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THESIS

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To the memory of my father.

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SUMMARY

We define nonextendible colored posets and zigzags of a poset. These notions are related to the earlier notions of gaps, holes, obstructions and zigzags considered by Duffus, Nevermann, Rival, Tardos and Wille. We establish some properties of zigzags. By using these properties it turns out, for example, that if we cancel a convex subset in a finite lattice we get a poset such that its only zigzags are gaps. This makes obvious a result of Rónyai that these posets admit near unanimity functions. We describe all the zigzags of fences and we present other examples with a full description of their zigzags.

The thesis contains a series of results related to order varieties. It turns out there is a strong connection between zigzags and the order varieties generated by finite posets. Zigzags yield information on finite, irreducible posets as well. As an application of the results on finite irreducible posets, we show that two different finite irreducible posets generate two different order varieties. We also show that there is a poset which has two different representations by irreducible posets. We thereby settle two open problems listed in a 1981 paper of Duffus and Rival.

We describe the finite bounded irreducible posets admitting a near unanimity function with arity at most six. Every finite bounded poset admitting a near unanimity function with arity at most six is obtained as a retract of a finite product of these irreducible posets.

The main result of the thesis is the proof of the famous conjecture, which states that any finite bounded poset which admits Jónsson operations, also admits a near unanimity function. The proof relies on an observation of Tardos, a result of McKenzie and the properties of zigzags studied in the thesis. We also provide an infinite poset that shows that we cannot drop the finiteness in the preceding conjecture.

Another famous question is when the clone of all monotone operations of a finite bounded poset is finitely generated. A finite bounded poset with a monotone clone that is not finitely generated was found by Tardos. In the thesis we present an infinite class of finite bounded posets such that their clones of monotone operations are not finitely generated.

1. Introduction

A *clone* on a set A is a set of finitary operations on A that contains the projection operations and is closed under composition of functions. A *monotone clone* consists of all monotone operations on a partially ordered set. Monotone clones have received a great deal of attention. A partially ordered set, briefly a poset, is called *bounded* if it has a largest and a smallest element. Martynjuk proved in [13] that the monotone clone of a finite bounded poset is maximal in the lattice of all clones on the underlying set. Later, in [24] Rosenberg showed that there are six classes of maximal clones on an arbitrary finite set. One of them is the class of monotone clones of finite bounded posets. The clones of the other five types were shown to be finitely generated, see [12]. The problem remained: Is the monotone clone of a finite bounded poset finitely generated? The answer is yes if the poset is lattice ordered, or if it has at most seven elements, see [12], or if it is obtained from a finite lattice ordered set by cancelling a convex subset of it, see [4]. But in 1986 Tardos answered the problem negatively in [26] by showing that the monotone clone of the eight element poset $\mathbf{T}$ shown in Figure 1 is not finitely generated.

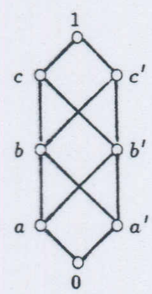


Figure 1. Poset $\mathbf{T}$

Each monotone clone of a finite bounded poset that is known to be finitely generated contains a special, n-ary operation, called a near unanimity function. For $n \geq 3$ an n-ary function f is called a *near unanimity function*, briefly a *nuf*, if it obeys the identity $f(x, \ldots, x, \underset{i}{\underbrace{y}}, x, \ldots, x) = x$ for every $1 \leq i \leq n$. If $n = 3$, then f is called a *majority* function. So the next question that occurs for finite bounded posets is whether a monotone clone is finitely generated if and only if it contains a near unanimity function. For any algebra that has a near unanimity function among its term operations, the clone of term operations is finitely generated, which proves one direction of the claim, see [15] or [25]. The hard part, to prove or disprove the other direction, is mentioned as an open problem in [2], [3] and [14].

In [26] Tardos uses certain special objects, zigzags, to prove that the clone of **T** is not finitely generated. In the same paper there is a remark that characterizes the finite posets having a monotone near unanimity function, in terms of their zigzags. Objects similar to zigzags called gaps [9] and [20], holes [19], and obstructions [18], have been studied from an order theoretical point of view. In [19] and [22] there is a characterization of finite posets having a monotone ternary near unanimity function. The proof establishes a connection, similar to Tardos's remark, between holes and ternary near unanimity functions. The ideas in the papers cited in this paragraph encouraged me to study the zigzags of arbitrary posets in connection with the structure theory of ordered sets as presented by Duffus and Rival in [9].

A variety of algebras is called *congruence modular* if the congruence lattice of every algebra in the variety satisfies the modular law. A variety of algebras is called *congruence distributive* if the congruence lattice of every algebra in the variety is distributive. It is well known, see [17], that if an algebra has a near unanimity term operation, then the variety generated by the algebra is congruence distributive and so it is congruence modular. We call an algebra a *monotone algebra* of a poset **P**, if the set of term operations of the algebra coincides with the set of all monotone operations of **P**. Davey showed in [2] that a monotone algebra of a bounded poset generates a congruence modular variety if and only if it generates a congruence distributive variety. In [14] McKenzie gave a useful characterization of finite bounded posets for which the corresponding monotone algebras generate congruence distributive varieties. It was conjectured in [2], [3] and [14], if a monotone algebra of a finite bounded poset generates a congruence distributive variety, then the algebra has a near unanimity term operation. Since the congruence distributivity of the variety generated by an algebra is equivalent to the algebra having some special term operations, called Jónsson operations, satisfying certain identities, we can rephrase the preceding conjecture as follows. A finite bounded poset **P** has monotone Jónsson operations if and only if there exists a monotone near unanimity operation on **P**. Zigzags turn out to be a powerful tool for settling this conjecture.

The remaining part of this chapter is a summary of the results contained in the thesis.

Chapter 2. In this chapter I give the basic definitions and show the connection between gaps, holes, obstructions and zigzags. Posets with the strong selection property and posets with near unanimity functions are characterized in terms of zigzags.

Chapter 3. I prove some claims about the shape of zigzags and I give an algorithm that constructs all zigzags of a poset. By using the characterization of finite posets with the strong selection property given in Chapter 2, I show if a convex subset of a finite lattice ordered set is deleted, then the resulting poset has the strong selection property.

Chapter 4. In this chapter I describe the zigzags of the antichains and fences. I prove a theorem by which one can determine all zigzags of certain other posets constructed from antichains and fences. Among these posets we find Tardos's eight element poset **T** and the class of locked fences defined in [14]. I give an easy proof of the fact mentioned in [14] that the monotone clone of any locked fence contains a near unanimity function.

Chapter 5. I characterize the order varieties generated by finite posets in terms of the zigzags of their generating elements. The theorems here are stated in a more general setting and apply not only to finite posets.

Chapter 6. By using the results of Chapter 5, I give a characterization of finite irreducible posets via zigzags. I answer two open questions from [9] asked by Duffus and Rival: I show that two different finite irreducible posets generate two different order varieties and that there is a poset that is not uniquely factorable by irreducible posets.

Chapter 7. With the help of the characterization of finite irreducible posets given in Chapter 6, I establish an interesting property of finite irreducible posets having a monotone near unanimity function. I give a complete list of all finite bounded irreducible posets with n-ary near unanimity functions, $n \leq 6$. It turns out that these posets have the strong selection property.

Chapter 8. In this chapter I present a proof of the above mentioned conjecture that states that for every finite bounded poset $\mathbf{P}$ there exist monotone Jónsson operations if and only if there exists a monotone near unanimity operation on $\mathbf{P}$. I present an example of an infinite bounded poset that has monotone Jónsson terms but has no monotone near unanimity function. So in the infinite case the conjecture does not hold.

Chapter 9. Finally, I generalize Tardos's theorem in [26] by proving that the monotone clone of every finite poset $\mathbf{P} = 1 + \sum_{i=1}^{n} \mathbf{A}_i + 1$, where $n \geq 3$ and each $\mathbf{A}_i$ is an antichain with at least two elements, is not finitely generated.

2. Zigzags and related concepts

The main concept of this thesis is the zigzag. To define it we need to clarify some basic concepts involving partially ordered sets. After we define the notion of a zigzag we show the correspondence between zigzags and other concepts such as gaps, obstructions and holes. In Proposition 2.3 we characterize via zigzags the finite posets with the strong selection property. In 2.4 we give a proof of Tardos's remark in [26] that describes, via zigzags, the finite posets with monotone near unanimity operations.

A *partially ordered set*, briefly, a *poset* $\mathbf{P}$ is a nonempty set P with a reflexive, transitive, antisymmetric relation $\leq_{\mathbf{P}}$ on it, i.e., $\mathbf{P} = (P, \leq_{\mathbf{P}})$. A poset with a largest and a smallest element is called *bounded.* For an arbitrary poset $\mathbf{P}$ we define $<_{\mathbf{P}} = \leq_{\mathbf{P}} \setminus \{(p,p) : p \in P\}$. We use boldface capital letters to denote a poset throughout this paper and when it is possible we leave off the subscript from the relational symbol. In a poset $\mathbf{P}$, $b \in P$ *covers* $a \in P$, i.e., $a \prec b$, if $a < b$ and there is no $c \in P$ such that $a < c < b$.

Let $\mathbf{P}$ be a poset and let T be a subset of $P \cup \prec_{\mathbf{P}}$ with $P \not\subseteq T$. We denote the poset $(P \setminus T, (\leq_{\mathbf{P}}|_{P\setminus T}) \setminus T)$ by $\mathbf{P} \setminus T$ and we say that T is *cancelled* from $\mathbf{P}$. For two posets $\mathbf{P}$ and $\mathbf{Q}$ with $P \cap Q = \emptyset$ let $\mathbf{P}+\mathbf{Q}$ denote the poset $(P \cup Q, \leq_{\mathbf{P}} \cup \leq_{\mathbf{Q}} \cup \{(p,q) : p \in P, q \in Q\})$. Let I be an index set and let $\mathbf{P}_i$, $i \in I$, be posets. Then the *product* $\prod_{i \in I} \mathbf{P}_i$ is a poset with the base set $\prod_{i \in I} P_i$ and the ordering $(a_i)_{i \in I} \leq (b_i)_{i \in I}$ if and only if $a_i \leq_{\mathbf{P}_i} b_i$ for every $i \in I$. If $\mathbf{P}_i = \mathbf{P}$ for every $i \in I$ then the product is called a *power* of $\mathbf{P}$ and is denoted $\mathbf{P}^I$.

A poset $\mathbf{Q}$ is a *subposet* of $\mathbf{P}$, i.e., $\mathbf{Q} \leq \mathbf{P}$, if $Q \subseteq P$ and $\leq_{\mathbf{Q}} = \leq_{\mathbf{P}}|_Q$. We say that a poset $\mathbf{Q}$ is *contained* in $\mathbf{P}$ if $Q \subseteq P$ and $\leq_{\mathbf{Q}} \subseteq \leq_{\mathbf{P}}|_Q$. If $\mathbf{Q}$ is contained in $\mathbf{P}$ we write $\mathbf{Q} \subseteq \mathbf{P}$. We say that $\mathbf{Q}$ is *properly contained* in $\mathbf{P}$ if $\mathbf{Q} \subseteq \mathbf{P}$ and $\mathbf{Q} \neq \mathbf{P}$. We note that each poset $\mathbf{P}$ contains any antichain defined on any nonempty subset of P.

An *up set* of $\mathbf{P}$ is a subset S of P for which $s \in S$ and $s \leq p \in P$ imply $p \in S$. A *down set* of $\mathbf{P}$ is defined dually. Let $S \subseteq P$. Then S^* denotes the set of all elements of P which are greater than or equal to every element of S in $\mathbf{P}$. S_* is defined dually. A set $S \subseteq P$ is called a *convex set* of $\mathbf{P}$ if $S = (S^*)_* \cap (S_*)^*$.

A map $f : Q \to P$ is called *monotone* with respect to $\mathbf{Q}$ and $\mathbf{P}$ if for every $a \leq_{\mathbf{Q}} b$ we have $f(a) \leq_{\mathbf{P}} f(b)$. For such a map we use the abbreviation that $f : \mathbf{Q} \to \mathbf{P}$ is monotone. We say an n-ary operation f on P *preserves* $\mathbf{P}$ or $\mathbf{P}$ *admits* f if and only if $f : \mathbf{P}^n \to \mathbf{P}$ is monotone.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A pair $(\mathbf{Q}, f)$ is called a $\mathbf{P}$-*colored poset* if f is a partially defined map from Q to P. If f can be extended to a fully defined monotone map $f' : \mathbf{Q} \to \mathbf{P}$ on Q then f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$-*extendible*, otherwise f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$-*nonextendible.* A $\mathbf{P}$-*zigzag* is a $\mathbf{P}$-nonextendible, $\mathbf{P}$-colored poset $(\mathbf{H}, f)$, where H is finite and for every $\mathbf{K}$, properly contained in $\mathbf{H}$, the $\mathbf{P}$-colored poset $(\mathbf{K}, f\mid_K)$ is $\mathbf{P}$-extendible. Roughly speaking, the $\mathbf{P}$-zigzags are the finite, minimal, nonextendible $\mathbf{P}$-colored posets. When it is clear what $\mathbf{P}$ is we leave it off from the terms like $\mathbf{P}$-zigzags, $\mathbf{P}$-extendible, etc.

For two $\mathbf{P}$-colored posets $(\mathbf{H}, f)$ and $(\mathbf{Q}, g)$ we say that $(\mathbf{H}, f)$ is *contained* in $(\mathbf{Q}, g)$ and we write $(\mathbf{H}, f) \subseteq (\mathbf{Q}, g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f = g\mid_H$. Observe that every finite nonextendible colored poset contains a zigzag. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored poset and let T be a subset of $H \cup \prec_{\mathbf{H}}$ with $H \not\subseteq T$. We denote the $\mathbf{P}$-colored poset $(\mathbf{H} \setminus T, f\mid_{H \setminus T})$ by $(\mathbf{H}, f) \setminus T$ and we say that T is *cancelled* from $(\mathbf{H}, f)$.

For a $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ we define $C(\mathbf{H}, f) = \{h \in H : \ f(h) \text{ exists}\}$ and $N(\mathbf{H}, f) = H \setminus C(\mathbf{H}, f)$. We call the elements of $C(\mathbf{H}, f)$ *colored elements* and the elements of $N(\mathbf{H}, f)$ *noncolored elements.* If $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$ are nonempty we define the posets $\mathbf{C}(\mathbf{H}, f)$ and $\mathbf{N}(\mathbf{H}, f)$ by the restriction of $\leq_{\mathbf{H}}$ to $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$, respectively.

Each poset $\mathbf{P}$ is associated with two undirected graphs on P. One is called the *comparability graph* of $\mathbf{P}$ that has an edge between a and b if and only if $a <_{\mathbf{P}} b$. The other is the *covering graph* of $\mathbf{P}$ that has an edge between a and b if and only if $a \prec_{\mathbf{P}} b$. Often, as an example, we shall draw a picture of a $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ for some particular $\mathbf{P}$. A picture like this consists of the covering graph of $\mathbf{H}$ and an element of $\mathbf{H}$ is drawn as a small shaded circle or a small empty circle according to whether f is defined or not defined on the given point. Every shaded point is labelled by the value of f. Sometimes unshaded points may have labels which always mean the names of the elements of $\mathbf{H}$.

EXAMPLE 2.1. *Let* $\mathbf{P}$ *be the poset shown in Figure 2. Then the* $\mathbf{P}$*-colored poset* $(\mathbf{H}, f)$ *shown in Figure 2 is a* $\mathbf{P}$*-zigzag.*

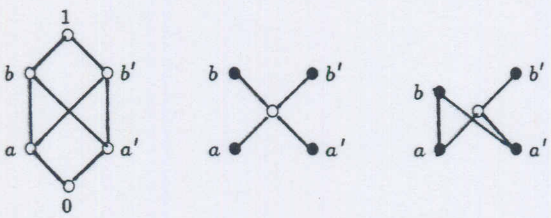


Figure 2. Poset $\mathbf{P}$, a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$, and $(\mathbf{H}, f) \setminus \{(h_1, h_2)\}$

PROOF: The $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ clearly is $\mathbf{P}$-nonextendible. Let h_1 be the noncolored element and let h_2 be the element colored by b in $(\mathbf{H}, f)$. Then $(\mathbf{H}, f) \setminus \{(h_1, h_2)\}$, which is a maximal $\mathbf{P}$-colored poset properly contained in $(\mathbf{H}, f)$, is extendible by coloring h_1 by b'. The other three maximal $\mathbf{P}$-colored posets properly contained in $(\mathbf{H}, f)$ are $\mathbf{P}$-extendible by symmetric arguments. Thus, by definition, $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag. □

A $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ is called *monotone* if f is a monotone map on its domain, otherwise $(\mathbf{H}, f)$ is *nonmonotone*. A monotone $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ is called an *extension* of the $\mathbf{P}$-colored poset $(\mathbf{H}, g)$ if f is an extension of g. Observe that for any poset $\mathbf{P}$ the $\mathbf{P}$-colored two element chain in which the top is colored by a and the bottom is colored by b, where $b \not\leq a \in P$, is a nonmonotone $\mathbf{P}$-zigzag and every nonmonotone $\mathbf{P}$-zigzag is of this form. So for every monotone zigzag $(\mathbf{H}, f)$, f is monotone on its domain and there is at least one element of H, where f is not defined.

A poset determined by the ordering of a complete lattice is called a *complete lattice ordered set.*

EXAMPLE 2.2. *There are no monotone* $\mathbf{L}$*-zigzags for a complete lattice ordered set* $\mathbf{L}$.

The reason is that every monotone $\mathbf{L}$-colored poset $(\mathbf{Q}, f)$ is extendible by $f' : a \mapsto$

$\wedge\{f(b) : b \in \overset{Q}{\cancel{L}}, a \leq b\}$. In particular, no finite lattice ordered set possesses monotone zigzags.

The notion of a zigzag is related to that of the gap [9], hole [19], obstruction [18] and the zigzags defined in [26]. So many results in the literature involving these concepts can be described in terms of zigzags. We concentrate on the study of finite posets hence we examine the related concepts for the case of finite posets.

A pair (D, U), where $D, U \subseteq P$ is called a *gap* of the poset $\mathbf{P}$ if $D \subseteq U_*$, equivalently $U \subseteq D^*$, and $D^* \cap U_* = \emptyset$.

Let $\mathbf{P}$ be a finite poset with a gap (D, U). Then we can certainly find elements $a_1, \ldots, a_n \in D$ and $b_1, \ldots, b_m \in U$ such that the colored poset in Figure 3 is a $\mathbf{P}$-zigzag. We note that this $\mathbf{P}$-zigzag need not be unique. Nevertheless, every $\mathbf{P}$-zigzag of the above form corresponds to the gap $(\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\})$ in $\mathbf{P}$.

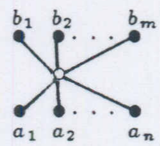


Figure 3. A $\mathbf{P}$-colored poset

Let us fix a finite poset $\mathbf{P}$. Let $B_{\mathbf{P}} = \{(D, U) : D$ is a down set of $\mathbf{P}$, U is an up set of $\mathbf{P}$ and $D^* \cap U_* \neq \emptyset\}$ and let $\mathbf{B_P}$ be the poset on $B_{\mathbf{P}}$ given by the ordering $(D_1, U_1) \leq (D_2, U_2)$ if and only if $D_1 \subseteq D_2$ and $U_2 \subseteq U_1$. We say $\mathbf{P}$ has the *strong selection property* if and only if there exists a monotone map $g : \mathbf{B_P} \to \mathbf{P}$ such that for every $(D, U) \in B_{\mathbf{P}}$, $g((D, U)) \in D^* \cap U_*$.

In the following proposition we describe via zigzags the finite posets with the strong selection property. Their monotone zigzags turn out to be the ones with exactly one noncolored element. It is not hard to get a description of the zigzags with one noncolored element, see Proposition 3.12.

PROPOSITION 2.3. *A finite poset* $\mathbf{P}$ *has the strong selection property if and only if every* $\mathbf{P}$*-zigzag has at most one noncolored element.*

PROOF: Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag, where $\mathbf{P}$ has the strong selection property. With every $h \in H$ we can associate a pair (D_h, U_h) defining $D_h = \{a \in P :$ there exists $h' \in C(\mathbf{H}, f)$, $h' \leq h$ and $a \leq f(h')\}$ an d $U_h = \{a \in P :$ such that there exists $h' \in C(\mathbf{H}, f)$, $h \leq h'$ and $f(h') \leq a\}$. We claim that $(\mathbf{H}, f)$ cannot contain two or more noncolored elements. For otherwise, if $h \in C(\mathbf{H}, f)$ then $f(h) \in {D_h}^* \cap U_{h*}$ and if $h \in N(\mathbf{H}, f)$ then by cancelling a noncolored point $h_0 \neq h$ from $(\mathbf{H}, f)$ the resulting colored poset will be extendible, so ${D_h}^* \cap U_{h*} \neq \emptyset$. Thus, by $f'(h) = g(D_h, U_h)$, where g is obtained from the definition of the strong selection property, we would get a monotone extension of f to $\mathbf{H}$.

Now, let us suppose that $\mathbf{P}$ is a finite poset having $\mathbf{P}$-zigzags with at most one noncolored element. We want to show that the above map g exists. Let h be the partial map from $B_{\mathbf{P}}$ to P defined by $h((\{p\}_*, \{p\}^*)) = p$, $p \in P$. The map h is, clearly, monotone from its domain to $\mathbf{P}$ and it is $\mathbf{P}$-extendible otherwise there would be a $\mathbf{P}$-zigzag as in Figure 3 with exactly one noncolored element (D, U), which is contained in the finite colored poset $(\mathbf{B_P}, h)$. But (D, U) cannot be in $B_{\mathbf{P}}$ since $a_1, \ldots, a_n \in D$ and $b_1, \ldots, b_m \in U$ and so $D^* \cap U_*$ is empty. Thus, $(\mathbf{B_P}, h)$ is an extendible colored poset. Let g be a monotone extension of h to $\mathbf{B_P}$. The map g clearly satisfies the requirements in the definition of the strong selection property. □

We define an obstruction for a finite poset $\mathbf{P}$ similarly to Nevermann [18]. Let $C_{\mathbf{P}} = \{(D, U) : D$ is a down set of $\mathbf{P}$, U is an up set of $\mathbf{P}\}$. First, we define the poset $\mathbf{C_P}$ by ordering $C_{\mathbf{P}}$ by $(D_1, U_1) \leq (D_2, U_2)$ if and only if $D_1 \subseteq D_2$ and $U_2 \subseteq U_1$. For a finite poset $\mathbf{H}$, a monotone map $v : \mathbf{H} \to \mathbf{C_P}$, $v : h \mapsto (D_h, U_h)$, is called an $\mathbf{H}$-*obstruction* if there is no order preserving map v' from $\mathbf{H}$ to $\mathbf{P}$ such that for every $h \in H$, $v'(h) \in {D_h}^* \cap U_{h*}$, but from any proper subposet $\mathbf{H}'$ of $\mathbf{H}$ there exists such a v'. Nevermann noted that if $\mathbf{H} = (\{0\}, \leq)$, then $v : H \to C_{\mathbf{P}}$ is an $\mathbf{H}$-obstruction if and only if $v(0)$ is a gap of $\mathbf{P}$.

Again as in the case of a gap we can associate a $\mathbf{P}$-zigzag to every $\mathbf{H}$-obstruction. First, we can create a $\mathbf{P}$-colored poset, where we define the poset of noncolored elements by $\mathbf{H}$ and for every $h \in H$, for every $a \in D_h$ we put a covered element colored by a below

h and for every $b \in U_h$ we put a covering element colored by b above h. The resulting colored poset is not $\mathbf{P}$-extendible and so it contains a $\mathbf{P}$-zigzag. We note that the set of the noncolored elements of an above zigzag is just H.

On the other hand, if we have a monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ then we can define $v : N(\mathbf{H}, f) \to C_{\mathbf{P}}$ by $v(h) = (D_h, U_h)$, where D_h and U_h are the same as in the proof of Proposition 2.3. Now, from the definition of a zigzag, it follows that v is an $\mathbf{N}(\mathbf{H}, f)$-obstruction.

An n-ary function $f : P^n \to P$, $n \geq 3$, is called a *near unanimity function*, briefly an n-nuf, if and only if $f(a, \ldots, a, \underbrace{b}_{i}, a, \ldots, a) = a$ for every $a, b \in P$ and for every $1 \leq i \leq n$. If $n = 3$, then f is called a *majority function*. The role of near unanimity functions in finite algebras will be highlighted in Chapter 7.

As we will see in the next remark the number of colored elements in the zigzags of a finite poset has a great importance in dealing with near unanimity functions preserving the poset. This remark is mentioned by Tardos in [26] without proof.

REMARK 2.4. *Let $n \geq 3$. A finite poset $\mathbf{P}$ admits an n-ary near unanimity function if and only if in every $\mathbf{P}$-zigzag the number of colored elements is at most $n-1$.*

PROOF: Let us suppose $\mathbf{P}$ admits an n-nuf. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored poset and let $C(\mathbf{H}, f) = \{h_1, \ldots, h_l\}$ such that $n \leq l = |C(\mathbf{H}, f)|$. Furthermore, let us suppose that for every $\mathbf{H}'$ properly contained in $\mathbf{H}$, $(\mathbf{H}', f\mid_{H'})$ is extendible. So for any $h \in C(\mathbf{H}, f)$ the colored poset $(\mathbf{H}_h, f\mid_{H_h})$, where $\mathbf{H}_h = (H \setminus \{h\}, \leq_{\mathbf{H}}\mid_{H\setminus\{h\}})$, is extendible. Let us take a function $f_h : H \to P$ for each $h \in C(\mathbf{H}, f)$ such that $f_h\mid_{H_h}$ is a $\mathbf{P}$-extension of $f\mid_{H_h}$ to H_h. Then by the hypothesis there is an l-nuf M_l preserving $\mathbf{P}$. Hence $M_l(f_{h_1}, \ldots, f_{h_l})$ is a fully defined monotone map from $\mathbf{H}$ to $\mathbf{P}$ that extends f. Thus, every $\mathbf{P}$-zigzag must have at most $n-1$ colored elements.

Now, let us suppose that in every $\mathbf{P}$-zigzag the number of colored elements is at most $n-1$. We look at the partial map $M_n : P^n \to P$ defined by $M_n(a, b, \ldots, b) = \cdots =$

$M_n(b,\ldots,b,a) = b$. Let us suppose M_n is not extendible to $\mathbf{P}^n$ as a monotone map from $\mathbf{P}^n$ to $\mathbf{P}$. Then the colored poset $(\mathbf{P}^n, M_n)$ contains a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$. We know that $|C(\mathbf{H}, f)| \leq n-1$. Hence there exists an $1 \leq i \leq n$ such that f takes on the i-th component for each element of $C(\mathbf{H}, f)$. But then the i-th projection from H to P is a $\mathbf{P}$-extension of f to $\mathbf{H}$, which is a contradiction. $\square$

We note that the only if part of the proof is valid for any poset $\mathbf{P}$. Remark 2.4 and Proposition 2.3 show that by making some assumptions on the number of colored or noncolored elements of the zigzags we get interesting classes of posets.

A finite poset is called a *fence* if its comparability graph is a path. A finite poset is called a *crown* if it is of the from $(F, \leq_{\mathbf{F}} \cup \{(a,b)\})$, where $\mathbf{F}$ is a fence with $2n \geq 4$ elements, a is the minimal endpoint of $\mathbf{F}$ and b is the maximal endpoint of $\mathbf{F}$. A poset is *connected* if its comparability graph is connected. In a connected poset $\mathbf{P}$ we define the *distance* $d(a,b)$ between a and b as $n-1$, where n is the smallest integer for which there exists an n-element subfence connecting a and b in $\mathbf{P}$. So $d(a,a) = 0$. The *diameter* of $\mathbf{P}$ is the supremum of $d(a,b)$, where $a, b \in P$. We define the *up distance* from a to b as the least positive integer n such that there is a subset $\{a_0, \ldots, a_n\} \subseteq P$ with $a = a_0$, $b = a_n$ and $a_0 \leq a_1 \geq a_2 \leq \ldots$. We define the *down distance* from a to b dually. Let $\uparrow(a,b)$ and $\downarrow(a,b)$ denote the up and down distance from a to b, respectively. Let $\leftrightarrow(a,b) = (\uparrow(a,b), \downarrow(a,b))$. Clearly, $|\uparrow(a,b) - \downarrow(a,b)| \leq 1$, if the up or down distance from a to b is defined. We note that by the definition $\leftrightarrow(a,a) = (1,1)$.

In [19] Nevermann and Rival define the concept of a hole in a poset. The definition for a finite poset $\mathbf{P}$ is the following. Let $V = \{(n,m) \in N^2 : |n-m| \leq 1\}$. We define the *disc* of $\mathbf{P}$ with center $c \in P$ and radius $r \in V$ as $D_{\mathbf{P}}(c,r) = \{a \in P : \leftrightarrow(c,a) \leq r\}$. Let T be a finite set with at least two elements. A sequence $(c_t, r_t)_{t \in T}$, where $c_t \in P$ and $r_t \in V$, is called a *hole* of $\mathbf{P}$ if $\bigcap_{t \in T} \{D_{\mathbf{P}}(c_t, r_t)\} = \emptyset$ and for every proper subset T' of T we have $\bigcap_{t \in T'} \{D_{\mathbf{P}}(c_t, r_t)\} \neq \emptyset$. A hole with $|T| = m$ is called an *m-hole*.

Now, we describe a one-to-one correspondence between holes and the elements of a class of certain nonextendible colored posets. Let $\mathbf{P}$ be a finite poset. Let us consider a hole of $\mathbf{P}$ defined in the previous paragraph. Let $r_1(t)$ and $r_2(t)$ be the first and second components of r_t. We create one colored fence for those t where $r_1(t) \neq r_2(t)$ and two colored fences for those t where $r_1(t) = r_2(t)$ in the following way. If $r_1(t) \leq r_2(t)$ we take a fence with $r_1(t)$ elements that starts upward from one of its endpoints, called a, and if $r_2(t) \leq r_1(t)$ we take a fence with $r_2(t)$ elements that starts downward from one of its endpoints, called a, and in both cases we color a by c_t. Let us stick together all these fences at the noncolored end. In the resulting colored poset let us stick together those colored points which correspond to the same t. In this way we get a nonextendible colored poset $(\mathbf{Q}, g)$ in which, if we cancel any colored point, we get an extendible colored poset.

We call a poset a *double fence* if it is obtained from a fence with at least three elements by taking a copy of it and a copy of its dual and sticking them together at their corresponding endpoints. The two points, where we stick together the fences are called the endpoints of the double fence. Now, let $(\mathbf{Q}, g)$ be a nonextendible $\mathbf{P}$-colored poset such that if we cancel any colored point from $(\mathbf{Q}, g)$ we get an extendible colored poset. Moreover, let $\mathbf{F}_i$, $i = 1, \ldots, m$, be a fence or a double fence with endpoints a_i and b_i. Let us suppose that $\mathbf{Q}$ is the poset obtained from $\mathbf{F}_i$, $i = 1, \ldots, m$ by sticking the $\mathbf{F}_i$ together at b_i, $i = 1, \ldots, m$, and $C(\mathbf{Q}, g) = \{a_1, \ldots, a_m\}$. Then the sequence $(g(a_i), \leftrightarrow (a_i, b_i))$, $i = 1, \ldots, m$, is an m-hole.

Observe that any zigzag contained in a nonextendible colored poset $(\mathbf{Q}, g)$ that corresponds to an m-hole must contain all colored points of $(\mathbf{Q}, g)$. This fact and Remark 2.4 imply the result contained in [22] that the arity of every nuf that preserves a finite poset $\mathbf{P}$ is greater than m for every m-hole of $\mathbf{P}$.

3. Properties of zigzags

In the previous section we defined zigzags. Now, we will explore the properties of these objects. In Proposition 3.1 we give a useful characterization of zigzags. Then in 3.2 through 3.11 we prove some claims concerning the shape of zigzags. In Claim 3.12 we describe all zigzags with fewer than three noncolored elements. By using Claim 3.11 we show that by cancelling a convex subset of a lattice ordered set we get a poset with the strong selection property, see Proposition 3.14. In Proposition 3.15 we show that every zigzag of a finite poset $\mathbf{P}$ is a monotone image of a zigzag of height less than the height of $\mathbf{P}$. For a finite poset $\mathbf{P}$, by Proposition 3.16 we get a recursive algorithm that creates all $\mathbf{P}$-zigzags with m noncolored elements for every m.

PROPOSITION 3.1. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored poset, where H is finite. Then $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag if and only if $\mathbf{H}$ is connected, $(\mathbf{H}, f)$ is not $\mathbf{P}$-extendible and by cancelling any covering pair of $(\mathbf{H}, f)$ the resulting colored poset is $\mathbf{P}$-extendible.*

PROOF: The only if part should be clear by the definition of a zigzag. To show the if part, let $\mathbf{H}' \subseteq \mathbf{H}$ and let us suppose $(\mathbf{H}', f\mid_{H'})$ is not $\mathbf{P}$-extendible. We will show $\mathbf{H}' = \mathbf{H}$. By the assumption $\mathbf{H}'$ has to contain all covering pairs of $\mathbf{H}$ otherwise $\mathbf{H}'$ would be extendible. Then H' has to contain every point of H which is in a connected component C of the comparability graph of $\mathbf{H}$, where C has at least two elements. By the assumption $\mathbf{H}$ has only one component and this component has more than one element since $(\mathbf{H}, f)$ is not $\mathbf{P}$-extendible. So $\mathbf{H} \subseteq \mathbf{H}'$, hence $\mathbf{H} = \mathbf{H}'$. □

Now, we list some facts concerning the shape of a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$.

CLAIM 3.2. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. The subgraph spanned by $N(\mathbf{H}, f)$ in the covering graph of $\mathbf{H}$ is connected.*

PROOF: Let us suppose the claim is not true. Then if we cancel the elements of one component of the subgraph spanned by $N(\mathbf{H}, f)$ from $(\mathbf{H}, f)$ we get a colored poset that is extendible. If we cancel all the other elements of $N(\mathbf{H}, f)$ from $(\mathbf{H}, f)$ we get another extendible colored poset. Because of the assumption, by taking the union of two extensions which extend the above two colored posets we would get an extension of f to $\mathbf{H}$. $\square$

CLAIM 3.3. *Let $(\mathbf{H}, f)$ be a monotone zigzag and let $a \in C(\mathbf{H}, f)$. For every $b \in H$ which satisfies $a \prec b$ or $b \prec a$ we have $b \in N(\mathbf{H}, f)$.*

PROOF: Without loss of generality we can assume $b \prec a$. If $b \in C(\mathbf{H}, f)$ then cancelling (b, a) in $(\mathbf{H}, f)$ we get an extendible colored poset and since $(\mathbf{H}, f)$ is monotone putting back (b, a) we still have an extendible colored poset. This contradicts the fact that $(\mathbf{H}, f)$ is not extendible. $\square$

A *monotone map* between two $\mathbf{P}$-colored posets means a monotone map between the two base posets which maps each a-colored element to an a-colored element and each noncolored element to a noncolored element. We say that a $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ is a *monotone image* of a $\mathbf{P}$-colored poset $(\mathbf{H}', f')$, if there exists a monotone map from $(\mathbf{H}', f')$ onto $(\mathbf{H}, f)$.

CLAIM 3.4. *For every $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ there exists a $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ such that $N(\mathbf{H}, f) = N(\mathbf{H}', f')$, $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$ and every colored element of $(\mathbf{H}', f')$ occurs in exactly one covering pair of $\mathbf{H}'$.*

PROOF: For nonmonotone $\mathbf{P}$-zigzags the claim is obvious. For a monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ the $\mathbf{P}$-colored poset $(\mathbf{H}', f')$ is defined as follows. The poset $\mathbf{N}(\mathbf{H}', f')$ is contained in $\mathbf{N}(\mathbf{H}, f)$ in such a way that the covering graph of $\mathbf{N}(\mathbf{H}', f')$ is the subgraph spanned by $N(\mathbf{H}, f)$ in the covering graph of $(\mathbf{H}, f)$. For every $s \in C(\mathbf{H}, f)$ and $h \in N(\mathbf{H}, f)$ with $s \prec_{\mathbf{H}} h$ there is a single element $s' \in C(\mathbf{H}', f')$ such that h is the unique element covering s' in $(\mathbf{H}', f')$ and $f'(s') = f(s)$. For every $h \in N(\mathbf{H}, f)$ and $s \in C(\mathbf{H}, f)$ with $h \prec_{\mathbf{H}} s$ there is single element $s' \in C(\mathbf{H}', f')$ such that h is the unique element covered by s' in $(\mathbf{H}', f')$ and $f'(s') = f(s)$. By Proposition 3.1 the so defined colored poset $(\mathbf{H}', f')$ is a zigzag which obviously satisfies the requirements of the claim. $\square$

A colored poset in which every colored element occurs in exactly one covering pair is called a *standard colored poset.*

CLAIM 3.5. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag and let a and b be two different elements of $C(\mathbf{H}, f)$. Let us suppose that there exists $c \in N(\mathbf{H}, f)$ with $c \prec a, b$. Then $f(a) \not\leq f(b)$.*

PROOF: Let us suppose the claim is not true. Then $f(a) \leq f(b)$. If we cancel (c, b) we get an extendible colored poset for which any extension extends $(\mathbf{H}, f)$, too. □

CLAIM 3.6. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag and let $a, b \in C(\mathbf{H}, f)$, where $a < b$. Then $f(a) \neq f(b)$.*

PROOF: The claim is obvious for nonmonotone zigzags. Now, let $(\mathbf{H}, f)$ be a monotone zigzag. By Claim 3.3 there is a $c \in N(\mathbf{H}, f)$ such that $a < c < b$. By cancelling c from $(\mathbf{H}, f)$ the resulting colored poset has an extension f'. If $f(a) = f(b)$ then f' together with the coloring of c by $f(a)$ is an extension of f to $\mathbf{H}$. □

In 3.4 we split colored points to obtain a new zigzag. In certain cases we can do the reverse.

CLAIM 3.7. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. Let $a, b \in C(\mathbf{H}, f)$ be two different maximal elements of $\mathbf{H}$ for which $f(a) = f(b)$. Then there exists a zigzag $(\mathbf{H}', f')$ for which $N(\mathbf{H}', f') = N(\mathbf{H}, f)$ and there is an onto monotone map from $(\mathbf{H}, f)$ to $(\mathbf{H}', f')$ which identifies only a and b.*

PROOF: We define $\mathbf{H}'$ as follows. Above every element of $\mathbf{H} \setminus \{a, b\}$ covered by a or b in $\mathbf{H}$ we put the covering element $c \notin H$. The coloring f' of $\mathbf{H}'$ is defined by f on $\mathbf{H} \setminus \{a, b\}$ and by $f'(c) = f(a)$. By Proposition 3.1, $(\mathbf{H}', f')$ is a zigzag which satisfies the requirements of the claim. □

CLAIM 3.8. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. Every monotone map $g : \mathbf{H} \to \mathbf{H}$ that is the identity map on $C(\mathbf{H}, f)$ has to be onto, i.e., an automorphism of $\mathbf{H}$.*

PROOF: Let us suppose g is a monotone map that is the identity on $C(\mathbf{H}, f)$ and maps H into a proper subset H' of H. Since $(\mathbf{H}, f)$ is a zigzag there is a $\mathbf{P}$-extension f' of $f\mid_{H'}$ to $\mathbf{H}'$. So $f' \circ g$ is a $\mathbf{P}$-extension of f to $\mathbf{H}$ which contradicts that $(\mathbf{H}, f)$ is not extendible. □

Let $\mathbf{Q}$ be a finite poset. Then $a \in \mathbf{Q}$ is called *retractable* if there is a non-onto monotone map on $\mathbf{Q}$ that fixes each element different from a. An element $a \in \mathbf{Q}$ is called *irreducible* if there is a unique $b \in Q$ with $a \prec b$ or $b \prec a$. Observe that every irreducible element is retractable.

CLAIM 3.9. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. Then $N(\mathbf{H}, f)$ has no retractable element of $\mathbf{H}$.*

PROOF: Apply Claim 3.8. □

CLAIM 3.10. *If $\mathbf{P} = \mathbf{Q}+1$, then every maximal element of a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is colored.*

PROOF: Let us suppose h is a maximal element of $\mathbf{H}$ and $h \in N(\mathbf{H}, f)$. By cancelling h in $(\mathbf{H}, f)$ we get an extendible colored poset. Now, an extension of f to this colored poset together with the coloring of h by 1 extends f to $\mathbf{H}$. □

CLAIM 3.11. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag with $|N(\mathbf{H}, f)| \geq 2$. Then for every $a \in N(\mathbf{H}, f)$, $a \notin \{b, c\} \subseteq H$ with $b \prec_{\mathbf{H}} c$ there exists a monotone $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ such that $\mathbf{H}' \subseteq \mathbf{H}$, $a, b, c \in \mathbf{H}'$, $f'\mid_{H' \setminus \{a\}} = f\mid_{H' \setminus \{a\}}$ and $f'(a)$ is defined in such a way that $f(d) \leq f'(a) \leq f(e)$ for every $d, e \in C(\mathbf{H}, f)$ with $d <_{\mathbf{H}} a <_{\mathbf{H}} e$.*

PROOF: Since $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag and $|N(\mathbf{H}, f)| \geq 2$ there exist $p \in P$ and a monotone partial map f'' from $\mathbf{H}$ to $\mathbf{P}$ given by $f''\mid_{C(\mathbf{H},f)} = f$ and $f''(a) = p$. For every such p we select one $\mathbf{P}$-zigzag contained in $(\mathbf{H}, f'')$ and we denote it by $(\mathbf{H}_p, f_p)$. Clearly, $a \in H_p$. Let us suppose there is no $p \in P$ such that $(b, c) \in \leq_{\mathbf{H}_p}$. Let $\mathbf{H}_{b,c} = \mathbf{H} \setminus \{(b, c)\}$. Then $\mathbf{H}_{b,c} \subseteq \mathbf{H}$ and $(\mathbf{H}_{b,c}, f)$ is not $\mathbf{P}$-extendible because every monotone extension of $(\mathbf{H}_{b,c}, f)$ to a contains a zigzag $(\mathbf{H}_p, f_p)$ for some p. But this contradicts that $(\mathbf{H}, f)$ is a zigzag. Thus there is a $p_0 \in P$ such that $(b, c) \in \leq_{\mathbf{H}_{p_0}}$. Taking $(\mathbf{H}', f') = (\mathbf{H}_{p_0}, f_{p_0})$ we get the claim. □

Claim 3.12. *For a* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *the following hold.*

(1) *If* $|N(\mathbf{H}, f)| = 0$, *then* $(\mathbf{H}, f)$ *is a two element nonmonotone zigzag.*

(2) *If* $|N(\mathbf{H}, f)| = 1$, *then* $(\mathbf{H}, f)$ *is the first colored poset shown in Figure 4, where* m *and* n *are nonnegative integers such that* $m + n > 0$ *and* $n, m \neq 1$. *Moreover,* f *is an order isomorphism on its domain.*

(3) *If* $|N(\mathbf{H}, f)| = 2$, *then* $(\mathbf{H}, f)$ *is the second colored poset shown in Figure 4, where* $k, l \geq 1$ *and* m *and* n *are nonnegative integers for which* $m, n \neq 1$. *Moreover, any comparable pair in* $Range(f)$ *not shown in the picture is of the form* $d_i < c_j$, $c_j < b_s$ *or* $a_t < d_i$ *for some* $1 \leq i \leq k$, $1 \leq j \leq l$, $1 \leq s \leq m$ *and* $1 \leq t \leq n$.

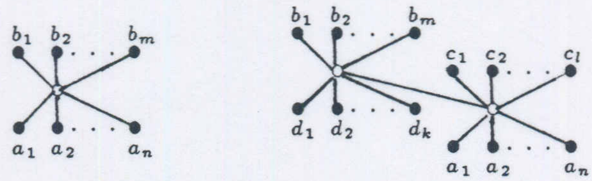


Figure 4. Monotone zigzags with one and two noncolored elements

Proof: We showed (1) before Example 2.2. First we prove (2). By Claim 3.3 we get the picture of $(\mathbf{H}, f)$. Obviously, $m + n > 0$. Claim 3.9 gives $n, m \neq 1$. Claim 3.5 gives that f is an order isomorphism on its domain. Next we prove (3). By using Claim 3.2 and Claim 3.3 we can see that $(\mathbf{H}, f)$ has to be a standard zigzag as shown in the picture. Claim 3.9 gives $k, l \geq 1$ and $m, n \neq 1$. To prove the last claim use Claim 3.5, Claim 3.6 and the definition of zigzag. □

Let $\mathbf{S}_i$ be the poset that we obtain by cancelling the top element from a Boolean lattice ordered set with i atoms. Let $\mathbf{S}_i^d$ be the dual of $\mathbf{S}_i$. Let $\mathbf{T}_{l,k}$ be the poset $(\mathbf{S}_l + 1 + \mathbf{S}_k^d) \times (1 + 1)$ without its top and bottom element. Then the reader can easily check the following claims. The poset $\mathbf{S}_n + \mathbf{S}_m^d$ has a zigzag of the form in (2) of Claim 3.12. The poset $\mathbf{S}_n + \mathbf{T}_{l,k} + \mathbf{S}_m^d$ has a zigzag of the form in (3) of Claim 3.12. For more on these posets see [18] and [20].

CLAIM 3.13. *For a bounded poset* $\mathbf{P}$ *the integers* m *and* n *are never zero in (2) and (3) of Claim 3.12.*

PROOF: Use Claim 3.10. $\square$

Let $\mathbf{L}$ be a finite lattice ordered set. For the next proposition only, we redefine the notion of a convex set. A subset S of L is called a *convex subset* of $\mathbf{L}$ if $a, b \in S$, $c \in L$ and $a \leq_{\mathbf{L}} c \leq_{\mathbf{L}} b$ imply $c \in S$. There is a well known result [4] of Demetrovics, Hannák and Rónyai which states that for a finite lattice ordered set $\mathbf{L}$ the poset $\mathbf{P} = \mathbf{L} \setminus S$, where S is a proper, convex subset of $\mathbf{L}$, admits a nuf. We will show that $\mathbf{P}$, in fact, has the strong selection property. Then by Proposition 2.3 every $\mathbf{P}$-zigzag has at most one noncolored element. So by (1) and (2) of Claim 3.12 the number of $\mathbf{P}$-zigzags is finite. Thus by Remark 2.4 the poset $\mathbf{P}$ admits a nuf. We note that the preceding argument gives a proof of the result in [19] that a finite poset with the strong selection property admits a nuf.

PROPOSITION 3.14. *Let* $\mathbf{L}$ *be a finite lattice ordered set. Let* S *be a proper, convex subset of* $\mathbf{L}$. *Then* $\mathbf{P} = \mathbf{L} \setminus S$ *has the strong selection property.*

PROOF: By Proposition 2.3 it is enough to show that for every monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ we have $|N(\mathbf{H}, f)| = 1$. Let us suppose this is not true. Then there exists a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that $|N(\mathbf{H}, f)|$ is minimal with respect to $|N(\mathbf{H}, f)| \geq 2$. Let $a \neq b \in N(\mathbf{H}, f)$. By Claim 3.11 there are two $\mathbf{P}$-zigzags $(\mathbf{H}_1, f_1)$ and $(\mathbf{H}_2, f_2)$ such that $\mathbf{H}_1, \mathbf{H}_2 \subseteq \mathbf{H}$, $\{a, b\} \subseteq \mathbf{H}_1, \mathbf{H}_2$ and $f_1 \mid_{H_1 \setminus \{a\}} = f \mid_{H_1 \setminus \{a\}}$, $f_2 \mid_{H_2 \setminus \{b\}} = f \mid_{H_2 \setminus \{b\}}$. Moreover $f_1(a)$ and $f_2(b)$ are defined in such a way that $f(c) \leq f_1(a) \leq f(d)$ for every $c, d \in C(\mathbf{H}, f)$ with $c < a < d$ and $f(c) \leq f_2(b) \leq f(d)$ for every $c, d \in C(\mathbf{H}, f)$ with $c < b < d$. The minimality of $|N(\mathbf{H}, f)| \geq 2$ implies that $|N(\mathbf{H}_1, f_1)| = |N(\mathbf{H}_2, f_2)| = 1$. Hence (2) in Claim 3.12 applies to $(\mathbf{H}_1, f_1)$ and $(\mathbf{H}_2, f_2)$. So by $a, b \in H_1$, the elements a and b have to be comparable in $\mathbf{H}$. Let us say $a < b$. Let $\wedge$ and $\vee$ be the join and meet operation of $\mathbf{L}$. We define $t = \wedge\{f(p) : p \in C(\mathbf{H}_1, f_1),\ b < p\}$ and $u = \vee\{f(p) : p \in C(\mathbf{H}_2, f_2),\ p < a\}$. Now, clearly, $f_1(a) \leq t$ and by the above inequalities for $f_1(a)$, $u \leq f_1(a)$. Observe that t

and u have to be in S otherwise $(\mathbf{H}_1, f_1)$ or $(\mathbf{H}_2, f_2)$ would be extendible $\mathbf{P}$-colored posets. Since S is convex we get that $f_1(a) \in S$, which contradicts $f_1(a) \in P$. $\square$

We remark that not every finite poset $\mathbf{P}$ with the strong selection property can be obtained from a finite lattice ordered set in the above way. Let $\mathbf{P}' = 1+2+2+1$, the poset in Figure 2, and let $\mathbf{P} = \mathbf{P}' \times \mathbf{P}'$. Then $\mathbf{P}$ has the strong selection property since $\mathbf{P}'$ has it by Proposition 3.14 and if $\mathbf{H}$ and $\mathbf{K}$ have the strong selection property then $\mathbf{H} \times \mathbf{K}$ has it, too, see [20]. In [6] it is shown that $\mathbf{P}$ cannot be obtained from a finite lattice ordered set by cancelling a convex subset.

In a finite poset $\mathbf{P}$ we define the *length* between two elements $a \leq_{\mathbf{P}} b$ as the maximum cardinality of a chain between a and b. The length between a and b is denoted by $\ell_{\mathbf{P}}(a, b)$.

PROPOSITION 3.15. *Let $\mathbf{P}$ be a finite, bounded poset. Then for every monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ there is a standard zigzag $(\mathbf{H}', f')$ such that $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$ and for every maximal chain $a = a_1 < a_2 < \cdots < a_n = b$ of $\mathbf{H}'$, $n \leq \ell_{\mathbf{P}}(f'(a), f'(b)) + 1$.*

PROOF: First of all, recall Claim 3.10 to see that for every maximal chain of any $\mathbf{P}$-zigzag the bottom and top elements are colored. The proof will proceed by induction on the cardinality of the set of noncolored elements of a zigzag. By (2) in Claim 3.12, the zigzags with one noncolored element satisfy the claim. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag with $|N(\mathbf{H}, f)| = m \geq 2$ and let us suppose that for every $\mathbf{P}$-zigzag with $m-1$ noncolored elements we have the claim. Let $h \in N(\mathbf{H}, f)$ be a maximal element of $\mathbf{N}(\mathbf{H}, f)$. Let us color h by $p \in P$ in $(\mathbf{H}, f)$. For every $p \in P$ the resulting colored poset contains a zigzag $(\mathbf{H}_p, f_p)$ such that $h \in H_p$. We select $(\mathbf{H}_p, f_p)$ nonmonotone whenever this is possible. If $(\mathbf{H}_p, f_p)$ is monotone then by the induction hypothesis there exists a standard zigzag $(\mathbf{Q}_p, g_p)$ which has $(\mathbf{H}_p, f_p)$ as its monotone image and its maximal chains satisfy the desired property in the claim. Observe that under the monotone map from $(\mathbf{Q}_p, g_p)$ onto $(\mathbf{H}_p, f_p)$ the preimage of h contains only maximal elements of $\mathbf{Q}_p$. If $(\mathbf{H}_p, f_p)$ is nonmonotone then we make $(\mathbf{Q}_p, g_p)$ a copy of $(\mathbf{H}_p, f_p)$. Now, we can construct a standard colored poset $(\mathbf{Q}, g)$ from $(\mathbf{Q}_p, g_p)$, $p \in P$, by gluing all elements of the preimages of h into one single noncolored point, called h', meanwhile preserving the coloring of the other points. This colored poset is not extendible since $(\mathbf{Q}_p, g_p)$, $p \in \mathbf{P}$, is not extendible. So it contains a zigzag $(\mathbf{Q}', g')$ in which the colored elements are exactly the extremal elements. Let us construct a standard zigzag $(\mathbf{H}', f')$ for $(\mathbf{Q}', g')$ as in Claim 3.4.

Clearly, there is a monotone map from $(\mathbf{H}', f')$ to $(\mathbf{H}, f)$ and this map must be onto otherwise $(\mathbf{H}', f')$ would be extendible. Let $a = a_1 < a_2 < \cdots < a_n = b$ be a maximal chain of $\mathbf{H}'$. If this chain does not contain h' then we are done by the induction hypothesis. Otherwise $a_{n-1} = h' \prec_{\mathbf{H}'} b$. The chain $a = a_1 < a_2 < \cdots < a_{n-1} = h'$ has to be in the preimage of H_p for some p. If $(\mathbf{H}_p, f_p)$ is monotone then by applying the induction hypothesis to $(\mathbf{Q}_p, g_p)$ we get $n - 1 \leq \ell_{\mathbf{P}}(g_p(a), p) + 1$. Since $(\mathbf{H}_p, f_p)$ is monotone our construction guarantees that, if we color h by p in $(\mathbf{H}, f)$ we get a monotone colored poset. Now, $f'(b)$ must be the color of an element above h in $(\mathbf{H}, f)$. Hence, by Claim 3.9 and Claim 3.5 we have $p < f'(b)$. Since $g_p(a) = f'(a)$ we have $\ell_{\mathbf{P}}(g_p(a), p) + 1 \leq \ell_{\mathbf{P}}(f'(a), f'(b))$. By combining the preceding two inequalities we get the claim. If $(\mathbf{H}_p, f_p)$ is nonmonotone, then $n = 3$ and the claim is obvious. $\square$

PROPOSITION 3.16. *Let* $\mathbf{P}$ *be a finite poset. Let* $(\mathbf{H}, f)$ *be a monotone* $\mathbf{P}$*-zigzag. Then for every* $a \in N(\mathbf{H}, f)$ *there exist* $\mathbf{P}$*-colored posets* $(\mathbf{H}_i, f_i) \subseteq (\mathbf{H}, f)$, $i \in I$, *for which* $a \in N(\mathbf{H}_i, f_i)$, *and there exist* $p_i \in P$, $i \in I$, *such that if* a *is colored by* p_i *in* $(\mathbf{H}_i, f_i)$ *the resulting colored poset is a* $\mathbf{P}$*-zigzag and if* a *is colored by* p_j *in* $(\mathbf{H}_i, f_i)$, $j \in I \setminus \{i\}$, *the resulting colored poset is* $\mathbf{P}$*-extendible. Moreover, for every* $p \in P$ *there exists* $i \in I$ *such that, if* a *is colored by* p *in* $(\mathbf{H}_i, f_i)$ *the resulting colored poset is not* $\mathbf{P}$*-extendible.*

PROOF: As in the previous proposition, if we color a by an element of $\mathbf{P}$ in $(\mathbf{H}, f)$ the resulting colored poset is still nonextendible so it contains some $\mathbf{P}$-zigzags which must contain a. Let $X = \{(\mathbf{G}_t, g_t) : t \in T\}$ be the set of all zigzags which can be obtained in this way. We assign a subset $S_t \subseteq P$ to every $(\mathbf{G}_t, g_t)$, $t \in T$, so that S_t contains $g_t(a)$

and all the elements of $\mathbf{P}$ by which recoloring a in $(\mathbf{G}_t, g_t)$, the resulting colored poset is nonextendible. We select a subset I of T as follows.

(1) I is a minimal set with respect to $\cup_{i \in I} S_i = P$.

(2) I has the maximal cardinality with respect to (1).

(3) I satisfies $S_t \cup (\cup_{j \in I \setminus \{i\}} S_j) \neq P$ for every S_t that is a proper subset of S_i, where $t \in T$ and $i \in I$.

For every $i \in I$ we select $(\mathbf{G}_t, g_t)$ such that $S_t = S_i$ and $\mathbf{G}_t$ is minimal with respect to the containment of posets. For simplicity, we can assume $i = t$. Then we define $\mathbf{H}_i = \mathbf{G}_i$ and $f_i = g_i|_{G_i \setminus \{a\}}$ for $i \in I$.

Let $T_i = S_i \setminus (\cup_{j \in I \setminus \{i\}} S_j)$ for $i \in I$. These sets are nonempty by (1). We claim that every T_i, $i \in I$, contains an element p_i such that if a is colored by p_i in $(\mathbf{H}_i, f_i)$, then we get a zigzag. Let us suppose this is not true. So there exists an i such that for every $p \in T_i$, if a in $(\mathbf{H}_i, f_i)$ is colored by p the resulting colored poset is not extendible but also not a zigzag. Hence it properly contains some zigzags. Let these zigzags be $(\mathbf{G}_v, g_v)$, where $v \in V \subseteq T$. Observe that $S_v \subseteq S_i$ and $\cup_{v \in V} S_v \cup (\cup_{j \in I \setminus \{i\}} S_j) = P$. Let us take a subset V_0 of V minimal with respect to $\cup_{v \in V_0} S_v \cup (\cup_{j \in I \setminus \{i\}} S_j) = P$. Clearly, $V_0 \cup (I \setminus \{i\})$ is minimal in the sense of (1). Since $|V_0| \geq 1$ and I satisfies (2), $V_0 = \{v_0\}$ for some v_0. Then by (3), $S_i = S_{v_0}$. But $\mathbf{G}_{v_0}$ is properly contained in $\mathbf{H}_i$ which contradicts the fact that $\mathbf{H}_i$ is minimal. Now, $(\mathbf{H}_i, f_i)$ and p_i, $i \in I$, clearly satisfy the claims of the lemma. □

Proposition 3.16 gives a procedure for constructing all $\mathbf{P}$-zigzags of a finite poset $\mathbf{P}$. Let us suppose we have determined all $\mathbf{P}$-zigzags with fewer than m noncolored elements. Then any $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ with m noncolored elements can be obtained as a monotone image of a $\mathbf{P}$-zigzag $(\mathbf{H}', f')$. The zigzag $(\mathbf{H}', f')$ is obtained from the $\mathbf{P}$-zigzags $(\mathbf{G}_i, g_i)$, $i \in I$, with at most $m - 1$ noncolored elements, by deleting the color of an $a_i \in C(\mathbf{G}_i, g_i)$ in each $(\mathbf{G}_i, g_i)$, $i \in I$, and sticking together the resulting colored posets at a_i, $i \in I$. By Proposition 3.16 we can choose $(\mathbf{G}_i, g_i)$ and a_i, $i \in I$ in such a way that the colored poset $(\mathbf{H}', f')$ indeed is a zigzag and $(\mathbf{H}, f)$ is the monotone image of $(\mathbf{H}', f')$.

4. Examples of zigzags of special posets

First we describe the zigzags of fences and antichains. Then in Theorem 4.3 and in Proposition 4.4 we present two constructions of posets from smaller posets and show that for these constructions it is easy to describe all zigzags if we know the zigzags of the smaller posets. For example, by these constructions we get locked fences, defined in [14], and Tardos's eight element poset in [26] from fences and the two element antichain, respectively. The description of the zigzags of locked fences yields an easy proof that each locked fence admits a near unanimity function.

We define the *middle element* of three elements of a fence as the one which is on the path connecting the other two. It was noticed, see for example [10], that for every fence the ternary function which assigns the middle element to each 3-tuple is a monotone, majority function. So by Remark 2.4 every zigzag of a fence has two colored elements. In the next proposition we describe all the zigzags of a fence.

PROPOSITION 4.1. *Let $\mathbf{P}$ be a fence. Then a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is a $\mathbf{P}$-colored fence satisfying the following properties. If a and b denote the endpoints of the colored fence, then a and b are the only colored points and at least one of the inequalities $\uparrow(a,b) <\uparrow(f(a),f(b))$ and $\downarrow(a,b) <\downarrow(f(a),f(b))$ holds.*

PROOF: Certainly, a $\mathbf{P}$-colored fence of the above form is a $\mathbf{P}$-zigzag. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. We want to prove that $(\mathbf{H}, f)$ is of the above form. As we mentioned above $|C(\mathbf{H}, f)| = 2$. Let $C(\mathbf{H}, f) = \{a, b\}$. Then $f(a)$ and $f(b)$ must be different otherwise $(\mathbf{H}, f)$ would be extendible. If there is a path $\mathbf{F}$ between a and b such that at least one of $\uparrow(a,b) <\uparrow(f(a),f(b))$ and $\downarrow(a,b) <\downarrow(f(a),f(b))$ holds then $(\mathbf{H}, f) = (\mathbf{F}, f|_F)$ by the minimality of $(\mathbf{H}, f)$ and we have the claim.

For otherwise, let both $\uparrow(a,b) \geq\uparrow(f(a),f(b))$ and $\downarrow(a,b) \geq\downarrow(f(a),f(b))$. In this case we show that $(\mathbf{H}, f)$ is extendible, thereby obtaining a contradiction. Let the path between

$f(a)$ and $f(b)$ in $\mathbf{P}$ be given by $f(a) = a_1 \prec a_2 \succ \ldots a_n = f(b)$, $n \geq 2$. We define the nonempty sets $B_i = \{c \in H : \downarrow(a, c) = i\}$, $1 \leq i \leq n-1$, and $B_n = H \setminus \cup_{i=1}^{n-1} B_i$. Let $f' : H \to P$ be the function that takes on the value a_i for every element of B_i, $1 \leq i \leq n$. Now, f' is clearly a monotone function from $\mathbf{H}$ to $\mathbf{P}$. Observe that $a \in B_1$ and $b \in B_n$ by the assumption on the distance between a and b. So f' extends f to $\mathbf{H}$. □

For example, by Proposition 4.1 the four element fence has the five monotone zigzags shown in Figure 5.

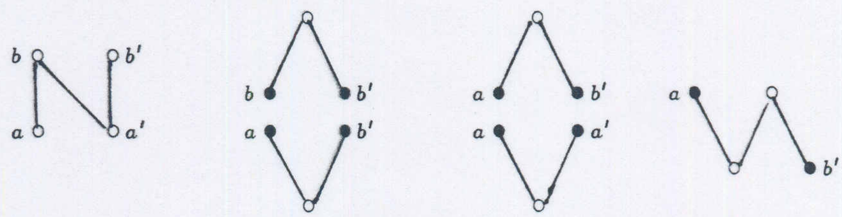


Figure 5. The four element fence and its monotone zigzags

Proposition 4.2. *Let $\mathbf{P}$ be an antichain. Then the $\mathbf{P}$-zigzags are the colored fences which are colored on the two endpoints by different elements of $\mathbf{P}$.*

Proof: Clearly, every colored fence of the above type is a $\mathbf{P}$-zigzag. On the other hand, every zigzag has to contain at least two colored points. Since every $\mathbf{P}$-zigzag is connected every $\mathbf{P}$-zigzag contains a copy of a colored fence mentioned in the claim. Hence, by the minimality of zigzags we have the claim. □

By the use of the following theorem we can easily describe the zigzags of certain posets constructed from fences and antichains.

Theorem 4.3. *Let $\mathbf{P}$ be a finite poset and let $\mathbf{A}$ be an antichain with $|A| \geq 2$. Let $\mathbf{P}' = \mathbf{P} + \mathbf{A} + 1$, where $+$ is the ordinal sum and 1 is the one element poset. Then every monotone $\mathbf{P}'$-zigzag can be obtained from a monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ by putting two covering elements, which are colored by different colors from $\mathbf{A}$, above every noncolored maximal element of $(\mathbf{H}, f)$. Moreover, every $\mathbf{P}'$-colored poset constructed in this way will be a $\mathbf{P}'$-zigzag.*

PROOF: First, we prove the last claim. Let $(\mathbf{H}', f')$ be a $\mathbf{P}'$-colored poset created from a $\mathbf{P}$ zigzag $(\mathbf{H}, f)$ in the above way. Observe that $\mathbf{H}'$ is connected since $\mathbf{H}$ is. The colored poset $(\mathbf{H}', f')$ is not $\mathbf{P}'$-extendible since any $\mathbf{P}'$ extension of f' restricted to H must lie in P and so it would be a $\mathbf{P}$-extension of f to $\mathbf{H}$. Then by Proposition 3.1 we have to show that by cancelling any covering edge in $(\mathbf{H}', f')$, the resulting colored poset is $\mathbf{P}'$-extendible. The only problem occurs if we cancel a new covering edge (a, b). By the construction there is a $b' \in H'$ that also covers a in $\mathbf{H}'$. But then coloring a by $f'(b')$ and coloring the other elements of $N(\mathbf{H}', f')$ by a $\mathbf{P}$-extension of $(\mathbf{H}, f) \setminus \{a\}$ restricted to $N(\mathbf{H}', f')$, we get a $\mathbf{P}'$-extension of $(\mathbf{H}', f') \setminus \{(a, b)\}$.

Let $(\mathbf{H}', f')$ be a monotone $\mathbf{P}'$-zigzag. We show that we can get $(\mathbf{H}', f')$ from a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ by the above procedure. By Claim 3.10 every maximal element of $(\mathbf{H}', f')$ is colored and so by Claim 3.9 and the remarks preceding it, above every noncolored element of $(\mathbf{H}', f')$ there are at least two colored elements. If $(\mathbf{H}', f')$ has no element colored from $\mathbf{A}$, then by Proposition 3.1 it follows immediately that $(\mathbf{H}', f')$ is a $\mathbf{P}$-zigzag, too.

Let us suppose now, $(\mathbf{H}', f')$ has some elements colored from $\mathbf{A}$. Let $a \in C(\mathbf{H}', f')$ and $f'(a) \in A$. We will show that a is maximal in $\mathbf{H}'$ and every element covered by a is maximal in $\mathbf{N}(\mathbf{H}', f')$. If a is not maximal in $\mathbf{H}'$ then there is a colored maximal element b above it. By the monotonicity of $(\mathbf{H}', f')$ and by Claim 3.6, $f'(a) < f'(b)$, hence $f'(b) = 1$ which contradicts Claim 3.5. So a is maximal in $\mathbf{H}'$. Now, let $b \prec a$ in $\mathbf{H}'$. By Claim 3.3, $b \in N(\mathbf{H}', f')$. Let us suppose b is not maximal in $\mathbf{N}(\mathbf{H}', f')$. Then there exists $b' \in N(\mathbf{H}', f')$ such that $b < b'$ and b' is maximal in $\mathbf{N}(\mathbf{H}', f')$. So there exist $c_1, c_2 \in C(\mathbf{H}', f')$ which cover b' in $(\mathbf{H}', f')$. By Claim 3.5, $1 \neq f'(c_1) \neq f'(c_2) \neq 1$. If we cancel (b, a) in $(\mathbf{H}', f')$ we get a $\mathbf{P}'$-extension of f' in which the color of b is smaller than $f'(a) \in A$, but then putting back (b, a), the resulting colored poset $(\mathbf{H}', f')$ is extendible, which is a contradiction.

Let b be a maximal element of $\mathbf{N}(\mathbf{H}', f')$ such that there exists an $a_1 \succ b$,where $a_1 \in C(\mathbf{H}', f')$ and $f'(a_1) \in A$. By Claim 3.5 every element which covers b is colored from $\mathbf{A}$.

We know that there is another element $a_2 \in C(\mathbf{H}', f')$ which covers b. But there is no third one since otherwise cancelling (b, a_1) in $(\mathbf{H}', f')$ we get an extension of f' which would be a monotone extension of $(\mathbf{H}', f')$ as well. So we have proved that every maximal element of $\mathbf{N}(\mathbf{H}', f')$ is covered by exactly two $\mathbf{A}$-colored elements or none.

Now, let us cancel every $\mathbf{A}$-colored point of $(\mathbf{H}', f')$. So we obtain a $\mathbf{P}$-colored poset $(\mathbf{H}, f)$. By Claim 3.2, $\mathbf{H}$ is connected and $\mathbf{P}$-nonextendible and if we cancel a covering edge in $(\mathbf{H}, f)$ we get a $\mathbf{P}$-extendible poset because this is true if we cancel the same edge in $(\mathbf{H}', f')$. So by Proposition 3.1, $(\mathbf{H}, f)$ has to be a $\mathbf{P}$-zigzag. By the above claims we get $(\mathbf{H}', f')$ from $(\mathbf{H}, f)$ in the desired way. $\square$

We can use the above theorem, its dual and the previous propositions to obtain the zigzags of certain posets like the famous poset 1+2+2+2+1, or 1+2+$\mathbf{P}$+2+1, where $\mathbf{P}$ is a fence. The latter posets are called *locked fences* in [14] and McKenzie mentions, without a proof, that they admit near unanimity functions. In the case of a four element fence a 7-nuf is given in [22]. In general, it follows from Proposition 4.1, Theorem 4.3 and Remark 2.4 that a locked fence with $|P| = n \geq 2$ admits a $(2n-1)$-nuf and $2n-1$ is the smallest possible arity.

The following simple proposition is similar in nature to Theorem 4.3.

PROPOSITION 4.4. *Let $\mathbf{P}$ be a finite poset and let $\mathbf{P}' = \mathbf{P} + 1$. Then the set of all monotone $\mathbf{P}'$-zigzags equals the set of those monotone $\mathbf{P}$-zigzags in which every maximal element is colored.*

PROOF: It is easy to see that every $\mathbf{P}$-zigzag in which every maximal element is colored is a $\mathbf{P}'$-zigzag. Conversely, let $(\mathbf{H}', f')$ be a monotone $\mathbf{P}'$-zigzag. Then by Claim 3.10, every maximal element of $(\mathbf{H}', f')$ is colored. These maximal elements must have colors from P because of Claim 3.9 and Claim 3.5. Since $(\mathbf{H}', f')$ is monotone it is a $\mathbf{P}$-colored poset. Now, $\mathbf{H}'$ is connected and $(\mathbf{H}', f')$ is $\mathbf{P}$-nonextendible. If we cancel any covering pair in $(\mathbf{H}', f')$ we get a $\mathbf{P}'$-extension g of f. By Claim 3.9 and by the remarks preceding it, there are at least two colored elements above every noncolored element in $(\mathbf{H}', f')$. Hence g also is a $\mathbf{P}$ extension of $(\mathbf{H}', f')$. So by Proposition 3.1 we have the claim. $\square$

To illustrate the claims in this chapter we present some examples. In Figure 6 we have two posets with all of their monotone zigzags. These posets are obtained from the four element fence in such a way that Theorem 4.3 and its dual apply. Since they have finitely many zigzags they admit a near unanimity function.

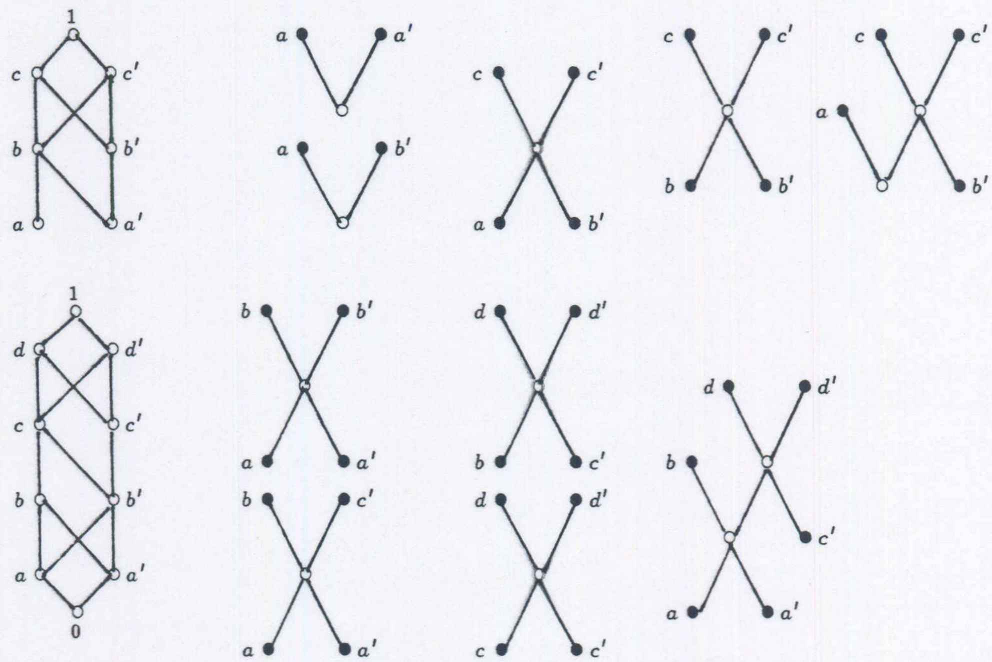


Figure 6. Posets admitting a nuf and their monotone zigzags

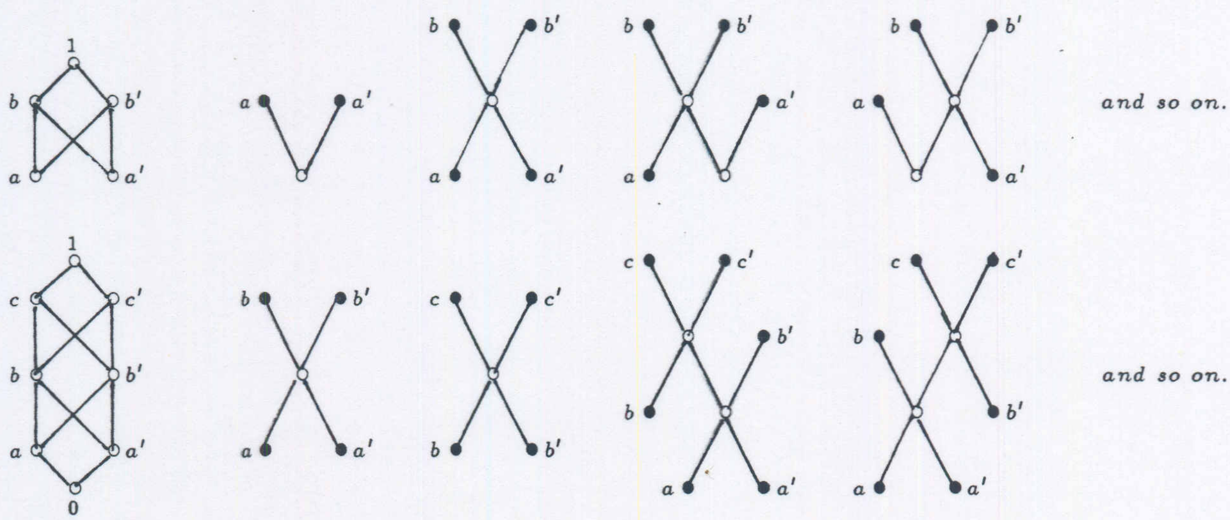


Figure 7. Posets admitting no nuf and their monotone zigzags

In Figure 7 we have two posets constructed from the two element antichain. All of their zigzags can be determined by applying Theorem 4.3 and its dual. They have infinitely many zigzags. In fact, they admit no near unanimity function.

The poset in Figure 8 is different. We had to use Proposition 3.11 and Proposition 3.16 to prove that the list of the monotone zigzags in Figure 8 is complete. The calculations are tedious and the proof is omitted. There are sixteen possibilities for $(\{p,q\},\{r,s\})$ in Figure 8. If $\{p,q\} = \{a,a'\}$, then $\{r,s\}$ is $\{b,b'\}$, $\{b,c\}$, $\{c',c\}$ or $\{c',b'\}$. Dually, if $\{r,s\} = \{d,d'\}$, then $\{p,q\}$ is $\{c,c''\}$, $\{c,b\}$, $\{b'',b\}$ or $\{b'',c''\}$. If $(p,r) = (b'',c')$, then (q,s) is (b,d), (b,d'), (a,d), (a,d'), (a,c), (a',d), (a',d') or (a',c). For (u,t) there are four possibilities, (u,t) equals (c,b), (c,c'), (b'',b) or (b'',c').

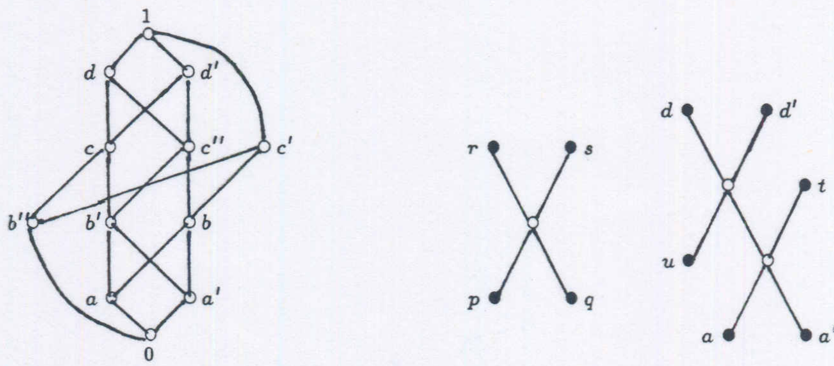


Figure 8. A poset with its monotone zigzags

5. Order varieties

In Chapter 2 we defined the product of posets. Another important notion for posets is the retract. Let $\mathbf{P}$ and $\mathbf{R}$ be two posets. We say that $\mathbf{R}$ is a *retract* of $\mathbf{P}$ if there are monotone functions $r : \mathbf{P} \to \mathbf{R}$ and $e : \mathbf{R} \to \mathbf{P}$ such that $r \circ e$ is equal to the identity function of R. The maps r and e are called *retraction* and *coretraction*, respectively. Let K be a class of posets. The *order variety* generated by K is the smallest class of posets containing K and closed under the retract and product constructions. In [9] it is shown that the order variety generated by K exists and is equal to $\mathcal{RP}(K)$, where $\mathcal{R}$ is the operator of taking retracts of posets and $\mathcal{P}$ is the operator of taking products of posets.

In [22] it was shown that a finite poset $\mathbf{P}$ admits a majority function if and only if $\mathbf{P}$ is a retract of a finite product of fences and the two element antichain. In general, the question arises, what "nice" set of finite posets generates, by taking retract and finite product, the set of all finite posets admitting an n-nuf. In Chapter 7 we answer this question for finite bounded posets, if $n \leq 6$. Before this we need to establish a connection between order varieties and nonextendible colored posets. In the first part of Chapter 5 we define the posets of finite type and show that they form an order variety that contains the order variety generated by finite posets. The order variety of posets of finite type is convenient to work with when we study order varieties generated by finite posets. In Proposition 5.9 we give a characterization of the members of an order variety generated by an arbitrary set of posets. In Proposition 5.10 we provide a similar characterization of the members of an order variety generated by an arbitrary class of posets of finite type. In Chapter 6, Proposition 5.10 will be used for a description of finite irreducible posets via zigzags.

First we want to introduce a particular order variety that turns out to be useful when working with finite posets. We define a poset $\mathbf{P}$ to be of *finite type* if and only if every $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ is extendible if every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is extendible. Since every finite nonextendible colored poset contains a zigzag we have the following simple proposition.

PROPOSITION 5.1. *Let* $\mathbf{P}$ *be a poset of finite type. A* $\mathbf{P}$*-colored poset is* $\mathbf{P}$*-extendible if and only if it does not contain a* $\mathbf{P}$*-zigzag.*

One expects the following result.

PROPOSITION 5.2. *Every finite poset* $\mathbf{P}$ *is of finite type.*

PROOF: Let $\mathbf{P}$ be a finite poset and let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored poset. Suppose that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\mathbf{P}$-extendible. We want to show that f is $\mathbf{P}$-extendible to $\mathbf{H}$. We will use Tikhonov's theorem, which states that a product of compact topological spaces is compact. For every $h \in H$ we define a compact topological space T_h, namely, if f is defined on h then T_h is the one element set $\{f(h)\}$, otherwise T_h is P with the discrete topology. Then $\prod_{h\in H} T_h$ can be considered to be the set of all functions from H to P which extend f to H. For $h_1 < h_2 \in H$ and $a \not\leq b$, where $a, b \in P$, we define an open set of $\prod_{h\in H} T_h$ in the product topology consisting of those elements of $\prod_{h\in H} T_h$ whose h_1-component is a and whose h_2-component is b. Let S denote the set of all open sets obtained in this way. Observe that every nonmonotone $\mathbf{P}$-extension of f is in one of the open sets of S. Let us suppose that f has no monotone extension to $\mathbf{H}$. Then the open sets of S cover $\prod_{h\in H} T_h$. But since $\prod_{h\in H} T_h$ is compact there only finitely many members of S suffice to cover $\prod_{h\in H} T_h$. Now, we can find a finite set $M \subseteq H$ which contains all the elements of H which occur in the definition of this finite cover. Then we have that every element of $\prod_{h\in H} T_h$ is nonmonotone on the finite subposet of $\mathbf{H}$ having base set M. But this contradicts the assumption. □

PROPOSITION 5.3. *The posets of finite type form an order variety.*

PROOF: First we show that a product $\prod_{i\in I} \mathbf{P}_i$ of posets $\mathbf{P}_i$, $i \in I$, of finite type is of finite type. Let $(\mathbf{H}, f)$ be a $\prod_{i\in I} \mathbf{P}_i$-colored poset such that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is

$\prod_{i\in I} \mathbf{P}_i$-extendible. Looking at the i-th component of f as a coloring on $\mathbf{H}$ the so obtained $\mathbf{P}_i$-colored poset is $\mathbf{P}_i$-extendible since $\mathbf{P}_i$ is of finite type. Now, the i-th component of a $\prod_{i\in I} \mathbf{P}_i$-extension of f is defined by the above extension.

Second we show that a retract $\mathbf{R}$ of a poset $\mathbf{P}$ of finite type is of finite type. Let $r : \mathbf{P} \to \mathbf{R}$ be an onto retraction and let $e : \mathbf{R} \to \mathbf{P}$ be a corresponding coretraction. Let $(\mathbf{H}, f)$ be an $\mathbf{R}$-colored poset such that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\mathbf{R}$-extendible. The $\mathbf{P}$ colored poset $(\mathbf{H}, e \circ f)$ is $\mathbf{P}$-extendible since $\mathbf{P}$ is of finite type. So there exists a monotone map $g : \mathbf{H} \to \mathbf{P}$ which extends $e \circ f$ to $\mathbf{H}$. But then $r \circ g : \mathbf{H} \to \mathbf{R}$ is an $\mathbf{R}$-extension of f to $\mathbf{H}$ since $r \circ e \circ f = f$. □

COROLLARY 5.4. *The order variety generated by all finite posets is a subvariety of the order variety of all posets of finite type.*

PROOF: Apply Proposition 5.2 and 5.3. □

We do not know if these two varieties are different.

Next we shall prove some general theorems concerning order varieties and, in particular, order varieties generated by finite posets.

For a poset $\mathbf{P}$ let $E(\mathbf{P})$, $N(\mathbf{P})$ and $Z(\mathbf{P})$ denote the class of all $\mathbf{P}$-extendible $\mathbf{P}$-colored posets, the class of all nonextendible $\mathbf{P}$-colored posets and the class of all $\mathbf{P}$-zigzags, respectively.

Throughout the following proofs we frequently use the fact that, if $(\mathbf{H}, f) \in Z(\mathbf{P})$ and $g : \mathbf{P} \to \mathbf{Q}$ is a monotone map then $(\mathbf{H}, g \circ f) \in Z(\mathbf{Q}) \cup E(\mathbf{Q})$.

PROPOSITION 5.5. *Let $\mathbf{P}$ and $\mathbf{R}$ be two posets. Then there exists a retraction r from $\mathbf{P}$ onto $\mathbf{R}$ if and only if there is a monotone map e from $\mathbf{R}$ to $\mathbf{P}$ such that for every $\mathbf{R}$-colored poset $(\mathbf{H}, f)$ in $N(\mathbf{R})$ the $\mathbf{P}$-colored poset $(\mathbf{H}, e \circ f)$ is in $N(\mathbf{P})$.*

PROOF: Let r be a retraction from $\mathbf{P}$ onto $\mathbf{R}$ with a corresponding coretraction e. Let $(\mathbf{H}, f) \in N(\mathbf{R})$. Now, let us suppose $(\mathbf{H}, e \circ f)$ is $\mathbf{P}$-extendible. By applying r to this

P-extension of $(\mathbf{H}, e \circ f)$ we get an **R**-extension of $(\mathbf{H}, r \circ e \circ f) = (\mathbf{H}, f)$ contradicting $(\mathbf{H}, f) \in N(\mathbf{R})$.

Conversely, let us suppose that there is a monotone map e from **R** to **P** such that for every **R**-colored poset $(\mathbf{H}, f) \in N(\mathbf{R})$ we have $(\mathbf{H}, e \circ f) \in N(\mathbf{P})$. For every $a \not\leq b$, $a, b \in R$, the two element chain colored by a at the bottom and colored by b at the top, is in $N(\mathbf{R})$. Hence the two element chain colored by $e(a)$ at the bottom and by $e(b)$ at the top, is in $N(\mathbf{P})$ and so $e(a) \not\leq e(b)$. Thus e preserves the relations $\leq$ and $\not\leq$. So e is an order embedding of **R** into **P**. Let us look at the **R**-colored poset $(\mathbf{P}, e^{-1})$. This is **R**-extendible by a monotone function r, for otherwise, $(\mathbf{P}, e^{-1}) \in N(\mathbf{R})$ and then by the hypothesis, $(\mathbf{P}, e \circ e^{-1}) = (\mathbf{P}, id_{e(R)}) \in N(\mathbf{P})$. This is impossible since id_P extends $id_{e(R)}$ to **P**. Clearly, $r \circ e = id_R$ showing that r is a retraction of **P** onto **R**. □

COROLLARY 5.6. *Let* **P** *be a poset and let* **R** *be a poset of finite type. There exists a retraction* r *from* **P** *onto* **R** *if and only if there is a monotone map* e *from* **R** *to* **P** *such that* $(\mathbf{H}, e \circ f) \in Z(\mathbf{P})$ *whenever* $(\mathbf{H}, f) \in Z(\mathbf{R})$.

PROOF: Use Proposition 5.1 and Proposition 5.5. □

We note that Corollary 5.6 easily implies that there exists an idempotent map from a poset **P** onto a subposet **R** of finite type if and only if every **R**-zigzag is also a **P**-zigzag.

PROPOSITION 5.7. *Let* $\mathbf{P}_i$, $i \in I$, *be a set of posets. Then a* $\prod_{i \in I} \mathbf{P}_i$*-colored poset* $(\mathbf{H}, f) \in N(\prod_{i \in I} \mathbf{P}_i)$ *if and only if there exist two subsets* A *and* B *of* I, *with* $A \neq \emptyset$ *and* $A \cup B = I$, *such that* $(\mathbf{H}, f_i) \in N(\mathbf{P}_i)$ *for every* $i \in A$, $(\mathbf{H}, f_i) \in E(\mathbf{P}_i)$ *for every* $i \in B$, *where* f_i *is the* i*-th component of* f.

PROOF: The claim is obvious. □

COROLLARY 5.8. *Let* $\mathbf{P}_i$, $i \in I$, *be a set of posets. Then a* $\prod_{i \in I} \mathbf{P}_i$*-colored poset* $(\mathbf{H}, f) \in Z(\prod_{i \in I} \mathbf{P}_i)$ *if and only if there exist two subsets* A *and* B *of* I, *with* $A \neq \emptyset$ *and* $A \cup B = I$, *such that* $(\mathbf{H}, f_i) \in Z(\mathbf{P}_i)$ *for every* $i \in A$, $(\mathbf{H}, f_i) \in E(\mathbf{P}_i)$ *for every* $i \in B$, *where* f_i *is the* i*-th component of* f.

PROOF: Use Proposition 5.7. □

PROPOSITION 5.9. *Let $\mathbf{P}$ be a poset and let K be a set of posets. Then $\mathbf{P} \in \mathcal{RP}(K)$ if and only if for every $\mathbf{P}$-colored poset $(\mathbf{H}, f) \in N(\mathbf{P})$ there is a $\mathbf{Q} \in K$ and a monotone map $g : \mathbf{P} \to \mathbf{Q}$ such that $(\mathbf{H}, g \circ f) \in N(\mathbf{Q})$.*

PROOF: Let $\mathbf{P}$ be a retract of the product of $\prod_{i \in I} \mathbf{P}_i$, where $\mathbf{P}_i \in K$ for every $i \in I$ and let $(\mathbf{H}, f) \in N(\mathbf{P})$. Then by Proposition 5.5 there exists a monotone map $e : \mathbf{P} \to \prod_{i \in I} \mathbf{P}_i$ which sends $(\mathbf{H}, f) \in N(\mathbf{P})$ to $(\mathbf{H}, e \circ f) \in N(\prod_{i \in I} \mathbf{P}_i)$. Then by Proposition 5.7 there exists an i such that $(\mathbf{H}, \pi_i \circ e \circ f) \in N(\mathbf{P}_i)$, where π_i is the i-th projection map. So taking $\mathbf{P}_i \in K$ as $\mathbf{Q}$ and $\pi_i \circ e$ as g we get one direction of the claim.

To prove the other direction, by Proposition 5.5 it suffices to show that there exist a set I, posets $\mathbf{P}_i \in K$, $i \in I$, and a monotone map $e : \mathbf{P} \to \prod_{i \in I} \mathbf{P}_i$ such that for every $(\mathbf{H}, f) \in N(\mathbf{P})$ we have $(\mathbf{H}, e \circ f) \in N(\prod_{i \in I} \mathbf{P}_i)$. Since K is a set we may define a set I which contains all pairs $(g, \mathbf{Q})$, where $\mathbf{Q} \in K$ and g is a monotone map from $\mathbf{P}$ to $\mathbf{Q}$. Now we define a map e from $\mathbf{P}$ to $\prod_{(g,\mathbf{Q}) \in I} \mathbf{Q}$ by $e_{(g,\mathbf{Q})}(a) = g(a)$, $a \in P$. Now, e is trivially monotone. Moreover, if $(\mathbf{H}, f) \in N(\mathbf{P})$ then $(\mathbf{H}, e \circ f) \in N(\prod_{(g,\mathbf{Q}) \in I} \mathbf{Q})$ since by the hypothesis there exists some $(g, \mathbf{Q}) \in I$ such that $(\mathbf{H}, g \circ f) \in N(\mathbf{Q})$, and we can apply Proposition 5.7. □

We have an unpleasant hypothesis in the previous proposition, namely that K must be a set rather than a class of posets. This could be avoided if we knew that for any poset $\mathbf{P}$, there exists a cardinal $\kappa(\mathbf{P})$ such that any $\mathbf{P}$-nonextendible colored poset $(\mathbf{H}, f)$ contains a $\mathbf{P}$-nonextendible one of size less than $\kappa(\mathbf{P})$. Alan Mekler has shown, [16], if for every cardinal λ there exists a strongly compact cardinal greater than λ, then $\kappa(\mathbf{P})$ exists. It is known that the existence of a strongly compact cardinal cannot be proven from ZFC, see Theorem 80 in [11], and the assumption used by Mekler is considered to be a very strong one in set theory. Observe, for every poset $\mathbf{P}$ of finite type we can let $\kappa(\mathbf{P}) = \omega$. So we state the following proposition.

Proposition 5.10. *Let* $\mathbf{P}$ *be a poset of finite type and* K *be a class of posets. Then* $\mathbf{P} \in \mathcal{RP}(K)$ *if and only if for every* $(\mathbf{H}, f) \in Z(\mathbf{P})$ *there is a* $\mathbf{Q} \in K$ *and a monotone map* $g : \mathbf{P} \to \mathbf{Q}$ *such that* $(\mathbf{H}, g \circ f) \in Z(\mathbf{Q})$.

PROOF: We get the proof copying the proof of Proposition 5.9, replacing $N(\dots)$ by $Z(\dots)$, Proposition 5.5 and 5.7 by Corollary 5.6 and Proposition 5.8, and changing the definition of I to a set of pairs $(g, \mathbf{Q})$ such that for every $(\mathbf{H}, f) \in Z(\mathbf{P})$ there exists a monotone $g : \mathbf{P} \to \mathbf{Q}$ with $(\mathbf{H}, g \circ f) \in Z(\mathbf{Q})$. □

By a similar argument we can derive Theorem 8 in [18]. From Proposition 5.9 we easily get a well known result of [9].

Corollary 5.11. *Let a finite poset* $\mathbf{P} \in \mathcal{RP}(K)$, *where* K *is a finite set of finite posets. Then* $\mathbf{P}$ *is a retract of a finite product of some members of* K.

PROOF: It follows from the only if part of Proposition 5.9 and from the proof of the if part of Proposition 5.9 since I will be finite. □

6. Finite irreducible posets

The following definitions can be found in [9]. A *representation* of a poset $\mathbf{P}$ is a family $(\mathbf{P}_i \mid i \in I)$ of posets such that $\mathbf{P}_i$ is a retract of $\mathbf{P}$ for each $i \in I$, and $\mathbf{P}$ is a retract of $\prod_{i \in I} \mathbf{P}_i$. A poset $\mathbf{P}$ is *irreducible* if for every representation $(\mathbf{P}_i \mid i \in I)$ of $\mathbf{P}$, $\mathbf{P}$ is a retract of $\mathbf{P}_i$ for some $i \in I$. If $\mathbf{P}$ is not irreducible then it is called *reducible*. For example, the two element antichain, fences and crowns are known to be irreducible posets, see [9].

In this chapter, first we present a poset that has two different representations by irreducible posets, see Claim 6.3 and Example 6.4. This answers a problem in [9]. Then, in Proposition 6.6 we give a characterization of finite irreducible posets by using Proposition 5.10. Via this characterization, in Theorem 6.9 we show that two nonisomorphic, finite irreducible posets generate two different order varieties, which also settles a problem in [9]. Proposition 6.6 is useful for proving the irreducibility of finite posets as demonstrated in Example 6.7.

The following problem on the unique representation of posets is mentioned as Problem 2 in [9].

PROBLEM 6.1. *Let $(\mathbf{P}_i \mid i \in I)$ and $(\mathbf{Q}_j \mid j \in J)$ be two representations of $\mathbf{P}$ such that $\mathbf{P}_i$ and $\mathbf{Q}_j$ are irreducible for every $i \in I$ and $j \in J$. For every $i \in I$ is there a $j \in J$ such that $\mathbf{P}_i$ is a retract of $\mathbf{Q}_j$?*

We will show that this problem is equivalent to the following.

PROBLEM 6.2. *Does there exist an irreducible poset $\mathbf{Q}$ which is a retract of $\prod_{j \in J} \mathbf{Q}_j$, where for all $j \in J$, $\mathbf{Q}_j$ is an irreducible poset and $\mathbf{Q}$ is not a retract of $\mathbf{Q}_j$?*

CLAIM 6.3. *If the answer is no for Problem 6.1 then the answer is yes for Problem 6.2 and conversely.*

PROOF: Let us suppose that the answer is no for Problem 6.1. Then there exists $\mathbf{P}$ which has two representations $(\mathbf{P}_i \mid i \in I)$ and $(\mathbf{Q}_j \mid j \in J)$ and there exists an $i \in I$ such that

$\mathbf{P}_i$ is not a retract of $\mathbf{Q}_j$ for any $j \in J$. Now, $\mathbf{P}_i$ is a retract of $\mathbf{P}$ and hence of $\prod_{j\in J}\mathbf{Q}_j$. So, in Problem 6.2 taking $\mathbf{Q} = \mathbf{P}_i$ we get the answer yes.

For the converse, let us suppose that we answered Problem 6.2 affirmatively, i.e., there exists an irreducible poset $\mathbf{Q}$ which is a retract of $\prod_{j\in J}\mathbf{Q}_j$, where $\mathbf{Q}_j$, $j \in J$, are irreducible posets, such that $\mathbf{Q}$ is not a retract of $\mathbf{Q}_j$ for any $j \in J$. Then let $\mathbf{P} = \prod_{j\in J}\mathbf{Q}_j$. Now, $(\mathbf{Q}_j \mid j \in J)$ is a representation of $\mathbf{P}$ by irreducibles since the j-th projection map on $\mathbf{P}$ is a retraction. An other representation of $\mathbf{P}$ by irreducible posets is given by $\mathbf{Q}$ and $\mathbf{Q}_j$, $j \in J$, since we can define a retraction from $\mathbf{Q} \times \prod_{j\in J}\mathbf{Q}_j$ onto $\mathbf{P}$ by $f(q, q_1, q_2, \dots) = (q_1, q_2, \dots)$, where $q \in Q$ and $q_j \in Q_j$, $j \in J$. Thus, we have the above two representations of $\mathbf{P}$ by irreducibles, and because $\mathbf{Q}$ is not a retract of $\mathbf{Q}_j$ for any $j \in J$, the answer is no for Problem 6.1. $\square$

The following example, see footnote 4, p. 85 in [9], shows that Problem 6.2 has the answer yes and so by the previous claim Problem 6.1 has the answer no.

EXAMPLE 6.4. *Let $\mathbf{Q}$ be the two element antichain and let $\mathbf{Q}_j$ be a j-element fence, $j \in J = \omega$. Then $\mathbf{Q}$ is a retract of the product $\prod_{j\in J}\mathbf{Q}_j$, but $\mathbf{Q}$ is not a retract of any $\mathbf{Q}_j$, $j \in J$.*

PROOF: Since $J = \omega$, $\prod_{j\in J}\mathbf{Q}_j$ is not connected so $\mathbf{Q}$ is a retract of $\prod_{j\in J}\mathbf{Q}_j$. On the other hand, a retraction preserves the connectedness of posets. So the nonconnected poset $\mathbf{Q}$ cannot be a retract of a connected poset $\mathbf{Q}_j$, $j \in J$. $\square$

We note that Problem 6.2 turns into Problem 5 in [9] if we assume that $\{\mathbf{Q}_j \mid j \in J\}$ is a finite set of finite posets. That problem remains unsolved.

Now we characterize finite irreducible posets in terms of their zigzags. We begin with a consequence of Proposition 5.10.

PROPOSITION 6.5. *A finite poset $\mathbf{P}$ is irreducible if and only if there exists a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that for every monotone map $g : \mathbf{P} \to \mathbf{P}$ either the range of g is not a subset of a proper retract of $\mathbf{P}$ or $(\mathbf{H}, g \circ f)$ is $\mathbf{P}$-extendible.*

PROOF: Let us suppose that $\mathbf{P}$ is irreducible and for every $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ there is a monotone map $g : \mathbf{P} \to \mathbf{P}$ such that the range of g is a subset of a proper retract $\mathbf{R}$ of $\mathbf{P}$ and $(\mathbf{H}, g \circ f)$ is not $\mathbf{P}$-extendible. Since $\mathbf{R}$ is a subposet of $\mathbf{P}$, $(\mathbf{H}, g \circ f)$ is also not $\mathbf{R}$-extendible. Hence $(\mathbf{H}, g \circ f)$ is an $\mathbf{R}$-zigzag. Then by Proposition 5.10, $\mathbf{P}$ is a retract of a product of its proper retracts, which contradicts the fact that $\mathbf{P}$ is irreducible.

Now, let $\mathbf{P}$ be reducible. Then there exists a representation of $\mathbf{P}$ by $(\mathbf{P}_i \mid i \in I)$ where each $\mathbf{P}_i$ is a retract of $\mathbf{P}$ with $|P_i| < |P|$. We can assume that each $\mathbf{P}_i$, $i \in I$, is the image of a monotone, idempotent map $r_i : \mathbf{P} \to \mathbf{P}$. Then by Proposition 5.10, for every $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ there is an $i \in I$ and a monotone $g : \mathbf{P} \to \mathbf{P}_i$ such that $(\mathbf{H}, g \circ f)$ is a $\mathbf{P}_i$-zigzag. Observe that g maps $\mathbf{P}$ to $\mathbf{P}$ and $(\mathbf{H}, g \circ f)$ is not $\mathbf{P}$-extendible otherwise $(\mathbf{H}, r_i \circ g \circ f) = (\mathbf{H}, g \circ f)$ would be $\mathbf{P}_i$-extendible. So the range of g is in a proper retract of $\mathbf{P}$ and $(\mathbf{H}, g \circ f)$ is not $\mathbf{P}$-extendible, which proves the claim. □

For a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ and a monotone map $g : \mathbf{P} \to \mathbf{P}$, if the $\mathbf{P}$-colored poset $(\mathbf{H}, g \circ f)$ is $\mathbf{P}$-extendible we say that g *collapses* $(\mathbf{H}, f)$.

We can strengthen the result of Proposition 6.5.

PROPOSITION 6.6. *Let $\mathbf{P}$ be a finite poset. Then $\mathbf{P}$ is irreducible if and only if there exists a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that every monotone, non-onto map $g : \mathbf{P} \to \mathbf{P}$ collapses $(\mathbf{H}, f)$.*

PROOF: Let $\mathbf{P}$ be a finite, irreducible poset. For every finite poset $\mathbf{H}$ let $S_{\mathbf{H}}$ be the set of all the $\mathbf{P}$-zigzags $(\mathbf{H}, f)$ such that for every monotone map $g : \mathbf{P} \to \mathbf{P}$ the range of g is not a subset of a proper retract of $\mathbf{P}$ or g collapses $(\mathbf{H}, f)$. By Proposition 6.5 we can select a poset $\mathbf{H}$ for which $S_{\mathbf{H}}$ is not empty. We define a relation $<$ on $S_{\mathbf{H}}$ by $(\mathbf{H}, f_1) < (\mathbf{H}, f_2)$ if and only if there is a monotone, non-onto map $g : \mathbf{P} \to \mathbf{P}$ such that $f_1 = g \circ f_2$. Clearly, $<$ is transitive. We claim that $<$ is irreflexive. Let us suppose instead that $(\mathbf{H}, f) < (\mathbf{H}, f)$ in $S_{\mathbf{H}}$. Then there exists a monotone, non-onto $g : \mathbf{P} \to \mathbf{P}$ such that $(\mathbf{H}, f) = (\mathbf{H}, g \circ f)$. This equality implies $(\mathbf{H}, f) = (\mathbf{H}, g^n \circ f)$ for any finite n. Let us select an n for which g^n is an idempotent map. Since g is not onto and g^n is idempotent, the range of g^n is in a proper retract of $\mathbf{P}$. Because $(\mathbf{H}, f) = (\mathbf{H}, g^n \circ f)$ the map g^n does not collapse $(\mathbf{H}, f)$. This contradicts $(\mathbf{H}, f) \in S_{\mathbf{H}}$.

Let $(\mathbf{H}, f)$ be a minimal element of $(S_{\mathbf{H}}, <)$. Let $g : \mathbf{P} \to \mathbf{P}$ an arbitrary monotone, non-onto map. We show that g collapses $(\mathbf{H}, f)$. Let us suppose this is not true. Then we claim that the $\mathbf{P}$-zigzag $(\mathbf{H}, g \circ f) \in S_{\mathbf{H}}$. Let $g' : \mathbf{P} \to \mathbf{P}$ be a monotone map. Since $(\mathbf{H}, f) \in S_{\mathbf{H}}$ the range of $g' \circ g$ is not in a proper retract of $\mathbf{P}$ or $g' \circ g$ collapses $(\mathbf{H}, f)$. Hence either the range of g' is not in a proper retract of $\mathbf{P}$ or g' collapses $(\mathbf{H}, g \circ f)$. So $(\mathbf{H}, g \circ f) \in S_{\mathbf{H}}$, which contradicts the minimality of $(\mathbf{H}, f)$ in $(S_{\mathbf{H}}, <)$. The other direction of the proof obviously follows from Proposition 6.5. $\square$

Proposition 6.6 is a useful tool for proving irreducibility of particular posets.

EXAMPLE 6.7. *The poset* $\mathbf{P}$ *in Figure 9 is irreducible.*

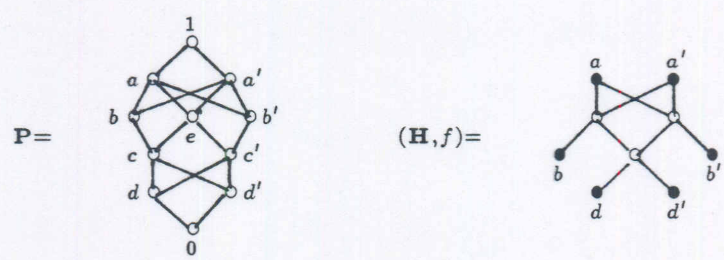


Figure 9. Poset $\mathbf{P}$ and the $\mathbf{P}$-zigzag $(\mathbf{H}, f)$

PROOF: It is easily seen that the $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ in Figure 9 is a $\mathbf{P}$-zigzag. We show that $(\mathbf{H}, f)$ is collapsed by every monotone, non-onto map on $\mathbf{P}$. Let us suppose this is not true. Then there exists a monotone non-onto map $g : \mathbf{P} \to \mathbf{P}$ such that $(\mathbf{H}, g \circ f)$ is a $\mathbf{P}$-zigzag. First, observe that $g(a) \| g(a')$, $g(b) \| g(b')$ and $g(d) \| g(d')$, otherwise $(\mathbf{H}, g \circ f)$ would be $\mathbf{P}$-extendible. Now, $g(a) < a', a$ is impossible. Otherwise $g(a') < a', a$ as well and we cannot find the six element subposet determined by $g(a)$, $g(a')$, $g(b)$, $g(b')$, $g(d)$ and $g(d')$ in $\mathbf{P}$. Thus $g(\{a, a'\}) = \{a, a'\}$. Dually, $g(\{d, d'\}) = \{d, d'\}$. Hence $g(\{b, b'\}) = \{b, b'\}$, otherwise $(\mathbf{H}, g \circ f)$ would be extendible. Let r be an idempotent power of g. Then r is monotone and fixes a, a', b, b', d and d'. These properties of r imply that r is the identity map on $\mathbf{P}$. But this contradicts the fact that g is non-onto. So by Proposition 6.6, $\mathbf{P}$ is irreducible. $\square$

By the same argument one can prove that the posets of the form $1+2+\mathbf{F}+2+1$, where $\mathbf{F}$ is a fence with at least four elements, are irreducible. We can prove this fact more easily, if we use the complete list of zigzags of $1+2+\mathbf{F}+2+1$ described by Proposition 4.1 and Theorem 4.3. The point is that Proposition 6.6 can be used for showing the irreducibility of a finite poset $\mathbf{P}$ even in the case when we do not have a full description of the $\mathbf{P}$-zigzags.

Similarly, as in Example 6.7 one can prove that $1+2+2+2+1$ is irreducible. So for those who do not like Example 6.4, because $\mathbf{Q}$ is not connected, we can replace it by the following example.

EXAMPLE 6.8. *Let* $\mathbf{Q} = 1+2+2+2+1$ *and let* $\mathbf{Q}_j = 1+2+\mathbf{F}_j+2+1$, *where* $\mathbf{F}_j$ *is a* j*-element fence and* $j \in J = \{4,5,6,\dots\}$. *Then* $\mathbf{Q}$ *is a retract of* $\prod_{j\in J}\mathbf{Q}_j$ *but* $\mathbf{Q}$ *is not a retract of any* $\mathbf{Q}_j$, $j \in J$.

PROOF: Recall that in Chapter 4 we described the $\mathbf{Q}$-zigzags and from Theorem 4.1 by applying Theorem 4.3 we get the $\mathbf{Q}_j$-zigzags, $j \in J$. Let $(\mathbf{H}, f)$ be an arbitrary $\mathbf{Q}$ zigzag. By Proposition 5.10 the only thing we have to prove is that for $(\mathbf{H}, f)$ there exist a $j \in J$ and a monotone map $g : \mathbf{Q} \to \mathbf{Q}_j$ such that $(\mathbf{H}, g \circ f)$ is $\mathbf{Q}_j$-zigzag. A large enough j and a monotone embedding g of $\mathbf{Q}$ into $\mathbf{Q}_j$ such that the elements at the middle level of $\mathbf{Q}$ go to the endpoints of $\mathbf{F}_j$, will be a suitable choice. By Proposition 5.6, $\mathbf{Q}$ is not a retract of any $\mathbf{Q}_j$, $j \in J$, since $\mathbf{Q}$ has infinitely many zigzags and each $\mathbf{Q}_j$ has only finitely many of them. $\square$

With the help of Proposition 6.6 we can answer Problem 4 in [9].

THEOREM 6.9. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be two nonisomorphic, finite, irreducible posets. Then the order varieties generated by them are different.*

PROOF: Let us suppose the claim is not true. By Proposition 6.6 there exists a $\mathbf{P}$-

zigzag $(\mathbf{H}, f)$ such that every monotone, non-onto map on $\mathbf{P}$ collapses $(\mathbf{H}, f)$. Since $\mathbf{P} \in \mathcal{RP}(\{\mathbf{Q}\})$, by Proposition 5.10 there exists a monotone map $g : \mathbf{P} \to \mathbf{Q}$ such that $(\mathbf{H}, g \circ f)$ is $\mathbf{Q}$-zigzag. Since $\mathbf{Q} \in \mathcal{RP}(\{\mathbf{P}\})$ we can apply Proposition 5.10 again. So there exists a monotone map $h : \mathbf{Q} \to \mathbf{P}$ such that $(\mathbf{H}, h \circ g \circ f)$ is $\mathbf{P}$-zigzag. But then the map $h \circ g : \mathbf{P} \to \mathbf{P}$, which does not collapse $(\mathbf{H}, f)$, has to be onto. So h is an onto map from $\mathbf{Q}$ to $\mathbf{P}$. Similarly we can get a monotone, onto map from $\mathbf{P}$ to $\mathbf{Q}$. Hence, by finiteness we get that $\mathbf{P}$ is isomorphic to $\mathbf{Q}$, which is a contradiction. $\square$

7. Finite bounded posets admitting an n-ary near unanimity function

A finite algebra A with a near unanimity term operation has a lot of nice algebraic properties. For example, the clone of term operations on A is finitely generated. The variety generated by A is congruence distributive or equivalently among the term operations on A there are Jónsson operations, which are defined in Chapter 8. During the study of monotone algebras, in the papers [2], [3], [5], [14], [22] and [26], the question occurred whether for a monotone algebra A of a finite bounded poset these seemingly weaker properties imply that A has a near unanimity term operation. The answer for this question has turned out to be hard. The exploration of problems of this kind made clear the central role the posets admitting a near unanimity function have in the study of monotone algebras.

Remark 2.4 gives a characterization of finite posets admitting a n-ary near unanimity function. We would like to find a somewhat more constructive description, similar to the result of Quackenbush, Rival and Rosenberg in [22], which states that every finite poset admitting a majority function is a retract of a finite product of fences and the two element antichain. Observe, the building elements here, i.e., the fences and the two element antichain, are irreducible posets admitting a majority function. In general, we can expect a similar characterization of the finite posets admitting an n-ary near unanimity function as the next proposition states.

PROPOSITION 7.1. *The class of the finite posets admitting an n-ary near unanimity operation coincides with the class of retracts of finite products of irreducible posets which admit an n-ary near unanimity operation.*

PROOF: The retract and the product of posets preserve the existence of an n-ary near unanimity operation. On the other hand, every finite poset has a finite representation by irreducibles, see [9]. These facts imply the claim. □

We note that a similar claim is true for finite bounded posets admitting an n-ary near unanimity function.

In Proposition 7.3 we give a property of finite irreducible posets with a nuf that helps us to decide for a particular finite poset with a nuf whether it is irreducible or not. In Proposition 7.4 we determine all irreducible finite bounded posets admitting an n-ary near unanimity function, where $n \leq 6$.

For a finite irreducible poset $\mathbf{P}$, a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is called a *prime* $\mathbf{P}$*-zigzag* if every non-onto monotone map on $\mathbf{P}$ collapses $(\mathbf{H}, f)$. Since $\mathbf{P}$ is irreducible, by Proposition 6.6, $\mathbf{P}$ has at least one prime zigzag. Recall, we defined retractable and irreducible elements for a finite poset in Chapter 3. For a finite poset $\mathbf{P}$ let $Re(\mathbf{P})$ denote the set of retractable elements of $\mathbf{P}$ and let $Ir(\mathbf{P})$ denote the set of irreducible elements of $\mathbf{P}$.

First we prove an easy corollary of Proposition 6.6.

COROLLARY 7.2. *Let* $\mathbf{P}$ *be a finite irreducible poset. Then for every prime* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *we have* $Re(\mathbf{P}) \subseteq Range(f)$.

PROOF: Let us suppose the claim is not true. Then there exists $a \in Re(\mathbf{P}) \setminus Range(f)$. So there is an idempotent monotone map r from $\mathbf{P}$ onto $\mathbf{P} \setminus \{a\}$. But then $(\mathbf{H}, r \circ f) = (\mathbf{H}, f)$. Thus r does not collapse $(\mathbf{H}, f)$. Since $(\mathbf{H}, f)$ is prime and r is non-onto this is a contradiction. $\square$

Let $\mathbf{P}$ be a finite poset and let $\mathbf{Q}$ be the poset of noncolored elements in a standard $\mathbf{P}$-zigzag. We define a poset $\mathbf{M_Q}$ on the set of standard $\mathbf{P}$-zigzags that have $\mathbf{Q}$ as their poset of noncolored elements as follows. For two standard zigzags $(\mathbf{H}, f)$ and $(\mathbf{H}', f')$ with $\mathbf{N}(\mathbf{H}, f) = \mathbf{N}(\mathbf{H}', f') = \mathbf{Q}$ we write $(\mathbf{H}, f) \leq (\mathbf{H}', f')$ if the following hold. For every $a \in C(\mathbf{H}', f')$ with $h \prec_{\mathbf{H}'} a$ there is $b \in C(\mathbf{H}, f)$ with $h \prec_{\mathbf{H}} b$ such that $f(b) \leq f'(a)$, and dually, for every $a \in C(\mathbf{H}', f')$ with $h \succ_{\mathbf{H}'} a$ there is $b \in C(\mathbf{H}, f)$ with $h \succ_{\mathbf{H}} b$ such that $f(b) \geq f'(a)$. It is easy to check that $\leq$ indeed is a partial order. We note that antisymmetry can be verified by Claim 3.5. A maximal element of $\mathbf{M_Q}$ is called a *stretched zigzag*. So a stretched zigzag is always standard. In the proof of the following proposition we make good use of the stretched zigzags.

PROPOSITION 7.3. *Every finite connected irreducible poset* $\mathbf{P}$ *that admits a nuf has a prime* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *such that* $Range(f) = Ir(\mathbf{P})$. *In particular,* $Re(\mathbf{P}) = Ir(\mathbf{P})$.

PROOF: If $\mathbf{P}$ is lattice ordered the claim is obvious. So in the proof we assume that $\mathbf{P}$ is not lattice ordered. This easily implies that every prime zigzag is monotone. First we note that $Re(\mathbf{P}) = Ir(\mathbf{P})$ follows from $Ir(\mathbf{P}) \subseteq Re(\mathbf{P})$ and the previous corollary if we prove the existence of a prime $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ with $Range(f) = Ir(\mathbf{P})$. Let us suppose there is no prime $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ with $Range(f) = Ir(\mathbf{P})$. So every prime $\mathbf{P}$-zigzag contains an element colored from $\mathbf{P} \setminus Ir(\mathbf{P})$. Let us take a prime zigzag $(\mathbf{H}_0, f_0)$. We can obviously choose $(\mathbf{H}_0, f_0)$ to be a stretched zigzag. Let $h \in C(\mathbf{H}_0, f_0)$ with $f_0(h) \in \mathbf{P} \setminus Ir(\mathbf{P})$. Since $(\mathbf{H}_0, f_0)$ is standard h is minimal or maximal in $\mathbf{H}_0$. Without loss of generality we can assume that h is maximal. Since $f_0(h) \in \mathbf{P} \setminus Ir(\mathbf{P})$, $f_0(h)$ is maximal or covered by at least two elements. If $f_0(h)$ is maximal we define a new zigzag $(\mathbf{H}'_1, f'_1)$ as follows. We delete the color of h in $(\mathbf{H}_0, f_0)$ and put an element below h that is covered by only h and is colored by $f_0(h)$. The resulting colored poset $(\mathbf{H}'_1, f'_1)$ clearly is a zigzag. If $f_0(h)$ is not maximal, then it is covered by two elements a and b in $\mathbf{P}$. Then we define the following zigzag $(\mathbf{H}'_1, f'_1)$. We delete the color of h in $(\mathbf{H}_0, f_0)$ and we put an element below h that is covered by only h and is colored by $f_0(h)$. We put two elements above h that cover only h and are colored by a and b. Since $(\mathbf{H}_0, f_0)$ is stretched, by Proposition 3.1 the resulting colored poset $(\mathbf{H}'_1, f'_1)$ is a $\mathbf{P}$-zigzag. One can see that in both definitions $(\mathbf{H}'_1, f'_1)$ is prime and standard. We define $(\mathbf{H}_1, f_1)$ to be a stretched zigzag above $(\mathbf{H}'_1, f'_1)$ in $\mathbf{M}_{\mathbf{N}(\mathbf{H}'_1, f'_1)}$. Note that $(\mathbf{H}_1, f_1)$ is prime.

We can iterate the above method to get an infinite sequence of prime, stretched zigzags $(\mathbf{H}_i, f_i)$, $i = 0, 1, \ldots$. Let c_i be the number of colored elements in $(\mathbf{H}_i, f_i)$. Clearly, $c_i \leq c_{i+1}$ for every $i \geq 0$. Observe that $c_i < c_{i+1}$ holds for infinitely many i. For otherwise there exists an i_0 such that $c_i = c_{i_0}$ if $i \geq i_0$. Hence, when we constructed $(\mathbf{H}_{i+1}, f_{i+1})$ from $(\mathbf{H}_i, f_i)$ for $i \geq i_0$ we had to choose a colored element h of $(\mathbf{H}_i, f_i)$ that was colored by a maximal or minimal element of $\mathbf{P}$. This implies that for a large enough $j \geq i_0$, $\mathbf{H}_j$

contains a fence $\mathbf{F}$ with the following properties. The fence $\mathbf{F}$ has a diameter greater than the diameter of $\mathbf{P}$. Moreover, if a and b denote the two endpoints of $\mathbf{F}$, then none of the elements of $F \setminus \{a\}$ are comparable with any element of $H_j \setminus F$ in $\mathbf{H}_j$, and the only colored element of $\mathbf{F}$ in $(\mathbf{H}_j, f_j)$ is b. Now, if we cancel $F \setminus \{a\}$ in $(\mathbf{H}_j, f_j)$ the resulting colored poset is extendible. Let f' be an extension of this colored poset. Then f' can easily be extended to $(\mathbf{H}_j, f_j)$ since $\mathbf{F}$ has a diameter larger than the diameter of $\mathbf{P}$. This contradicts the fact that $(\mathbf{H}_j, f_j)$ is a zigzag. We can finish the proof by applying Remark 2.4. $\square$

We cannot prove the converse of Corollary 7.3. Namely, we do not know, if every finite connected irreducible poset $\mathbf{P}$ which has a prime zigzag $(\mathbf{H}, f)$ with $Range(f) = Ir(\mathbf{P})$ admits a nuf.

PROPOSITION 7.4. *The list of the finite, irreducible bounded posets admitting a* 6*-nuf is the following:* 1, $1+1$ *with* 3*-nufs,* $\mathbf{S}_2+\mathbf{S}_2^d$ *with a* 5*-nuf,* $\mathbf{S}_2+\mathbf{S}_3^d$, $\mathbf{S}_3+\mathbf{S}_2^d$ *with* 6*-nufs, where* $\mathbf{S}_n$ *is the poset given by the Boolean lattice of* n *atoms without its top element and* $\mathbf{S}_n^d$ *is the dual of* $\mathbf{S}_n$.

PROOF: Let $\mathbf{P}$ be a finite, irreducible bounded poset with a 6-nuf. We show that for a monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$, $|N(\mathbf{H}, f)| = 1$. It suffices to show this for a standard zigzag. So let us assume $(\mathbf{H}, f)$ is a standard zigzag and $|N(\mathbf{H}, f)| \geq 2$. By Remark 2.4 we have $|C(\mathbf{H}, f)| \leq 5$. By Claim 3.9 and 3.10 every maximal element of $\mathbf{N}(\mathbf{H}, f)$ is covered by at least two colored elements of $(\mathbf{H}, f)$ and every minimal element of $\mathbf{N}(\mathbf{H}, f)$ covers at least two colored elements of $(\mathbf{H}, f)$. These facts imply that $N(\mathbf{H}, f)$ has one maximal and one minimal element. Since $|N(\mathbf{H}, f)| \geq 2$ there is a noncolored element which covers the bottom element of $\mathbf{N}(\mathbf{H}, f)$. This noncolored element must also cover a colored element, for otherwise it would be irreducible, which contradicts Claim 3.9. Dually, there is a noncolored element covered by the top element of $\mathbf{N}(\mathbf{H}, f)$ which is also covered by a colored element. But then $|C(\mathbf{H}, f)| \geq 6$ which is a contradiction.

So $|N(\mathbf{H}, f)| = 1$ and by Proposition 2.3, $\mathbf{P}$ has the strong selection property. Now, we can invoke a result of Nevermann and Wille in [20] which gives a complete list of the finite irreducible posets having the strong selection property. By using this result and Remark 2.4 we get the list of posets mentioned in the claim. □

We note that there are finite, irreducible bounded posets admitting a 7-nuf that do not have the strong selection property. For example, the bounded posets in Figure 6 and Figure 8 in Chapter 4 are easily shown to be irreducible by Proposition 6.6. Both posets have a zigzag with two noncolored elements. So, by Proposition 2.3, they do not have the strong selection property. We conjecture that there are infinitely many finite, irreducible bounded posets which admit a 7-nuf.

8. Finite bounded posets admitting Jónsson operations

Ternary operations d_i, $0 \le i \le n$, on a set are called *Jónsson operations* if they satisfy the so called *Jónsson identities* given by

$$\begin{array}{llr} & d_0(x,y,z) = d_n(z,y,x) = d_i(x,y,x) = x & \text{for } 0 \le i \le n, \\ & d_{2i}(x,x,y) = d_{2i+1}(x,x,y) & \text{for } 0 \le i \le (n-1)/2 \\ \text{and} & d_{2i+1}(x,y,y) = d_{2i+2}(x,y,y) & \text{for } 0 \le i \le (n-2)/2. \end{array}$$

As we mentioned in Chapter 1 an algebra has Jónsson operations among its term operations if and only if the variety generated by the algebra is congruence distributive. It is well known that, if an algebra has a near unanimity function among its term operations, then it also has Jónsson operations, see for example [17]. The converse of this claim for monotone algebras of finite bounded posets is conjectured in [2], [3] and [14]. The main result of this thesis is the proof of this conjecture in Theorem 8.1. In Theorem 8.1 we give a list of equivalent properties for finite bounded posets. The first property on the list is that a finite bounded poset **P** admits a near unanimity function and the second one is that **P** admits Jónsson operations. We also provide an example of an infinite bounded poset which admits Jónsson operations but admits no nuf, see Example 8.3. In Proposition 8.4 we give an interesting property of finite bounded posets admitting Jónsson operations.

We define the *diameter* of a colored poset to be the diameter of its base poset.

THEOREM 8.1. *For a finite bounded poset* **P** *the following are equivalent:*

(1) **P** *admits a near unanimity function.*

(2) **P** *admits Jónsson operations.*

(3) **P** *admits ternary operations $D_1, \ldots, D_{n'}$, for an $n' \ge 1$, satisfying*

$$\begin{array}{llr} & D_1(x,x,y) = D_{n'}(y,x,x) = D_i(x,y,x) = x & \text{for } 1 \le i \le n' \\ \text{and} & D_i(x,y,y) = D_{i+1}(x,x,y) & \text{for } 1 \le i \le n'-1. \end{array}$$

(4) *There exists a partially defined, monotone n-nuf for some n, which is fully defined on the set of n-tuples $A_n = \{(a, \ldots, a, \underset{i}{\underbrace{b}}, c, \ldots, c) : a, b, c \in P,\ 1 \le i \le n\}$.*

(5) *There exists a finite m such that every* **P***-zigzag has a diameter at most m.*

(6) *The number of* **P***-zigzags is finite.*

PROOF: (1) implies (2): This is well known. One can prove it easily as follows. Let $f : \mathbf{P}^s \to \mathbf{P}$ be a monotone nuf. Then we define $d_{2i-1}(x,y,z) = f(z,\ldots,z,\underbrace{y}_{i+1},x,\ldots,x)$ and $d_{2i}(x,y,z) = d_{2i-1}(x,z,z)$ for $1 \leq i \leq s-1$. Let $d_0(x,y,z) = x$. So the operations $d_j(x,y,z)$, $0 \leq j \leq 2(s-2)$, are Jónsson operations.

(2) implies (3): In [14] McKenzie proves that $\mathbf{P}$ admits Jónsson operations if and only if $\mathbf{P}$ admits operations $b_0(x,y),\ldots,b_{m'}(x,y)$ which satisfy

$$\begin{array}{rcl} & x = b_0(x,y) = b_i(x,x) = b_{m'}(y,x) & \text{for } 0 \leq i \leq m', \\ & b_{2i}(x,y) \leq b_{2i+1}(x,y) & \text{for } 0 \leq i \leq (m'-1)/2 \\ \text{and} & b_{2i+1}(x,y) \geq b_{2i+2}(x,y) & \text{for } 0 \leq i \leq (m'-2)/2. \end{array}$$

In fact, in his proof, when he proves the if part (Theorem 2.3), he uses the b_i to construct monotone Jónsson operations $d_0(x,y,z),\ldots,d_{2n'-1}(x,y,z)$, where the operations with even indices do not depend on their second variable. With the help of the d_{2i-1}, $1 \leq i \leq n'$, we define

$$D_1(x,y,z) = d_1(x,y,z),\ldots,D_{n'}(x,y,z) = d_{2n'-1}(x,y,z).$$

For these operations the first line of identities in (3) immediately follows from the Jónsson identities, and the second line of identities in (3) follows from

$$D_i(x,y,y) = d_{2i-1}(x,y,y) = d_{2i}(x,y,y) = d_{2i}(x,x,y) = d_{2i+1}(x,x,y) = D_{i+1}(x,x,y).$$

Thus, if $\mathbf{P}$ admits Jónsson operations then $\mathbf{P}$ admits operations defined in (3).

(3) implies (4): Let $B_i = \{(a,\ldots,a,\underbrace{b}_{i},c,\ldots,c) : a,b,c \in P\} \subseteq P^{n'+2}$ for $1 \leq i \leq n'+2$. Note $B_1 \subseteq B_2$ and $B_{n'+2} \subseteq B_{n'+1}$. So $A_{n'+2} = \cup_{i=2}^{n'+1} B_i$. Let $D_1,\ldots,D_{n'}$ be the ternary operations given in (3). We define an $(n'+2)$-nuf f on $A_{n'+2}$. Let $f(a,\ldots,a,\underbrace{b}_{i},c,\ldots,c) = D_{i-1}(c,b,a)$ be defined on B_i for $2 \leq i \leq n'+1$. Observe, if $\mathbf{d} = (a,\ldots,a,b,c,\ldots,c) \in B_i \cap B_j$, where $2 \leq i < j \leq n'+1$, then either $\mathbf{d}$ is a constant vector or $a = b \neq c$ or $a \neq b = c$. In the last two cases j has to be $i+1$. Since $f(a',\ldots,\underbrace{a'}_{i},b',\ldots,b') = D_{i-1}(b',a',a') = D_i(b',b',a')$ for $2 \leq i \leq n'$, $f(\mathbf{d})$ is defined the same on B_i and B_{i+1}.

Thus f is a well defined function on $A_{n'+2}$. Also, f is a nuf on $A_{n'+2}$ because $D_1(x,x,y) = D_{n'}(y,x,x) = D_i(x,y,x) = x$ for $1 \le i \le n'$.

Lastly, we show that f is monotone on $A_{n'+2}$. Let $\mathbf{d} = (a,\ldots,a,\underbrace{b}_{i},c,\ldots,c) < \mathbf{e} = (a',\ldots,a',\underbrace{b'}_{j},c',\ldots,c')$, where $\mathbf{d},\mathbf{e} \in A_{n'+2}$. We want to show $f(\mathbf{d}) \le f(\mathbf{e})$. If $i = j$ the proof is obvious. The $j < i$ case is the dual of the $i < j$ case. So let $i < j$. Then

$$\begin{aligned} f(\mathbf{d}) &= D_{i-1}(c,b,a) \le D_{i-1}(c,a',a') = D_i(c,c,a') \\ &\le D_i(c,a',a') = \cdots \le D_{j-2}(c,a',a') = D_{j-1}(c,c,a') \le D_{j-1}(c',b',a') = f(\mathbf{e}), \end{aligned}$$

which proves the claim.

(4) implies (5): Let n be as in (4) and let us suppose there is a $\mathbf{P}$-zigzag with a diameter at least $n+2$. Then, as in Proposition 3.4, from this zigzag we can construct a standard zigzag $(\mathbf{H},f)$ which still has a diameter at least $n+2$. Hence $\mathbf{N}(\mathbf{H},f)$ has a diameter $d \ge n$. Let us select two points $a,b \in N(\mathbf{H},f)$ such that their distance is d in $\mathbf{N}(\mathbf{H},f)$. We know that cancelling a in $(\mathbf{H},f)$ leaves a $\mathbf{P}$-colored poset that is $\mathbf{P}$-extendible. Let f_a be such a monotone extension. Similarly, let f_b be a monotone extension corresponding to the cancellation of b in $(\mathbf{H},f)$. Let $B_i = \{h : h \in N(\mathbf{H},f)$ and h has down distance i from a in $\mathbf{N}(\mathbf{H},f)\}$, where $1 \le i \le d+1$. We note that by definition $a \in B_1$. Let $d_0 = d$ if $B_{d+1} = \emptyset$ otherwise $d_0 = d+1$.

Now, we make some observations. Since b has a distance d from a, $B_i \ne \emptyset$ for $1 \le i \le d_0$. The sets $B_1,\ldots,B_{d_0}$ give a partition of $N(\mathbf{H},f)$ with $a \in B_1$ and $b \in B_{d_0}$. For any $1 \le i \le d_0$, B_i is a down set if i is odd and B_i is an up set if i is even. Moreover, $\cup_{j=1}^{i-1} B_j$ and $N(\mathbf{H},f) \setminus \cup_{j=1}^{i} B_j$ span two subposets of $\mathbf{N}(\mathbf{H},f)$ which cannot be connected in $\mathbf{N}(\mathbf{H},f) \setminus B_i$.

We define a function g_i on H for every $1 \le i \le d_0$. Let g_i be f_b on $\cup_{j=1}^{i-1} B_j$ and let g_i be f_a on $N(\mathbf{H},f) \setminus \cup_{j=1}^{i} B_j$. On B_i let g_i be equal to 0 if i is odd and 1 if i is even. Since $(\mathbf{H},f)$ is standard every element of $C(\mathbf{H},f)$ is connected to $N(\mathbf{H},f)$ by a single covering edge. Depending on if i is odd or i is even we define g_i to be 0 or 1 on those elements of

$C(\mathbf{H}, f)$ which are connected to some element of B_i by covering edges. For the remaining elements of $C(\mathbf{H}, f)$ the function g_i is defined by the corresponding values of f. By the previous observations g_i, clearly, is a monotone function from $\mathbf{H}$ to $\mathbf{P}$.

Since $d_0 \geq n$, by the hypothesis there exists M_{d_0}, a monotone partial d_0-nuf, which is fully defined on A_{d_0}. Now, $M_{d_0}(g_1(x), \ldots, g_{d_0}(x))$ is a monotone map from $\mathbf{H}$ to $\mathbf{P}$ which extends f to H. This contradicts the fact that $(\mathbf{H}, f)$ is a zigzag.

(5) implies (6): For a finite poset $\mathbf{Q}$ let $\ell(\mathbf{Q})$ denote the number of elements in a subchain of maximum cardinality. For an $a \in Q$ let $\ell_{\mathbf{Q}}(a)$ denote the maximum number of elements in a subchain with a top element a. Of course, we always have $\ell_{\mathbf{Q}}(a) \leq \ell(\mathbf{Q})$ for every $a \in Q$.

Let us suppose (5) is true and $\mathbf{P}$ has infinitely many zigzags. Let k be the cardinality of $\mathbf{P}$. Since $\mathbf{P}$ is finite there is $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that $|H| \geq \sum_{i=0}^{m+1} k^{ik}$. By Proposition 3.15 we can assume that $\ell(\mathbf{H})$ is at most $k-1$. The basic idea of the proof is simple. Starting from $(\mathbf{H}, f)$ we create a sequence of zigzags $(\mathbf{H}_i, f_i)$, $1 \leq i \leq m+1$, such that each $(\mathbf{H}_i, f_i)$ has diameter at least i. The large size of $|H|$ will guarantee that we can construct these zigzags. The existence of $(\mathbf{H}_{m+1}, f_{m+1})$ contradicts (5) and so we get the claim. In order to create the $(\mathbf{H}_i, f_i)$, $1 \leq i \leq m+1$ we need to prove the following two claims.

Claim 1. *Let $\mathbf{P}$ be a finite poset of cardinality k. Let $(\mathbf{H}, f)$ be a monotone $\mathbf{P}$-zigzag and let D be a down set of $\mathbf{H}$. Then there exist a $\mathbf{P}$-zigzag $(\mathbf{H}', f')$, a down set D' of $\mathbf{H}'$ and a monotone map g from $(\mathbf{H}', f')$ onto $(\mathbf{H}, f)$ such that the following hold.*

(a) $\mathbf{H}' \setminus D' = \mathbf{H} \setminus D$, $g(u) = u$ *for every* $u \in H' \setminus D'$ *and* $g(D') = D$.

(b) $|\{d'\}_*| < k^{\ell_{\mathbf{H}}(g(d'))}$ *for every* $d' \in D'$.

(c) $\ell(\mathbf{H}') \leq \ell(\mathbf{H})$.

Proof of Claim 1: Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag and let D be a down set of $\mathbf{H}$. We prove the claim by induction on $|D|$. If $|D| = 0$ there is nothing to prove. Let us suppose $|D| \geq 1$. Let $d \in D$ be maximal in the poset $\mathbf{D}$ spanned by the elements of D in $\mathbf{H}$. We apply the induction hypothesis for $(\mathbf{H}, f)$ and $D \setminus \{d\}$. Thus, there exist a $\mathbf{P}$ zigzag $(\mathbf{H}_0, f_0)$, a down set D_0 of $\mathbf{H}_0$ and a monotone map g_0 from $(\mathbf{H}_0, f_0)$ to $(\mathbf{H}, f)$ such that the following hold.

(a') $\mathbf{H}_0 \setminus D_0 = \mathbf{H} \setminus (D \setminus \{d\})$, $g_0(u) = u$ for every $u \in H_0 \setminus D_0$ and $g_0(D_0) = D \setminus \{d\}$.

(b') $|\{d_0\}_*| < k^{\ell_{\mathbf{H}}(g_0(d_0))}$ for every $d_0 \in D_0$.

(c') $\ell(\mathbf{H}_0) \leq \ell(\mathbf{H})$.

Observe that the properties of g_0 guarantee $\{d\}_* \setminus \{d\} \subseteq D_0$. Now, we create a new $\mathbf{P}$-colored poset $(\mathbf{H}_1, f_1)$ from $(\mathbf{H}_0, f_0)$ by replacing d in $(\mathbf{H}_0, f_0)$ by elements $d_1, \ldots, d_t$ as follows. For each antichain in $\{d\}_* \setminus \{d\}$ with at most k elements we pick a new element d_i that covers those elements, and d_i itself is covered by all elements that cover d in $\mathbf{H}_0$. We leave d_i noncolored, if d is noncolored in $(\mathbf{H}_0, f_0)$, otherwise $f_1(d_i) = f_0(d)$. The so obtained colored poset $(\mathbf{H}_1, f_1)$ is $\mathbf{P}$-nonextendible. For otherwise let f_1' be a monotone extension of f_1 to $\mathbf{H}_1$. Since $(\mathbf{H}_1, f_1) \setminus \{d_1, \ldots, d_t\} = (\mathbf{H}_0, f_0) \setminus \{d\}$, $f_1' \,|_{H_0 \setminus \{d\}}$ is a monotone extension of f_0 to $\mathbf{H}_0 \setminus \{d\}$. Note that the colored poset $(\mathbf{H}_0, f_1' \,|_{H_0 \setminus \{d\}})$ is nonextendible since $(\mathbf{H}_0, f_0)$ is nonextendible. So it contains a zigzag $(\mathbf{Q}, g)$. Observe that $d \in Q$. Also $(\mathbf{Q}, g)$ is a nonmonotone zigzag, when d is colored and d is the only noncolored element of $(\mathbf{Q}, g)$, when d is noncolored. By (1) and (2) of Claim 3.12, $(\mathbf{Q}, g)$ has its colored elements in $\{d\}_* \cup \{d\}^*$ and the elements of $(\mathbf{Q}, g)$ in $\{d\}_* \setminus \{d\}$ form an antichain with at most k elements. This is impossible, otherwise $(\mathbf{Q}, g)$ would also be contained in the colored poset $(\mathbf{H}_1, f_1' \,|_{H_1 \setminus \{d_1, \ldots, d_t\}})$ that is assumed to be extendible. Thus $(\mathbf{H}_1, f_1)$ is nonextendible. Hence it contains a zigzag $(\mathbf{H}', f')$.

There is a monotone map g_1 from $(\mathbf{H}_1, f_1)$ to $(\mathbf{H}_0, f_0)$ that is the identity map on $H_1 \setminus \{d_1, \ldots, d_t\}$ and sends the elements $d_1, \ldots, d_t$ to d. Observe that there does not exist a nonempty set T of points and covering pairs in $\mathbf{H}_0$ such that $(\mathbf{H}_0, f_0) \setminus T$ is a monotone image of $(\mathbf{H}', f')$, otherwise by composing an extension of f_0 to $(\mathbf{H}_0, f_0) \setminus T$ with $g_1 \,|_{H'}$ we would get a monotone extension of f' to $\mathbf{H}'$. Hence $\mathbf{H}' \setminus \{d_1, \ldots, d_t\} = \mathbf{H}_1 \setminus \{d_1, \ldots, d_t\}$ and H' contains at least one of $d_1, \ldots, d_t$. Let $g = g_0 \circ g_1 \,|_{H'}$ and let $D' = g^{-1}(D)$. Now, clearly, g is a monotone map onto $(\mathbf{H}, f)$ and D' is a down set of $\mathbf{H}'$.

We want to show that $(\mathbf{H}', f')$, D' and g satisfy (a), (b) and (c). First of all, by $\mathbf{H}' \setminus \{d_1, \ldots, d_t\} = \mathbf{H}_1 \setminus \{d_1, \ldots, d_t\}$ and (a'), (a) is satisfied. Let $d' \in D'$. We show (b) holds even in $\mathbf{H}_1$. If $d' \in D_0$, then by (b') we have the claim. If $d' = d_i$ for some i and d' is minimal in $\mathbf{H}_1$, then (b) is obvious. In the remaining case $d' = d_i$ for some i, and the number of elements covered by d' in $\mathbf{H}_1$ is at least one and at most k. The elements covered by d' are in D_0. So by (b') we have $|\{d'\}_*| < kk^{\ell_{\mathbf{H}}(g(d_0))}$ for an element $d_0 \in D_0$ covered by d'. Now, notice that $\ell_{\mathbf{H}}(g(d_0)) \leq \ell_{\mathbf{H}}(g(d')) - 1$ since $g(d_0) \in D \setminus \{d\}$ and g is monotone. Thus we have (b). Finally, (c) is obvious from (c') and the construction of $(\mathbf{H}', f')$. $\square$

Let $\mathbf{Q}$ be a connected poset. Let $a \in Q$ and $B \subseteq Q$. Then $d_{\mathbf{Q}}(a, B)$ denotes the minimum of $d_{\mathbf{Q}}(a, b)$, $b \in B$, where $d_{\mathbf{Q}}(a, b)$ is the distance between a and b in $\mathbf{Q}$.

CLAIM 2. *Let $\mathbf{P}$ be a finite poset of cardinality k. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag with $w \in H$. Let us suppose that $(\mathbf{H}, f)$ and w satisfy the following properties.*

(A) *$H = A \cup B \cup C$, where A, B and C are pairwise disjoint sets.*

(B) *B and C are not empty and B is an up set in $\mathbf{H}$.*

(C) *For every $a \in A$ and $c \in C$ we have $a \| c$ in $\mathbf{H}$.*

(D) *$w \in A \cup B$.*

Then there exist a $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ and $w' \in H'$ with the following properties.

(a) *$H' = A' \cup B' \cup C'$, where A', B' and C' are pairwise disjoint sets.*

(b) *B' is a nonempty up set in $\mathbf{H}'$, $|B'| \leq |B|$ and $|C| \leq |C'|$.*

(c) *For every $a' \in A'$ and $c' \in C'$ we have $a' \| c'$ in $\mathbf{H}'$.*

(d) *$w' \in A' \cup B'$.*

(e) *$d_{\mathbf{H}'}(w', B') \geq d_{\mathbf{H}}(w, B)$*

(f) *$\ell(\mathbf{H}') \leq \ell(\mathbf{H})$.*

(g) *The number of elements $c' \in C'$ with $c' < b'$ for some $b' \in B'$ is at most $kd|B|$, where d is the maximum of $|\{c\}_*|$ for $c \in C$.*

Proof of Claim 2: We note, if the number of covering pairs between B and C is at most $k|B|$, then there is nothing to do. In any case, we construct a nonextendible colored poset $(\mathbf{Q}, g)$ from $(\mathbf{H}, f)$ as follows. Let $\mathbf{C} = \mathbf{H} \setminus (A \cup B)$. For every monotone extension t of $f\mid_C$ to $\mathbf{C}$ there exists a zigzag in $(\mathbf{H}, f \cup t)$. Let $(\mathbf{Q}_t, g_t)$ be one such a zigzag for each t. Observe that $\emptyset \neq Q_t \cap C \subseteq C(\mathbf{Q}_t, g_t)$. If $(\mathbf{Q}_t, g_t)$ is monotone, then by Claim 3.3 every element of $Q_t \cap C$ is covered from $Q_t \cap (A \cup B)$, hence by property (C), from $Q_t \cap B$. Note, if $(\mathbf{Q}_t, g_t)$ is nonmonotone we also have that every element of $Q_t \cap C$ is covered from $Q_t \cap B$. Let us take disjoint copies, one for $\mathbf{C}$ and one for each $\mathbf{Q}_t$. Then let us stick together the copies of the $\mathbf{Q}_t$ to the copy of $\mathbf{C}$ along the elements that were common in $Q_t \cap C$. In this way we get a poset $\mathbf{Q}$. We refer to the copy of $\mathbf{Q}_t$ in $\mathbf{Q}$ as $\mathbf{Q}_{t,0}$ and to the copy of $\mathbf{C}$ in $\mathbf{Q}$ as $\mathbf{C}_0$. The coloring g on $\mathbf{Q}$ inherits $f\mid_C$ on C_0 and $f\mid_{Q_t \setminus C}$ on $Q_{t,0}$ for all t. Now, $(\mathbf{Q}, g)$ is not extendible since for every monotone extension t of g to $\mathbf{C}_0$ there exists a copy of $(\mathbf{Q}_t, g_t)$ contained in $(\mathbf{Q}, g)$.

Let $(\mathbf{H}', f')$ be a zigzag contained in $(\mathbf{Q}, g)$. There is a monotone map h between the colored posets $(\mathbf{H}', f')$ and $(\mathbf{H}, f)$, where h maps an element $u \in H'$ to the element of H from which u is copied. Observe that h must be onto, otherwise $(\mathbf{H}', f')$ would be extendible. We define w', A', B' and C' as follows. Let w' be an element of H' with $h(w') = w$. So there is an s with $w' \in Q_{s,0}$. Let A' be $h^{-1}(A) \cap Q_{s,0}$. Let B' be $h^{-1}(B) \cap Q_{s,0}$. Finally, let $C' = H' \setminus (A' \cup B')$.

Then we obviously have (a). Since h is onto we have $C_0 \subseteq H'$. So $C_0 \subseteq C'$. Hence $|C| \leq |C'|$. By the definition, B' is an up set, and clearly $|B'| \leq |B|$. Since $\mathbf{H}'$ is connected w' is connected by a fence to an element of C_0 in $\mathbf{H}'$ and this fence must use a copy of an element of B by properties (C) and (D). Hence B' is not empty and (b) holds. By (C) we get (c) and by the definition of w' we get (d). Since

$$d_{\mathbf{H}'}(w', B') \geq d_{\mathbf{Q}_{s,0}}(w', B') \geq d_{\mathbf{Q}_s}(w, B \cap Q_s) \geq d_{\mathbf{H}}(w, B),$$

(e) also holds. Clearly, $\ell(\mathbf{H}') \leq \ell(\mathbf{Q}) \leq \ell(\mathbf{H})$, which gives (f). By Proposition 3.3 and Proposition 3.5 every element of a $\mathbf{P}$-zigzag covers at most k colored elements. Let us apply this to the zigzag $(\mathbf{Q}_s, g_s)$. By property (C) we get that the elements of $B \cap Q_s$ together cover at most $k|B \cap Q_s|$ colored elements of $(\mathbf{Q}_s, g_s)$. So in $\mathbf{Q}$ the copy of $B \cap Q_s$ covers at most $k|B \cap Q_s|$ elements of C_0. Hence the number of elements of C_0 dominated by some elements of B' is at most $kd|B|$, where d is the maximum of $|\{c\}_*|$ for $c \in C$. Thus (g) is satisfied in $(\mathbf{H}', f')$. □

With the help of the preceding two claims, for $0 \leq i \leq m+1$, we give a recursive definition of $(\mathbf{H}_i, f_i)$, $A_i, B_i, C_i \subseteq H_i$ and $a_i \in H_i$ that satisfy the following properties.

(a_i) $H_i = A_i \cup B_i \cup C_i$, where A_i, B_i and C_i are pairwise disjoint sets.

(b_i) If i is even then B_i is a nonempty up set of $\mathbf{H}_i$. If i is odd then B_i is a nonempty down set of $\mathbf{H}_i$. In both cases $|B_i| \leq k^{ik}$ and $|C_i| \geq |H| - \sum_{j=0}^{i} k^{jk}$.

(c_i) For every $a \in A_i$ and $c \in C_i$ we have $a \| c$ in $\mathbf{H}_i$.

(d_i) $a_i \in A_i \cup B_i$.

(e_i) $d_{\mathbf{H}_i}(a_i, B_i) \geq i$

(f_i) $\ell(\mathbf{H}_i) \leq k-1$

We define $(\mathbf{H}_0, f_0) = (\mathbf{H}, f)$, $A_0 = \emptyset$, $B_0 = \{a_0\}$ and $C_0 = H \setminus \{a_0\}$, where a_0 is a maximal element of $\mathbf{H}$. Observe that $(\mathbf{H}_0, f_0), A_0, B_0, C_0$ and a_0 satisfy (a_0)-(f_0). We define $(\mathbf{H}_i, f_i), A_i, B_i, C_i$ and a_i for $i \geq 1$. We only do this for an odd i. For an even i one can define and prove everything dually.

So let $i \geq 1$ be odd. Then $(\mathbf{H}_{i-1}, f_{i-1}), A_{i-1}, B_{i-1}, C_{i-1}$ and a_{i-1} are defined already and satisfy (a_{i-1})-(f_{i-1}). Since $i-1$ is even B_{i-1} is an up set in $\mathbf{H}_{i-1}$. Let us apply Claim 1 to $(\mathbf{H}_{i-1}, f_{i-1})$ with $D = C_{i-1}$. The resulting zigzag $(\mathbf{H}'_{i-1}, f'_{i-1})$ with $A_{i-1}, B_{i-1}, D' = C'_{i-1}$ and a_i still satisfies (a_{i-1})-(f_{i-1}), and we have gained the property that for every $c \in C'_{i-1}$, $|\{c\}_*| \leq k^{k-1}$. Now we apply Claim 2 to $(\mathbf{H}'_{i-1}, f'_{i-1})$ with $A = A_{i-1}$, $B = B_{i-1}$, $C = C'_{i-1}$ and $w = a_{i-1}$. Let $(\mathbf{H}_i, f_i)$ be the resulting zigzag. We define $A_i = A' \cup B'$. Let B_i be the set of elements in C' that are dominated by some elements of B' and let $C_i = H_i \setminus (A_i \cup B_i)$. Finally let $a_i = w'$.

Let us check the properties (a_i)-(f_i). First of all, (a_i), (c_i), (d_i) and (f_i) are obvious.

Moreover, (e_i) is obvious, if we show $B_i \neq \emptyset$. So the property that really needs a proof is (b_i). By (f) and (g) of Claim 2 and (b) of Claim 1 we have $|B_i| \leq k k^{k-1} k^{(i-1)k} = k^{ik}$. By (b) of Claim 2 and (a) of Claim 1, $|C'| \geq |C'_{i-1}| \geq |C_{i-1}|$. Since $C_i = C' \setminus B_i$ and $B_i \subseteq C'$, by (b_{i-1}) we have

$$|C_i| = |C'| - |B_i| \geq |C_{i-1}| - |B_i| \geq |H| - \sum_{j=0}^{i-1} k^{jk} - k^{ik} = |H| - \sum_{j=0}^{i} k^{jk}.$$

Finally, B_i is nonempty since B' and C' are nonempty and $(\mathbf{H}_i, f_i)$ is connected. So we have (5) implies (6).

(6) implies (1): Use Remark 2.4. □

We remark that (4) implies (2) for any poset $\mathbf{P}$ since by the proof of (1) implies (2) we can also obtain Jónsson operations from a partial n-nuf that is fully defined on A_n.

PROPOSITION 8.2. *Any finite poset* $\mathbf{P}$ *with the strong selection property has a partially defined, monotone* 4*-nuf that is fully defined on* A_4.

PROOF: Let $\mathbf{P}$ be an arbitrary finite poset with the strong selection property. Then by Proposition 2.3 every $\mathbf{P}$-zigzag has at most one noncolored element. We show that there is a monotone, partial 4-nuf t on $\mathbf{P}$ that is fully defined on $A_4 = \{(a, \ldots, a, \underset{j}{b}, c, \ldots, c) : a, b, c \in P,\ 1 \leq j \leq 4\} \subseteq P^4$. Suppose this is not true. This means that the colored poset $(\mathbf{A}_4, g)$, where g is given by $g(a, \ldots, a, \underset{j}{b}, a, \ldots, a) = a$, $1 \leq j \leq 4$, $a, b \in P$, is not $\mathbf{P}$-extendible. So it contains a monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$. Let h be the only noncolored element of $(\mathbf{H}, f)$. Recall, the zigzag $(\mathbf{H}, f)$ is of the form described in (2) of Claim 3.12. Note that $h \in A_4$ has at least two coordinates which are the same, say $a \in P$. But then $f(h') \leq a \leq f(h'')$ for any $h', h'' \in H$ with $h' < h < h''$. Hence the coloring of h by a yields a monotone extension of $(\mathbf{H}, f)$, which is a contradiction. □

Now, we are prepared to give an example of an infinite bounded poset that admits Jónsson operations but does not admit a near unanimity function.

EXAMPLE 8.3. *Let* $\mathbf{P}_n = \mathbf{S}_n + \mathbf{S}_n^d$, *where* $\mathbf{S}_n$ *is the poset given by the Boolean lattice of* n *atoms without its top element and* $\mathbf{S}_n^d$ *is the dual of* $\mathbf{S}_n$. *Let* $\mathbf{P} = \prod_{i \in I} \mathbf{P}_i$, *where* $I = \{2, 3, \dots\}$. *Then* $\mathbf{P}$ *admits Jónsson operations and* $\mathbf{P}$ *admits no nuf.*

PROOF: By [20], each $\mathbf{P}_i$, $i \in I$, has the strong selection property. Then, by Proposition 8.2, for each $i \in I$ there exists a partial 4-nuf t_i on $\mathbf{P}_i$ that is fully defined on $B_i = \{(a, \dots, a, \underset{j}{b}, c, \dots, c) : a, b, c \in P_i,\ 1 \le j \le 4\} \subseteq P_i^4$. With the help of the t_i, $i \in I$, we can also define a monotone, partial 4-ary near unanimity operation on $\{(a, \dots, a, \underset{j}{b}, c, \dots, c) : a, b, c \in P, 1 \le j \le 4\}$ coordinatewise. So $\mathbf{P}$ admits Jónsson operations by the remark following Theorem 8.1. After the proof of Claim 3.12 we noted that each $\mathbf{P}_i$, $i \in I$, has a $\mathbf{P}_i$-zigzag $(\mathbf{H}_i, f_i)$ with $2i$ colored elements. By Corollary 5.8, for each $(\mathbf{H}_i, f_i)$ we can create a $\mathbf{P}$-zigzag with the same number of colored elements. Hence, by the note after Remark 2.4, $\mathbf{P}$ admits no nuf. $\square$

Although Theorem 2.4 in [14] and Theorem 8.1 here yield many equivalent conditions for a finite bounded poset that does not admit Jónsson operations, these conditions are hard to check for particular posets. In the following proposition we give an interesting condition that forces finite bounded posets not to admit Jónsson operations.

We define dismantlable posets as in [7]. A finite poset $\mathbf{P}$ is *dismantlable* if $P = \{a_0, \dots, a_n\}$, where a_0 is an irreducible element of $\mathbf{P}$ and a_i is an irreducible element of $\mathbf{P} \setminus \{a_0, \dots, a_{i-1}\}$ for $1 \le i \le n-1$.

PROPOSITION 8.4. *Let* $\mathbf{P}$ *be a finite bounded poset. If* $\mathbf{P}$ *has a convex subposet that is not dismantlable, then* $\mathbf{P}$ *does not admit Jónsson operations.*

PROOF: Let $\mathbf{Q}$ be a convex subposet of $\mathbf{P}$ that is not dismantlable. Then, by [8], $\mathbf{Q}$ has an idempotent image $\mathbf{R}$ that contains more than one element and does not contain any irreducible element. Since $\mathbf{R}$ is not a lattice it has a monotone $\mathbf{R}$-zigzag $(\mathbf{H}, f)$. Similarly as in Proposition 7.3, starting from $(\mathbf{H}, f)$ we can create an $\mathbf{R}$-zigzag $(\mathbf{H}', f')$ with an arbitrarily large diameter as follows.

We can obviously choose $(\mathbf{H}, f)$ to be a stretched zigzag. Let $h \in C(\mathbf{H}, f)$. Since $(\mathbf{H}, f)$ is standard, h is minimal or maximal in $\mathbf{H}$. Without loss of generality we can assume that h is maximal. Since $f(h)$ is reducible, $f(h)$ is maximal or covered by at least two elements. If $f(h)$ is maximal we define a new $\mathbf{R}$-zigzag $(\mathbf{H}_0, f_0)$ as follows. We delete the color of h in $(\mathbf{H}, f)$ and put an element below h that is covered by only h and is colored by $f(h)$. The resulting colored poset $(\mathbf{H}_0, f_0)$ clearly is a zigzag. If $f(h)$ is not maximal, then it is covered by two elements a and b in $\mathbf{R}$. Then we define the following $\mathbf{R}$-zigzag $(\mathbf{H}_0, f_0)$. We delete the color of h in $(\mathbf{H}, f)$ and we put an element below h that is covered by only h and is colored by $f(h)$. We put two elements above h that cover only h and are colored by a and b. Since $(\mathbf{H}, f)$ is stretched, by Proposition **3**.1 the resulting colored poset $(\mathbf{H}_0, f_0)$ is an $\mathbf{R}$-zigzag. Then we take a stretched $\mathbf{R}$-zigzag above $(\mathbf{H}_0, f_0)$.

Obviously, as a result of iterating the above procedure sufficiently many times we can get an $\mathbf{R}$-zigzag $(\mathbf{H}', f')$ with an arbitrarily large diameter. Since $\mathbf{R}$ is an idempotent image of $\mathbf{Q}$, $(\mathbf{H}', f')$ is a $\mathbf{Q}$-zigzag as well, see the remark following Proposition 5.5. We construct a $\mathbf{P}$-colored poset as follows. For every $a \in Q^*$ we put a covering element colored by a above each maximal element of $N(\mathbf{H}', f')$ and for every $b \in Q_*$ we put a covered element colored by b below each minimal element of $N(\mathbf{H}', f')$. The resulting colored poset $(\mathbf{H}'', f'')$ is not $\mathbf{P}$-extendible, otherwise $(\mathbf{H}', f')$ would be $\mathbf{Q}$-extendible. So $(\mathbf{H}'', f'')$ contains a $\mathbf{P}$-zigzag $(\mathbf{G}, g)$. The zigzag $(\mathbf{G}, g)$ must contain $(\mathbf{H}', f')$, otherwise $(\mathbf{G}, g)$ would be extendible. Hence $(\mathbf{G}, g)$ has the same diameter as $(\mathbf{H}', f')$, which contradicts (5) in Theorem 8.1. □

So by Proposition 8.4 it is enough to exhibit a nondismantlable convex subposet of a finite bounded poset $\mathbf{P}$ in order to show that $\mathbf{P}$ does not admit Jónsson operations. To prove that a poset is not dismantlable is easy, since for the dismantling of a poset the greedy algorithm works, see [8]. For example, any locked crown, i.e., any poset of the form $\mathbf{P} = \mathbf{1} + \mathbf{2} + \mathbf{C} + \mathbf{2} + \mathbf{1}$, where $\mathbf{C}$ is a crown, admits no Jónsson operations since $\mathbf{C}$ is a nondismantlable convex subposet of $\mathbf{P}$. The converse of Proposition 8.4 may be true but we are not able to prove it. It is easy to show that the class of finite bounded posets for which all of their convex subposets are dismantlable is closed under taking retracts and finite products.

9. Monotone clones that are not finitely generated

In [26] Tardos proved that for the poset $\mathbf{T} = 1+2+2+2+1$, the clone of monotone operations is not finitely generated. In Theorem 9.10 we generalize his result by exhibiting an infinite class of posets with monotone clones that are not finitely generated. We note that each of these posets contains $\mathbf{T}$ as a retract.

One of the most intriguing problems on finite posets is the following. Is it true that if the clone of all monotone operations on a finite bounded poset is finitely generated then it contains a nuf? Since the property of admitting a nuf is preserved under retraction, if the answer were yes we would have the following claim. If the clone of monotone operations on a finite bounded poset is finitely generated, then the clone of monotone operations on a retract of this poset is also finitely generated. This would imply that for every finite bounded poset containing $\mathbf{T}$ as a retract, the clone of monotone operations is not finitely generated. But to prove even the claim in the previous sentence seems very difficult.

It is known that the answer is no for the question at the beginning of the preceding paragraph, if we allow unbounded posets. This was shown by Demetrovics and Rónyai who proved that a crown admits no nuf although its clone of monotone operations is finitely generated. In Lemma 9.1 through Theorem 9.4 we give our proof of this result.

LEMMA 9.1. *Let A be a finite set with $|A| = k \geq 3$. Let $f : A^n \to A$ be an onto operation that depends on at least two of its variables. Then there exist sets $A_i \subseteq A$, $i = 1, \ldots, n$, with a cardinality $|A_i| = k-1$ such that $f\,|_{A_1 \times \cdots \times A_n}$ is onto.*

PROOF: The claim is a special case of Jablonsky's Lemma on page 40 in [21]. □

PROPOSITION 9.2. *Let $\mathbf{C}_n$ be the crown with $2n$ elements and let C be its clone of monotone operations. Then C has no onto operation that depends on at least two of its variables.*

PROOF: Let $f : \mathbf{C}_n^m \to \mathbf{C}_n$ be a monotone onto function which depends on at least two of its variables. Then by the previous lemma there exist $(2n-1)$-element subfences $\mathbf{A}_1, \dots, \mathbf{A}_m$ of $\mathbf{C}_n$ such that the restriction of f to $\mathbf{A}_1 \times \cdots \times \mathbf{A}_m$ is a monotone map onto $\mathbf{C}_n$. Each $\mathbf{A}_i$ has an element c_i such that the distance between c_i and every (extremal) element of $\mathbf{A}_i$ is at most $n-1$. This implies that $f(c_1, \dots, c_m)$ has a distance at most $n-1$ from each extremal element in $f(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m) = \mathbf{C}_n$. This contradicts the fact that in $\mathbf{C}_n$ for every element there exists another element such that their distance is n. □

PROPOSITION 9.3. *If $f : \mathbf{C}_n^m \to \mathbf{C}_n$ is a monotone map depending on at least two of its variables, then $f(\mathbf{C}_n^m)$ is a fence with a diameter at most n.*

PROOF: Observe that $f(\mathbf{C}_n^m) \subseteq \mathbf{C}_n$ is connected. Hence the previous proposition implies that $f(\mathbf{C}_n^m)$ is a fence. Observe that in $\mathbf{C}_n$ the distance between two arbitrary extremal elements is at most n. This property is preserved for $f(\mathbf{C}_n^m)$, which gives the claim. □

By the use of Proposition 9.3 we can finish the proof in the same way as in [6].

THEOREM 9.4. *Let $\mathbf{C}_n$ be the crown with $2n$ elements and let C be its clone of monotone operation. Then C is finitely generated.*

PROOF: Observe that $C = H \cup (\cup_{\mathbf{F} \subseteq \mathbf{C}_n} H_{\mathbf{F}})$, where H is the set of essentially unary operations in C, $\mathbf{F}$ is a subfence of $\mathbf{C}_n$ with diameter at most n, and $H_{\mathbf{F}}$ is the set of all functions in C which have range in F. So it is enough to show that $H_{\mathbf{F}}$ is finitely generated. Observe that each fence with a diameter at most n is a retract of $\mathbf{C}_n$. Recall that every fence admits a majority function. So for each $\mathbf{F}$, the set $H_{\mathbf{F}}$ is a clone that contains the function $f(x_1, x_2, x_3) = m(r(x_1), r(x_2), r(x_3))$, where m is a majority function on $\mathbf{F}$ and r is an idempotent monotone map from $\mathbf{C}_n$ onto $\mathbf{F}$. Now, let us replace the majority function by f in the well known argument, see [15] or [25], which shows that a clone with a majority function is finitely generated. In this way we get a proof that $H_{\mathbf{F}}$ is finitely generated. □

We note that in [6] it was shown that even the binary operations generate C.

To find examples of finite bounded posets such that their clones of monotone operations are not finitely generated we begin with some claims that are valid for arbitrary finite algebras and serve as the basis of Tardos's proof in [26].

For an algebra A let $Clo(A)$ be the set of finitary term operations on A and let $Clo_m(A)$ be the set of m-ary term operations on A. For a set F of operations on A let $Inv(F)$ be the set of all finitary relations on A preserved by all operations from F and let $Inv_m(F)$ be the set of m-ary relations on A preserved by all operations from F. We note that for an algebra A, $Inv(Clo(A))$ coincides with the set of subuniverses of finite powers of A.

LEMMA 9.5. *Let A be a finite algebra. Let $r_1, r_2, \ldots, r_m$ be elements of A^n and let B be the universe of the subalgebra of A^n generated by $r_1, r_2, \ldots, r_m$. Then $B = \{f(r_1, \ldots, r_m) :$ where $f \in Clo_m(A)\}$.*

PROOF: See page 50 in [21]. □

Let A be a finite algebra. We characterize $Inv(Clo_m(A))$ with the help of relations from $Inv(Clo(A))$.

LEMMA 9.6. *An n-ary relation $R \in Inv_n(Clo_m(A))$ if and only if $R = \cup_{i=1}^{s} R_i$, where $R_i \in Inv_n(Clo(A))$, $1 \leq i \leq s$, and every m-element subset of R is contained in some R_i.*

PROOF: Let $R = \{r_1, \ldots, r_t\} \in Inv_n(Clo_m(A))$. Then we have

$$R = \bigcup_{\{i_1,\ldots,i_m\} \subseteq \{1,\ldots,t\}} \{f(r_{i_1}, \ldots, r_{i_m}) : \ f \in Clo_m(A)\}.$$

By using the previous lemma $R = \cup_{i=1}^{s} R_i$, where $R_i \in Inv_n(Clo(A))$ and, clearly, every m-element subset of R is contained in some R_i.

Now, let R be a set of n-tuples such that $R = \cup_{i=1}^{s} R_i$, where $R_i \in Inv_n(Clo(A))$ and every m-element subset of R is contained in some R_i. When applying any $f \in Clo_m(A)$ to R only m elements of R are involved but these elements are in R_i for some i and f preserves R_i. So f preserves R also. Thus $R \in Inv_n(Clo_m(A))$. □

A set of sets is called an *m-cover* if any m-element subset of their union is a subset of one of them.

Lemma 9.6 has the following corollaries.

COROLLARY 9.7. *$Clo(A)$ is generated by the elements of $Clo_m(A)$ if and only if for every $R = \cup_{i=1}^{s} R_i$, where $\{R_1, \ldots, R_s\} \subseteq Inv_n(Clo(A))$ is an m-cover, $R \in Inv_n(Clo(A))$.*

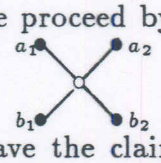
PROOF: $Clo(A)$ is generated by the elements of $Clo_m(A)$ if and only if $Inv(Clo(A)) = Inv(Clo_m(A))$. Hence the previous lemma gives the claim. □

By the negation of Corollary 9.7 we get the following claim.

COROLLARY 9.8. *$Clo(A)$ is not finitely generated if and only if for every m there exists an m-cover $\{R_1, \ldots, R_{s_m}\} \subseteq Inv_n(Clo(A))$ for some n such that $R = \cup_{i=1}^{s_m} R_i \notin Inv_n(Clo(A))$ i.e. R is smaller than the join of R_i, $1 \le i \le s_m$, in the subalgebra lattice of A^n.*

Now, we start proving the generalization of the main theorem in [26].

PROPOSITION 9.9. *Let $\mathbf{P} = 1 + \sum_{i=1}^{n} \mathbf{A}_i + 1$, where $n \ge 3$ and each A_i, $1 \le i \le n$, is a finite antichain with at least two elements. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag such that $\mathbf{N}(\mathbf{H}, f)$ is a fence. Then all the colored elements of $(\mathbf{H}, f)$ are of the form as follows. There exist two elements $a, b \in C(\mathbf{H}, f)$ which, together with the elements of $N(\mathbf{H}, f)$ form a colored fence in $(\mathbf{H}, f)$. The values $f(a)$ and $f(b)$ are incomparable and so $f(a), f(b) \in A_i$ for some i. Each minimal element of $\mathbf{N}(\mathbf{H}, f)$ covers exactly two elements $c, c' \in C(\mathbf{H}, f)$ and $f(c) \| f(c') \in A_{i-1}$. Dually, each maximal element of $\mathbf{N}(\mathbf{H}, f)$ is covered by exactly two elements $d, d' \in C(\mathbf{H}, f)$ and $f(d) \| f(d') \in A_{i+1}$.*

PROOF: We proceed by induction on $t = |N(\mathbf{H}, f)|$. If $t = 1$ then by Proposition 3.12, $(\mathbf{H}, f) =$ 

, where $a_1 \| a_2 \in A_{j+1}$ and $b_1 \| b_2 \in A_j$ for some $1 \le j \le n-1$. Thus we have the claim. Let $t \ge 2$ and let h_1 and h_2 be the endpoints of $\mathbf{N}(\mathbf{H}, f)$. Let us apply Proposition 3.11. Then there exist monotone $\mathbf{P}$-zigzags $(\mathbf{H}_j, f_j)$, $j = 1, 2$, such that $\mathbf{H}_j \subseteq \mathbf{H}$, $h_1, h_2 \in H_j$, $f_j |_{H_j \setminus \{h_j\}} = f |_{H_j \setminus \{h_j\}}$ and $f_j(h_j)$ is defined such that

$f(u) \leq f_j(h_j) \leq f(v)$ for every $u, v \in C(\mathbf{H}, f)$ with $u <_{\mathbf{H}} h_j <_{\mathbf{H}} v$. By Proposition 3.2 and by Proposition 3.3 each $\mathbf{H}_j$, $j = 1, 2$, contains $N(\mathbf{H}, f)$. Now, we can apply the induction hypothesis for $(\mathbf{H}_1, f_1)$. So we get that $(\mathbf{H}_1, f_1)$ is of the form described in the claim for some $1 \leq i \leq n$. This and the above inequalities for $f_2(h_2)$ imply that $f_2(h_2) \in A_i$. So applying the induction hypothesis to $(\mathbf{H}_2, f_2)$ we get that $(\mathbf{H}_2, f_2)$ is of the form described in the claim and it corresponds to the same i as $(\mathbf{H}_1, f_1)$. Let $(\mathbf{Q}, g) = (\cup_{j=1}^{2} \mathbf{H}_j, f\mid_{\cup_{j=1}^{2} \mathbf{H}_j})$. Clearly, $\mathbf{N}(\mathbf{Q}, g) = \mathbf{N}(\mathbf{H}, f)$ and $(\mathbf{Q}, g) \subseteq (\mathbf{H}, f)$. Observe that $(\mathbf{Q}, g)$ is nonextendible since in every extension of it h_1 has to be colored by $f_1(h_1)$ and $(\mathbf{H}_1, f_1)$ is nonextendible. So $(\mathbf{Q}, g) = (\mathbf{H}, f)$. Since $(\mathbf{Q}, g)$ is a zigzag $(\mathbf{H}, f)$ is of the required form. □

After this proposition we are ready to prove the main claim of this section. The proof is a slight modification of the proof contained in [26].

THEOREM 9.10. *Let C be the clone of monotone operations on $\mathbf{P} = 1 + \sum_{i=1}^{n} \mathbf{A}_i + 1$, where $n \geq 3$ and each $\mathbf{A}_i$, $1 \leq i \leq n$, is a finite antichain with at least two elements. Then C is not finitely generated.*

PROOF: Let A be the algebra given on the universe P whose fundamental operations are all the monotone operations on $\mathbf{P}$. We define some subalgebras of A^{m+5} with the help of the poset $\mathbf{Q}_m$, $m \geq 3$, in Figure 10.

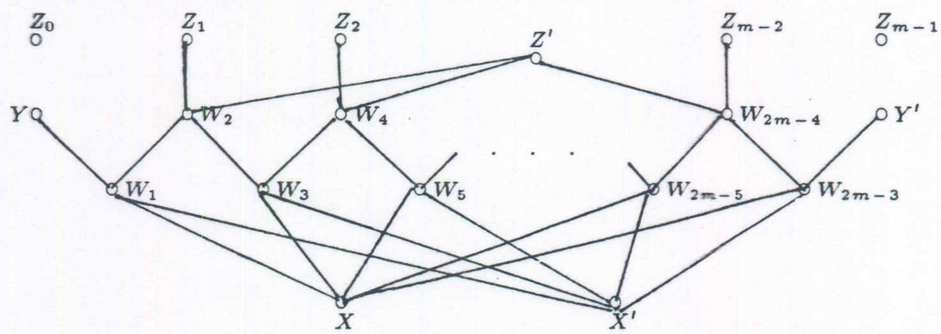


Figure 10. Poset $\mathbf{Q}_m$

Let $R_0 \subseteq P^{m+5}$ be the set of those $(m+5)$-tuples of the form $(a, a', b, b', c_1, \ldots, c_m, c')$ for which all the partial functions f_i, $1 \leq i \leq m$, given by

$$f_i(X) = a, f_i(X') = a', f_i(Y) = b, f_i(Y') = b', f_i(Z') = c',$$

$$f_i(Z_0) = c_i, \ldots, f_i(Z_{m-1}) = c_{(m-1)+i},$$

where the indices are considered by modulo m, are **P**-extendible when restricted to both $\mathbf{Q}_m \setminus \{Y\}$ and $\mathbf{Q}_m \setminus \{Y'\}$. Let R_i contain those elements of R_0, where f_i is **P**-extendible to $\mathbf{Q}_m$. We note that in the case of $\mathbf{P} = \mathbf{T}$ Tardos has the same definition in [**26**].

It is obvious that each R_i, $0 \leq i \leq m$, is a subalgebra of A^{m+5}. We show that the R_i, $1 \leq i \leq m$, form an $[(m-1)/2]$-cover. To this end first we let $u \in R_i$ and $S_u = \{l : u \in R_l\}$ and we prove that $|S_u| \geq m-2$. Let us suppose that $u \notin R_j$ for some $j \neq i$. Then the corresponding f_j is not **P**-extendible to $\mathbf{Q}_m$. So the colored poset $(\mathbf{Q}_m, f_j)$ contains a zigzag $(\mathbf{H}, f)$. Since $\mathbf{N}(\mathbf{H}, f)$ is connected it must form a fence. Since **P** is bounded and the restrictions of f_j to $\mathbf{Q}_m \setminus \{Y\}$ and $\mathbf{Q}_m \setminus \{Y'\}$ are extendible we get that $\mathbf{H} = \mathbf{Q}_m \setminus \{Z_0, Z_{m-1}\}$ and $f = f_j\mid_{\mathbf{H}}$. By Proposition 9.9 there exists $1 \leq t \leq n$ such that

$$f_j(X) \| f_j(X') \in A_{t-1}, f_j(Y) \| f_j(Y') \in A_t \text{ and } f_j(Z_l) \| f_j(Z') \in A_{t+1} \text{ for } 1 \leq l \leq m-2.$$

The fact that $u \in R_0$ gives that $f_j(Z_0)$ and $f_j(Z_{m-1})$ are above all the elements of A_t in **P**. One of $f_j(Z_0)$ and $f_j(Z_{m-1})$ is above $f_j(Z')$ otherwise both are incomparable with $f_j(Z')$ contradicting $u \in R_i$. Hence u is in at least $m-2$ of the R_l, where $1 \leq l \leq m$.

Let $R = \cup_{i=1}^m R_i$. Since any $[(m-1)/2]$ elements of R avoid at most $2[(m-1)/2] \leq m-1$ of the R_i, where $1 \leq i \leq m$, there is at least one R_i which contains these $[(m-1)/2]$ elements. So the R_i form an $[(m-1)/2]$-cover. By Corollary 9.8, it remains to show that R is not preserved by a monotone operation on **P**. Let us select a subposet **T** of **P** that is isomorphic to $1+2+2+2+1$. Observe that there exists an idempotent monotone map r from **P** onto **T**. By Lemma 5 in [26] there is a monotone function $g : \mathbf{T}^{2m} \to \mathbf{T}$ that does not preserve $\cup_{i=1}^m r(R_i)$. But then $g(r(x_1), \ldots, r(x_{2m}))$ does not preserve R. $\square$

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