THE JOIN OF TWO MINIMAL CLONES AND THE MEET OF TWO MAXIMAL CLONES

Gábor Czédli, Radomír Halaš, Keith A. Kearnes,
Péter P. Pálfy, and Ágnes Szendrei

Dedicated to László Szabó on his 50th birthday

Abstract. We construct two minimal clones on any finite set such that the join of the two clones contains all operations. Dually, we exhibit two maximal clones on any finite set with at least three elements such that the intersection of the two clones is the trivial clone containing projections only.

1. Introduction

In this paper we study a feature of the lattice $\mathbf{L}_A$ of all clones on a finite set A with $|A| > 1$. It is known that the lattice $\mathbf{L}_A$ has finitely many atoms (minimal clones) and finitely many coatoms (maximal clones), although the lattice itself is infinite; namely, $\mathbf{L}_A$ is countable if $|A| = 2$ and has cardinality of the continuum if $|A| \geq 3$. The list of all maximal clones was found by Rosenberg [7]. The members of this list are described as clones $\mathrm{Pol}\,(\rho)$ of all operations preserving a relation ρ where ρ runs over a specific list of relations (cf. [5], Section 4.3). In contrast, the classification of the minimal clones is far from being completed (cf. [6]).

Szabó [8] has shown that for every finite set A there exist three minimal clones whose join in $\mathbf{L}_A$ is the clone of all operations, and dually, there exist three maximal clones whose intersection is trivial, that is, contains projection operations only. For the case when $|A|$ is an odd prime number, he was able to show the existence of two minimal and two maximal clones with the respective properties, and asked whether the same conclusion is true for every finite set A with $|A| \geq 3$. In a recent paper [9] he proved that this is indeed the case if $|A| = 2p$ for a prime $p \geq 5$. Our goal in this paper is to show that the answer to Szabó's question is affirmative in general.

Theorem. (1) *On any finite set with at least two elements there exist two minimal clones such that their join contains all operations.*

(2) *Dually, on any finite set with at least three elements there exist two maximal clones whose intersection is trivial.*

It is easy to check that no two maximal clones intersect trivially if $|A| = 2$.

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We note that a result of similar nature was proved by Demetrovics and Rónyai [1], and also by Nozaki, Miyakawa, Pogosyan, and Rosenberg [3]: if A is a finite set with $|A| \geq 4$ then there are two linear orders $\leq_1$ and $\leq_2$ on A such that the intersection of the maximal clones $\mathrm{Pol}\,(\leq_1)$ and $\mathrm{Pol}\,(\leq_2)$ contains projections and constant operations only.

2. Two minimal clones with a large join

Our aim in this section is to show that on every finite set A with $|A| \geq 2$ there exist two minimal clones whose join is the clone of all operations on A. Several different constructions will be presented which can be found in Propositions 2.3–2.5 and Observation 2.6 below.

Throughout this section we will freely use interchangeable 'clone terminology' and 'algebra terminology' for the same objects, whichever is more convenient. In particular, we will use that for a set F of operations on A the clone $[F]$ generated by F is nothing else than the clone of term operations of the algebra $(A; F)$, and $[F]$ is the clone of all operations on A exactly when $(A; F)$ is primal. Recall also that two algebras — and also their clones of term operations — are said to be equivalent if there is a third algebra which is isomorphic to one of them and has the same clone of term operations as the other one.

The first lemma lists the minimal clones that we shall use in our constructions.

Lemma 2.1. *The following operations generate minimal clones:*

(1) *any permutation of prime order;*
(2) *any binary operation $*$ satisfying the identities $x * x = x$, $x * y = y * x$, $x * (x * y) = x * y$, a so-called 2-semilattice operation;*
(3) *any nontrivial rectangular band operation, i.e., any binary operation $\circ$ satisfying the identities $x \circ x = x$, $x \circ (y \circ z) = (x \circ y) \circ z = x \circ z$;*
(4) *the (lower) median operation $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ of any lattice;*
(5) *the following binary operation f on $\{1, 2, 3, 4\}$:*

f	1	2	3	4
1	1	1	1	1
2	2	2	2	3
3	2	3	3	3
4	2	4	4	4

All statements in Lemma 2.1 are known; see [5], 4.4.1 for (1); [2], 5.1(b) for (2); [5], 4.4.4(b) or [2], 5.2(b) for (3); [5], 4.4.5(ii) for (4); and [10], p. 83 for (5).

We want to define two operations f and g on a given finite set A such that the one-generated clones $[f]$ and $[g]$ are minimal, but the clone $[f, g]$ generated by the two operations together is the clone of all operations. Obviously, we may assume that every operation arising from f or g by identification of some variables is a projection. Hence each of these operations is either unary or it is a k-ary idempotent operation, where $k \geq 2$. If both f and g are idempotent operations, then they cannot generate all operations, so one of them, say, g must be unary. Two unary operations cannot generate all operations either, hence f must be a k-ary idempotent operation for some $k \geq 2$. Now observe that g cannot have any fixed points, since a fixed point would be a singleton subalgebra of $(A; f, g)$, and hence the algebra would not be primal.

This implies that g is a fixed-point-free permutation of prime order p (see [5], 4.4.1). Clearly, p divides $|A|$. If $p < |A|$, then the arity of f is at most p, since otherwise the elements of any cycle of g would form a subalgebra in $(A; f, g)$, and hence the algebra would not be primal. These considerations motivate our choice of operations.

For most finite sets A there are lots of very different pairs of suitable operations f, g. To emphasize this feature we will present three essentially different constructions which work for most, but not all, base sets. However, detailed proof will be given only for the first construction.

Obviously, our operations f and g are both surjective. Thus, in order to check that $[f, g]$ is the clone of all operations we can use the following completeness criterion which can be obtained directly from [12], Corollary 4.5.

Lemma 2.2. *If a finite algebra has surjective basic operations then it is either a primal algebra or one of the following conditions holds:*

(i) *the algebra is abelian, i.e., it satisfies the term condition* (see [11], p. 42)*;*
(ii) *the algebra has a proper subalgebra;*
(iii) *the algebra has a nontrivial automorphism;*
(iv) *the algebra has a proper nontrivial congruence.*

Our first construction uses a 2-semilattice operation for f.

Proposition 2.3. *Let A be a finite set with $|A| \neq 4$. There exist two minimal clones $[*]$ and $[g]$ on A with $*$ a 2-semilattice operation and g a fixed-point-free permutation of prime order such that $[*, g]$ is the clone of all operations.*

Proof. If $*$ is a 2-semilattice operation and g is a fixed-point-free permutation of prime order, then the minimality of $[*]$ and $[g]$ is guaranteed by Lemma 2.1 (2), (1). We have to choose $*$ and g so that the properties (i)–(iv) listed in Lemma 2.2 fail for the algebra $(A; f, g)$.

If $|A| = p$ is a prime number, then we identify A with $\{0, 1, \dots, p-1\}$ and let $g = (0\,1 \dots\ p-1)$. Then $(A; g)$ is simple and has no proper subalgebras. If we select $*$ to be the operation min, then $*$ does not satisfy the term condition and $(A; *)$ has no nontrivial automorphisms. Hence $(A; *, g)$ is a primal algebra by Lemma 2.2.

Now assume that $|A|$ is a composite number, and let p be any prime divisor of $|A|$ such that for the decomposition $|A| = pk$ we have $k \geq 3$. Notice that the smallest prime divisor of $|A|$ can always be chosen to be p, because our assumptions exclude the case $|A| = 4$. Let us arrange the elements of A in a $p \times k$ array, and let us define g so that it permutes the elements in each column cyclically as shown in Figure 1.

Figure 1

To describe the 2-semilattice operation notice first that a binary operation $*$ on A is a 2-semilattice operation if and only if $a * a = a$ for all $a \in A$, and for arbitrary distinct elements $a, b \in A$ one of the following conditions holds:

- $\{a, b\}$ is a 2-element semilattice, or
- $c = a*b = b*a$ is different from a, b, and $\{a, c\}$ as well as $\{b, c\}$ are semilattices with $a * c = c = b * c$.

If two distinct elements a, b of A form a semilattice such that $a * b = b$, then we will draw an arrow from a to b. A 2-semilattice where each pair of distinct elements forms a semilattice is called a tournament. With the arrow notation just introduced, a tournament becomes a complete graph where each edge is directed.

Now we are in a position to describe the specific 2-semilattice operation $*$ that will be used to prove the claim of the proposition. Beyond the trivial requirement that $*$ is idempotent, the definition of $*$ is given by Figure 2 which is to be interpreted as follows.

Figure 2

The only two-element subsets that are not subsemilattices are the first two elements in each column (denoted by dashed boxes in Figure 2), and the product of these two elements is the top element in the next column where 'next' is understood along the cycle in the first row in Figure 2. There is an arrow between any two other points, however, the direction of the arrows that are not shown in Figure 2 are irrelevant as long as:

(1) the tournaments T_1 and T_2 formed by the first row and by the second row — both considered as tournaments on the same base set whose elements are numbered by the columns — have no proper nontrivial common congruences; moreover
(2) the tournament T_2 has no nontrivial automorphisms.

Now we use Lemma 2.2 to show that the algebra $(A; *, g)$ is primal. A 2-semilattice is a non-abelian algebra, hence (i) fails. If S is a subalgebra of $(A; *, g)$ then g forces S to be a union of complete columns. Hence S contains an element from the first row. Whenever $i \in S$ is an element in the first row, then $i * g(i)$ is the next element in the first row along the cycle in T_1 as shown in Figure 2. This implies that S contains the full first row. But then $S = A$, showing that (ii) in Lemma 2.2 fails.

Notice that the first row can be described as the set of all elements $a \in A$ such that $a = a_1 * a_2$ for some $a_1 \neq a$ and $a_2 \neq a$. Thus the set R of elements in the first row is preserved by every automorphism of $(A; *, g)$. Hence the same holds for $g(R)$, which is the set of elements in the second row. This set is a subalgebra of $(A; *)$, therefore property (2) implies that every automorphism of $(A; *, g)$ fixes the elements in the second row. The fixed points of an automorphism form a subalgebra, but we have already established that $(A; *, g)$ has no proper subalgebras. Hence $(A; *, g)$ has no nontrivial automorphisms either. Thus (iii) of Lemma 2.2 fails.

Finally we prove that the algebra $(A; *, g)$ is simple. We will denote the elements of the first row by $1, 2, \ldots, k$; hence the elements of the second row are $g(1), g(2), \ldots, g(k)$. Let $\equiv$ be a congruence of $(A; *, g)$ which is distinct from the equality relation. There exist distinct $\equiv$-related elements i and a such that i is in the first row, because a repeated application of g can move any pair of distinct, $\equiv$-related elements in that position. We want to show that there exist two distinct $\equiv$-related elements in the first row. Clearly, we have $a * g(i) \equiv i * g(i) = i + 1$ where addition is understood modulo k. Since $a \neq i$, therefore $\{a, g(i)\}$ is a subsemilattice of $(A; *)$, hence $a * g(i) \in \{a, g(i)\}$. In the case when $a * g(i) = a$ we get that $i \equiv a \equiv i + 1$, so i and $i + 1$ are $\equiv$-related elements in the first row. In the remaining case when $a * g(i) = g(i)$ we get that $g(i) \equiv i + 1$. For $i = k$ this means that $g(k) \equiv 1$, which implies that $2 = 1 * g(1) \equiv g(k) * g(1) = g(1)$. Hence we may assume that $i < k$. Then we obtain that $i + 2 = (i + 1) * g(i + 1) \equiv g(i) * g(i + 1) = g(i) \equiv i + 1$, so $i + 1$ and $i + 2$ are $\equiv$-related elements in the first row.

Thus the first row contains two distinct $\equiv$-related elements. The restriction $\equiv|_{T_1}$ of $\equiv$ to the subalgebra T_1 of $(A;*)$ is a congruence of T_1, and similarly $\equiv|_{T_2} = g(\equiv|_{T_1})$ is a congruence of T_2, both distinct from the equality relation. Identifying the base sets of T_1 and T_2 via g we see that these congruences yield a common congruence of T_1 and T_2. Thus we conclude by property (1) that $\equiv|_{T_1}$ is the full relation in T_1, that is, the whole first row is in the same block of $\equiv$. Applying g we see that the same holds for all rows.

Now for the first two elements $u \equiv u'$ and $g(u) \equiv g(u')$ in any two consecutive rows we have $u' = g(u) * u' \equiv g(u') * u = g(u')$. This shows that all rows are in the same block of $\equiv$, that is, $\equiv$ is the full relation.

To conclude the proof we have to exhibit tournaments T_1 and T_2 on the base set $\{1, 2, \ldots, k\}$ such that T_1 and T_2 possess the arrows prescribed in Figure 2 and conditions (1)–(2) are satisfied. We will show that we get appropriate tournaments if we direct every arrow which hasn't been drawn yet so that $i \to j$ iff $i > j$. In this case T_2 is the k-element chain with the semilattice operation min, so (2) obviously holds.

If $k \neq 4$ then T_1 is simple. This is easy to check for $k = 3$. For $k > 4$ the following observations are enough to verify the claim. If $\equiv$ is a congruence of T_1 and $1 \leq i, j \leq k$ then

$$\begin{aligned}
1 < j < i,\ j \equiv i \quad &\Longrightarrow \quad j - 1 = i * (j-1) \equiv j * (j-1) = j, \\
1 < i < k,\ 1 \equiv i \quad &\Longrightarrow \quad 1 = 1 * (i+1) \equiv i * (i+1) = i + 1, \\
1 \equiv k \quad &\Longrightarrow \quad 1 = 1 * 3 \equiv k * 3 = 3 \quad (\text{if } k > 4), \\
1 \equiv 3 \quad &\Longrightarrow \quad 2 = 1 * 2 \equiv 3 * 2 = 3 \quad (\text{hence } 1 \equiv 2).
\end{aligned}$$

In the case when $k = 4$ the same observations yield that the only proper nontrivial congruence of T_1 has one nonsingleton block, namely $\{1, 4\}$. This is not a congruence of T_2. Therefore condition (1) is satisfied. $\square$

Proposition 2.4. *Let A be a finite set such that $|A|$ is a composite number and $|A| \neq 4, 8$. There exist two minimal clones $[\circ]$ and $[g]$ on A with $\circ$ a rectangular band operation and g a fixed-point-free permutation of prime order such that $[\circ, g]$ is the clone of all operations.*

Remark. It is well known that every rectangular band is isomorphic to a semigroup of the form $(I \times J; \circ)$ where I, J are arbitrary sets and the operation $\circ$ is defined by $(i, j) \circ (i', j') = (i, j')$. This shows that if $|A|$ is prime, then every rectangular band operation is a projection, hence the conclusion of Proposition 2.4 cannot hold. If $|A| = 4$ or 8 then $|I|$ or $|J|$ must be 2, and g must be of order 2, so by inspecting all possibilities for g one can check that none of them satisfies the requirements of Proposition 2.4.

Proof. We will present two different constructions to cover all cases that are allowed for $|A|$:

(I) $|A| = 16k$ for some $k \geq 1$, and g is of order 2;
(II) $|A| = pk$ for some odd prime p and $k \geq 2$, and g is of order p.

As the remark above shows, the rectangular band operation $\circ$ is determined by a rectangular array of the elements of A. We will think of an element labelled (i, j) as being in row i and column j. Once the rectangular band is so pictured, we will define g by drawing all arrows $a \to g(a)$ $(a \in A)$. In case g is of order 2, pairs of opposite arrows $a \to g(a)$ and $g(a) \to g^2(a) = a$ will be replaced by an undirected edge connecting a and $g(a)$.

Our construction for case (I) is shown in Figure 3.

Figure 3

Notice that if we delete the edges that connect the neighboring 4×4 blocks in a cyclic manner, then all the 4×4 blocks except the first one are identical, and the first one is the transpose of the others.

For the elements a, b, c, d indicated in Figure 3 and for the term $t(x, y) = x \circ g(x \circ y)$ we have $t(a, c) = t(a, d)$ and $t(b, c) \neq t(b, d)$. Therefore the algebra $(A; \circ, g)$ is not abelian. To verify that every element generates $(A; \circ, g)$, one can use the definition of g and the fact that whenever two elements u, v belong to a subalgebra, then $\circ$ forces all four vertices of the rectangle with diameter u, v to belong to the subalgebra. Thus we can check first that within one block each element generates all others, and then observe that generation spreads throughout the algebra via the edges connecting the blocks. This shows that $(A; \circ, g)$ has no proper subalgebras.

The proof that $(A; \circ, g)$ is simple is similar. Notice that the operation $\circ$ spreads congruences as follows: for a congruence $\equiv$, if u, v are arbirary $\equiv$-related elements, then in the rows of u and v each pair of elements in the same column are also $\equiv$-related, and dually, in the columns of u and v each pair of elements in the same row are also $\equiv$-related. Using this and the definition of g one can follow the next three steps to show that each congruence $\equiv$ of $(A; \circ, g)$ which is distinct from the equality relation must be the full relation.

(i) If there are two distinct $\equiv$-related elements in the same column then applying g and $\circ$ several times one can always find an element w in their 4×4 block so that $w \equiv g(w)$. Since $w \circ w = w$ also holds, it follows that the $\equiv$-class containing w is a subalgebra. But $(A; \circ, g)$ has no proper subalgebras, hence $\equiv$ is the full relation.

It remains to consider the case when there are two distinct $\equiv$-related elements u, v in the same row.

(ii) If u, v are in different 4×4 blocks, but their position within their blocks is the same, then we can apply the operations $\circ$ and g to u and v in a parallel way till we produce an $\equiv$-related pair such that one of the elements is in the first block. To see that this is possible we use the fact that each of the elements u, v generates the algebra.

(iii) All other cases can be reduced to step (i) because one can easily get from u, v two $\equiv$-related elements which are not in the same row. In most cases one application of g to $\equiv$-related elements in the columns of u and v suffices.

Finally we want to show that $(A; \circ, g)$ has no nontrivial automorphisms. Notice that the following properties are preserved by every automorphism:

$P(x)$ there are $2k$ distinct elements y in the row of x such that $g(x)$ and $g(y)$ are also in the same row;

$Q(x)$ x is the only element in its column for which $P(x)$ holds;

$R(x)$ x is the only element in its column for which $P(x)$ fails.

It is easy to check that the elements satisfying property Q, resp. R are exactly those labelled $q_1, q_2, \dots, q_k$, resp. $r_1, r_2, \dots, r_k$ in Figure 3. Thus q_1 is the only element satisfying Q whose row contains an element satisfying R. Hence q_1 is a fixed point of each automorphism. As in Proposition 2.3 we obtain that $(A; \circ, g)$ has no nontrivial automorphisms.

The construction for case (II) is depicted in Figures 4 and 5, according to whether k is even or odd.

Figure 4

Again, the $p \times 2$ building blocks, except the first one in Figure 4, are identical, and the first $p \times 2$ block in Figure 4 arises from the others by reversing the arrows in one cycle. The elements a, b, c, d shown in both figures and the terms $t_0(x, y) = x \circ g(x \circ y)$ and $t_1(x, y) = g^2(x \circ y) \circ x$, respectively, prove that the algebra $(A, \circ, g)$ is not abelian. The proof that $(A, \circ, g)$ has no proper subalgebras and is simple follows the same lines as in case (I); the details are omitted.

Figure 5

To prove that $(A; \circ, g)$ has no nontrivial automorphisms, it suffices to observe that — whether k is even or odd — the top left element is the only element x in A such that $g^{p-1}(x)$ is in the same row as x and $g(x)$ is in the same column as x. $\square$

Proposition 2.5. *Let A be a finite set such that $|A|$ is not a power of 2 and $|A| \neq 6$. There exist two minimal clones $[m]$ and $[g]$ on A with m the lower median operation of a lattice and g a fixed-point-free permutation of prime order such that $[m, g]$ is the clone of all operations.*

Remark. The argument preceding Lemma 2.2 together with an inspection of all six-element lattices show that the scope of the above statement cannot be extended to other values of $|A|$.

Proof. Let $|A| = n = pk$ for some prime $p \geq 3$ and $k \geq 1$. The case when $n = p \geq 3$ is a prime is settled in [8] as follows: the lattice producing m is a chain and g sends each element, except the top element, to its cover. The case when $n = 2p$ and $p \geq 5$ is included in [9], where the lattice is the direct product of the two-element lattice and the p-element modular lattice of length 2.

Here we will give three different constructions which work for many, but not all, values of $n = pk$. The three cases covered by the constructions are as follows:

(I) $n = pk \geq 9$ and $k \geq 2$ (a modified version works also for $k = 1$, $n = p \geq 5$);
(II) $p \geq 5$ and $k \geq 3$ (again, a modified version works also for $k = 1, 2$);
(III) $p = 3$ and $k \geq 3$.

We will write the base set A in the form $A = \bigcup_{i=1}^{k} A_i$ where $A_i = \{a_{i,1}, \dots, a_{i,p}\}$ $(i = 1, 2, \dots, k)$ denote the g-orbits; in some cases no more information on g will be relevant. Notice that the majority operation m does not satisfy the term condition. Furthermore, every congruence of $(A; m, g)$ is a lattice congruence, for it is preserved by $x \wedge y = m(x, y, 0)$ and $x \vee y = m(x, y, 1)$.

Figure 6

(I) For $n = pk \geq 9$ and $k \geq 2$ let L_1 be the lattice depicted in Figure 6. This lattice is obtained from a fence by adjoining a top and a bottom element. For the elements of the fence on the right which are not shown in Figure 6 the labels are irrelevant. Since L_1 is simple, so is the algebra $(A; m, g)$. For subsets X and Y of A let $m(X) \to Y$ stand for the property that some element of the form $m(x, y, z)$ with $x, y, z \in X$ belongs to Y. If $k \neq 3$ then the equalities

$$\begin{aligned} m(a_{k,1}, a_{k,2}, a_{k,p}) &= a_{k-1,1}, \\ m(a_{i,1}, a_{i,2}, a_{i,p}) &= a_{1,p} \qquad (2 \leq i \leq k-1), \\ m(a_{1,1}, a_{1,2}, a_{1,p}) &= a_{2,2}, \\ m(a_{i-1,1}, a_{i,1}, a_{1,p}) &= a_{i+1,2} \qquad (2 \leq i \leq k-1) \end{aligned}$$

show that $m(A_k) \to A_{k-1}$, $m(A_i) \to A_1$ for $2 \leq i \leq k-1$, and $m(A_1 \cup \ldots \cup A_i) \to A_{i+1}$ for $1 \leq i < k$. If $k = 3$ then step $m(A_2) \to A_1$ in the argument above may fail, but it can be replaced by $m(A_2) \to A_3$ and $m(A_2 \cup A_3) \to A_1$. Thus $(A; m, g)$ has no proper subalgebras.

Let us call an element $x \in A$ *m-irreducible* if $A \setminus \{x\}$ is closed with respect to m. There are exactly three m-irreducible elements: the top element $a_{k,p}$, the leftmost element $a_{1,p-1}$ which is a coatom, and the rightmost element which is an atom or coatom depending on the parity of n. Applying m to these three elements we get a common fixed point of all automorphisms. Like in the proof of Proposition 2.3, we conclude that $(A; m, g)$ has no nontrivial automorphisms.

It can be shown that a fence L_1 with bottom and top elements, combined with an appropriately chosen g works also for $k = 1$, $p \geq 5$. For $p \in \{5, 7\}$ the argument requires further modifications, because the bottom element in L_1 is also m-irreducible and the lattice is not simple.

Figure 7

(II) For $p \geq 5$ and $k \geq 3$ let L_2 be the lattice given in Figure 7. The previous argument works with slight modifications. Now $m(A_1) \to A_k$ and $m(A_{i+1}) \to A_i$ for $1 \leq i < k$. Only the bottom element $a_{1,1}$ has the property that it is the unique m-reducible element in its g-orbit, therefore every automorphism fixes $a_{1,1}$.

For $p \geq 5$ and $1 \leq k \leq 2$ the analogues of the lattice L_2 with appropriate changes in the labelling can be shown to work.

Figure 8

(III) For $p = 3$ and $k \geq 3$ let L_3 be the lattice in Figure 8, and let $g(a_{i,1}) = a_{i,2}$, $g(a_{i,2}) = a_{i,3}$ and $g(a_{i,3}) = a_{i,1}$ $(1 \leq i \leq k)$. This lattice is not simple. For lattice congruences α and β let $g(\alpha) \to \beta$ denote the property that $g(x) \wedge g(y) \leq u < v \leq g(x) \vee g(y)$ for some $(x, y) \in \alpha$ and $(u, v) \in \beta$. The atoms in the congruence lattice of L_3 are the principal congruences $\Theta(a_{k,2}, a_{1,1})$, $\Theta(a_{3,1}, a_{k,2})$ and $\Theta(a_{i-1,1}, a_{i,1})$ for $4 \leq i \leq k$. We have $g\big(\Theta(a_{i-1,1}, a_{i,1})\big) \to \Theta(a_{i,1}, a_{i+1,1})$ for $4 \leq i \leq k-1$, $g^2\big(\Theta(a_{k-1,1}, a_{k,1})\big) \to \Theta(a_{1,1}, a_{k,2})$, $g\big(\Theta(a_{1,1}, a_{k,2})\big) \to \Theta(a_{3,1}, a_{k,2})$ and $g\big(\Theta(a_{3,1}, a_{k,2})\big) \to \Theta(a_{3,1}, a_{4,1})$, therefore the simplicity of $(A; m, g)$ follows easily. Since $m(A_i) \to A_{i+1}$ for $1 \leq i < k$, $m(A_k) \to A_2$ and $m(A_2 \cup A_k) \to A_1$, we get that $(A; m, g)$ has no proper subalgebras. If $k \geq 4$ then $a_{k-1,1}$ is the only m-reducible element such that the other two elements in its g-orbit are m-irreducible while if $k = 3$ then $a_{2,3}$ is the only m-irreducible element. $\square$

The case $|A| = 4$ is not covered by any of the constructions in Propositions 2.3–2.5. In fact, it turns out that this case is very exceptional. As before, our task is to find a binary idempotent operation f generating a minimal clone and a fixed-point-free permutation g of order 2 such that $[f, g]$ is the clone of all operations. All minimal clones on the 4-element set which are generated by an idempotent binary operation have been determined by Szczepara [10]: up to equivalence there are 120 such clones. Going through this list one can observe the following surprising fact.

Observation 2.6. *Let A be a 4-element set. Up to equivalence there is only one pair of minimal clones $[f]$, $[g]$ on A with f a binary idempotent operation and g a fixed-point-free permutation of order 2 such that $[f, g]$ is the clone of all operations.*

Proof. To show the existence let f be the binary operation given in Lemma 2.1 (5). Since $\{1, 2\}$ and $\{2, 3\}$ are both subalgebras of $(A; f)$, the only unary operation that can be used is $g = (1\,3)(2\,4)$. The operation f clearly does not satisfy the term condition. It is straightforward and easy to check that $(A; f, g)$ is simple and it has neither proper subalgebras, nor nontrivial automorphisms. So Lemma 2.2 yields that $[f, g]$ is the clone of all operations.

In order to exlude the other 119 binary minimal clones one can use the following criteria:

(a) if every 2-element subset containing $a \in A$ is a subalgebra of $(A; f)$, then $\{a, g(a)\}$ is a subalgebra of $(A; f, g)$;
(b) if every 2-element subset not containing $a \in A$ is a subalgebra of $(A; f)$, then $A \setminus \{a, g(a)\}$ is a subalgebra of $(A; f, g)$;
(c) if a fixed-point-free permutation h of order 2 is an automorphism of $(A; f)$, then h is an automorphism of $(A; f, g)$ as well.

An application of these criteria leaves only two operations for further consideration, the one in Lemma 2.1 (5) and the following:

f	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	3	3	3	3
4	3	2	3	4

(see [10], p. 188). Since $\{1, 2\}$ and $\{2, 4\}$ are subalgebras of $(\{1, 2, 3, 4\}; f)$, only $g = (1\,4)(2\,3)$ can be taken in order to avoid subalgebras. However, in this case $(\{1, 2, 3, 4\}; f, g)$ has a nontrivial congruence with classes $\{1, 3\}$, $\{2, 4\}$. ☐

3. Two maximal clones with trivial intersection

On a 2-element set there are five maximal clones, and the intersection of any two of them is nontrivial; this can be easily checked using Post's lattice (see [5], p. 78). On finite sets with at least three elements the contrary is true, as we will prove now.

Proposition 3.1. *On every finite set with at least three elements there exist a linear order $\leq$ and a fixed-point-free permutation π such that the intersection of the maximal clones* $\mathrm{Pol}\,(\leq)$ *and* $\mathrm{Pol}\,(\pi)$ *is the trivial clone consisting of projections only.*

Proof. Let $A = \{0, 1, 2, \ldots, n-1\}$ with $n > 2$, let $\leq$ be the natural order on A, and for a fixed prime divisor p of n let π be the fixed-point-free permutation $\pi =$

$(0\ 1\ \dots\ p-1)(p\ p+1\ \dots\ 2p-1)\dots(n-p\ n-p+1\ \dots\ n-1)$. It is well known that the clones $\mathrm{Pol}\,(\leq)$ and $\mathrm{Pol}\,(\pi)$ are maximal; see [5], 4.3.7 and 4.3.9. We will show that the intersection of these clones is trivial.

Suppose the contrary, and let f be a nontrivial operation of minimum arity in the intersection $\mathrm{Pol}\,(\leq) \cap \mathrm{Pol}\,(\pi)$. Then any identification of variables turns f into a projection, hence by Świerczkowski's Lemma one of the following cases occurs (see [11], 1.12):

(i) f is a nontrivial unary operation;
(ii) f is a nontrivial binary idempotent operation;
(iii) f is a ternary majority operation;
(iv) f is a Mal'cev operation;
(v) f is a k-ary semiprojection for some $k \geq 3$.

We will show case-by-case that no operation of any of the given types belongs to $\mathrm{Pol}\,(\leq) \cap \mathrm{Pol}\,(\pi)$.

(i) First observe that f cannot be a permutation, since $\leq$ admits no nontrivial automorphisms, and hence $\mathrm{Pol}\,(\leq)$ contains no nonidentity permutations. Therefore a suitable power e of f is a nontrivial unary operation in $\mathrm{Pol}\,(\leq) \cap \mathrm{Pol}\,(\pi)$ which satisfies the identity $e(e(x)) = e(x)$. Clearly, the range R of e is the set of fixed points of e. Moreover, $e \in \mathrm{Pol}\,(\pi)$ implies that R is closed under π. Since e is nontrivial, there exist $i, j \in A$ such that $e(i) = j \neq i$. We will assume that $i < j$; the case $i > j$ can be handled similarly. Using monotonicity and the fact that $j \in R$ we conclude that $j = e(i) \leq e(j-1) \leq e(j) = j$. Hence $e(j-1) = j$, and so $j - 1 \notin R$. But R is closed under π, therefore if follows that j is the smallest element of a π-cycle. Thus

$$e(j-p) = e\big(\pi(j-1)\big) = \pi\big(e(j-1)\big) = \pi(j) = j+1 > j = e(j).$$

In view of $j - p < j$ this contradicts monotonicity.

(ii) For simplicity let us write $x * y = f(x, y)$. First we want to show that $*$ restricts to the set $\{0, 1\}$ as a projection. Since $*$ is idempotent and monotone, we have $0 = 0 * 0 \leq 0 * 1 \leq 1 * 1 = 1$. So $0 * 1 \in \{0, 1\}$, and similarly $1 * 0 \in \{0, 1\}$. Therefore it suffices to exclude the cases $0 * 1 = 1 * 0 = 0$ and $0 * 1 = 1 * 0 = 1$. The first case is impossible because it would imply that

$$\begin{aligned}(p-1) * 0 &= \pi^{p-1}(0) * \pi^{p-1}(1) = \pi^{p-1}(0 * 1) = \pi^{p-1}(0) = p-1 \\ &> p-2 = \pi^{p-2}(0) = \pi^{p-2}(1 * 0) = \pi^{p-2}(1) * \pi^{p-2}(0) = (p-1) * (p-2) \\ &\geq (p-1) * 0,\end{aligned}$$

and the second case is impossible because it would imply that

$$0 * 1 = 1 > 0 = \pi^{p-1}(1) = \pi^{p-1}(1 * 0) = \pi^{p-1}(1) * \pi^{p-1}(0) = 0 * (p-1) \geq 0 * 1.$$

Interchanging the variables of $*$ if necessary we may assume that $*$ restricts to $\{0, 1\}$ as the first projection.

We are going to show that $*$ is the first projection. To this end let us partition the operation table of $*$ into squares of size $p \times p$ according to the cycles of π. Since π is an automorphism of the algebra $(A; *)$, one can easily see that

(†) whenever $u * v = u$ holds throughout a row of a square, then it holds throughout the whole square.

We will use this property first to conclude that $u * v = u$ holds in the top left square. Indeed, we have

$$(p-1) * 0 = \pi^{p-1}(0) * \pi^{p-1}(1) = \pi^{p-1}(0 * 1) = \pi^{p-1}(0) = p-1 = (p-1) * (p-1),$$

so by monotonocity $(p-1) * j = p-1$ for all $0 \le j \le p-1$. Hence the claim follows by (†).

Next we argue by induction on the squares in the leftmost stripe of the operation table that $u * 0 = u$ for all $u < n$. Suppose that this has already been established up to some square, and the next square starts in row i. Then i is divisible by p. By the induction hypothesis $i-1 = (i-1) * 0 \le i * 0 \le i * i = i$, hence either $i * 0 = i-1$ or $i * 0 = i$. In the former case

$$\begin{aligned}(i+1) * 1 = \pi(i) * \pi(0) = \pi(i * 0) = \pi(i-1) = i-p \\ < i-1 = \pi(i-2) = \pi\big((i-2) * 0\big) = \pi(i-2) * \pi(0) = (i-1) * 1,\end{aligned}$$

a contradiction. So we obtain that $i * 0 = i$. By idempotence and monotonicity we get that $i * v = i$ throughout the first row of the square. Hence (†) yields, in particular, that $u * 0 = u$ throughout the first column of the square, as claimed.

In the last row of the operation table we have $(n-1) * 0 = n-1 = (n-1) * (n-1)$, therefore by monotonicity we conclude that every entry in the last row is $n-1$. But then (†) applies to the bottom right square and yields that $u * v = u$ throughout that square. Now we can proceed in the rightmost stripe from bottom to top, using induction the same way as before, to prove that $u * (n-1) = u$ for all $u < n$. Now the equalities $u * 0 = u = u * (n-1)$ established for all $u < n$, combined with monotonicity, show that $*$ is the first projection.

(iii) If f is a ternary majority operation in $\mathrm{Pol}\,(\le)$ and $a < b < c$ are arbitrary elements in A, then $b = f(a,b,b) \le f(a,b,c) \le f(b,b,c) = b$. Hence the result of f applied to a, b, c is the middle one (with respect to $<$) of the three elements. The same conclusion remains true even if we permute a, b, c. This property of f can be used to show that f does not belong to $\mathrm{Pol}\,(\pi)$. In fact, if $p > 2$ then we have

$$f\big(\pi(0), \pi(1), \pi(p-1)\big) = f(1, 2, 0) = 1 \ne 2 = \pi(1) = \pi\big(f(0, 1, p-1)\big),$$

while if $p = 2$ (and $|A| > 2$) then we have

$$f\big(\pi(0), \pi(1), \pi(2)\big) = f(1, 0, 3) = 1 \ne 0 = \pi(1) = \pi\big(f(0, 1, 2)\big).$$

(iv) and (v) It is not hard to verify that $\mathrm{Pol}\,(\le)$ does not contain any Mal'cev operation or semiprojection, see [4], Lemmas 5 and 7. □

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JATE Bolyai Institute, Szeged, Aradi vértanúk tere 1, H–6720 HUNGARY
E-mail address: `czedli@math.u-szeged.hu`

Department of Algebra and Geometry, Palacký University Olomouc, Tomkova 40, 779 00 Olomouc, CZECH REPUBLIC
E-mail address: `halas@risc.upol.cz`

Department of Mathematics, University of Louisville, Louisville, KY 40292, USA
E-mail address: `kearnes@louisville.edu`

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, P.O.Box 127, H–1364 HUNGARY
E-mail address: `ppp@math-inst.hu`

JATE Bolyai Institute, Szeged, Aradi vértanúk tere 1, H–6720 HUNGARY
E-mail address: `a.szendrei@math.u-szeged.hu`