Maps preserving geodesic and their connection with relative entropy and geometric mean

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The $n \times n$ positive definite complex matrices $\mathbb{P}_n$ form an open subset of the space $\mathbb{H}_n$ of all $n \times n$ Hermitian matrices, hence it naturally has a smooth Riemannian manifold structure so that the tangent space at any foot point can be identified. A Riemannian metric $K_D(H, K)$ is a family of inner products on $\mathbb{H}_n$ depending smoothly on the foot point D. Let $\psi\colon (0, \infty) \times (0, \infty) \to (0, \infty)$ be a kernel function. If D has the spectral decomposition $\sum_{i=1}^{k} \lambda_i P_i$, then a Riemannian metric can be defined as

$$K^{\psi}(H, K) = \sum_{i,j=1}^{k} \psi(\lambda_i, \lambda_j)^{-1} \operatorname{Tr} P_i H P_j K, \qquad D \in \mathbb{P}_n,\ H, K \in \mathbb{H}_n, \tag{1}$$

where Tr is the usual trace functional on matrices. Suppose that $\gamma : [0, 1] \to \mathbb{P}_n$ is a differential curve, then the length of γ with respect to the metric K^{ψ} is given by

$$L(\gamma) := \int_0^1 \sqrt{K^{\psi}_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt. \tag{2}$$

The geodesic distance $\delta(A, B)$ between $A, B \in \mathbb{P}_n$ is defined as the infimum of the length taken over all continuous, piecewise continuously differentiable path from A to B. A geodesic joining A to B is a curve γ from A to B such that $L(\gamma) = \delta(A, B)$ and it will be denoted by $\gamma^{\psi}_{A,B}(t)$.

In [1, 2] F. Hiai and D. Petz have determined geodesics on several classes of Riemannian manifolds defined by different classes of kernel functions. The aim of this talk is to present some structural results concerning geodesics appearing in [1] and [2]. Let $p > 1$ be fixed and ψ be a proper kernel function. We are going to describe the structure of all bijective transformations $\phi\colon \mathbb{P}_n \to \mathbb{P}_n$ that preserve the Schatten p-norm of all geodesics with respect to the Riemannian metric K^{ψ}, i.e. which satisfy that

$$\|\gamma^{\psi}_{A,B}(t)\|_p = \|\gamma^{\psi}_{\phi(A),\phi(B)}(t)\|_p$$

for all $t \in [0, 1]$ and $A, B \in \mathbb{P}_n$.

Moreover, we present the structure of all bijective transformations ϕ on $\mathbb{P}_n$ that preserve geodesics with respect to the Riemannian metric induced by ψ, i.e. which satisfy

$$\phi(\gamma^{\psi}_{A,B}(t)) = \gamma^{\psi}_{\phi(A),\phi(B)}(t)$$

for all $t \in [0, 1]$ and $A, B \in \mathbb{P}_n$.

Finally, for the kernel function $\psi(x, y) = xy$ we determine bijective transformations that preserve the length of differentiable path in $(\mathbb{P}_n, K^{\psi})$.

[1] F. Hiai, D. Petz, Riemannian metrics on positive definite matrices related to means, *Linear Algebra and Its Applications* **430** (2009), 3105–3130.

[2] F. Hiai, D. Petz, Riemannian metrics on positive definite matrices related to means II, *Linear Algebra and its Applications* **436** (2012), 2117–2136.