Term minimal algebras

Ágnes Szendrei

In [1] C. Bergman and R. McKenzie used a construction for strictly simple algebras which is analogous to taking the *induced minimal algebra* that plays a central role in tame congruence theory, however, polynomial operations are replaced by term operations. We call the resulting algebra an *induced minimal term algebra*. It follows that all induced minimal term algebras of a strictly simple algebra are strictly simple (Proposition 1.1); furthermore, they are isomorphic, up to term equivalence (Proposition 1.10), so their 'equivalence type' is a characteristic of the algebra.

Strictly simple term minimal algebras are (up to term equivalence) the algebras arising by this construction. According to the behaviour of their unary term operations, they are naturally divided into four classes (Theorem 1.9). Interestingly, the algebras belonging to three of these four classes were described earlier ([8], [10], [11]), so it remains to investigate the fourth class.

This class consists of all strictly simple algebras $\mathbf{A}$ such that $\mathrm{Clo}_1 \mathbf{A} = \{0\} \cup G$ for an element $0 \in A$ and for a permutation group G acting regularly on $A - \{0\}$. We prove that for $|A| > 2$, either $\mathbf{A}$ is term equivalent to a one-dimensional vector space, or the semilattice operation $\wedge$ with $a \wedge b = 0$ for all distinct elements $a, b \in A$ is a term operation of $\mathbf{A}$ (Theorem 2.2). Consequently, all these algebras generate minimal varieties (Corollary 2.6). Further, we characterize the algebras of types **3**, **4**, **5**, respectively, within this class (Theorems 3.6–3.8), and find that for each of these types, the clones of the corresponding algebras with a given unary part form an interval in the lattice of clones. Finally, we show that these intervals have cardinality $2^{\aleph_0}$ (Theorem 4.1) — a property indicating that this class differs essentially from the other three classes.

A survey of all four classes of strictly simple term minimal algebras is presented in Section 5.

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1. Types of strictly simple term minimal algebras

If not stated otherwise, algebras are denoted by boldface capitals and their universes by the corresponding letters in italics. Two algebras are called *term equivalent* [*polynomially equivalent*], if they have the same clone of term [polynomial] operations. The clone of term operations [the set of n-ary term operations] of an algebra $\mathbf{A}$ is denoted by $\mathrm{Clo}\,\mathbf{A}$ [resp., $\mathrm{Clo}_n\,\mathbf{A}$]. Similarly, the clone of polynomial operations [the set of n-ary polynomial operations] of $\mathbf{A}$ is denoted by $\mathrm{Pol}\,\mathbf{A}$ [resp., $\mathrm{Pol}_n\,\mathbf{A}$]. For a set F of operations on A, $[F]$ will stand for the clone generated by F, that is, $[F] = \mathrm{Clo}\,(A;F)$. An operation g on A can be restricted to a subset B of A if and only if $g(B,\dots,B) \subseteq B$; in this case the restriction is denoted by $g|_B$.

Recall that an operation f is called *idempotent* if it satisfies the identity $f(x,\dots,x) = x$. The clone of all idempotent term operations of an algebra $\mathbf{A}$ will be denoted by $\mathrm{Clo}_{\mathrm{id}}\,\mathbf{A}$. An algebra $\mathbf{A}$ is called *idempotent* if every operation of $\mathbf{A}$ is idempotent (or, equivalently, $\{a\}$ is a subalgebra of $\mathbf{A}$ for all $a \in A$).

The set of non-negative integers is denoted by $\mathbf{N}_0$. For a set A, let T_A, S_A, and C_A denote the full transformation monoid on A, the full symmetric group on A and the set of (unary) constant operations on A, respectively. It will cause no confusion if the unary constant operation on A with value a will be denoted by a. The identity mapping on A is denoted by id_A (or id if A is clear from the context).

A permutation group G acting on A is called *transitive* if the algebra $(A;G)$ has no proper subalgebras, and *primitive* if the algebra $(A;G)$ is simple and $|G| > 1$ (if $|A| = 2$). Clearly, primitivity implies transitivity. A transitive permutation group G is called *regular* if the identity permutation is the only member of G having fixed points. For an abstract group G, the unit of G will always be denoted by 1.

Recall that an algebra $\mathbf{A}$ is said to be *strictly simple* if $|A| \geq 2$, $\mathbf{A}$ is simple and has no nontrivial proper subalgebras. By a trivial algebra we always mean a one-element algebra. For an algebra $\mathbf{A}$ we denote by $Z_{\mathbf{A}}$ the set of all elements $u \in A$ such that $\{u\}$ is a trivial subalgebra of $\mathbf{A}$.

For a set A and for $k \geq 1$, the nonvoid subsets of A^k will be called *k-ary relations* (on A), and for an algebra $\mathbf{A}$ the universes of subalgebras of $\mathbf{A}^k$ will also be called *compatible relations* of $\mathbf{A}$. If ρ is a compatible relation of the algebra $(A;f)$, then we also say that *f preserves ρ*. It is easy to see that for a relation ρ on A, the operations on A preserving ρ form a clone, which will be denoted by $\mathcal{P}_\rho$. The composition of binary relations ρ, σ on A is denoted by $\rho\sigma$, and the converse of ρ is denoted by ρ^{-1}.

For $k \geq 1$ and for a subset $I = \{i_0,\dots,i_{l-1}\}$ of k with $i_0 < \dots < i_{l-1}$, we denote the projection mapping $A^k \to A^I$, $(x_0,\dots,x_{k-1}) \mapsto (x_{i_0},\dots,x_{i_{l-1}})$ by pr_I.

Now we introduce a construction analogous to the one used in tame congruence theory, however, polynomial operations are replaced by term operations. Let $\mathbf{A}$ be an algebra and B a subset of A. The *induced term algebra* of $\mathbf{A}$ on B is defined as follows:

$$\mathbf{A}||_B = (B;\ \{g|_B\colon\ g \in \mathrm{Clo}\,\mathbf{A},\ g(B,\dots,B) \subseteq B\}).$$

Furthermore, we use the following notation:

$$\mathrm{TE}_{\mathbf{A}} = \{e \in \mathrm{Clo}_1\,\mathbf{A}\colon\ e^2 = e\},$$
$$\mathrm{TU}_{\mathbf{A}} = \{e(A)\colon\ e \in \mathrm{TE}_{\mathbf{A}},\ |e(A)| > 1\},$$

and we let $\mathrm{TM}_{\mathbf{A}}$ be the set of all elements of $\mathrm{TU}_{\mathbf{A}}$ that are minimal with respect to inclusion.

Clearly, for a finite algebra $\mathbf{A}$, $\mathrm{TM}_{\mathbf{A}}$ is nonempty. We will repeatedly use the well-known fact that if $\mathbf{A}$ is finite, then for every unary term operation f of $\mathbf{A}$, some power f^k ($k \geq 1$) of f belongs to $\mathrm{TE}_{\mathbf{A}}$. Since each $e \in \mathrm{TE}_{\mathbf{A}}$ acts identically on $e(A)$, therefore for every set $B = e(A) \in \mathrm{TU}_{\mathbf{A}}$, $\mathbf{A}||_B$ can also be given in the form

$$\mathbf{A}||_B = (B;\ \{eg|_B\colon\ g \in \mathrm{Clo}\,\mathbf{A}\}).$$

For strictly simple algebras $\mathbf{A}$ and for $B \in \mathrm{TM}_{\mathbf{A}}$, the construction $\mathbf{A}||_B$ was first used by C. Bergman and R. McKenzie [1], and it proved also useful in [12] in studying simple Abelian algebras. If $B \in \mathrm{TM}_{\mathbf{A}}$, then $\mathbf{A}||_B$ will be called an *induced minimal term algebra* of $\mathbf{A}$.

Some elementary properties of induced minimal term algebras are the following.

Proposition 1.1. ([1], [12]) *Let $\mathbf{A}$ be a finite algebra, and let $B \in \mathrm{TM}_{\mathbf{A}}$.*
(i) *Every element $h \in \mathrm{TE}_{\mathbf{A}||_B}$ is either the identity or constant.*
(ii) *If $\mathbf{A}$ has no nontrivial proper subalgebras, then $Z_{\mathbf{A}} = Z_{\mathbf{A}||_B}$.*
(iii) *If $\mathbf{A}$ is strictly simple, then so is $\mathbf{A}||_B$.*

Definition 1.2. An algebra $\mathbf{A}$ will be called *term minimal* if $|A| \geq 2$, $\mathbf{A}$ is finite, and every element $e \in \mathrm{TE}_{\mathbf{A}}$ is either the identity or constant.

Thus Proposition 1.1 (i) states that for every finite algebra $\mathbf{A}$, the induced minimal term algebras of $\mathbf{A}$ are term minimal. The aim of this paper is to start a systematic study of term minimal algebras. As we shall see later (Proposition 1.10), for every finite strictly simple algebra $\mathbf{A}$, the "type", up to term equivalence, of its induced minimal term algebras does not depend on the choice of B, and hence it is a characteristic of $\mathbf{A}$.

Lemma 1.3. *If $\mathbf{A}$ is a term minimal algebra such that $|Z_{\mathbf{A}}| \geq 2$ and $\mathbf{A}$ has no nontrivial proper subalgebras, then $\mathbf{A}$ is idempotent.*

Proof. Since $f(u) = u$ for all $f \in \mathrm{Clo}_1\,\mathbf{A}$ and $u \in Z_{\mathbf{A}}$, therefore by term minimality, every element $e \in \mathrm{TE}_{\mathbf{A}}$ is the identity. However, by the finiteness, each $f \in \mathrm{Clo}_1\,\mathbf{A}$ has some power in $\mathrm{TE}_{\mathbf{A}}$, implying that f is a permutation. Thus $A - Z_{\mathbf{A}}$ is closed under all unary term operations of $\mathbf{A}$, implying that every element $a \in A - Z_{\mathbf{A}}$ generates a proper subalgebra of $\mathbf{A}$. By our assumptions on $\mathbf{A}$, this is impossible unless $A = Z_{\mathbf{A}}$. Hence $\mathbf{A}$ is idempotent.

Lemma 1.4. *If $\mathbf{A}$ is a simple term minimal algebra having no proper subalgebras, then one of the following conditions holds:*
(1.4) *$\mathrm{Clo}_1\,\mathbf{A}$ is a transitive permutation group on A,*
(1.4)′ *$\mathrm{Clo}_1\,\mathbf{A} = C_A \cup G$ for some permutation group G on A.*

Proof. If $\mathrm{Clo}_1\,\mathbf{A}$ contains no constant, then by term minimality the identity is the only element of $\mathrm{TE}_{\mathbf{A}}$. As in the previous proof, we get that $\mathrm{Clo}_1\,\mathbf{A}$ is a permutation group. It is transitive, since $\mathbf{A}$ has no proper subalgebras.

Now assume $\mathrm{Clo}_1\,\mathbf{A}$ contains at least one constant, say c ($c \in A$). Since $\{f(c)\colon\ f \in \mathrm{Clo}_1\,\mathbf{A}\}$ is a subalgebra of $\mathbf{A}$, it must be equal to A, implying that every constant belongs to $\mathrm{Clo}_1\,\mathbf{A}$. Thus $\mathrm{Pol}\,\mathbf{A} = \mathrm{Clo}\,\mathbf{A}$, and hence by tame congruence theory (see [5; 1.9(1), 2.11, 2.13(1)]), (1.4)′ holds.

Lemma 1.5. *If* $\mathbf{A}$ *is a term minimal algebra with no nontrivial proper subalgebras and a unique trivial subalgebra* $\{0\}$*, then* $\mathrm{Clo}_1\,\mathbf{A} = \{0\} \cup G$ *for some permutation group* G *on* A *which acts transitively on* $A - \{0\}$*. Moreover, if* $\mathbf{A}$ *is simple, then* G *acts regularly on* $A - \{0\}$*.*

Proof. Clearly, $f(0) = 0$ for all $f \in \mathrm{Clo}_1\,\mathbf{A}$. Thus $\mathrm{Clo}_1\,\mathbf{A}$ contains at most one constant, namely 0. Suppose $\mathrm{Clo}_1\,\mathbf{A}$ contains an operation f which is neither a permutation nor the constant 0. Let $a \in A$ be such that $f(a) \neq 0$. By assumption the element $f(a)$ generates $\mathbf{A}$, therefore there exists an $h \in \mathrm{Clo}_1\,\mathbf{A}$ such that $hf(a) = a$. Thus $hf \in \mathrm{Clo}_1\,\mathbf{A}$ is not a permutation, however, $hf(0) = 0$ and $hf(a) = a$ $(a \neq 0)$. So the appropriate power of hf in $\mathrm{TE}_{\mathbf{A}}$ is neither constant nor the identity. This contradiction shows that $\mathrm{Clo}_1\,\mathbf{A} \subseteq S_A \cup \{0\}$. The inclusion $\mathrm{Clo}_1\,\mathbf{A} \subseteq S_A$ cannot hold, since then we would get (as in Lemma 1.3) that $A = Z_{\mathbf{A}}$, which is impossible. Thus $0 \in \mathrm{Clo}_1\,\mathbf{A}$, proving that $\mathrm{Clo}_1\,\mathbf{A} = \{0\} \cup G$ for some permutation group G on A. Clearly, $g(0) = 0$ for all $g \in G$. Since every element $a \in A - \{0\}$ generates $\mathbf{A}$, therefore G is transitive on $A - \{0\}$.

To verify the second claim, consider the relations

$$\tau_{a,b} = \{(f(a), f(b)):\ f \in \mathrm{Clo}_1\,\mathbf{A}\} \quad (a, b \in A).$$

It is straightforward to check that they are compatible relations of $\mathbf{A}$.

Assume that $\mathrm{Clo}_1\,\mathbf{A}$ has the form described in the first claim of the lemma, and G is not regular on $A - \{0\}$. Let g be a nonidentity permutation from G fixing some element $a \in A - \{0\}$, and let $b \in A$ be such that $g(b) \neq b$ (hence $b \neq 0, a$). We use the binary relation $\rho = \tau_{a,b}$. Clearly, (a, b), $(a, g(b))$ are distinct pairs in ρ. Further, $(c, 0) \in \rho \Leftrightarrow c = 0 \Leftrightarrow (0, c) \in \rho$, as $f(b) = 0$ $[f(a) = 0]$ implies $f = 0$. Thus the relation $\sigma = \rho^{-1}\rho$ is symmetric, distinct from the equality relation, and is also reflexive. (For the latter we need that $\mathrm{pr}_{\{1\}}\,\rho = \{f(b):\ f \in \mathrm{Clo}_1\,\mathbf{A}\} = A$, which is ensured by the assumption on $\mathrm{Clo}_1\,\mathbf{A}$.) Moreover, for $d \in A$, $(0, d) \in \sigma$ if and only if $d = 0$. Thus the transitive closure of σ is a congruence of $\mathbf{A}$, distinct from the equality relation, such that $\{0\}$ forms a block. Hence $\mathbf{A}$ is not simple.

It will be more convenient to present the algebras discussed in the second claim of Lemma 1.5 in a different form. Let G be an (abstract) group. For an element $0 \notin G$ we set $G^0 = \{0\} \cup G$. For $g \in G$ we define mappings $l_g, r_g: G^0 \to G^0$ by

$$l_g(x) = \begin{cases} 0 & \text{if } x = 0 \\ gx & \text{if } x \in G \end{cases}, \quad r_g(x) = \begin{cases} 0 & \text{if } x = 0 \\ xg & \text{if } x \in G \end{cases},$$

and we put

$$L_G = \{l_g:\ g \in G\}, \quad R_G = \{r_g:\ g \in G\}.$$

Clearly, L_G and R_G are permutation groups on G^0, acting regularly on G.

Lemma 1.6. *For arbitrary group* G*, a transformation* $f: G^0 \to G^0$ *commutes with all members of* $\{0\} \cup L_G$ *if and only if it belongs to* $\{0\} \cup R_G$*, and symmetrically, a transformation* $f: G^0 \to G^0$ *commutes with all members of* $\{0\} \cup R_G$ *if and only if it belongs to* $\{0\} \cup L_G$*.*

The straightforward proof is omitted.

Corollary 1.7 *Let G be a group, and let $\mathbf{A}$ be an algebra on $A = G^0$ $(0 \notin G)$.*

(1.7) *If $\{0\} \cup L_G \subseteq \mathrm{Clo}_1 \mathbf{A}$, then $\mathrm{Aut}\,\mathbf{A} \subseteq R_G$.*

(1.7)′ *If $R_G \subseteq \mathrm{Aut}\,\mathbf{A}$ and $\{0\}$ is a subalgebra of $\mathbf{A}$, then $\mathrm{Clo}_1 \mathbf{A} \subseteq \{0\} \cup L_G$.*

(1.7)″ *The following conditions are equivalent:*

(i) $\mathrm{Clo}_1 \mathbf{A} = \{0\} \cup L_G$,

(ii) $\{0\} \cup L_G \subseteq \mathrm{Clo}_1 \mathbf{A}$, *$\{0\}$ is a subalgebra of $\mathbf{A}$, and $R_G \subseteq \mathrm{Aut}\,\mathbf{A}$.*

Proof. Obviously, a transformation $f: G^0 \to G^0$ commutes with the constant 0 if and only if $f(0) = 0$. Thus (1.7) and (1.7)′ are immediate consequences of Lemma 1.6.

To see (i)⇒(ii) in (1.7)″, we use the relations $\tau_{a,b}$ introduced in the proof of Lemma 1.5. Suppose (i) holds, and let $a \in G$. Then

$$\begin{aligned}\tau_{1,a} &= \{(f(1), f(a)):\ f \in \mathrm{Clo}_1 \mathbf{A}\} \\ &= \{(0,0)\} \cup \{(g, ga):\ g \in G\},\end{aligned}$$

that is $\tau_{1,a}$ is r_a, considered as a binary relation. Since $\tau_{1,a}$ is a compatible relation of $\mathbf{A}$, we get that $r_a \in \mathrm{Aut}\,\mathbf{A}$. This proves $R_G \subseteq \mathrm{Aut}\,\mathbf{A}$.

(ii)⇒(i) is immediate from (1.7)′.

Lemma 1.8. *If $\mathbf{A}$ is a strictly simple term minimal algebra with a unique trivial subalgebra $\{0\}$, then there exists a group G with $A = G^0$, $0 \notin G$, such that the equivalent conditions in (1.7)″ are satisfied.*

Conversely, for every finite group G and $0 \notin G$, each simple algebra $\mathbf{A}$ on $A = G^0$ having the equivalent properties in (1.7)″ is a strictly simple term minimal algebra with a unique trivial subalgebra $\{0\}$.

Proof. Assume $\mathbf{A}$ is a strictly simple term minimal algebra with a unique trivial subalgebra $\{0\}$. By Lemma 1.5 $\mathrm{Clo}_1 \mathbf{A} = \{0\} \cup G'$ for some permutation group G' on A such that G' acts regularly on $A - \{0\}$. Set $G = A - \{0\}$. It is well known that there is a group $G = (G, \cdot)$ such that the regular permutation group $G'|_G$ coincides with the group of left translations $x \mapsto gx$ $(g \in G)$ of G. Thus $G' = L_G$, and hence (1.7)″ (i) holds. The converse statement is obvious, using (1.7)″ (i).

Summarizing Lemmas 1.3–1.8 we get four basic types of strictly simple term minimal algebras.

Theorem 1.9. *Let $\mathbf{A}$ be a strictly simple term minimal algebra.*

If $Z_{\mathbf{A}} = \emptyset$ and $\mathbf{A}$ has no constant term operations, then

(O)(a) $\mathrm{Clo}_1 \mathbf{A}$ *is a transitive permutation group on A.*

If $Z_{\mathbf{A}} = \emptyset$ and $\mathbf{A}$ has a constant term operation, then

(O)(b) *$C_A \subseteq \mathrm{Clo}_1 \mathbf{A}$ and $\mathrm{Clo}_1 \mathbf{A} - C_A$ is a permutation group on A.*

If $Z_{\mathbf{A}} = \{0\}$, then

(I) *there exists a group G with $A = G^0$, $0 \notin G$, such that $\mathrm{Clo}_1 \mathbf{A} = \{0\} \cup L_G$.*

If $|Z_{\mathbf{A}}| \geq 2$, then

(II) $\mathrm{Clo}_1 \mathbf{A} = \{\mathrm{id}\}$, *that is, $\mathbf{A}$ is an idempotent algebra.*

All strictly simple term minimal algebras of types (O)(a), (O)(b), and (II) are known, up to term equivalence (see [11], [8], [10], respectively). In the survey in

Section 5 we present an explicit list of them. Hence it remains to study the strictly simple term minimal algebras of type (I). This is done in Sections 2–4.

We close this section by proving that the induced minimal term algebras of a strictly simple algebra $\mathbf{A}$ are term equivalent (up to isomorphism), so their "equivalence type" is a characteristic of the algebra $\mathbf{A}$.

For an algebra $\mathbf{A}$ and for subsets $B, C \subseteq A$, we say that B and C are *term isomorphic* if $\mathbf{A}$ has unary term operations f, g such that

$$f(B) = C, \quad g(C) = B, \quad gf|_B = \mathrm{id}_B, \quad \text{and} \quad fg|_C = \mathrm{id}_C.$$

Note that in this case the restriction of f to a mapping $\pi\colon B \to C$ induces a natural bijection between the term operations of $\mathbf{A}||_B$ and $\mathbf{A}||_C$ such that π becomes an isomorphism; namely this bijection is

$$\mathrm{Clo}\,\mathbf{A}||_B \to \mathrm{Clo}\,\mathbf{A}||_C, \quad h(x_0, \dots, x_{n-1}) \mapsto \pi h(\pi^{-1}(x_0), \dots, \pi^{-1}(x_{n-1})).$$

Proposition 1.10. *For a strictly simple algebra $\mathbf{A}$, any two sets in $\mathrm{TM}_\mathbf{A}$ are term isomorphic.*

Proof. If $|Z_\mathbf{A}| \geq 2$, then by Proposition 1.1 (ii) and Lemma 1.3 we have $\mathrm{TM}_\mathbf{A} = \{Z_\mathbf{A}\}$, so the claim is trivial. If $Z_\mathbf{A} = \emptyset$ and $\mathbf{A}$ has a constant term operation c, then every constant is a term operation of $\mathbf{A}$, since $\{f(c)\colon f \in \mathrm{Clo}_1\,\mathbf{A}\}$ is a subalgebra of $\mathbf{A}$ and hence it must be equal to A. Thus $\mathrm{Clo}\,\mathbf{A} = \mathrm{Pol}\,\mathbf{A}$, and the claim follows from tame congruence theory ([5; 2.8]).

Assume $Z_\mathbf{A} = \emptyset$ and $\mathbf{A}$ has no constant term operation. Let $B, C \in \mathrm{TM}_\mathbf{A}$ with $B = e(A)$, $C = f(A)$, $e, f \in \mathrm{TE}_\mathbf{A}$. Consider arbitrary elements $b \in B$, $c \in C$. Since c generates $\mathbf{A}$, there exists a $g \in \mathrm{Clo}_1\,\mathbf{A}$ with $g(c) = b$. Clearly, $egf(A) \subseteq B$, and $egf|_B \in \mathrm{Clo}_1\,\mathbf{A}||_B$. By Proposition 1.1, $\mathbf{A}||_B$ is a strictly simple term minimal algebra with no proper subalgebras. Furthermore, $\mathbf{A}||_B$ inherits the property of having no constant term operations; indeed, were $eh|_B$ constant for some $h \in \mathrm{Clo}_1\,\mathbf{A}$, then ehe would be a constant term operation of $\mathbf{A}$. Thus Lemma 1.4 for $\mathbf{A}||_B$ yields that $egf|_B$ is a permutation of B, implying $egf(A) = B$, whence $eg(C) = B$. Interchanging the roles of B and C we get a $g' \in \mathrm{Clo}_1\,\mathbf{A}$ such that $fg'(B) = C$. Thus, for $h = egfg' \in \mathrm{Clo}_1\,\mathbf{A}$ we have $h(B) = B$. By the finiteness, $(h|_B)^k = \mathrm{id}_B$ for some $k \geq 2$, proving that B and C are term isomorphic via fg' and $h^{k-1}eg$.

Finally, assume $|Z_\mathbf{A}| = 1$, say $\{0\}$ is the unique trivial subalgebra of $\mathbf{A}$. Again, let $B, C \in \mathrm{TM}_\mathbf{A}$ with $B = e(A)$, $C = f(A)$, $e, f \in \mathrm{TE}_\mathbf{A}$. Obviously, $0 \in B, C$. Now select $b \in B$, $c \in C$ so that $b, c \neq 0$. Since b, c generate $\mathbf{A}$, there exist $g, \bar{g} \in \mathrm{Clo}_1\,\mathbf{A}$ with $g(c) = b$, $\bar{g}(b) = c$. Clearly, $egf\bar{g}(A) \subseteq B$, $egf\bar{g}(b) = b$, and $egf\bar{g}|_B \in \mathrm{Clo}_1\,\mathbf{A}||_B$. By Proposition 1.1, $\mathbf{A}||_B$ is a strictly simple term minimal algebra with a unique trivial subalgebra $\{0\}$. Since $egf\bar{g}|_B$ is not the constant 0 (as it fixes $b \neq 0$), Lemma 1.5 for $\mathbf{A}||_B$ yields that $egf\bar{g}|_B$ is a permutation of B. Thus $B = egf\bar{g}(A) \subseteq eg(C) \subseteq B$, whence $eg(C) = B$. From now on we can repeat the foregoing argument.

2. Simple G^0-algebras

In what follows, G always denotes a finite group. We will call an algebra $\mathbf{A}$ a *G^0-algebra* if $0 \notin G$, $A = G^0$, and $\mathrm{Clo}_1\,\mathbf{A} = \{0\} \cup L_G$ (as in condition (I) in Theorem

1.9). Clearly, if $\mathbf{A}$ is a G^0-algebra, then $\mathbf{A}$ is term minimal, has no nontrivial proper subalgebras, and $Z_{\mathbf{A}} = \{0\}$; moreover, $\operatorname{Aut}\mathbf{A} = R_G$. We do not assume simplicity unless explicitly stated otherwise. We use the equivalence of conditions (i)–(ii) in $(1.7)''$ without further reference, as alternative characterizations of G^0-algebras.

An operation f on A will be called *absorptive* if the value of f is 0 whenever at least one of its arguments is 0. It is easy to see that for every $n \geq 1$, an n-ary absorptive operation is either constantly 0 or depends on all of its variables. We will call an absorptive operation *nontrivial* if it is not constantly 0.

Important role will be played by the binary absorptive operation $\wedge$ defined as follows:

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, $\wedge$ is a semilattice operation: 0 is the least element and any two distinct $x, y \in G$ are incomparable.

Proposition 2.1. *$(A;\ 0, L_G, \wedge)$ is a simple G^0-algebra.*

Proof. Since $\wedge$ admits the members of R_G as automorphisms, $(A;\ 0, L_G, \wedge)$ is a G^0-algebra. Its simplicity follows from Lemma 2.5 below.

The principal result of this section is that this algebra is a reduct of almost all simple G^0-algebras, provided $|A| \geq 3$.

Theorem 2.2. *For every simple G^0-algebra $\mathbf{A}$ with at least three elements, $\wedge$ is a term operation of $\mathbf{A}$ unless $\mathbf{A}$ is term equivalent to a one-dimensional vector space.*

An algebra is called *surjective* if its fundamental operations are surjective.

Corollary 2.3. *Every simple G^0-algebra with at least three elements is term equivalent to a surjective algebra.*

Proof. Let $\mathbf{A}$ be a simple G^0-algebra with at least three elements. Since $\{0\}$ is a subalgebra of $\mathbf{A}$ and $R_G \subseteq \operatorname{Aut}\mathbf{A}$, therefore the constant operations with value 0 are the only nonsurjective term operations of $\mathbf{A}$. It suffices to prove that the unary constant 0 is expressible from the surjective term operations of $\mathbf{A}$. If $\mathbf{A}$ is term equivalent to a one-dimensional vector space, then $0 = x - x$; otherwise, by Theorem 2.2, $\wedge$ is a term operation of $\mathbf{A}$, and $0 = x \wedge l_g(x)$ for arbitrary $g \in G - \{1\}$.

Theorem 2.2 and Proposition 2.1 imply that for every finite group G with at least two elements, the clones of simple G^0-algebras that are not term equivalent to one-dimensional vector spaces form an interval in the lattice of clones. Let $\mathcal{R}_0(R_G)$ denote the clone of all operations f on G^0 such that $f(0, \ldots, 0) = 0$ and f admits each member of R_G as an automorphism. (Equivalently, $\mathcal{R}_0(R_G) = \mathcal{P}_{\{0\}} \cap \bigcap_{g \in G} \mathcal{P}_{r_g}$.)

Corollary 2.4. *A finite algebra $\mathbf{A}$ with at least three elements is a simple G^0-algebra if and only if*

$$[L_G, \wedge] \subseteq \operatorname{Clo}\mathbf{A} \subseteq \mathcal{R}_0(R_G),$$

or $\mathbf{A}$ is term equivalent to a one-dimensional vector space.

For an algebra $\mathbf{A}$, $V(\mathbf{A})$ denotes the variety generated by $\mathbf{A}$. A variety V is called *minimal* if it has exactly two subvarieties: V and the trivial variety consisting of one-element algebras.

Lemma 2.5. *Let $\mathbf{A}$ be a G^0-algebra such that $\wedge$ is a term operation of $\mathbf{A}$. Then*

(i) *$\mathbf{A}$ is simple;*

(ii) *every nontrivial algebra in $V(\mathbf{A})$ has a subalgebra isomorphic to $\mathbf{A}$;*

(iii) *$V(\mathbf{A})$ is a minimal variety.*

Proof. It suffices to prove (ii), as (iii) is an immediate consequence of it, and (i) follows from (ii) since $\mathbf{A}$ is finite.

We may assume without loss of generality that 0, l_g ($g \in G$) and $\wedge$ are fundamental operations of $\mathbf{A}$. Clearly, $\mathbf{A}$ satisfies the following identities:

$$(2.5)\quad l_g(0) = 0, \quad l_{g^{-1}}(l_g(x)) = x, \quad l_g(x) \wedge l_h(x) = \begin{cases} 0 & \text{if } g \neq h \\ l_g(x) & \text{if } g = h \end{cases} \quad (g, h \in G),$$

moreover, since $\mathrm{Clo}_1 \mathbf{A} = \{0\} \cup L_G$, therefore for every $n \geq 1$, every n-ary fundamental operation f of $\mathbf{A}$ and for arbitrary $f_0, \ldots, f_{n-1} \in \{0\} \cup L_G$, there exists a unique $f_n \in \{0\} \cup L_G$ such that the following identity holds:

$$(2.5)' \qquad f(f_0(x), \ldots, f_{n-1}(x)) = f_n(x).$$

Obviously, $f_n = 0$ if $f_0 = \ldots = f_{n-1} = 0$.

Let $\mathbf{B}$ be a nontrivial algebra in $V(\mathbf{A})$. Since $\mathbf{B}$ satisfies the identities $(2.5)'$ and $l_1(x) = x$, therefore for any element $b \in B$ with $b \neq 0$, the subalgebra of $\mathbf{B}$ generated by b is

$$\{0\} \cup \{l_g(b)\colon\ g \in G\}.$$

Supposing $0 = l_g(b)$ for some $g \in G$, we would get by the identities (2.5) that $0 = l_{g^{-1}}(0) = l_{g^{-1}}(l_g(b)) = b$, a contradiction. Similarly, $l_g(b) = l_h(b)$ for some distinct $g, h \in G$ would imply $0 = l_g(b) \wedge l_h(b) = l_g(b) \wedge l_g(b) = l_g(b)$, leading, as before, to a contradiction. Hence the elements 0 and $l_g(b)$ ($g \in G$) are pairwise distinct, so for any $a \in A - \{0\}$, the assignment

$$A \to B, \quad 0 \mapsto 0,\ l_g(a) \mapsto l_g(b) \quad (g \in G)$$

defines an injective mapping, which by the identities $(2.5)'$ is an embedding $\mathbf{A} \to \mathbf{B}$.

Corollary 2.6. *Every simple G^0-algebra with at least three elements generates a minimal variety.*

Proof. Let $\mathbf{A}$ be a simple G^0-algebra with at least three elements. If $\mathbf{A}$ is term equivalent to a one-dimensional vector space, then the minimality of $V(\mathbf{A})$ is well known. Otherwise, by Theorem 2.2, $\wedge$ is a term operation of $\mathbf{A}$, and the claim follows from Lemma 2.5.

The rest of this section is devoted to the proof of Theorem 2.2, which consists of two main steps: first we show that every simple G^0-algebra $\mathbf{A}$ with at least three elements, which is not term equivalent to a one-dimensional vector space, has a

nontrivial binary absorptive term operation (Lemma 2.15), and then we prove that $\wedge$ is a term operation of $\mathbf{A}$ (Lemma 2.16).

Throughout Lemmas 2.7–2.16 G is a finite group with at least two elements, and $A = G^0$ ($0 \notin G$). For convenience, we extend the multiplication of G to G^0 so that 0 be a zero element of $(G^0; \cdot)$.

We will say that two n-ary operations f, f' on A are *similar* if there exist $g \in G$ and $g_0, \dots, g_{n-1} \in G$ such that

$$f'(x_0, \dots, x_{n-1}) = l_g(f(l_{g_0}(x_0), \dots, l_{g_{n-1}}(x_{n-1}))).$$

It is clear that similarity is an equivalence relation, and a number of properties of operations are invariant under similarity; such properties are the following: being absorptive, depending on a variable x_i, being a quasigroup operation, etc. (cf. also Lemma 2.7). If $\mathbf{A}$ is a G^0-algebra, then $\operatorname{Clo}\mathbf{A}$ obviously contains all operations that are similar to its members. Furthermore, for every binary term operation $\bullet$ of $\mathbf{A}$ we have $x \bullet 0, 0 \bullet x \in \{0\} \cup L_G$, so $\bullet$ is similar to one of the following kinds of binary term operations $\circ$ of $\mathbf{A}$:

(0) 0 is a zero element for $\circ$ (i.e. $\circ$ is absorptive);

(1) 0 is a left unit and right zero for $\circ$ (i.e. the identities $0 \circ x = x$, $x \circ 0 = 0$ hold), or dually

(1)$'$ 0 is a right unit and left zero for $\circ$;

(2) 0 is a unit element for $\circ$ (i.e. the identities $0 \circ x = x \circ 0 = x$ hold).

For a binary operation $*$ on A we denote

$$X(*) = \{a \in G\colon\ 1 * a = 0\}.$$

Lemma 2.7. *For arbitrary binary operations $\bullet, \circ$ on A, if*

(i) *$\circ$ is similar to $\bullet$, or*

(ii) *$\circ$ arises from $\bullet$ by interchanging variables,*

then $|X(\circ)| = |X(\bullet)|$.

Proof. In case (i), if $x \circ y = l_g(l_{g_0}(x) \bullet l_{g_1}(y))$, then it is easy to check that

$$1 \circ a = g(g_0 \bullet g_1 a) = g(r_{g_0}(1) \bullet r_{g_0}(g_1 a g_0^{-1})) = g(r_{g_0}(1 \bullet g_1 a g_0^{-1})),$$

so

$$a \in X(\circ) \Leftrightarrow 1 \circ a = 0 \Leftrightarrow 1 \bullet g_1 a g_0^{-1} = 0 \Leftrightarrow a \in g_1^{-1} X(\bullet) g_0.$$

In case (ii), if $x \circ y = y \bullet x$, then a similar easy computation yields that

$$a \in X(\circ) \Leftrightarrow a^{-1} \in X(\bullet).$$

For a subgroup H of G we denote the equivalence relation on A with blocks aH ($a \in A$) by ε_H. (In accordance with the extension of multiplication to G^0, $0H = \{0\}$.) Clearly, each permutation l_g ($g \in G$) in L_G induces a permutation of the blocks of ε_H by $l_g(aH) = gaH$ ($a \in A$). (Note that distinct permutations in L_G may induce the same permutation on the blocks.) So L_G induces a permutation group on the blocks of ε_H. If this induced permutation group is regular on the set $\{aH\colon\ a \in G\}$, we will say that L_G acts regularly on the blocks of ε_H distinct from

$\{0\}$. The claims in the following lemma are immediate consequences of well-known facts for regular permutation groups.

Lemma 2.8. (i) *An equivalence relation ρ on A, distinct from the full relation, is a congruence of the algebra $(A; L_G)$ if and only if $\rho = \varepsilon_H$ for some subgroup H of G.*

(ii) *L_G acts regularly on the blocks of ε_H distinct from $\{0\}$ if and only if H is a normal subgroup of G.*

Lemma 2.9. *Let $\mathbf{A}$ be a G^0-algebra. If $\mathbf{A}$ has a binary term operation which is a quasigroup operation, then $\mathbf{A}$ is term equivalent either to a one-dimensional vector space, or to the algebra $(A; \mathcal{R}_0(R_G))$.*

Proof. By Lemma 2.8, every congruence of $\mathbf{A}$, distinct from the full relation is of the form ε_H for some subgroup H of G. However, it is well known (and easy to verify) that quasigroups have uniform congruences. Since ε_H is uniform if and only if $|H| = 1$, we conclude that $\mathbf{A}$ is simple (and hence strictly simple). Now by McKenzie's theorem [7], $\mathbf{A}$ is affine or quasiprimal. If $\mathbf{A}$ is affine, then the description of simple affine algebras, up to term equivalence (Clark–Krauss [2]), and the assumption on $\mathrm{Clo}_1\,\mathbf{A}$ yield that $\mathbf{A}$ is a one-dimensional vector space. If $\mathbf{A}$ is quasiprimal, then the term operations of $\mathbf{A}$ are exactly the operations preserving the subalgebra $\{0\}$ of $\mathbf{A}$ and the automorphisms of $\mathbf{A}$ (as $\mathbf{A}$ has no nontrivial proper subalgebras). As the automorphism group of $\mathbf{A}$ is R_G, we get that $\mathbf{A}$ is term equivalent to $(A; \mathcal{R}_0(R_G))$.

Lemma 2.10. *Let $\mathbf{A}$ be a G^0-algebra, and H a subgroup of G. If ε_H is not a congruence of $\mathbf{A}$, then $\mathbf{A}$ has a binary term operation $\circ$ such that $(1 \circ a, 1 \circ b) \notin \varepsilon_H$ for some $(a, b) \in \varepsilon_H$.*

Proof. As ε_H is not a congruence of $\mathbf{A}$, $\mathbf{A}$ has a unary polynomial operation $p(x) = f(x, c_1, \ldots, c_{n-1})$ ($f \in \mathrm{Clo}_n\,\mathbf{A}$, $c_1, \ldots, c_{n-1} \in A$) not preserving ε_H, say, $(p(a), p(b)) \notin \varepsilon_H$ for some $(a, b) \in \varepsilon_H$. Since $f(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1})$ is a term operation of $\mathbf{A}$ for all $1 \leq i \leq n-1$, we can assume that $c_1, \ldots, c_{n-1} \in G$. Thus $x \circ y = f(y, l_{c_1}(x), \ldots, l_{c_{n-1}}(x))$ is a binary term operation of $\mathbf{A}$ satisfying the requirements.

Lemma 2.11. *Let $\mathbf{A}$ be a G^0-algebra. If $\mathbf{A}$ has a binary term operation $\circ$ such that $|X(\circ)| \geq 1$, $\circ$ is not constantly 0, and 0 is a one-sided zero with respect to $\circ$, then $\mathbf{A}$ has a nontrivial binary absorptive term operation.*

Proof. Let, say, 0 be a left zero for $\circ$. (By Lemma 2.7, the assumptions of the lemma are left-right symmetric.) If 0 is also a right zero, then $\circ$ is a nontrivial absorptive operation, and we are done. Otherwise we can assume (by similarity) that 0 is a right unit. Let $a \in X(\circ)$, and consider the term operation

$$x * y = x \circ (l_a(x) \circ y).$$

Clearly, 0 is a left zero for $*$. Further, $1 * 0 = 1 \circ (a \circ 0) = 1 \circ a = 0$, implying by the automorphisms in R_G that 0 is a right zero for $*$. Thus $*$ is absorptive. As $a^{-1} * a = a^{-1} \circ (1 \circ a) = a^{-1} \circ 0 = a^{-1} \neq 0$, $*$ is nontrivial.

The next three lemmas are concerned with term operations $\circ$ satisfying the following condition for some integer $m \geq 1$:

$(\dagger_m(\circ))$ $m \leq |X(\circ)| < |G|$ and 0 is a unit element for $\circ$.

Lemma 2.12. *Let* $\mathbf{A}$ *be a* G^0*-algebra. If* $\mathbf{A}$ *has a binary term operation* $\circ$ *with property* $(\dagger_1(\circ))$ *such that* $a \circ b = a$ $(= a \circ 0)$ *for some* $a, b \in G$*, then* $\mathbf{A}$ *has a nontrivial binary absorptive term operation.*

Proof. Let $c \in X(\circ)$, and consider the term operation

$$x * y = x \circ (l_c(x) \circ y).$$

We have

$$\begin{aligned} 1 * 0 &= 1 \circ (c \circ 0) = 1 \circ c = 0, \\ 1 * ba^{-1}c &= 1 \circ (c \circ ba^{-1}c) = 1 \circ r_{a^{-1}c}(a \circ b) = 1 \circ r_{a^{-1}c}(a) = 1 \circ c = 0, \\ 1 * c^2 &= 1 \circ (c \circ c^2) = 1 \circ r_c(1 \circ c) = 1 \circ 0 = 1. \end{aligned}$$

Hence 0 is a right zero for $*$ and $1 \leq |X(*)| < |G|$, therefore the claim follows from Lemma 2.11.

Lemma 2.13. *Let* $\mathbf{A}$ *be a* G^0*-algebra. If* $\mathbf{A}$ *has a binary term operation* $\circ$ *with property* $(\dagger_1(\circ))$ *such that* $a \circ b = a \circ c$ *for some* $a, b, c \in G$*,* $b \neq c$*, then* $\mathbf{A}$ *has either a nontrivial binary absorptive term operation, or a binary term operation* $\bullet$ *satisfying* $(\dagger_2(\bullet))$.

Proof. Let $e = a \circ b = a \circ c$ and $d \in X(\circ)$. If $e = 0$, then $ba^{-1}, ca^{-1} \in X(\circ)$, so $\circ$ is appropriate for $\bullet$. Assume $e \neq 0$, and consider the term operation

$$x * y = l_a(x) \circ l_u(l_a(x) \circ y) \quad \text{with} \quad u = dae^{-1}.$$

We have

$$\begin{aligned} 1 * b &= a \circ u(a \circ b) = a \circ ue = a \circ da = r_a(1 \circ d) = 0, \\ 1 * c &= a \circ u(a \circ c) = a \circ ue = 0, \\ 1 * da &= a \circ u(a \circ da) = a \circ u(r_a(1 \circ d)) = a \circ 0 = a, \end{aligned}$$

proving that $2 \leq |X(*)| < |G|$. If 0 is a one-sided zero for $*$, then by Lemma 2.11, $\mathbf{A}$ has a nontrivial binary absorptive term operation. Otherwise, by Lemma 2.7, an operation $\bullet$ similar to $*$ satisfies $(\dagger_2(\bullet))$.

Lemma 2.14. *Let* $\mathbf{A}$ *be a* G^0*-algebra, and assume* $\mathbf{A}$ *has a binary term operation* $\circ$ *with property* $(\dagger_2(\circ))$*. If every binary absorptive term operation of* $\mathbf{A}$ *of the form*

$$(2.14) \quad l_w(x \circ y) \circ (l_v(x) \circ l_u(y)) \quad \text{or} \quad l_w(x \circ y) \circ (l_u(y) \circ l_v(x)) \quad (u, v, w \in G)$$

is constantly zero, then

(i) $X(\circ) = Nt$ *for some normal subgroup* N *of* G *and for some element* $t \in G$ *such that* $t^2 \in N$ *and the coset* Nt *is also closed under conjugation;*

(ii) ε_N *is a congruence of the algebra* $(A; \circ)$*;*

(iii) *if* ε_N *is not a congruence of* $\mathbf{A}$*, then* $\mathbf{A}$ *has a term operation* $*$ *with* $1 * a = 0$*,* $1 * b \neq 0$ *for some* $(a, b) \in \varepsilon_N$.

Proof. (i) The value of both term operations in (2.14) for $(x,y)=(1,0)$ and $(x,y)=(0,1)$ is $w\circ v=r_w(1\circ vw^{-1})$ and $w\circ u=r_w(1\circ uw^{-1})$, respectively. Thus these operations are absorptive if and only if $vw^{-1},uw^{-1}\in X(\circ)$. For $(x,y)=(1,a)$ $(a\in G)$, the two term operations in (2.14) take the values

$$w(1\circ a)\circ(v\circ ua)=w(1\circ a)\circ(1\circ uav^{-1})v,$$
$$w(1\circ a)\circ(ua\circ v)=w(1\circ a)\circ(1\circ v(ua)^{-1})ua,$$

respectively. Assume all term operations in (2.14) with $vw^{-1},uw^{-1}\in X(\circ)$ are constantly zero. Putting $w=1$ we get that

$$(2.14)_1\qquad (1\circ a)\circ(1\circ uav^{-1})v=0\quad\text{for all}\quad a\in G,\ u,v\in X(\circ),$$

while for $a\in X(\circ)$, this assumption means that

$$(2.14)_2\qquad a,vw^{-1},uw^{-1}\in X(\circ)\quad\Rightarrow\quad uav^{-1},va^{-1}u^{-1}\in X(\circ).$$

In particular, for $w=1$,

$$(2.14)_3\qquad a,v,u\in X(\circ)\quad\Rightarrow\quad uav^{-1},va^{-1}u^{-1}\in X(\circ).$$

Putting $a=u=v$ in $(2.14)_3$ shows that $X(\circ)$ is closed under taking inverses, and hence the following modified version of $(2.14)_3$ also holds:

$$(2.14)_4.\qquad a,v,u\in X(\circ)\quad\Rightarrow\quad ua^{-1}v\in X(\circ).$$

By $(2.14)_4$, $X(\circ)$ is a coset Nt of a subgroup N of G $(t\in G)$. Since $X(\circ)$ is closed under taking inverses, we have $t^{-1}N=Nt$, or equivalently, $N=tNt$. Hence $t^2\in N$ and $Nt=tN$.

Choose any $a\in X(\circ)$ and $w\in G$. Applying $(2.14)_2$ with $u=v=aw$ we get that $(aw)a(aw)^{-1}\in X(\circ)$. Now $(2.14)_3$ implies that

$$waw^{-1}=a^{-1}((aw)a(aw)^{-1})a\in X(\circ),$$

that is, $X(\circ)=Nt$ is closed under conjugation. Consequently, for arbitrary $w\in G$,

$$wNw^{-1}=w(tN)t^{-1}w^{-1}=(tN)wt^{-1}w^{-1}\subseteq(tN)(w(Nt)w^{-1})=(tN)(Nt)=N,$$

showing that N is a normal subgroup of G.

(ii) Observe that for arbitrary $a,b\in G$ and $v\in Nt=X(\circ)$,

$$(a,b)\in\varepsilon_N\quad\Leftrightarrow\quad bva^{-1}\in a(Nt)a^{-1}=Nt\quad\Leftrightarrow\quad 1\circ bva^{-1}=0\quad\Leftrightarrow\quad a\circ bv=0.$$

Now assume $(a,b)\in\varepsilon_N$, and let $c\in A$ be an arbitrary element. We have to show that $(c\circ a,c\circ b)\in\varepsilon_N$ and $(a\circ c,b\circ c)\in\varepsilon_N$. This is trivial if at least one of a,b,c is 0, so assume $a,b,c\in G$. Since

$$\begin{aligned}
(c\circ a,c\circ b)\in\varepsilon_N\quad&\Leftrightarrow\quad((1\circ ac^{-1})c,(1\circ bc^{-1})c)\in\varepsilon_N\\
&\Leftrightarrow\quad(1\circ ac^{-1},1\circ bc^{-1})\in\varepsilon_N\\
&\Leftrightarrow\quad(1\circ ac^{-1})\circ(1\circ bc^{-1})t=0,\\
(a\circ c,b\circ c)\in\varepsilon_N\quad&\Leftrightarrow\quad((1\circ ca^{-1})a,(1\circ cb^{-1})b)\in\varepsilon_N\\
&\Leftrightarrow\quad(1\circ ca^{-1},(1\circ cb^{-1})ba^{-1})\in\varepsilon_N\\
&\Leftrightarrow\quad(1\circ ca^{-1})\circ(1\circ cb^{-1})ba^{-1}t=0,
\end{aligned}$$

and $(ac^{-1}, bc^{-1}) \in \varepsilon_N$, $(ca^{-1}, cb^{-1}) \in \varepsilon_N$, $t \in Nt$, $ba^{-1}t \in Nt$, therefore it suffices to show that

$$(a, b) \in \varepsilon_N,\ v \in Nt \quad \Rightarrow \quad (1 \circ a) \circ (1 \circ b)v = 0.$$

However, here $b = an$ for some $n \in N$, and in view of $nv \in Nt = a^{-1}(Nt)a$, we have $nv = a^{-1}ua$ for some $u \in Nt$, implying that $b = an = a(a^{-1}ua)v^{-1} = uav^{-1}$, and now the equality to be proved follows from $(2.14)_1$.

(iii) Since ε_N is not a congruence of $\mathbf{A}$, therefore by Lemma 2.10, $\mathbf{A}$ has a binary term operation $\bullet$ such that $(1 \bullet a, 1 \bullet b) \notin \varepsilon_N$ for some $(a, b) \in \varepsilon_N$. Clearly, $a \neq b$, so $a, b \in G$. If $1 \bullet a = 0$, we are done, so assume $1 \bullet a \in G$, and define

$$x * y = l_w(x) \circ (x \bullet y)$$

where $w \in G$ is selected so that $1 \bullet a \in Ntw$. Clearly, then $1 \bullet b \notin Ntw$, so

$$1 * a = w \circ (1 \bullet a) = (1 \circ (1 \bullet a)w^{-1})w = 0,$$
$$1 * b = w \circ (1 \bullet b) = (1 \circ (1 \bullet b)w^{-1})w \neq 0,$$

completing the proof.

Lemma 2.15. *Let $\mathbf{A}$ be a simple G^0-algebra. If $\mathbf{A}$ is not term equivalent to a one-dimensional vector space, then $\mathbf{A}$ has a nontrivial binary absorptive term operation.*

Proof. Suppose that $\mathbf{A}$ is a simple G^0-algebra such that $\mathbf{A}$ has no nontrivial binary absorptive term operation. By Lemma 2.10 for $H = G$, $\mathbf{A}$ has a binary term operation $\circ$ with $1 \leq |X(\circ)| < |G|$. By Lemma 2.11, 0 cannot be a one-sided zero element for $\circ$, so by similarity (and Lemma 2.7) we can assume that 0 is a unit element for $\circ$. If $\circ$ is a quasigroup operation, then by Lemma 2.9 $\mathbf{A}$ is term equivalent to a one-dimensional vector space. (It cannot be term equivalent to the algebra $(A; \mathcal{R}_0(R_G))$, since the latter has a nontrivial binary absorptive term operation.) So we are done if $\circ$ is a quasigroup operation.

Assume now that $\circ$ is not a quasigroup operation. Taking the operation arising from $\circ$ by interchanging its variables if necessary (and using Lemma 2.7) we see that there is no loss of generality in assuming that $a \circ b = a \circ c$ for some $a, b, c \in A$ with $b \neq c$. Clearly, $a \neq 0$, and since the hypotheses of Lemma 2.12 cannot hold, we have $b, c \neq 0$. Thus, by Lemma 2.13, $\mathbf{A}$ has a binary term operation $\bullet$ with property $(\dagger_2(\bullet))$. By Lemma 2.14 (i), (ii), there exists a normal subgroup N of G such that $X(\bullet)$ is a coset of N and ε_N is a congruence of the algebra $(A; \bullet)$.

However, ε_N is not a congruence of $\mathbf{A}$, therefore by Lemma 2.14 (iii), $\mathbf{A}$ has a term operation $*$ with $1 * a = 0$, $1 * b \neq 0$ for some $(a, b) \in \varepsilon_N$. Clearly $a \neq b$, so $a, b \in G$. Consider now the following term operation of $\mathbf{A}$:

$$x \odot y = (x \bullet y) * l_a(x \bullet l_{a^{-1}b}(y)).$$

As $a^{-1}b \in N$, we have

$$1 \odot v = (1 \bullet v) * a(1 \bullet a^{-1}bv) = 0 * 0 = 0 \quad \text{for all} \quad v \in X(\bullet),$$

yielding $X(\bullet) \subseteq X(\odot)$, whence $|X(\odot)| \geq 1$. Furthermore,

$$1 \odot 0 = (1 \bullet 0) * a(1 \bullet 0) = 1 * a = 0,$$
$$0 \odot 1 = (0 \bullet 1) * a(0 \bullet a^{-1}b) = 1 * b \neq 0,$$

showing that 0 is a right zero and not a left zero for $\odot$. By Lemma 2.11, $\mathbf{A}$ has a nontrivial binary absorptive term operation. This contradiction completes the proof of the lemma.

For the proof of the last lemma we require some notation. Let A be a set, $n \geq 1$, and $\rho \subseteq A^n$ an n-ary relation on A. For arbitrary permutations $\pi_i \in S_A$ $(0 \leq i \leq n-1)$, we define an n-ary relation $\rho[\pi_0, \ldots, \pi_{n-1}]$ on A by

$$\rho[\pi_0, \ldots, \pi_{n-1}] = \{(x_0, \ldots, x_{n-1}) \in A^n\colon\ (\pi_0(x_0), \ldots, \pi_{n-1}(x_{n-1})) \in \rho\}.$$

Furthermore, for $1 \leq l < n$ and $a_l, \ldots, a_{n-1} \in A$, we define the l-ary relation arising from ρ by 'fixing the jth component at a_j for $j = l, \ldots, n-1$' as follows:

$$\rho(x_0, \ldots, x_{l-1}, a_l, \ldots, a_{n-1})$$
$$= \{(x_0, \ldots, x_{l-1}) \in A^l\colon\ (x_0, \ldots, x_{l-1}, a_l, \ldots, a_{n-1}) \in \rho\}.$$

It is straightforward to check that if for some algebra $\mathbf{A}$, ρ is a compatible relation of $\mathbf{A}$ (i.e. the universe of a subalgebra of $\mathbf{A}^n$), π_i $(0 \leq i \leq n-1)$ are automorphisms of $\mathbf{A}$ and $\{a_i\}$ $(l \leq i < n)$ are trivial subalgebras of $\mathbf{A}$, then $\rho[\pi_0, \ldots, \pi_{n-1}]$ and $\rho(x_0, \ldots, x_{l-1}, a_l, \ldots, a_{n-1})$ are also compatible relations of $\mathbf{A}$.

Lemma 2.16. *The following conditions are equivalent for a simple G^0-algebra $\mathbf{A}$:*

(i) *$\mathbf{A}$ has a nontrivial binary absorptive term operation;*

(ii) *for every $k \geq 2$, $\mathbf{A}$ has a k-ary idempotent absorptive term operation;*

(iii) *$\wedge$ preserves every compatible relation of $\mathbf{A}$;*

(iv) *$\wedge$ is a term operation of $\mathbf{A}$.*

Proof. (i)$\Rightarrow$(ii). Let $\bullet$ be a nontrivial binary absorptive term operation of $\mathbf{A}$. Since $\bullet$ is not constantly 0, there exist elements $a, b \in A$ such that $a \bullet b \in G$. Clearly, $a, b \in G$. For the term operation $x \circ y = l_{(a \bullet b)^{-1}}(l_a(x) \bullet l_b(y))$ of $\mathbf{A}$ we have $1 \circ 1 = 1$, therefore $\circ$ is idempotent. For $l \geq 1$ define 2^l-ary term operations f_l of $\mathbf{A}$ as follows: $f_1(x_0, x_1) = x_0 \circ x_1$, and for $l \geq 1$,

$$f_{l+1}(x_0, \ldots, x_{2^{l+1}-1}) = f_l(x_0, \ldots, x_{2^l-1}) \circ f_l(x_{2^l}, \ldots, x_{2^{l+1}-1}).$$

Obviously, each f_l $(l \geq 1)$ is idempotent and absorptive. Identifying variables in them we get the term operations required in (ii).

(ii)$\Rightarrow$(iii). Assume (ii) holds for $\mathbf{A}$, and let ρ be a k-ary compatible relation of $\mathbf{A}$ $(k \geq 1)$. Note first that $(0, \ldots, 0) \in \rho$ since 0 is a term operation of $\mathbf{A}$. We have to prove that $\wedge$ preserves ρ. This is trivial for $k = 1$, so we show it first for $k = 2$. Let $(a, b), (c, d) \in \rho$. We want to prove that $(a \wedge c, b \wedge d) \in \rho$. We may assume $a = c \neq 0$, $b \neq d$, $b, d \neq 0$, since otherwise the claim is trivial, or symmetric to this case. Taking $\rho[\mathrm{id}, r_g]$ in place of ρ for appropriate $r_g \in \mathrm{Aut}\,\mathbf{A}$, we may assume $c = d$. (Note that since $\wedge$ admits r_g as an automorphism, therefore $\wedge$ preserves ρ if

and only if it preserves $\rho[\mathrm{id}, r_g]$.) As ρ is closed under the operations in $\{0\} \cup L_G$, we get that ρ is reflexive. Further, it contains a pair (a, b) with $a \neq b$, $a, b \neq 0$. Suppose $\rho(x, 0) = \{0\}$. For the transitive closure $\bar{\rho}$ of ρ we have $(x, 0) \in \bar{\rho}$ if and only if $x = 0$, moreover, for $g = ba^{-1} \in G$ $(g \neq 1)$ we have

$$(l_g^m(a), l_g^m(b)) = (l_g^m(a), l_g^{m+1}(a)) \in \rho \quad \text{for all} \quad m \geq 0,$$

implying $(l_g^m(a), l_g^n(a)) \in \bar{\rho}$ for all $n \geq m \geq 0$. Since $l_g^{|G|} = \mathrm{id}$, therefore $\bar{\rho} \cap \bar{\rho}^{-1}$ is a nontrivial equivalence relation such that $\{0\}$ is a block and $\{l_g^m(a) : m \geq 0\}$ belongs to one block. This contradicts the simplicity of $\mathbf{A}$. Thus $\rho(x, 0) = A$, implying $(a, 0) \in \rho$, what was to be proved.

Now let $k \geq 3$, and let $(a_0, \ldots, a_{k-1}), (b_0, \ldots, b_{k-1}) \in \rho$. Assume without loss of generality that there exist $0 \leq l \leq m \leq n \leq k$ such that $b_i = 0$, $a_i \neq 0$ for $0 \leq i < l$, $a_i = 0$, $b_i \neq 0$ for $l \leq i < m$, $a_i \neq b_i$, $a_i, b_i \neq 0$ for $m \leq i < n$, and $a_i = b_i$ for $n \leq i < k$. We need to prove that $(a_0 \wedge b_0, \ldots, a_{k-1} \wedge b_{k-1}) = (0, \ldots, 0, a_n, \ldots, a_{k-1})$ belongs to ρ. If, say, $a_{k-1} = b_{k-1} = 0$, then we can take $\rho(x_0, \ldots, x_{k-2}, 0)$ instead of ρ. So we may assume without loss of generality that $a_n, \ldots, a_{k-1} \neq 0$. It is easy to check that

$$\sigma = \{(x_0, \ldots, x_{n-1}, x_n) : (x_0, \ldots, x_{n-1}, x_n, r_{a_n^{-1}a_{n+1}}(x_n), \ldots, r_{a_n^{-1}a_{k-1}}(x_n)) \in \rho\}$$

is a compatible relation of $\mathbf{A}$ and

$$(a_0, \ldots, a_{l-1}, 0, \ldots, 0, a_m, \ldots, a_{n-1}, a_n) \in \sigma,$$
$$(0, \ldots, 0, b_l, \ldots, b_{m-1}, b_m, \ldots, b_{n-1}, a_n) \in \sigma.$$

Using the case $k = 2$ for the binary compatible relations $\mathrm{pr}_{\{i,n\}}\, \sigma$ of $\mathbf{A}$ with $m \leq i < n$, we see that σ has elements of the form

$$(\ldots, \overbrace{0}^{i\text{th component}}, \ldots, a_n) \quad \text{for} \quad m \leq i < n.$$

Now applying to these $n - m + 2$ elements of σ an $(n - m + 2)$-ary idempotent absorptive term operation of $\mathbf{A}$, we get that $(0, \ldots, 0, a_n) \in \sigma$, whence $(0, \ldots, 0, a_n, \ldots, a_{k-1}) \in \rho$. (If $n = k$, then of course the last component of σ is missing, and we have to take $\mathrm{pr}_{\{i\}}\, \sigma$ instead of $\mathrm{pr}_{\{i,n\}}\, \sigma$.)

(iii)$\Rightarrow$(iv) follows from the well-known fact that every operation f defined on the base set of a finite algebra $\mathbf{A}$ such that f preserves all compatible relations of $\mathbf{A}$, is a term operation of $\mathbf{A}$.

(iv)$\Rightarrow$(i) is obvious.

3. Characterizing types 3,4, and 5

As in the previous section, G always denotes a finite group having at least two elements, and $A = G^0$ with $0 \notin G$. Our aim in this section is to describe simple G^0-algebras according to their types **1** up to **5**, assigned to them by tame congruence theory. It is clear that for a simple G^0-algebra $\mathbf{A}$, if $\mathbf{A}$ is term equivalent to a one-dimensional vector space, then $\mathbf{A}$ is of type **2**; otherwise, by Theorem 2.2, $\wedge$ is

a term operation of $\mathbf{A}$, so $\mathbf{A}$ has type **3**, **4**, or **5**. We will need some results from tame congruence theory establishing properties that distinguish these three types. Recall that a minimal set of a simple algebra $\mathbf{A}$ is a set $N \subseteq A$ such that $|N| > 1$, $N = f(A)$ for some $f \in \mathrm{Pol}_1 \mathbf{A}$, and N is minimal (with respect to inclusion) among the subsets of $\mathbf{A}$ satisfying these conditions.

Theorem 3.1. ([5; 1.9(1), 2.11; 5.26(1); 2.13(3), 4.10]) *Let $\mathbf{A}$ be a non-Abelian simple algebra (that is, $\mathbf{A}$ is of type **3**, **4**, or **5**).*

(i) *$\mathbf{A}$ is of type **4** or **5** if and only if it has a connected compatible partial order.*

(ii) *$\mathbf{A}$ is of type **3** or **4** if and only if it has a two-element minimal set $N = \{0, 1\}$ and binary polynomial operations p_1, p_2 such that $p_1|_N$ and $p_2|_N$ are the two distinct semilattice operations on N.*

Lemma 3.2. *Let $\mathbf{A}$ be a G^0-algebra. If $\mathbf{A}$ has a connected compatible partial order $\leq$, then $\leq$ is the semilattice order corresponding to $\wedge$, or its dual.*

Proof. Assume $a \leq b$ for some $a, b \in G$. Then for $g = ba^{-1}$ we have

$$a \leq b = l_g(a) \leq l_g(b) = l_{g^2}(a) \leq l_{g^2}(b) \leq \ldots \leq l_{g^{k-1}}(a) \leq l_{g^{k-1}}(b) \leq l_{g^k}(a) \leq \ldots \ .$$

Since G is finite, $g^k = 1$ for some $k \geq 1$, proving $a = b$. Thus, in every pair of distinct comparable elements, one member is 0. Connectedness implies that 0 is comparable to each element in G. Clearly, there are no elements $a, b \in G$ with $a \leq 0 \leq b$, so $\leq$ is one of the orders described in the lemma.

From now on, $\leq$ will always denote the semilattice order corresponding to $\wedge$, and we call a G^0-algebra *ordered* if the operations of $\mathbf{A}$ are monotone with respect to $\leq$. The following claim is straightforward to prove.

Lemma 3.3. *Every absorptive operation on A is monotone with respect to $\leq$.*

The next lemma is an easy necessary and sufficient condition for a G^0-algebra to be ordered.

Lemma 3.4. *A G^0-algebra $\mathbf{A}$ is ordered if and only if every binary term operation of $\mathbf{A}$ is either essentially unary or absorptive.*

Proof. Suppose first that $\mathbf{A}$ is ordered, and let $\circ$ be a non-absorptive binary term operation of $\mathbf{A}$. By similarity we can assume that 0 is a one-sided unit element for $\circ$, say a right unit. Then for arbitrary elements $a, b \in A$, we have $a \circ b \geq a \circ 0 = a$, implying $a \circ b = a$ if $a \neq 0$. For $a = 0$ and $b \in A$ we have $0 \circ b \leq c \circ b = c$ for all $c \in G$, so since $|G| > 1$, we conclude that $0 \circ b = 0$. Thus $\circ$ does not depend on its second variable.

Conversely, suppose $\mathbf{A}$ is not ordered. Then $\mathbf{A}$ has a unary polynomial operation which is not monotone with respect to $\leq$. As in the proof of Lemma 2.10, we get a binary term operation $\circ$ of $\mathbf{A}$ such that $1 \circ a \not\leq 1 \circ b$ for some $a \leq b$ in A. Clearly, $a \neq b$, so $a = 0$. Thus $1 \circ 0 \not\leq 1 \circ b$, implying $1 \circ 0 \neq 0$. This shows that $\circ$ is not absorptive. It has to depend on both variables, as all unary term operations of $\mathbf{A}$ (i.e. all operations in $\{0\} \cup L_G$) are monotone with respect to $\leq$.

Now we are in a position to characterize simple G^0-algebras of types **3**, **4**, and **5**. For this purpose we will need two operations $\diamond$, h_0, and a relation μ_0 on A:

$$x \diamond y = \begin{cases} y & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}, \quad h_0(x,y,z) = \begin{cases} z & \text{if } z = x \text{ or } z = y \\ 0 & \text{otherwise} \end{cases},$$

and

$$\mu_0 = \{(a,a,0,0), (a,0,a,0), (a,0,0,0), (a,a,a,a) \colon a \in A\}.$$

The claims in the next proposition are easy consequences of the definitions and Proposition 2.1. (Here $|G| = 1$ is also allowed.)

Proposition 3.5. (i) *$(A;\ 0, L_G, \wedge, \diamond)$ is a simple G^0-algebra.*
(ii) *$(A;\ 0, L_G, h_0)$ is a simple ordered G^0-algebra with $h_0(y,y,x) = x \wedge y$.*
(iii) *Every absorptive operation on A preserves μ_0; in particular, μ_0 is a compatible relation of $(A;\ 0, L_G, \wedge)$.*

We show that among simple G^0-algebras, those of type **3** are characterized by the property of having the algebra in (i) as a reduct. Furthermore, among ordered simple G^0-algebras, those of type **4** are characterized by the property of having the algebra in (ii) as a reduct, and those of type **5** by the property of admitting μ_0 as a compatible relation.

Theorem 3.6. *Let $\mathbf{A}$ be a simple G^0-algebra. The following conditions are equivalent:*
(i) *$\mathbf{A}$ is of type **3**;*
(ii) *$\mathbf{A}$ has a binary term operation $*$ satisfying the identities*

$$x * 0 = 0, \quad 0 * x = x, \quad x * x = 0;$$

(iii) *$\diamond$ is a term operation of $\mathbf{A}$.*

Proof. (i)$\Rightarrow$(ii). Suppose $\mathbf{A}$ is of type **3**. By Theorem 2.2, $\wedge$ is a term operation of $\mathbf{A}$. Furthermore, by Theorem 3.1, $\mathbf{A}$ has no connected compatible partial order. Thus $\mathbf{A}$ is not ordered, and hence, as we have seen in the proof of Lemma 3.4, $\mathbf{A}$ has a binary term operation $\circ$ such that $0 \neq 1 \circ 0 \neq 1 \circ b$ for some element $b \in G$. Let $a = (1 \circ 0)^{-1}$, and consider the term operation

$$x * y = y \wedge l_a(y \circ l_b(x)).$$

Clearly,

$$\begin{aligned} 1 * 0 &= 0 \wedge a(0 \circ b) = 0, \\ 0 * 1 &= 1 \wedge a(1 \circ 0) = 1 \wedge 1 = 1, \\ 1 * 1 &= 1 \wedge a(1 \circ b) = 0, \end{aligned}$$

as $a(1 \circ b) = (1 \circ 0)^{-1}(1 \circ b) \neq 1$. By the automorphisms in R_G this implies that $*$ satisfies the identities required in (ii).

(ii)$\Rightarrow$(iii). Suppose $\mathbf{A}$ has a binary term operation $*$ described in (ii). Obviously, $*$ cannot be the term operation of a one-dimensional vector space, therefore by Theorem 2.2, $\wedge$ is a term operation of $\mathbf{A}$. We prove (iii) by showing that $\diamond$ preserves all compatible relations of $\mathbf{A}$.

Let ρ be an n-ary compatible relation of $\mathbf{A}$ for some $n \geq 1$, and let $a, b \in \rho$. The ith component of a will be denoted by a_i, and similarly for b. We may assume

without loss of generality that there exist integers $0 \leq j \leq k \leq n$ such that $a_i = b_i$ for $0 \leq i \leq j-1$, $0 = a_i \neq b_i$ for $j \leq i \leq k-1$, and $0 \neq a_i \neq b_i$ for $k \leq i \leq n-1$. Put

$$\begin{aligned} b' &= (b_0, \dots, b_{j-1}, 0, \dots, 0, 0, \dots, 0), \\ b'' &= (0, \dots, 0, b_j, \dots, b_{k-1}, 0, \dots, 0), \\ \bar{b} &= (b_0, \dots, b_{j-1}, b_j, \dots, b_{k-1}, 0, \dots, 0), \\ \hat{a} &= (0, \dots, 0, 0, \dots, 0, a_k, \dots, a_{n-1}). \end{aligned}$$

We have to verify that $a \diamond b = b''$ belongs to ρ. Since ρ is closed under $\wedge$ and $*$, we have $b' = a \wedge b \in \rho$ and $\hat{a} = b' * a \in \rho$. As $a_k, \dots, a_{n-1} \neq 0$, for each $k \leq i \leq n-1$ the n-tuple

$$c^{(i)} = \begin{cases} l_{b_i a_i^{-1}}(\hat{a}) * b & \text{if } b_i \neq 0 \\ b & \text{if } b_i = 0 \end{cases}$$

has ith component 0, while the first k components coincide with those of b. Further, since ρ is closed under the operations in L_G, these n-tuples belong to ρ. Thus $\bar{b} = \wedge_{i=k}^{n-1} c^{(i)} \in \rho$, whence it follows that $b'' = b' * \bar{b} \in \rho$.

(iii)$\Rightarrow$(i). Assume (iii) holds for $\mathbf{A}$. As in the previous step, we get that $\wedge$ is a term operation of $\mathbf{A}$. Thus $\mathbf{A}$ is of type **3**, **4**, or **5**. By Theorem 3.1, if it were of type **4** or **5**, then it would have a connected compatible partial order, and hence by Lemma 3.2 it would be ordered. However, this is excluded by the term operation $\diamond$.

Theorem 3.7. *Let $\mathbf{A}$ be an ordered simple G^0-algebra. The following conditions are equivalent:*

(i) $\mathbf{A}$ *is of type* **4**;

(ii) $\mathbf{A}$ *has a ternary term operation h satisfying the identities*

$$h(x,y,y) = h(y,x,y) = y, \quad h(0,0,x) = h(x,x,0) = 0;$$

(iii) h_0 *is a term operation of* $\mathbf{A}$.

Proof. Note in advance that since $\mathbf{A}$ is simple and ordered, therefore by Theorem 2.2, $\wedge$ is a term operation of $\mathbf{A}$.

(i)$\Rightarrow$(ii). Assume $\mathbf{A}$ is of type **4**. The range $\{0,1\}$ of the unary polynomial operation $x \wedge 1$ is a minimal set, and $\wedge|_{\{0,1\}}$ is the meet operation on $\{0,1\}$. Since $\mathbf{A}$ is of type **4**, by Theorem 3.1 it has a polynomial operation $p(x,y) \in \mathrm{Pol}_2 \mathbf{A}$ such that

$$p(0,0) = 0, \quad p(0,1) = p(1,0) = p(1,1) = 1.$$

For some $n \geq 2$, some $f \in \mathrm{Clo}_n \mathbf{A}$ and some elements $a_2, \dots, a_{n-1} \in A$ we have $p(x,y) = f(x,y,a_2,\dots,a_{n-1})$. Since $f(x_0,\dots,x_{i-1},0,x_{i+1},\dots,x_{n-1})$ is a term operation of $\mathbf{A}$ for all $2 \leq i \leq n-1$, we may assume $a_2, \dots, a_{n-1} \in G$. Let

$$h(x,y,z) = f(x,y,l_{a_2}(z),\dots,l_{a_{n-1}}(z)).$$

Clearly, $p(x,y) = h(x,y,1)$. Hence

$$h(0,0,1) = 0, \quad h(0,1,1) = 1, \quad h(1,0,1) = 1. \tag{3.7}$$

By the second equality in (3.7),

$$h(x,1,1) \geq h(0,1,1) = 1 \quad \text{for all} \quad x \in A,$$

implying $h(x,1,1)=1$ for all $x\in A$. Using that R_G is the automorphism group of $\mathbf{A}$, we get that

$$h(x,y,y)=y \quad \text{for all} \quad x\in A,\ y\in G.$$

Now

$$h(x,0,0)\le h(x,y,y)=y \quad \text{for all} \quad x\in A,\ y\in G,$$

implying $h(x,0,0)=0$ for all $x\in A$ (as $|G|\ge 2$). Thus h satisfies the identity $h(x,y,y)=y$. Similarly, from the third equality in (3.7) we derive the identity $h(y,x,y)=y$.

Now consider the binary term operation $x\circ y=h(y,y,x)$ of $\mathbf{A}$. By Lemma 3.4, $\circ$ is essentially unary or absorptive. If it is absorptive, then all identities in (ii) hold. Suppose $\circ$ is essentialy unary. Since it is idempotent and by the first equality in (3.7) $1\circ 0=0$, therefore $\circ$ satisfies the identity $x\circ y=y$, and h is a majority operation. It is easy to check that the term operation

$$h'(x,y,z)=h(h(x,y,z),x\wedge z,y\wedge z)$$

of $\mathbf{A}$ satisfies all identities in (ii).

(ii)$\Rightarrow$(iii). Suppose $\mathbf{A}$ has a term operation h satisfying the identities in (ii). It will follow that h_0 is a term operation of $\mathbf{A}$, if we show that h_0 preserves every compatible relation of $\mathbf{A}$.

Let ρ be an n-ary compatible relation of $\mathbf{A}$ for some $n\ge 1$, and let $a,b,c\in\rho$. The ith component of a will be denoted by a_i, and similarly for b,c. We may assume without loss of generality that there exist integers $0\le j\le k\le l\le n$ such that $a_i=b_i=c_i$ for $0\le i\le j-1$, $a_i\ne b_i=c_i$ for $j\le i\le k-1$, $b_i\ne a_i=c_i$ for $k\le i\le l-1$, and $a_i,b_i\ne c_i$ for $l\le i\le n-1$. Put

$$\begin{aligned}
c' &= (c_0,\dots,c_{j-1},0,\dots,0,c_k,\dots,c_{l-1},0,\dots,0),\\
c'' &= (c_0,\dots,c_{j-1},c_j,\dots,c_{k-1},0,\dots,0,0,\dots,0),\\
\bar{c} &= (c_0,\dots,c_{j-1},c_j,\dots,c_{k-1},c_k,\dots,c_{l-1},0,\dots,0).
\end{aligned}$$

We have to verify that $h_0(a,b,c)=\bar{c}$ belongs to ρ. However, since ρ is closed under $\wedge$, we have $c'=a\wedge c\in\rho$ and $c''=b\wedge c\in\rho$, implying $\bar{c}=h(c',c'',c)\in\rho$.

(iii)$\Rightarrow$(i). Suppose h_0 is a term operation of $\mathbf{A}$. As we have seen before, $\{0,1\}$ is a minimal set. Clearly, $\wedge|_{\{0,1\}}$, $h_0(x,y,1)|_{\{0,1\}}$ are two distinct semilattice operations on $\{0,1\}$. Since $\mathbf{A}$ has a connected compatible partial order, it is of type **4** by Theorem 3.1.

Theorem 3.8. *Let $\mathbf{A}$ be an ordered simple G^0-algebra. The following conditions are equivalent:*

(i) $\mathbf{A}$ *is of type* **5**;

(ii) $\mathbf{A}$ *has a quaternary reflexive compatible relation μ such that*

(1) $(a,a,0,0),\ (a,0,a,0)\in\mu$ *for all* $a\in A$,

(2) $(a,a,a,0)\in\mu$ *if and only if* $a=0$.

(iii) μ_0 *is a compatible relation of* $\mathbf{A}$.

Proof. Note in advance that since $\mathbf{A}$ is simple and ordered, therefore by Theorem 2.2, $\wedge$ is a term operation of $\mathbf{A}$.

(i)⇒(ii). Assume $\mathbf{A}$ is of type **5**. By Theorem 3.7, h_0 is not a term operation of $\mathbf{A}$, so for some $n \geq 1$, $\mathbf{A}$ has an n-ary compatible relation ρ not preserved by h_0. Select elements $a, b, c \in \rho$ such that $h_0(a, b, c) \notin \rho$. We use the same notation as in part (ii)⇒(iii) of the proof of Theorem 3.7. Thus $\bar{c} \notin \rho$, while as ρ is closed under $\wedge$, we have $c', c'' \in \rho$.

Taking $\rho(x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{n-1})$ instead of ρ if some $c_i = 0$, we can assume that $c_0, \dots, c_{n-1} \neq 0$. Define

$$\mu = \{(x, y, z, u) \in A^4 \colon (r_{c_0}(x), \dots, r_{c_{j-1}}(x), r_{c_j}(y), \dots, r_{c_{k-1}}(y), \\ r_{c_k}(z), \dots, r_{c_{l-1}}(z), r_{c_l}(u), \dots, r_{c_{n-1}}(u)) \in \rho\}.$$

Clearly, μ is a compatible relation of $\mathbf{A}$. Moreover, since $c'', c', c \in \rho$ and $\bar{c} \notin \rho$, we have

$$(1, 1, 0, 0) \in \mu, \quad (1, 0, 1, 0) \in \mu, \quad (1, 1, 1, 1) \in \mu, \quad (1, 1, 1, 0) \notin \mu.$$

Since μ is closed under the unary operations in $\{0\} \cup L_G$, therefore it has the properties required in (ii).

(ii)⇒(iii). Let σ be the subalgebra of $\mathbf{A}^4$ generated by the quadruples $(a, a, 0, 0)$, $(a, 0, a, 0)$, (a, a, a, a) $(a \in A)$. Clearly, $\sigma \subseteq \mu$. Since $\wedge$ is a term operation of $\mathbf{A}$, therefore $(a, 0, 0, 0) = (a, a, 0, 0) \wedge (a, 0, a, 0) \in \sigma$ for all $a \in A$, and hence $\mu_0 \subseteq \sigma$. Observe that for every element $(a, b, c, d) \in \sigma$ we have $a \geq b \geq d$ and $a \geq c \geq d$, since this holds for all the generating elements of σ (and since the term operations of $\mathbf{A}$ are monotone with respect to $\leq$). It is easy to check that the quadruples satisfying this condition are exactly the elements of μ_0 and $(a, a, a, 0)$ for $a \in G$. Since the latter do not belong to μ, therefore $\sigma = \mu_0$, completing the proof of (iii).

(iii)⇒(i). Assume μ_0 is a compatible relation of $\mathbf{A}$. Again, the range $\{0, 1\}$ of the unary polynomial operation $x \wedge 1$ is a minimal set, and $\wedge|_{\{0,1\}}$ is the meet operation on $\{0, 1\}$. Therefore by Theorem 3.1 it suffices to exclude the existence of a binary polynomial operation $p(x, y) \in \mathrm{Pol}_2 \mathbf{A}$ with

$$p(0, 0) = 0, \quad p(0, 1) = p(1, 0) = p(1, 1) = 1.$$

As μ_0 is reflexive, it is preserved by every polynomial operation of $\mathbf{A}$. So if a polynomial p described above existed, then we would get

$$p((1, 1, 0, 0), (1, 0, 1, 0)) = (1, 1, 1, 0) \in \mu_0,$$

a contradiction.

Theorems 3.6, 3.7, and 3.8 show that within the interval described in Corollary 2.4, the clones of simple algebras of types **3**, **4**, and **5** form three disjoint subintervals.

Corollary 3.9. *Let* $\mathbf{A}$ *be a* G^0*-algebra.*

(i) $\mathbf{A}$ *is a simple algebra of type* **3** *if and only if*

$$[L_G, \wedge, \diamond] \subseteq \mathrm{Clo}\, \mathbf{A} \subseteq \mathcal{R}_0(R_G),$$

(ii) $\mathbf{A}$ *is a simple algebra of type* **4** *if and only if*

$$[L_G, h_0] \subseteq \mathrm{Clo}\, \mathbf{A} \subseteq \mathcal{R}_0(R_G) \cap \mathcal{P}_{\leq},$$

and

(iii) **A** *is a simple algebra of type* **5** *if and only if*

$$[L_G, \wedge] \subseteq \operatorname{Clo} \mathbf{A} \subseteq \mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0}.$$

For the bounds of these intervals we have

$$\begin{array}{ccccc} \mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0} & \subseteq & \mathcal{R}_0(R_G) \cap \mathcal{P}_{\leq} & \subseteq & \mathcal{R}_0(R_G) \\ \cup\mid & & \cup\mid & & \cup\mid \\ [L_G, \wedge] & \subseteq & [L_G, h_0] & \subseteq & [L_G, \wedge, \diamond] \end{array}.$$

Proof. The claims in (i)–(iii) are immediate consequences of Corollary 2.4, Theorems 3.6–3.8, Theorem 3.1, and the inclusions in the last statement.

In view of Proposition 3.5 and the equality $\geq = \operatorname{pr}_{\{0,1\}} \mu_0$, the only nontrivial inclusion in the last statement is the right one at the bottom. To prove this, observe that for the ternary operation $f(x,y,z) = ((x \diamond y) \diamond (x \diamond z)) \diamond z$ we have

$$f(x,y,z) = \begin{cases} 0 & \text{if } x = y = 0 \\ z & \text{otherwise,} \end{cases}$$

and hence $h_0(x,y,z) = f(x \wedge z, y \wedge z, z)$.

Corollary 3.10. *Every simple G^0-algebra of type* **3** *or* **4** *generates a congruence distributive variety.*

Proof. Let **A** be a simple G^0-algebra of type **3** or **4**. By the preceding corollary, h_0 is a term operation of **A**. It is easy to verify that the term operations

$$\begin{aligned} d_1(x,y,z) &= h_0(z,y,x), \\ d_2(x,y,z) &= h_0(z,y,x) \wedge h_0(x,y,z), \\ d_3(x,y,z) &= h_0(x,y,z) \end{aligned}$$

satisfy the identities $d_i(x,y,x) = x$ $(i = 1,2,3)$, $x = d_1(x,x,y)$, $d_1(x,y,y) = d_2(x,y,y)$, $d_2(x,x,y) = d_3(x,x,y)$, $d_3(x,y,y) = y$. Hence by Jónsson's theorem [6], $V(\mathbf{A})$ is congruence distributive.

Corollary 3.11. *Every simple G^0-algebra of type* **3** *generates a congruence 3-permutable variety.*

Proof. Let **A** be a simple G^0-algebra of type **3**. By Corollary 3.9, $\wedge$ and $\diamond$ are term operation of **A**. It is easy to verify that the term operations

$$\begin{aligned} p_1(x,y,z) &= ((z \wedge y) \diamond y) \diamond x, \\ p_2(x,y,z) &= p_1(z,y,x) \end{aligned}$$

satisfy the identities $x = p_1(x,y,y)$, $p_1(x,x,y) = p_2(x,y,y)$, $p_2(x,x,y) = y$. Hence by the theorem of Hagemann and Mitschke [4], $V(\mathbf{A})$ is congruence 3-permutable.

In this claim congruence 3-permutability cannot be replaced by congruence permutability. To see this, observe that for the simple G^0-algebra $\mathbf{A} = (A; L_G, \wedge, \diamond)$ of type **3** $(A \times \{0\}) \cup (\{0\} \times A)$ is a compatible relation of **A**.

4. $2^{\aleph_0}$ inequivalent G^0-algebras

We show that all three intervals in Corollary 3.9 contain $2^{\aleph_0}$ clones.

Theorem 4.1. *For arbitrary finite group G with at least two elements and for every type $\mathbf{i} \in \mathbf{3}, \mathbf{4}, \mathbf{5}$, there exist simple G^0-algebras $\mathbf{A}^{\mathbf{i}}_J$ $(J \subseteq \mathbf{N}_0)$ of type $\mathbf{i}$ such that for distinct sets $I, J \subseteq \mathbf{N}_0$, the algebras $\mathbf{A}^{\mathbf{i}}_I$ and $\mathbf{A}^{\mathbf{i}}_J$ are not term equivalent.*

It was observed by R. McKenzie that on each finite set with at least three elements, there are $2^{\aleph_0}$ pairwise nonequivalent algebras generating congruence distributive varieties. In view of Corollary 3.10, Theorem 4.1 also implies this fact.

To construct algebras satisfying the requirements in Theorem 4.1, fix an element $a \in G - \{1\}$, and for $n \geq 2$, $0 \leq i \leq n-1$, put

$$\eta^R_{n,i} = \{(g, \dots, g, \overbrace{ag}^{i\text{th component}}, g, \dots, g) \colon g \in G\},$$
$$\eta^R_n = \bigcup_{0 \leq i \leq n-1} \eta^R_{n,i},$$
$$\eta^L_{n,i} = \{(g, \dots, g, \overbrace{ga}^{i\text{th component}}, g, \dots, g) \colon g \in G\},$$
$$\eta^L_n = \bigcup_{0 \leq i \leq n-1} \eta^L_{n,i},$$

and

$$\chi^0_n = \{(a_0, \dots, a_{n-1}) \in A^n \colon a_i = 0 \text{ for at least one } i,\ 0 \leq i \leq n-1\}.$$

Clearly, each $\eta^R_{n,i}$ is an orbit of R_G acting on G^n, and each $\eta^L_{n,i}$ is an orbit of L_G acting on G^n. For $n \geq 3$ define an n-ary operation f_n and an n-ary relation β_n on $A = G^0$ as follows:

$$f_n(x_0, \dots, x_{n-1}) = \begin{cases} x_i & \text{if } (x_0, \dots, x_{n-1}) \in \eta^R_{n,i} \text{ for some } 0 \leq i \leq n-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta_n = \chi^0_n \cup \eta^L_n.$$

We note that a similar construction was used in Demetrovics–Hannák [3].

The following properties of f_n and β_n are clear from the definitions.

Lemma 4.2. (i) *β_n is preserved by the operations 0, $l_g \in L_G$ $(g \in G)$, $\wedge$, h_0, and $\diamond$.*

(ii) *f_n admits the members of R_G as automorphisms.*

Lemma 4.3. *For $k, n \geq 5$, f_n preserves β_k if and only if $k \neq n$.*

Proof. Clearly, the property 'f_n preserves β_k' means that for every $n \times k$ matrix whose rows belong to β_k, the k-tuple of column values of f_n also belongs to β_k. For an $n \times k$ matrix $C = (c_{ij})_{n \times k}$ the rows and columns of C will be denoted by c_{i-} and c_{-j} $(0 \leq i \leq n-1, 0 \leq j \leq k-1)$, respectively. The transpose of a vector will be denoted by $^{\mathrm{T}}$.

The rows of the matrix

$$\begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ 1 & 1 & a & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a \end{pmatrix}$$

belong to β_n while the row of column values of f_n, namely $(a, a, a, \dots, a)$, does not, therefore f_n does not preserve β_n.

Now let $k \neq n$, and consider an arbitrary $n \times k$ matrix $C = (c_{ij})_{n\times k}$ whose rows belong to β_k, and let $d = (f_n(c_{-0}^{\mathrm{T}}), \dots, f_n(c_{-\,k-1}^{\mathrm{T}}))$. We have to prove that $d \in \beta_k$. If at least one component of d is 0, then $d \in \chi_k^0 \subseteq \beta_k$ and we are done. Therefore we may assume

(4.3) $\quad c_{-j}^{\mathrm{T}} \in \eta_n^R$ for all j $(0 \leq j \leq k-1)$,

so C has no entries equal to 0. Hence by the assumption on the rows

(4.3)′ $\quad c_{i-} \in \eta_k^L$ for all i $(0 \leq i \leq n-1)$.

Since β_k is invariant under permuting its components and f_n is invariant under permuting its variables, we can permute the columns and rows of C without restricting generality. In particular, we can assume that

$$c_{0-} = (ga, g, \dots, g)$$

for some $g \in G$. Then by (4.3) each c_{-j}^{T} $(1 \leq j \leq k-1)$ is one of the following:

(1) $\quad v^{\mathrm{T}} = (g, a^{-1}g, \dots, a^{-1}g)$, $\quad$ or

(2) $\quad w_i^{\mathrm{T}} = (g, g, \dots, g, \overbrace{ag}^{i\text{th component}}, g, \dots, g)$ $\quad$ for some $\quad 1 \leq i \leq n-1$.

If $c_{-j} = v$ for all $1 \leq j \leq k-1$, then C is of the form

$$\begin{pmatrix} ga & g & \dots & g \\ & a^{-1}g & \dots & a^{-1}g \\ & a^{-1}g & \dots & a^{-1}g \\ & \vdots & \ddots & \vdots \\ & a^{-1}g & \dots & a^{-1}g \end{pmatrix},$$

implying by (4.3)′ that $c_{i0} = a^{-1}ga$ for all $1 \leq i \leq n-1$, whence $d = c_{0-} \in \beta_k$.

If at least two of the columns c_{-j} $(1 \leq j \leq k-1)$ are of type (1) and at least one of them is of type (2), then we may assume without loss of generality that $c_{-1} = c_{-2} = v$ and $c_{-\,k-1} = w_1$. Now for $2 \leq i \leq n-1$,

$$c_{i1} = c_{i2} = a^{-1}g \neq g = c_{i\,k-1},$$

yielding by $(4.3)'$ that $a^{-1}ga = g$, or equivalently, $ga = ag$, and

$$c_{i-} = (a^{-1}g, \ldots, a^{-1}g, g) \quad \text{for} \quad 2 \leq i \leq n-1.$$

Further, $c_{11} = c_{12} = a^{-1}g$ and $c_{1\,k-1} = ag$. However, $(a^{-1}g)a = g \neq ag$, therefore by $(4.3)'$ we must have $a^{-1}g = ag$, that is, $a^2 = 1$. Thus C is of the form

$$\begin{pmatrix} ga & g & g & g & \ldots & g & g \\ & a^{-1}g & a^{-1}g & & & & ag \\ a^{-1}g & a^{-1}g & a^{-1}g & a^{-1}g & \ldots & a^{-1}g & g \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{-1}g & a^{-1}g & a^{-1}g & a^{-1}g & \ldots & a^{-1}g & g \end{pmatrix} \quad \text{with} \quad a = a^{-1},\ ga = ag.$$

So, looking at the columns c_{-j} $(3 \leq j \leq k-2)$ we get from (4.3) and $(4.3)'$ that c_{1-} must be the row $(a^{-1}ga, a^{-1}g, \ldots, a^{-1}g)$. Thus $d = (a^{-1}ga, g, \ldots, g, ag) = (g, g, \ldots, g, ga) \in \beta_k$.

Assume v occurs among the columns c_{-j} $(1 \leq j \leq k-1)$ exactly once. If there is a repetition among the columns of type (2), then we can assume without loss of generality that $c_{-1} = v$ and $c_{-2} = c_{-3} = w_1$. Now for $2 \leq i \leq n-1$,

$$c_{i1} = a^{-1}g \neq g = c_{i2} = c_{i3},$$

implying by $(4.3)'$ that $a^{-1}g = ga$. Further,

$$c_{11} = a^{-1}g \quad \text{and} \quad c_{i2} = c_{i3} = ag.$$

Since $(ga =)a^{-1}g = (ag)a$ is now impossible, we must have $a^{-1}g = ag$. Thus $ag = ga$ and $a^2 = 1$. Hence

$$c_{i-} = (g, a^{-1}g, g, \ldots, g) \quad \text{for all} \quad 2 \leq i \leq n-1,$$

yielding that C is of the form

$$\begin{pmatrix} ga & g & g & g & g & \ldots & g \\ & a^{-1}g & ag & ag & & & \\ g & a^{-1}g & g & g & g & \ldots & g \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g & a^{-1}g & g & g & g & \ldots & g \end{pmatrix} \quad \text{with} \quad a = a^{-1},\ ga = ag.$$

Now looking at the columns c_{-j} $(4 \leq j \leq k-1)$ and using (4.3), $(4.3)'$ we see that $c_{1-} = (aga, a^{-1}g, ag, ag, \ldots, ag) = (g, ga^{-1}, ga^{-1}, ga^{-1}, \ldots, ga^{-1})$, whence $d = (ga, g, ag, \ldots, ag) = (ga^{-1}, g, ga^{-1}, \ldots, ga^{-1}) \in \beta_k$.

If there is no repetition among the columns of type (2) in c_{-j} $(1 \leq j \leq k-1)$, then $n-1 \geq k-2$ and we can assume without loss of generality that $c_{-1} = v$ and $c_{-j} = w_{j-1}$ for $2 \leq j \leq k-1$. Now for the components of c_{1-} we have

$$c_{11} = a^{-1}g \neq g = c_{13} = \ldots = c_{1\,k-1} = g \neq ag = c_{12}.$$

In view of $k \geq 5$ there are at least two g's, so this contradicts $(4.3)'$.

Finally, suppose every column c_{-j} $(1 \leq j \leq k-1)$ is of type (2). If there is a repetition among these columns , then we can assume without loss of generality that $c_{-1} = c_{-2} = w_1$. By symmetry there are two essentially different possibilities for c_{1-}:

$$c_{1-} = (aga, ag, ag, \ldots, ag) \quad \text{or} \quad c_{1-} = (ag, ag, \ldots, ag, aga).$$

In the first case the assumptions on the columns imply $c_{-j} = w_1$ for all $1 \leq j \leq k-1$, that is, C is of the form

$$\begin{pmatrix} ga & g & g & g & \cdots & g \\ aga & ag & ag & ag & \ldots & ag \\ & g & g & g & \cdots & g \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & g & g & g & \cdots & g \end{pmatrix}.$$

Thus by $(4.3)'$ we conclude that $c_{-0}^{\mathrm{T}} = (ga, aga, ga, \ldots, ga)$. Now $d = c_{1-} \in \beta_k$. In the second case we can conclude $c_{-j} = w_1$ for $1 \leq j \leq k-2$ only, so C has the form

$$\begin{pmatrix} ga & g & g & \cdots & g & g \\ ag & ag & ag & \ldots & ag & aga \\ & g & g & \cdots & g & \\ & \vdots & \vdots & \ddots & \vdots & \\ & g & g & \cdots & g & \end{pmatrix}.$$

Since $c_{-\,k-1} \in \{w_1, \ldots, w_{n-1}\}$ and $aga \neq ag$, therefore we have $aga = g$, and we may assume $c_{-\,k-1} = w_2$. Thus, applying $(4.3)'$, we see from c_{2-} that $ga = ag$ (whence also $a^2 = 1$) and

$$\begin{aligned} c_{2-} &= (g, g, g, \ldots, g, ga), \\ c_{i-} &= (ga, g, g, \ldots, g, g) \quad \text{for} \quad 3 \leq i \leq n-1. \end{aligned}$$

Thus $d = (g, ag, \ldots, ag, ga) = (g, ga^{-1}, \ldots, ga^{-1}, ga^{-1}) \in \beta_k$.

If there is no repetition among the columns c_{-j} $(1 \leq j \leq k-1)$, then $n \geq k$, and since $n \neq k$, we have $n > k$. So we may assume without loss of generality that C is of the form

$$\begin{pmatrix} ga & g & g & g & \cdots & g \\ & ag & g & g & \cdots & g \\ & g & ag & g & \ldots & g \\ & g & g & ag & \ldots & g \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & g & g & g & \ldots & ag \\ & g & g & g & \cdots & g \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & g & g & g & \ldots & g \end{pmatrix}.$$

By $(4.3)'$ we have $ag = ga$ and $c_{i0} = g$ for $1 \leq i \leq k-1$ and $c_{i0} = ga$ for $k \leq i \leq n-1$. Thus $c_{-0} \notin \eta_n^R$, contradicting (4.3).

This completes the proof of the lemma.

Proof of Theorem 4.1. For arbitrary set $J \subseteq \mathbf{N}_0$, let

$$\mathbf{A}_J^{\mathbf{5}} = (G^0;\ L_G, \wedge, \{f_{n+5}\colon n \in J\}),$$
$$\mathbf{A}_J^{\mathbf{4}} = (G^0;\ L_G, h_0, \{f_{n+5}\colon n \in J\}),$$
$$\mathbf{A}_J^{\mathbf{3}} = (G^0;\ L_G, \wedge, \diamond, \{f_{n+5}\colon n \in J\}).$$

By Lemmas 4.2 (i) and 4.3, β_{k+5} ($k \in \mathbf{N}_0$) is a compatible relation of $\mathbf{A}_J^{\mathbf{i}}$ if and only if $k \notin J$, proving the claim on non-equivalence. The other claims of the theorem follow from Corollary 3.9, Lemma 4.2 (ii), Lemma 3.3, and Proposition 3.5 (iii).

5. Survey of strictly simple term minimal algebras

We need some more notation, in addition to what was introduced at the beginning of Sections 1 and 2, and before Proposition 3.5.

Let A be a finite set. For a permutation group G acting on A, let $\mathcal{R}(G)$ denote the clone of all operations f on A such that f admits each member of G as an automorphism, and $\mathcal{R}_{\mathrm{id}}(G)$ its subclone consisting of all idempotent operations in $\mathcal{R}(G)$. If $|A| = 2$ and $G = \{\mathrm{id}\}$, then we write $\mathcal{R}$, $\mathcal{R}_{\mathrm{id}}$ instead of $\mathcal{R}(G)$, $\mathcal{R}_{\mathrm{id}}(G)$, respectively. Similarly, if $A = G^0$ for the one-element group G (and hence $R_G = \{\mathrm{id}\}$), then we write $\mathcal{R}_0$ instead of $\mathcal{R}_0(R_G)$ (see Section 2).

For an element $0 \in A$ and for $n \geq 2$, let $\mathcal{F}_n^0$ denote the clone of all operations f on A preserving the relation χ_n^0 (introduced in Section 4). Furthermore, we put $\mathcal{F}_\omega^0 = \bigcap_{k=2}^{\infty} \mathcal{F}_k^0$.

For a vector space ${}_K\widehat{A} = (A; +, K)$ over a field K, $\mathrm{End}\,{}_K\widehat{A}$ stands for the endomorphism ring of ${}_K\widehat{A}$ and $T(\widehat{A})$ for the group $\{x + a\colon\ a \in A\}$ of translations of $\widehat{A}$.

In case $|A| = 2$, the two distinct semilattice operations on A will be denoted by $\wedge$ and $\vee$.

By Theorem 1.9, there are four basic types of strictly simple term minimal algebras. In the discussion below we follow this classification. Within each class, we list the algebras according to their types **1**–**5** by tame congruence theory.

(O)(a) $\mathbf{A}$ *is a strictly simple term minimal algebra such that* $\mathrm{Clo}_1\,\mathbf{A}$ *is a transitive permutation group on* A

These algebras were determined, up to term equivalence, in [11]. There are the following possibilities:

(**1**) $\mathrm{Clo}\,\mathbf{A} = [G]$ for a primitive permutation group G on A;

(**2**) $\mathrm{Clo}\,\mathbf{A} = [\mathrm{Clo}_{\mathrm{id}}\,({}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A}),\ T(\widehat{A})]$ for some vector space ${}_K\widehat{A} = (A; +, K)$ over a finite field K;

(**3**) $\mathrm{Clo}\,\mathbf{A} = \mathcal{R}(G)$ for a regular permutation group G on A.

In this class, there are no algebras of type **4** or **5**.

(O)(b) $\mathbf{A}$ *is a strictly simple term minimal algebra such that* $C_A \subseteq \mathrm{Clo}_1 \mathbf{A}$ *and* $\mathrm{Clo}_1 \mathbf{A} - C_A$ *is a permutation group on* A

These are the so-called minimal algebras (with $\mathrm{Clo}\,\mathbf{A} = \mathrm{Pol}\,\mathbf{A}$) that play a central role in tame congruence theory. We have the following cases:

(**1**) $\mathrm{Clo}\,\mathbf{A} = [G \cup C_A]$ for a permutation group G on A which is primitive if $|A| > 2$;

(**2**) $\mathrm{Clo}\,\mathbf{A} = \mathrm{Pol}_K \widehat{A}$ for a one-dimensional vector space ${}_K\widehat{A}$ over some finite field K;

(**3**) $|A| = 2$ and $\mathrm{Clo}\,\mathbf{A} = \mathcal{R}$;

(**4**) $|A| = 2$ and $\mathrm{Clo}\,\mathbf{A} = \mathcal{P}_{\leq}$;

(**5**) $|A| = 2$ and $\mathrm{Clo}\,\mathbf{A} = [\wedge,\ C_A]$ or $[\vee,\ C_A]$.

The fact that for $|A| > 2$ every algebra $\mathbf{A}$ in class (O)(b) is as decribed in (**1**) or (**2**) above, follows from a more general theorem of P. P. Pálfy [8]. For $|A| = 2$, the possibilities can be seen by inspecting Post's description [9] of all clones on a two-element set. (For another proof, cf. [5; 4.7, 4.8].) It is worth noting here that Hobby and McKenzie [5; 4.32, 13.9] described all term minimal algebras $\mathbf{A}$ of type **2**–**5** with $C_A \subseteq \mathrm{Clo}_1 \mathbf{A}$, whether simple or not.

(I) $\mathbf{A}$ *is a strictly simple term minimal algebra such that* $\mathrm{Clo}_1 \mathbf{A} = \{0\} \cup L_G$ *for some group* G *with* $A = G^0$, $0 \notin G$

The investigation of these algebras occupied Sections 2–4 of this paper. The main results are that for $|A| > 2$ and for a fixed finite group G ($|G| > 1$), in class (I) there is no algebra of type **1**, and an algebra is of type **2** if and only if it is term equivalent to a one-dimensional vector space (therefore such an algebra exists if and only if $|A| = |G| + 1$ is a prime power and G is cyclic); furthermore, the clones of algebras of types **3**, **4**, and **5** form intervals in the lattice of clones of all algebras on $A = G^0$ with unary part $\{0\} \cup L_G$, as shown in Figure 1. Each of the intervals corresponding to types **3**, **4**, **5** have cardinality $2^{\aleph_0}$.

Figure 1

Remarks. (1) Every G^0-algebra $\mathbf{A}$ which is not simple, has a congruence ε_N where N is a normal subgroup of G with $|N| > 1$. Indeed, by Lemma 2.8 (i), ε_H is a congruence of $\mathbf{A}$ for some subgroup H of G such that $|H| > 1$. Selecting H so that $|H|$ be maximal, we see that $\mathbf{A}/\varepsilon_H$ is a simple term minimal algebra having no nontrivial proper subalgebra and a single trivial subalgebra. Thus, by Lemmas 1.5 and 2.8 (ii) we conclude that H is a normal subgroup of G.

This implies that for every G^0-algebra $\mathbf{A}$ which is not simple, $\operatorname{Clo}\mathbf{A} \subseteq \mathcal{R}_0(R_G) \cap \mathcal{P}_{\varepsilon_N}$ for some normal subgroup N of G with $|N| > 1$. It can be proved that each of these clones $\mathcal{R}_0(R_G) \cap \mathcal{P}_{\varepsilon_N}$ is covered by $\mathcal{R}_0(R_G)$.

(2) It is easy to see that the operations in L_G are automorphisms of the semilattice $(A; \wedge)$, therefore every operation in $[L_G, \wedge]$ is either constantly 0 or of the form $l_{g_0}(x_{i_0}) \wedge \ldots \wedge l_{g_{k-1}}(x_{i_{k-1}})$ for some $g_0, \ldots, g_{k-1} \in G$ and some pairwise distinct variables $x_{i_0}, \ldots, x_{i_{k-1}}$. Thus $[L_G, \wedge]$ covers $[0, L_G]$ in the lattice of clones.

(3) Among the six bounds of the intervals of clones of simple G^0-algebras of types **3**, **4**, **5**, respectively, there are no other inclusions than those suggested by Figure 1 (and established in Corollary 3.9). In view of Theorems 3.6–3.8, the only

inclusion to be excluded is $\mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0} \subseteq [L_G, \wedge, \diamond]$. As we have seen in the proof of Lemma 4.3, the absorptive operation $f_3 \in \mathcal{R}_0(R_G)$ does not preserve the relation β_3. By Proposition 3.5 (iii), we have $f_3 \in \mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0}$, while by Lemma 4.2 (i), β_3 is a compatible relation of the algebra $(A;\ L_G, \wedge, \diamond)$, whence $f_3 \notin [L_G, \wedge, \diamond]$. Thus

$$\mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0} \not\subseteq [L_G, \wedge, \diamond].$$

(4) Let Vect_G denote the family of all one-dimensional vector spaces that are G^0-algebras. Clearly, if ${}_K\widehat{A} \in \mathrm{Vect}_G$, then $\{0\} \cup L_G = \mathrm{Clo}_1\, {}_K\widehat{A}$ is the set of scalar multiplications of ${}_K\widehat{A}$. This shows that such a vector space exists if and only if $|A| = |G^0|$ is a prime power and G is a cyclic group. In this case, it follows that for each ${}_K\widehat{A} \in \mathrm{Vect}_G$, we have ${}_K\widehat{A} = (A; \oplus, 0, L_G)$ for some Abelian group operation $\oplus$, and hence $\mathrm{Clo}\, {}_K\widehat{A} = [L_G, \oplus]$. Now an argument similar to that used in (2) implies that $[0, L_G]$ is a lower cover of $\mathrm{Clo}\, {}_K\widehat{A}$, and by Lemma 2.9 we get that $\mathcal{R}_0(R_G)$ is an upper cover of $\mathrm{Clo}\, {}_K\widehat{A}$.

(5) Let G be a cyclic group of order $q-1$ ($q = p^k$, p prime, $k \geq 1$) and $A = G^0$. It is not hard to determine the exact number of distinct clones $\mathrm{Clo}\, {}_K\widehat{A}$ with ${}_K\widehat{A} \in \mathrm{Vect}_G$. Obviously K is the q-element field. So we can assume without loss of generality that $A = K$ and G is the multiplicative group of K. The addition of K will be denoted by $+$, and a generator of G by a. Obviously, ${}_K\widehat{K} = (K; +, 0, L_G) \in \mathrm{Vect}_G$.

Now let ${}_K\widehat{A} = (K; \oplus, 0, L_G)$ be an arbitrary vector space in Vect_G. Clearly, ${}_K\widehat{A}$ is isomorphic to ${}_K\widehat{K}$. Moreover, if $\pi\colon\ {}_K\widehat{K} \to {}_K\widehat{A}$ is an isomorphism, then the unary operation of ${}_K\widehat{A}$ corresponding to the operation l_a of ${}_K\widehat{K}$ belongs to L_G, that is,

$$\pi l_a \pi^{-1} = \pi(1\ a\ a^2\ \ldots\ a^{q-2})\pi^{-1} = (\pi(1)\ \pi(a)\ \pi(a^2)\ \ldots\ \pi(a^{q-2})) \in L_G.$$

Hence π is one of the polynomial functions of K in

$$P = \{cx^i\colon\ c \in G,\ 1 \leq i \leq q-2,\ \gcd(i, q-1) = 1\}.$$

It is easy to see that P is a permutation group on K, and the converse of the above claim also holds: each permutation $\pi \in P$ yields a vector space ${}_K\widehat{A} \in \mathrm{Vect}_G$ with addition

$$x \oplus y = \pi(\pi^{-1}(x) + \pi^{-1}(y)).$$

Two permutations $\pi, \pi' \in P$ yield the same operation $\oplus$ exactly when $\pi^{-1}\pi' \in \mathrm{Aut}\,(K;+)$. For $\sigma(x) = cx^i \in P$ we have

$$\sigma \in \mathrm{Aut}\,(K;+) \quad \text{iff} \quad \tau(x) = x^i \in \mathrm{Aut}\,(K;+) \quad \text{iff} \quad \tau \in \mathrm{Aut}\,(K;+,\cdot),$$

and the latter is well known to be true if and only if i is a power of p. Thus

$$P \cap \mathrm{Aut}\,(K;+) = \{cx^{p^l}\colon\ c \in G,\ 0 \leq l \leq k-1\}.$$

So the number of distinct operations $\oplus$ yielding vector spaces ${}_K\widehat{A} \in \mathrm{Vect}_G$ is the index of $P \cap \mathrm{Aut}\,(K;+)$ in P, namely $\frac{1}{k}\varphi(p^k - 1)$ (φ is Euler's function). Since

$\mathrm{Clo}_K \widehat{A} = [L_G, \oplus]$ and $\oplus$ is the only Abelian group operation in $\mathrm{Clo}_K \widehat{A}$, therefore the number of clones $\mathrm{Clo}_K \widehat{A}$ in question is also $\frac{1}{k}\varphi(p^k - 1)$.

Using Post's result [9], one can easily draw the corresponding lattice for $|A| = 2$, see Figure 2. Of course, in case $|A| = 2$ we have $|G| = 1$, $L_G = \{\mathrm{id}\}$, and every algebra is simple; however, since the analogue of Corollary 2.3 is not true, therefore the constant 0 cannot be omitted from the fundamental operations. Furthermore, there are only countably many clones. In addition, there are two essential differences between the cases $|A| > 2$ and $|A| = 2$:

(a) In case $|A| = 2$ the interval corresponding to type **5** collapses into one element, implying $\mathcal{R}_0 \cap \mathcal{P}_{\mu_0} \subseteq [0, h_0]$ (cf. Remark (3) for $|A| > 2$).

(b) However, there is another clone of type **5**, namely $[0, \vee]$, which has no counterpart in case $|A| > 2$.

Figure 2

Now we can summarize the possibilities as follows:

(**1**) $|A| = 2$ and $\mathrm{Clo}\,\mathbf{A} = [0, \mathrm{id}]$;

(**2**) $\mathrm{Clo}\,\mathbf{A} = \mathrm{Clo}_K \widehat{A}$ for a one-dimensional vector space ${}_K\widehat{A}$ over some finite field K;

(**3**) $[0, L_G, \wedge, \diamond] \subseteq \operatorname{Clo}\mathbf{A} \subseteq \mathcal{R}_0(R_G)$ for a finite group G with $A = G^0$, $0 \notin G$; for $|A| > 2$, this interval has cardinality $2^{\aleph_0}$, while for $|A| = 2$, it is a descending $\omega + 1$-chain with elements

$$\mathcal{R}_0, \quad \mathcal{R}_0 \cap \mathcal{F}_k^0 \quad (2 \leq k < \omega), \quad \mathcal{R}_0 \cap \mathcal{F}_\omega^0 = [0, \wedge, \diamond];$$

(**4**) $[0, L_G, h_0] \subseteq \operatorname{Clo}\mathbf{A} \subseteq \mathcal{R}_0(R_G) \cap \mathcal{P}_\leq$ for a finite group G with $A = G^0$, $0 \notin G$; for $|A| > 2$, this interval has cardinality $2^{\aleph_0}$, while for $|A| = 2$, it is a descending $\omega + 1$-chain with elements

$$\mathcal{R}_0 \cap \mathcal{P}_\leq, \quad \mathcal{R}_0 \cap \mathcal{P}_\leq \cap \mathcal{F}_k^0 \quad (2 \leq k < \omega), \quad \mathcal{R}_0 \cap \mathcal{P}_\leq \cap \mathcal{F}_\omega^0 = [0, h_0];$$

(**5**) $|A| = 2$ and $\operatorname{Clo}\mathbf{A} = [0, \vee]$, or
$[0, L_G, \wedge] \subseteq \operatorname{Clo}\mathbf{A} \subseteq \mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0}$ for a finite group G with $A = G^0$, $0 \notin G$; for $|A| > 2$, this interval has cardinality $2^{\aleph_0}$, while for $|A| = 2$, it has exactly one element.

(II) **A** *is an idempotent strictly simple term minimal algebra*

For $|A| > 2$ these algebras were determined in [10], up to term equivalence. Again, the case $|A| = 2$ can be settled by referring to Post's lattice [9]. Thus we get the following possibilities:

(**1**) $|A| = 2$ and $\operatorname{Clo}\mathbf{A} = [\mathrm{id}]$;

(**2**) $\operatorname{Clo}\mathbf{A} = \operatorname{Clo}_{\mathrm{id}}\left({}_{(\operatorname{End}_K \widehat{A})}\widehat{A}\right)$ for some vector space ${}_K\widehat{A} = (A; +, K)$ over a finite field K;

(**3**) $\operatorname{Clo}\mathbf{A} = R_{\mathrm{id}}(G)$ for a permutation group G on A such that every nonidentity member of G has at most one fixed point; or
$\operatorname{Clo}\mathbf{A} = \mathcal{R}_{\mathrm{id}}(G) \cap \mathcal{F}_k^0$ for some k $(2 \leq k \leq \omega)$, some element $0 \in A$, and some permutation group G on A such that 0 is the unique fixed point of every nonidentity member of G;

(**4**) $|A| = 2$ and $\operatorname{Clo}\mathbf{A} = \mathcal{R}_{\mathrm{id}}(G) \cap \mathcal{P}_\leq$ for a permutation group G on A; or
$|A| = 2$ and $\operatorname{Clo}\mathbf{A} = \mathcal{R}_{\mathrm{id}} \cap \mathcal{P}_\leq \cap \mathcal{F}_k^0$ for some k $(2 \leq k \leq \omega)$ and some element $0 \in A$;

(**5**) $|A| = 2$ and $\operatorname{Clo}\mathbf{A} = [\wedge]$ or $[\vee]$.

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Bolyai Institute
Aradi vértanúk tere 1
6720 Szeged, Hungary

$\mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0}$ $\mathcal{R}_0(R_G) \cap \mathcal{P}_{\leq}$ $\mathcal{R}_0(R_G)$

$[L_G, \wedge]$ $[L_G, h_0]$ $[L_G, \wedge, \diamond]$

$|G| \geq 2$ $\mathcal{R}_0(R_G) \cap \mathcal{P}_{\varepsilon_N}$ $[0, L_G]$ $\mathrm{Clo}_K \widehat{A}$

$(N \triangleleft G,\ |N|>1)$ (for appropriate G)

1 **2** **3** **4** **5** $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$

$|G| = 1$ $\mathcal{R}_0 \cap \mathcal{P}_{\mu_0} = [0, \wedge]$ $\mathcal{R}_0 \cap \mathcal{P}_{\leq}$ $\mathcal{R}_0$

$[0, h_0]$ $[0, \wedge, \diamond]$ $[0, \vee]$ $[0]$ $[0, +]$

1 **2** **3** **4** **5** $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$

$\mathcal{R}_0(R_G) \cap \mathcal{P}_{\mu_0}$ $\mathcal{R}_0(R_G) \cap \mathcal{P}_{\leq}$ $\mathcal{R}_0(R_G)$

$[L_G, \wedge]$ $[L_G, h_0]$ $[L_G, \wedge, \diamond]$

$|G| \geq 2$ $\mathcal{R}_0(R_G) \cap \mathcal{P}_{\varepsilon_N}$ $[0, L_G]$ $\mathrm{Clo}_K \widehat{A}$

$(N \triangleleft G,\ |N|>1)$ (for appropriate G)

1 **2** **3** **4** **5** $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$

$|G| = 1$ $\mathcal{R}_0 \cap \mathcal{P}_{\mu_0} = [0, \wedge]$ $\mathcal{R}_0 \cap \mathcal{P}_{\leq}$ $\mathcal{R}_0$

$[0, h_0]$ $[0, \wedge, \diamond]$ $[0, \vee]$ $[0]$ $[0, +]$

1 **2** **3** **4** **5** $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$