to \mathbf{Q}$ such that $(\mathbf{H}, g \circ f)$ is **Q**-zigzag. Since $\mathbf{Q} \in \mathscr{RP}(\{\mathbf{P}\})$ we can apply Proposition 10 again. So there exists a monotone map $h: \mathbf{Q} \to \mathbf{P}$ such that $(\mathbf{H}, h \circ g \circ f)$ is **P**-zigzag. But then the map $h \circ g: \mathbf{P} \to \mathbf{P}$, which does not collapse $(\mathbf{H}, f)$, has to be onto. So h is an onto map from **Q** to **P**. Similarly we can get a monotone, onto map from **P** to **Q**. Hence, by finiteness we get that **P** is isomorphic to **Q**, which is a contradiction.

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