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SYMMETRIC ALGEBRAS

Dedicated to the memory of Professor W. Nöbauer

1. Introduction and statement of main results

Homogeneous algebras, that is algebras $\mathbf{A} = (A; F)$ whose automorphism group $\mathrm{Aut}\,\mathbf{A}$ is the full symmetric group on A, were first investigated by E. Marczewski [6]. B. Csákány, T. Gavalcová [2] and S. S. Marchenkov [4] described all homogeneous algebras, up to term equivalence. For an n-element set A with $n \geq 5$ the lattice of clones of homogeneous algebras on A turned out to be

Figure 1

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(with n "levels"). It is easy to see that if $\mathbf{A}$ is a finite homogeneous algebra, then the family $\operatorname{Iso}\mathbf{A}$ of internal isomorphisms (i.e., isomorphisms between subalgebras) of $\mathbf{A}$ is either the set I_0 of all bijections $B \to C$ with $B, C \subseteq A,\ |B| = |C|$, or the set I_1 of all such bijections with $|B| \neq |A|-1$. In Figure 1 the encircled points correspond to the largest clones of algebras with these sets of internal isomorphisms. Thus the possible clones of homogeneous algebras $\mathbf{A}$ with a given set $\operatorname{Iso}\mathbf{A}$ of internal isomorphisms form two chains. On closer examination one can see that the left one of these chains consists of the clones containing the dual discriminator

$$d(x,y,z) = \begin{cases} x & \text{if } x = y \\ z & \text{otherwise} \end{cases} \qquad (x,y,z \in A),$$

while the right one of those not containing d. Moreover, the "levels" correspond to the existence of crosses of various sizes among the subalgebras of $\mathbf{A}^2$. Recall that for $k, l \geq 2$ a $k \times l$ *cross* on A is a subset of A^2 of the form

$$X(B_1, B_2, b_1, b_2) = (B_1 \times \{b_2\}) \cup (\{b_1\} \times B_2)$$

$$\text{with} \quad b_1 \in B_1, b_2 \in B_2, |B_1| = k, |B_2| = l,$$

and the size of X is $\max\{k, l\}$.

The aim of this paper is to show that, apart from a few exceptional algebras which occur also among the homogeneous algebras provided the base set is small, the situation is similar under much weaker symmetry conditions than homogeneity. It suffices to require that

(S1) for every subalgebra $\mathbf{B} = (B; F)$ of $\mathbf{A}$, the automorphism group of $\mathbf{B}$ acts primitively on B, and

(S2) for every subalgebra $\mathbf{B} = (B; F)$ of $\mathbf{A}$ and for any set $C \subseteq A$ with $|C| = |B|$, C supports a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}$.

Thus our results generalize also S. S. Marchenkov's theorem [5] on finite algebras $\mathbf{A}$ with $\operatorname{Aut}\mathbf{A}$ the alternating group on A ($|A| \geq 4$), and some results of S. S. Marchenkov, J. Demetrovics, L. Hannák [3] concerning the case $|A| = 3$. (See Section 5.)

In this paper we call an algebra $\mathbf{A}$ *symmetric* if (S1) and (S2) are satisfied. Clearly, (S2) implies that

(S3) if $\mathbf{B} = (B; F)$ is a proper subalgebra of $\mathbf{A}$, then $(C; F)$ is a subalgebra of $\mathbf{A}$ whenever C is a subset of A with $|C| \leq |B|$.

Therefore, to every finite symmetric algebra $\mathbf{A}$ there corresponds a natural number $\nu(\mathbf{A})$, $0 \leq \nu(\mathbf{A}) \leq |A| - 1$, such that a proper subset B of A is the universe of a subalgebra of $\mathbf{A}$ if and only if $|B| \leq \nu(\mathbf{A})$. Obviously, $\nu(\mathbf{A}) \geq 1$ if and only if $\mathbf{A}$ is idempotent.

For the description of symmetric algebras we need some notation. We write $\operatorname{Clo}\mathbf{A}$ for the clone of (term operations of) $\mathbf{A}$. If I is a family of bijections between subsets of A, let $\mathcal{Q}_I$

denote the clone of all operations on A admitting the members of I as internal isomorphisms. Let $\mathcal{D}_1$ denote the clone of all idempotent operations on A, and $\mathcal{E}_1$ the subclone of $\mathcal{D}_1$ consisting of all operations preserving every set

$$L_{a,b} = \{(a,a,a),\ (a,b,b),\ (b,a,b),\ (b,b,a)\} \quad (a,b \in A, a \neq b).$$

For $2 \leq m \leq |A|$ let $\mathcal{D}_m$ be the clone of all operations in $\mathcal{D}_1$ preserving every $m \times 2$ cross, and $\mathcal{E}_m$ the clone consisting of all operations $f \in \mathcal{D}_1$ for which there exists a projection p agreeing with f on every m-element subset B of A. If $\widehat{A} = (A;+)$ is an Abelian group and ${}_R\widehat{A} = (A;+,R)$ is an R-module, we write $\mathcal{I}(\widehat{A})$, resp. $\mathcal{I}({}_R\widehat{A})$, for the clone of all idempotent term operations (or, what is the same, the clone of all idempotent polynomial operations) of $\widehat{A}$, resp. ${}_R\widehat{A}$, $\mathcal{P}(\widehat{A})$, resp. $\mathcal{P}({}_R\widehat{A})$, for the clone of all polynomial operations of $\widehat{A}$, resp. ${}_R\widehat{A}$, and $T(\widehat{A})$ for the group of all translations $x+a$ $(a \in A)$ of $\widehat{A}$.

The first observation is that almost all symmetric algebras are idempotent.

1.1. Proposition. *Let $\mathbf{A} = (A;F)$ be a finite symmetric algebra which is not idempotent. Then $|A|$ is prime and there is a (cyclic) group $\widehat{A} = (A;+)$ such that one of the following holds:*

(1.1.1) $\operatorname{Clo}\mathbf{A} = \mathcal{Q}_{T(\widehat{A})}$;

(1.1.2) $\operatorname{Clo}\mathbf{A} = \mathcal{Q}_{T(\widehat{A})} \cap \mathcal{P}(\widehat{A})$, *which is generated by* $\mathcal{I}(\widehat{A}) \cup T(\widehat{A})$;

(1.1.3) $\operatorname{Clo}\mathbf{A}$ *is generated by* $T(\widehat{A})$.

Thus we can concentrate on idempotent symmetric algebras. The main result of the paper is

1.2. Theorem. *For every finite idempotent symmetric algebra $\mathbf{A} = (A;F)$ one of the following conditions are satisfied:*

(1.2.1) $\operatorname{Clo}\mathbf{A} = \mathcal{Q}_{\operatorname{Iso}\mathbf{A}} \cap \mathcal{D}_m$ *for some m with $1 \leq m \leq \nu(\mathbf{A})$ or $m = |A|$;*

(1.2.2) $\operatorname{Clo}\mathbf{A} = \mathcal{Q}_{\operatorname{Iso}\mathbf{A}} \cap \mathcal{E}_m$ *for some m with $1 \leq m \leq \nu(\mathbf{A})$ or $m = |A|$;*

(1.2.3) $\operatorname{Clo}\mathbf{A} = \mathcal{I}({}_K\widehat{A})$ *for a 1-dimensional vector space ${}_K\mathbf{A} = (A;+,K)$ over a finite field K;*

(1.2.4) $\operatorname{Clo}\mathbf{A} = \mathcal{I}(\widehat{A})$ *for a 4-element abelian group $\widehat{A} = (A;+)$ of exponent 2.*

The proofs will be given in Section 3, following some preparations in Section 2. The short Section 4 is devoted to the question of the existence of the various types of symmetric algebras with given internal isomorphisms. Finally, in Section 5 we present some applications; in particular, we derive the results of [2], [3], [4], [5] mentioned above.

2. Preliminaries

We follow the convention that algebras are denoted by boldface capitals, and their universes by the corresponding letters in italics. The universes of subalgebras of an algebra are called subuniverses. Let A be a nonempty set. We write $|A|$ for the cardinality of A. Recall that an operation f on A is said to *preserve* a subset B of A^k $(k \geq 1)$ if B is a subuniverse of the algebra $(A; f)^k$. It is well known that the clone $\operatorname{Clo} \mathbf{A}$ of term operations of a finite algebra $\mathbf{A}$ is determined by the subuniverses of finite powers of $\mathbf{A}$ in the following sense.

2.1. Theorem. *Let $\mathbf{A}$ be a finite algebra. For an operation f on A we have $f \in \operatorname{Clo} \mathbf{A}$ if and only if f preserves the subuniverses of $\mathbf{A}^k$ for all integers $k \geq 1$.*

We now introduce some notation for constructions that will be used to produce subuniverses from subuniverses of finite powers of arbitrary, or in the last case for idempotent, algebras $\mathbf{A}$. As usual, for $C, C' \subseteq A^2$, $C \circ C'$ denotes the relation product of C, C', and C^{-1} the converse of C. Let B be a subset of A^k $(k \geq 1)$. We will write $\mathbf{k}$ for the set $\{1, \ldots, k\}$ indexing the components of B. For an l-tuple $(i_1, \ldots, i_l) \in \mathbf{k}^l$ the projection of B onto its components $i_1, \ldots, i_l$ is denoted by $\operatorname{pr}_{i_1,\ldots,i_l} B$. If $I = \{i_1, \ldots, i_l\}$ is a nonvoid subset of $\mathbf{k}$ with $i_1 < \ldots < i_l$, we let $\operatorname{pr}_I B$ stand for $\operatorname{pr}_{i_1,\ldots,i_l} B$. The symbol $B \preceq B_1 \times \ldots \times B_k$ will be used to designate that $\operatorname{pr}_i B = B_i$ for all $i \in \mathbf{k}$. For $B \preceq B_1 \times \ldots \times B_k$ and for arbitrary bijections $\pi_i \colon B_i \to C_i$ $(C_i \subseteq A,\ i \in \mathbf{k})$ we set

$$B[\pi_1, \ldots, \pi_k] = \{(x_1\pi_1, \ldots, x_k\pi_k) \colon\ (x_1, \ldots, x_k) \in B\}.$$

If $1 \leq l < k$ and $(a_{l+1}, \ldots, a_k) \in A^{k-l}$, then we define the subset of A^l arising from B by substituting the constants $a_{l+1}, \ldots, a_k$ for the $(l+1)$-st up to the k-th components as follows:

$$B(x_1, \ldots, x_l, a_{l+1}, \ldots, a_k) = \{(x_1, \ldots, x_l) \in A^l \colon\ (x_1, \ldots, x_l, a_{l+1}, \ldots, a_k) \in B\}.$$

For arbitrary bijection $\pi \colon C \to C'$ $(C, C' \subseteq A)$ the set $\{(c, c\pi) \colon c \in C\}$ will also be called a bijection. We say that a subset B of A^k $(k \geq 1)$ is *directly decomposable* if, for some partition $\{I, \bar{I}\}$ of $\mathbf{k}$, B coincides with $(\operatorname{pr}_I B) \times (\operatorname{pr}_{\bar{I}} B)$ up to the order of its components. Recall that B is said to be *reduced* if it is directly indecomposable and no projection $\operatorname{pr}_{i,j} B$ $(1 \leq i < j \leq k)$ of B is a bijection. The *size* of B is $\max\{|\operatorname{pr}_i B| \colon 1 \leq i \leq k\}$. In describing the idempotent symmetric algebras we will need the following result from [11] (see Theorem 4.3 and the remark on p. 98).

2.2. Theorem. [11] *Let $\mathbf{A}$ be a finite idempotent algebra. For any integer $k \geq 2$ and for arbitrary reduced subuniverse $B \preceq B_1 \times \ldots \times B_k$ of $\mathbf{A}^k$ one of the following conditions holds:*

(2.2.1) $\mathbf{A}^2$ *has a reduced subuniverse of the same size as* B;

(2.2.2) *the subalgebras* $\mathbf{B}_i$ $(1 \le i \le k)$ *of* $\mathbf{A}$ *are pairwise isomorphic, moreover, there exist a finite field* K *and a vector space* ${}_K\widehat{B}_1 = (B_1; +, K)$ *such that* $\operatorname{Clo}\mathbf{B}_1 = \mathcal{I}({}_{(\operatorname{End}{}_K\widehat{B}_1)}\widehat{B}_1)$. *For arbitrary isomorphisms* $\pi_i \colon \mathbf{B}_i \to \mathbf{B}_1$ $(1 \le i \le k)$, *the subuniverse* $B[\pi_1, \ldots, \pi_k]$ *of* $\mathbf{A}^k$ *is, up to the order of its components, of the form*

$$\{(y_1, \ldots, y_{l-1}, g_l(y_1, \ldots, y_{l-1}), \ldots, g_k(y_1, \ldots, y_{l-1})) \colon y_1, \ldots, y_{l-1} \in B_1\}$$

with $2 \le l \le k$ *and* $g_l, \ldots, g_k \in \mathcal{P}({}_K\widehat{B}_1)$.

In this paper an algebra is called *trivial* if all its basic operations (hence all term operations) are projections. By a result of I. G. Rosenberg [9] every nontrivial algebra has a term operation of one of five well-determined types. Recall that a *majority,* resp. *minority, operation* is a ternary operation f satisfying the identities

$$f(x, y, y) = f(y, x, y) = f(y, y, x) = y,$$

or

$$f(x, y, y) = f(y, x, y) = f(y, y, x) = x,$$

respectively, and for $n \ge 3$, $1 \le i \le n$, an n-ary operation g is called an *i-th n-ary semiprojection* if it satisfies all identities

$$g(x_{j_1}, \ldots, x_{j_n}) = x_{j_i} \quad \text{with} \quad j_1, \ldots, j_n \in \{1, \ldots, n-1\}.$$

2.3. Theorem. [9] *For every nontrivial algebra* $\mathbf{A}$, $\operatorname{Clo}\mathbf{A}$ *contains either*

(2.3.1) *a nontrivial unary operation, or*

(2.3.2) *a binary idempotent operation distinct from the projections, or*

(2.3.3) *a majority operation, or*

(2.3.4) *for some* $n \ge 3$, *an* n*-ary semiprojection which is not a projection, or*

(2.3.5) *a minority operation of the form* $x + y + z$ *for some Abelian group* $(A; +)$ *of exponent* 2.

K. A. Baker and A. F. Pixley [1] have shown that the condition in Theorem 2.1 can be considerably simplified if $\operatorname{Clo}\mathbf{A}$ contains a majority operation.

2.4. Theorem. [1] *Let* $\mathbf{A}$ *be a finite algebra such that* $\operatorname{Clo}\mathbf{A}$ *contains a majority operation. For an operation* f *on* A *we have* $f \in \operatorname{Clo}\mathbf{A}$ *if and only if* f *preserves the subuniverses of* $\mathbf{A}^2$.

In order not to have to interrupt some arguments in the next section we mention two additional results.

2.5. Proposition. *Let* $\mathbf{A}$ *be a 4-element algebra such that every equivalence on* A *with two 2-element blocks is a congruence of* $\mathbf{A}$. *Then* $\operatorname{Clo}\mathbf{A} \subseteq \mathcal{P}(\widehat{A})$ *for an Abelian group* $\widehat{A} = (A; +)$ *of exponent* 2.

Proof. Let $\mathcal{C}$ denote the clone of all operations on A preserving the three equivalences with two 2-element blocks. Clearly, $\mathcal{C}$ contains the constants, and every nonconstant unary operation in $\mathcal{C}$ is a permutation. Furthermore, $\mathcal{C}$ contains the minority operation q on A with $q(a,b,c)$ the unique element of $A - \{a,b,c\}$ whenever $a,b,c \in A$ are distinct. Thus, by the theorem of P. P. Pálfy [7], $\mathcal{C} = \mathcal{P}({}_K\widehat{A})$ for a vector space ${}_K\widehat{A}$ over a finite field K. By the assumptions ${}_K\widehat{A}$ must be 2-dimensional with $|K| = 2$, whence $\widehat{A}$ is of exponent 2 and $\mathcal{C} = \mathcal{P}(\widehat{A})$, as required.

A direct proof could be based on the observation that for $n \geq 2$ and for every n-ary operation $f \in \mathcal{C}$ depending on all of its variables the unary operations $f(x,x,a_3,\ldots,a_n)$ $(a_3,\ldots,a_n \in A)$ are constant. The details are left to the reader. Note also that in Proposition 2.5 the clone $\mathcal{P}(\widehat{A})$ does not depend on the choice of $\widehat{A}$, and the operation q occurring in the proof, which is sometimes called Swierczkowski's operation, is $q(x,y,z) = x + y + z$.

2.6. Proposition. *For a 2-element idempotent homogeneous algebra* $\mathbf{A}$, *either* $\operatorname{Clo}\mathbf{A}$ *contains the (unique) majority operation on* A, *or* $\operatorname{Clo}\mathbf{A}$ *is contained in the clone generated by the (unique) minority operation on* A.

Proof. Assume $\mathbf{A}$ is nontrivial. Applying the assumptions on $\mathbf{A}$ and the fact that the projections are the only semiprojections on a 2-element set, we see from Theorem 2.3 that $\operatorname{Clo}\mathbf{A}$ contains either the unique majority operation d or the unique minority operation $p(x,y,z) = x + y + z$ (which is independent of the choice of $\widehat{A}$). Supposing $d \notin \operatorname{Clo}\mathbf{A}$ we get from Theorem 2.1 that some finite power $\mathbf{A}^k$ of $\mathbf{A}$ contains a subuniverse B not preserved by d. Select B so that k be minimal. Then, clearly, $k \geq 2$, B is reduced, it is of size 2, and hence $B \preceq A^k$. Since $p \in \operatorname{Clo}\mathbf{A}$ and p preserves none of the reduced subsets of A^2, therefore by Theorem 2.2 $\operatorname{Clo}\mathbf{A} = \mathcal{I}({}_{(\operatorname{End}_K \widehat{A})}\widehat{A})$ for a vector space ${}_K\widehat{A} = (A; +, K)$. As $|A| = 2$, we have $|K| = 2$, yielding that ${}_K\widehat{A}$ is 1-dimensional and $\mathcal{I}({}_{(\operatorname{End}_K \widehat{A})}\widehat{A}) = \mathcal{I}({}_K\widehat{A}) = \mathcal{I}(\widehat{A})$. It is well known and easy to see that $\mathcal{I}(\widehat{A})$ is generated by p. This completes the proof.

The claim of Proposition 2.6 could also be checked by making use of E. L. Post's description of all clones on a 2-element set [8] (see e.g. [11]).

Recall that a permutation group G acting on a finite set A is called *primitive* if, for arbitrary subset S of A with $1 < |S| < |A|$, there exists a $\pi \in G$ such that $\emptyset \neq S \cap S\pi \subset S$, moreover, $|G| > 1$ if $|A| = 2$. It is easy to see that every primitive permutation group G on A is transitive, that is, for arbitrary elements $a, b \in A$ there is a $\pi \in G$ with $a\pi = b$. A transitive permutation group G on A is said to be *regular* if no nonidentity permutation in G has fixed points.

In addition to (S3), the following consequences of (S1)–(S2) will be frequently used.

$(\mathrm{S}1)_2$ For every 2-element subuniverse $B = \{a, b\}$ of $\mathbf{A}$, the transposition $(a\ b)$ on B is an internal isomorphism of $\mathbf{A}$.

$(\mathrm{S}1)_3$ For every 3-element subuniverse $B = \{a, b, c\}$ of $\mathbf{A}$, the cycle $(a\ b\ c)$ on B is an internal isomorphism of $\mathbf{A}$.

$(\mathrm{S}1)^*$ For every subuniverse B of $\mathbf{A}$ and for every subset B_0 of B with $1 \leq |B_0| < |B|$, there exists a set H of internal isomorphisms $B \to B$ of $\mathbf{A}$ such that

$$|\bigcap_{\pi \in H} B_0\pi| = 1.$$

(This follows by induction on $|B_0|$ from the definition of primitivity.)

$(\mathrm{S}2)^*$ For arbitrary subuniverses B, C of $\mathbf{A}$ with $|B| = |C|$, and for any elements $b \in B$, $c \in C$, there exists an internal isomorphism $\pi \colon B \to C$ of $\mathbf{A}$ such that $b\pi = c$. (This is an immediate consequence of (S2) and the transitivity of $\operatorname{Aut} \mathbf{C}$.)

3. Proofs

Proof of Proposition 1.1. By assumption $\nu(\mathbf{A}) = 0$, that is $\mathbf{A}$ has no proper subalgebras. By (S1) $\operatorname{Aut} \mathbf{A}$ is primitive. Furthermore, since the fixed points of each automorphism of $\mathbf{A}$ form a subuniverse in $\mathbf{A}$, we get that the nonidentity automorphisms of $\mathbf{A}$ have no fixed points. Thus $\operatorname{Aut} \mathbf{A}$ is a regular permutation group of prime degree, whence $|A|$ is prime and there exists a cyclic group $\widehat{A} = (A; +)$ such that $\operatorname{Aut} \mathbf{A} = T(\widehat{A})$. Now the claim follows from the Corollary to Theorem 1 in [10].

The rest of this section is devoted to the proof of Theorem 1.2. From now on $\mathbf{A}$ denotes a finite idempotent symmetric algebra. In Lemmas 3.1–3.4 we show that there are a lot of crosses among the subuniverses of $\mathbf{A}^2$.

3.1. Lemma. *If there is a $k \times l$ cross among the subuniverses of $\mathbf{A}^2$, then every $k \times l$ cross is a subuniverse of $\mathbf{A}^2$.*

Proof. Let $B = X(B_1, B_2, b_1, b_2)$ and $B' = X(B'_1, B'_2, b'_1, b'_2)$ be two $k \times l$ crosses on A such that B is a subuniverse of $\mathbf{A}^2$. Then for $i = 1, 2$, $B_i = \mathrm{pr}_i\, B$ is a subuniverse of $\mathbf{A}$, hence by (S3) and (S2)* there exists an internal isomorphism $\pi_i\colon B_i \to B'_i$ of $\mathbf{A}$ such that $b_i\pi_i = b'_i$. Now, clearly, $B' = B[\pi_1, \pi_2]$, therefore B' is a subuniverse of $\mathbf{A}^2$.

3.2. Lemma. *If $\mathbf{A}^2$ has a 2×2 cross among its subuniverses as well as a reduced subuniverse of size m $(2 < m \leq |A|)$, then there is an $m \times 2$ cross among the subuniverses of $\mathbf{A}^2$.*

Proof. We proceed by induction. Suppose $2 \leq k < m$ and there is a $k \times 2$ cross among the subuniverses of $\mathbf{A}^2$. We prove that for some n $(k < n \leq m)$, there is also an $n \times 2$ cross among the subuniverses of $\mathbf{A}^2$.

Let $B \preceq B_1 \times B_2$ be a reduced subuniverse of $\mathbf{A}^2$ with $|B_1| = m$, $m \geq |B_2| \geq 2$. Since B is not a bijection and $|B_1| = m > k$, therefore there exist pairwise distinct elements $a_1, \ldots, a_k \in A$ such that

$$\left|\bigcup_{i=1}^{k} B(x, a_i)\right| > k.$$

(Note that $B(x, a_i)$ is empty if $a_i \notin B_2$.) Moreover, since $B \neq B_1 \times B_2$, we can select $a_1, \ldots, a_k$ so that not all nonempty sets $B(x, a_i)$ $(1 \leq i \leq k)$ are equal, say

$$\bigcup_{i=1}^{k} B(x, a_i) \supset B(x, a_1) \ (\neq \emptyset). \tag{3.2.1}$$

Let C_1 and C_0 denote the left and right hand sides of (3.2.1), respectively, and let $n = |C_1|$. Then we have $n > k$, furthermore, by assumption and by Lemma 3.1, the cross

$$X = \{(a_1, v'), (a_1, v), (a_2, v), \ldots, (a_k, v)\}$$

is a subuniverse of $\mathbf{A}^2$ for arbitrary elements $v, v' \in A$, $v \neq v'$. Thus $C = B \circ X$ is a subuniverse of $\mathbf{A}^2$. It is easy to see that

$$C = (C_0 \times \{v'\}) \cup (C_1 \times \{v\}), \quad C_1 = \mathrm{pr}_1\, C,$$

so by property (S1)* applied to the subuniverse C_1 of $\mathbf{A}$ and its subset C_0 we conclude that an intersection of subuniverses of $\mathbf{A}^2$ of the form $C[\sigma, \mathrm{id}]$ for some internal isomorphisms $\sigma\colon C_1 \to C_1$ of $\mathbf{A}$ is an $n \times 2$ cross.

3.3. Lemma. *If $\mathbf{A}^2$ has a reduced subuniverse of size m $(2 \leq m \leq |A|)$, then*

(3.3.1) $\nu(\mathbf{A}) \geq 2$ *or* $|A| = 2$ *(i.e., every 2-element subset of A is a subuniverse of $\mathbf{A}$), and*

(3.3.2) *either there is an* $m \times 2$ *cross among the subuniverses of* $\mathbf{A}^2$, *or* $|A| = m = 4$ *and* $\mathbf{A}$ *has a congruence with two* 2*-element blocks.*

Proof. Let $B \preceq B_1 \times B_2$ be a reduced subuniverse of $\mathbf{A}^2$ with $|B_1| = m,\ m \geq |B_2| \geq 2$. In view of (S3), if $m < |A|$ or some of the subuniverses $B(x, b_2), B(b_1, x)$ $(b_1 \in B_1, b_2 \in B_2)$ are neither 1-element nor equal to A, then (3.3.1) is evident. Otherwise B is an $|A| \times |A|$ cross, say $B = X(A, A, a_1, a_2)$. Using (S1) we get that there exist $\pi_1, \pi_2 \in \operatorname{Aut} \mathbf{A}$ with $a_1\pi_1 \neq a_1, a_2\pi_2 \neq a_2$. Then

$$\{a_1, a_1\pi_1\} = \operatorname{pr}_1 (B \cap B[\pi_1, \pi_2])$$

is a subuniverse of $\mathbf{A}$, whence (3.3.1) follows.

To verify (3.3.2), it suffices to show by the previous lemma that $\mathbf{A}^2$ has a 2×2 cross among its subuniverses unless $|A| = m = 4$ and $\mathbf{A}$ has a congruence with two 2-element blocks. Since B is reduced and $|B_1| \geq |B_2|$, the subuniverses $B(x, b_2)$ $(b_2 \in B_2)$ of $\mathbf{A}$ are neither all equal to B_1, nor all singletons. We distinguish two cases.

If there exist two distinct sets $B(x, b_2)$ and $B(x, b_2')$ $(b_2, b_2' \in B_2)$ which are not disjoint, say $b_1 \in B(x, b_2) \cap B(x, b_2')$ and $b_1' \in B(x, b_2) - B(x, b_2')$, then $B \cap (\{b_1, b_1'\} \times \{b_2, b_2'\})$ is a 2×2 cross and a subuniverse of $\mathbf{A}^2$ by (3.3.1).

If any two distinct sets of the form $B(x, b_2)$ $(b_2 \in B_2)$ are disjoint, then B_1 has a partition $\{U_1, \ldots, U_s\}$ and B_2 has a partition $\{V_1, \ldots, V_s\}$ such that

$$B = \bigcup_{i=1}^{s} (U_i \times V_i).$$

We may assume that $|U_1| \geq |U_2| \geq \ldots \geq |U_s|$. Since B is reduced and $|U_1| + |U_2| + \ldots + |U_s| = |B_1| \geq |B_2| = |V_1| + |V_2| + \ldots + |V_s|$, we have $s \geq 2$ and $|U_1| \geq 2$.

We show that either $\mathbf{A}$ has a 3-element subuniverse, or

$$(3.3.3) \qquad |A| = m = 4, \quad s = 2, \quad |U_1| = |U_2| = 2, \quad \text{and} \quad |V_1| = |V_2| = 1 \text{ or } 2.$$

Observe first that U_i and V_i $(1 \leq i \leq s)$ are subuniverses of $\mathbf{A}$, since $U_i = B(x, b_2)$ for $b_2 \in V_i$, and similarly $V_i = B(b_1, x)$ for $b_1 \in U_i$. Moreover, they are proper since $s \geq 2$. Therefore, if $|U_i| \geq 3$ or $|V_i| \geq 3$ for some $1 \leq i \leq s$, we are done by property (S3). Suppose now that $|U_i| \leq 2$ and $|V_i| \leq 2$ for all $1 \leq i \leq s$. Making use of (3.3.1) we get that for arbitrary $b_2 \in V_i,\ b_2' \in V_j$ $(1 \leq i < j \leq s)$ the set

$$U_i \cup U_j = \operatorname{pr}_1 \left(B \cap (A \times \{b_2, b_2'\})\right)$$

is a subuniverse of $\mathbf{A}$, and similarly for $V_i \cup V_j$. This together with (S3) again yields a 3-element subuniverse unless (3.3.3) holds.

Assume that $\mathbf{A}$ has a 3-element subuniverse, that is, $\nu(\mathbf{A}) \geq 3$ or $|A| = 3$. Select elements $u_1, u_1' \in U_1$ $(u_1 \neq u_1')$, $u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$, and let $U = \{u_1, u_1', u_2\}$. Then

$$C = B \cap (U \times \{v_1, v_2\})$$

is a subuniverse of $\mathbf{A}^2$. Clearly, $C = \{(u_1, v_1), (u_1', v_1), (u_2, v_2)\}$, so $E = C \circ C^{-1}$ is the equivalence relation with blocks $\{u_1, u_1'\}$, $\{u_2\}$. By (S1)$_3$ the cycle $\gamma = (u_1\ u_1'\ u_2)\colon U \to U$ is an internal isomorphism of $\mathbf{A}$. Thus

$$E[\gamma, \gamma] \circ E = U^2 - \{(u_1, u_2)\}$$

is a subuniverse of $\mathbf{A}^2$ for which we can apply the first case settled above, yielding the existence of a 2×2 cross among the subuniverses of $\mathbf{A}^2$.

Finally, assume that we have (3.3.3). Select $v_1 \in V_1$, $v_2 \in V_2$. Then

$$D = B \cap (A \times \{v_1, v_2\})$$

is a subuniverse of $\mathbf{A}^2$, whence the equivalence relation $D \circ D^{-1}$ with blocks U_1, U_2 is a congruence of $\mathbf{A}$.

3.4. Lemma. *Assume there is a cross of size m $(m \geq 2)$ among the subuniverses of $\mathbf{A}^2$.*

(3.4.1) *If $m < |A|$, then every cross of size at most m is a subuniverse of $\mathbf{A}^2$.*

(3.4.2) *If $m = |A|$, then every $k \times l$ cross such that $k, l \neq \nu(\mathbf{A}) + 1, \ldots, |A| - 1$ is a subuniverse of $\mathbf{A}^2$.*

Proof. Let $B = X(B_1, B_2, b_1, b_2)$ be an $m \times n$ $(m \geq n \geq 2)$ cross among the subuniverses of $\mathbf{A}^2$. What we have to show is that every $k \times l$ cross with $k, l \leq m$ such that $\mathbf{A}$ has both k-element and l-element subuniverses is a subuniverse of $\mathbf{A}^2$. A $k \times 2$ cross can be constructed from B as follows: $C = B \cap (C_1 \times C_2)$ where C_1 is a k-element subset of B_1 with $b_1 \in C_1$ while C_2 is a 2-element subset of B_2 with $b_2 \in C_2$. By (S3) and Lemma 3.3 C_1 and C_2 are subuniverses of $\mathbf{A}$, whence C is a subuniverse of $\mathbf{A}^2$. Now Lemma 3.1 implies that every $k \times 2$ cross is a subuniverse of $\mathbf{A}^2$. Similarly for $l \times 2$ crosses. Finally, for arbitrary $k \times l$ cross $D = X(D_1, D_2, d_1, d_2)$ we have $D = \bar{D} \circ \bar{\bar{D}}^{-1}$ where $\bar{D} = X(D_1, \{c, c'\}, d_1, c)$ is a $k \times 2$ cross and $\bar{\bar{D}} = X(D_2, \{c, c'\}, d_2, c')$ is an $l \times 2$ cross (c, c' are arbitrary distinct elements of A). Hence D is a subuniverse of $\mathbf{A}^2$.

The next lemma reveals the structure of the exceptional 4-element algebras occurring in (3.3.2).

3.5. Lemma. *If $|A| = 4$ and $\mathbf{A}$ has a congruence with two 2-element blocks, then either $\mathbf{A}$ is trivial, or it satisfies* (1.2.4).

Note that for a 4-element Abelian group $\widehat{A} = (A; +)$ of exponent 2, $\mathcal{I}(\widehat{A})$ is generated by the ternary minority operation $x + y + z$, which does not depend on the choice of the group $\widehat{A}$.

Proof. Since by (S1) Aut $\mathbf{A}$ is primitive, it contains the alternating group on A. Therefore all three equivalences with two 2-element blocks are congruences of $\mathbf{A}$. Proposition 2.5 implies now that Clo $\mathbf{A}$ is a subclone of $\mathcal{I}(\widehat{A})$ for an Abelian group $\widehat{A} = (A; +)$ of exponent 2. Hence the claim of the lemma follows from the well-known fact that $\mathcal{I}(\widehat{A})$ is minimal.

Now we will see that the 2-element subalgebras of $\mathbf{A}$ already determine whether the dual discriminator is a term operation of $\mathbf{A}$.

3.6. Lemma. *Assume $\nu(\mathbf{A}) \geq 2$ or $|A| = 2$. Then the following conditions are equivalent:*

(3.6.1) *the discriminator d on A is a term operation of* $\mathbf{A}$,

(3.6.2) $\mathbf{A}$ *has a majority term operation, and*

(3.6.3) *for some (or equivalently, for arbitrary) 2-element subalgebra* $\mathbf{S}$ *of* $\mathbf{A}$, Clo $\mathbf{S}$ *contains the (unique) majority operation on S.*

Proof. The claim is trivial if $|A| = 2$, so assume $|A| \geq 3$. Since d is a majority operation, (3.6.1)$\Rightarrow$(3.6.2)$\Rightarrow$(3.6.3) is obvious. The equivalence of the two claims in (3.6.3) and (3.6.3)$\Rightarrow$(3.6.2) are immediate consequences of the fact that the two-element subalgebras of $\mathbf{A}$ are isomorphic. To prove (3.6.2)$\Rightarrow$(3.6.1), suppose Clo $\mathbf{A}$ contains a majority operation f, and $d \notin$ Clo $\mathbf{A}$. By Theorem 2.4 $\mathbf{A}^2$ has a subuniverse $B \preceq B_1 \times B_2$ not preserved by d. Since d does not preserve B, interchanging the components of B if necessary we get that there exist distinct elements $a_1, b_1 \in B_1$ and pairwise distinct elements $a_2, a_2', b_2 \in B_2$ such that

$$(a_1, a_2), (a_1, a_2'), (b_1, b_2) \in B \quad \text{and} \quad (a_1, b_2) \notin B.$$

Consider the subuniverse $C = B \cap (\{a_1, b_1\} \times A)$ of $\mathbf{A}^2$. Clearly, $C(a_1, x) = B(a_1, x)$, $C(b_1, x) = B(b_1, x)$, and d does not preserve C. If $|A| = 4$ and C has the form

$$C = \{(a_1, a_2), (a_1, a_2'), (b_1, b_2), (b_1, b_2')\}$$

with $\{a_2, a_2', b_2, b_2'\} = A$, then the equivalence $C^{-1} \circ C$ with blocks $\{a_2, a_2'\}, \{b_2, b_2'\}$ is a congruence of $\mathbf{A}$. Hence the previous lemma yields that $f \notin$ Clo $\mathbf{A}$, contradicting our assumption.

We prove that in all other cases $\mathbf{A}$ has a 3-element subuniverse, and hence $\nu(\mathbf{A}) \geq 3$ or $|A| = 3$. Let $|A| > 3$. In view of (S3) our claim is obvious if $|C(a_1, x)| > 2$ (as $b_2 \notin C(a_1, x)$), or $2 < |C(b_1, x)| < |A|$, or $|\mathrm{pr}_2\, C| < |A|$ (as $a_2, a_2', b_2 \in \mathrm{pr}_2\, C$). If none of these conditions holds, then

$$C = \{(a_1, a_2), (a_1, a_2')\} \cup \{(b_1, x) \colon x \in A\}.$$

Applying $(\mathrm{S1})_2$ we get that the transposition $\tau = (a_1\ b_1)$ is an internal isomorphism of $\mathbf{A}$. By (S1) $\mathbf{A}$ has an automorphism π such that $\{a_2, a_2'\} \cap \{a_2\pi, a_2'\pi\}$ is a 1-element set. Thus

$$\mathrm{pr}_2\, (C \cap C[\tau, \pi]) = \{a_2, a_2', a_2\pi, a_2'\pi\}$$

is a 3-element subuniverse of $\mathbf{A}$.

Now it follows that

$$D = B \cap (\{a_1, b_1\} \times \{a_2, a_2', b_2\})$$

is a subuniverse of $\mathbf{A}^2$ such that

$$(a_1, a_2), (a_1, a_2'), (b_1, b_2) \in D \quad \text{and} \quad (a_1, b_2) \notin D.$$

Hence

$$(a_1, f(a_2, a_2', b_2)) = f((a_1, a_2), (a_1, a_2'), (b_1, b_2)) \in D$$

implies $f(a_2, a_2', b_2) \neq b_2$. Similarly, $f(a_2', a_2, b_2) \neq b_2$, $f(b_2, a_2, a_2') \neq b_2$, and so on. Since $\{a_2, a_2', b_2\}$ is a subuniverse of $\mathbf{A}$, and the roles of a_2 and a_2' are symmetric, we may assume that $f(a_2, a_2', b_2) = a_2$. However, by $(\mathrm{S1})_3$ the cycle $\gamma = (a_2\ b_2\ a_2')$ is an internal isomorphism of $\mathbf{A}$, whence

$$f(b_2, a_2, a_2') = f(a_2\gamma, a_2'\gamma, b_2\gamma) = a_2\gamma = b_2,$$

a contradiction. This completes the proof of the lemma.

3.7. Lemma. *If the dual discriminator d is not a term operation of $\mathbf{A}$, then either* (1.2.3) *holds, or*

(3.7.1) *for arbitrary distinct elements $a, b \in A$, $L_{a,b}$ is a subuniverse of $\mathbf{A}^3$.*

Proof. Since $d \notin \mathrm{Clo}\, \mathbf{A}$, therefore by Theorem 2.1 some finite power $\mathbf{A}^k$ of $\mathbf{A}$ has a subuniverse $B \preceq B_1 \times \ldots \times B_k$ not preserved by d. Selecting B so that k be minimal one can easily see that $k \geq 2$ and B is reduced. By Theorem 2.2 and Lemma 3.3 there are two possibilities: either all 2-element subsets of A are subuniverses of $\mathbf{A}$ (this holds also when $|B_1| = 2$), or $|B_1| > 2$ and there exist a finite field K and a vector space ${}_K\widehat{B}_1 = (B_1; +, K)$ such that $\mathrm{Clo}\, \mathbf{B}_1 = \mathcal{I}(_{(\mathrm{End}\, {}_K\widehat{B}_1)}\widehat{B}_1)$. In the latter case $\mathbf{B}_1$ has no nonsingleton proper subalgebras, therefore by (S3) $B_1 = A$ and $\mathbf{B}_1 = \mathbf{A}$; accordingly, we write ${}_K\widehat{A}$ for ${}_K\widehat{B}_1$.

Also, by (S1), $\operatorname{Aut}\mathbf{A}$ is primitive, hence ${}_K\widehat{A}$ is 1-dimensional and $\mathcal{I}({}_{(\operatorname{End}_K \widehat{A})}\widehat{A}) = \mathcal{I}({}_K\widehat{A})$. Thus $\operatorname{Clo}\mathbf{A} = \mathcal{I}({}_K\widehat{A})$.

Now suppose that the 2-element subsets of A are subuniverses of $\mathbf{A}$, and let $a, b \in A$, $a \neq b$, $S = \{a, b\}$. By the previous lemma $\operatorname{Clo}\mathbf{S}$ does not contain the majority operation. However, $\mathbf{S}$ is obviously an idempotent algebra; moreover, by $(\mathrm{S}1)_2$ $\mathbf{S}$ is homogeneous. Thus we see from Proposition 2.6 that $\operatorname{Clo}\mathbf{S}$ must be contained in the clone generated by the minority operation on S. Since this operation preserves $L_{a,b}$, $L_{a,b}$ is a subuniverse of $\mathbf{S}^3$, and hence of $\mathbf{A}^3$, too.

3.8. Lemma. *If a set of the form $L_{a,b}$ $(a, b \in A,\ a \neq b)$ as well as an $|A| \times 2$ cross is among the subuniverses of $\mathbf{A}^3$ and $\mathbf{A}^2$, respectively, then $\mathbf{A}$ is trivial.*

Proof. Suppose $\operatorname{Clo}\mathbf{A}$ is nontrivial. Then by Theorem 2.3 it contains an operation f of one of the types (2.3.1)–(2.3.5). Since $\mathbf{A}$ is idempotent, case (2.3.1) cannot occur.

Assume (2.3.2) holds for f. Since $L_{a,b}$ is a subuniverse of $\mathbf{A}^3$, $\{a, b\}$ is a subuniverse of $\mathbf{A}$. Hence $f(a, b) \in \{a, b\}$, say $f(a, b) = a$. Now for arbitrary $c, d \in A$, $c \neq d$, $(\mathrm{S}2)^*$ implies the existence of an internal isomorphism $\pi\colon \{a, b\} \to \{c, d\}$ of $\mathbf{A}$ such that $a\pi = c$, $b\pi = d$. Thus

$$f(c, d) = f(a\pi, b\pi) = f(a, b)\pi = a\pi = c.$$

Taking into account the idempotency of f we get that f is a projection, which was excluded.

If f were a majority operation, we would have

$$(b, b, b) = f((a, b, b), (b, a, b), (b, b, a)),$$

contradicting the assumption that $L_{a,b}$ is a subuniverse of $\mathbf{A}^3$.

Assume now f is an n-ary $(n \geq 3)$, say first semiprojection, and let

$$C = X(A, \{a_2, a_2'\}, a_1, a_2) \quad (a_2 \neq a_2')$$

be an $|A| \times 2$ cross among the subuniverses of $\mathbf{A}^2$. Then, for arbitrary elements $b_2, \ldots, b_n \in A$,

$$(f(a_1, b_2, \ldots, b_n), a_2') = f((a_1, a_2'), (b_2, a_2), \ldots, (b_n, a_2)) \in C,$$

implying $f(a_1, b_2, \ldots, b_n) = a_1$. Since $\operatorname{Aut}\mathbf{A}$ is transitive, it follows that f is a projection, hence f does not satisfy (2.3.4).

Finally, if f were of type (2.3.5), then for arbitrary element $c \in A$ we would get

$$(c, a_2') = f((c, a_2), (a_1, a_2), (a_1, a_2')) \in C,$$

a contradiction.

After these preparations we are in a position to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose first the dual discriminator d is a term operation of $\mathbf{A}$. It is well known and straightforward to check (see B. Csákány and T. Gavalcová [2]) that d preserves only three types of subsets of A^2: $B_1 \times B_2$ $(B_1, B_2 \subseteq A)$, bijections $B_1 \to B_2$ $(B_1, B_2 \subseteq A,\ |B_1| = |B_2|)$, and crosses. On the other hand, by Theorem 2.4, $\operatorname{Clo} \mathbf{A}$ is determined by the subuniverses of $\mathbf{A}^2$. Let m denote the maximum of the sizes of crosses among the subuniverses of $\mathbf{A}^2$ if such crosses exist, and let $m = 1$ otherwise. Combining the preceding facts with Lemma 3.4 we see that (1.2.1) holds.

Assume from now on that d is not a term operation of $\mathbf{A}$, and apply Lemma 3.7. If $\mathbf{A}$ satisfies (1.2.3), we are done. Suppose therefore that (3.7.1) holds for $\mathbf{A}$, and let m denote the maximum of the sizes of reduced subuniverses of $\mathbf{A}^2$ if such subuniverses exist, and let $m = 1$ otherwise. If $m > 1$, then one of the two possibilities in (3.3.2) holds for $\mathbf{A}$. If $|A| = m = 4$, $\mathbf{A}$ has a congruence with two 2-element blocks, and $\mathbf{A}$ is nontrivial, then by Lemma 3.5 $\mathbf{A}$ is of type (1.2.4). Otherwise, there is an $m \times 2$ cross among the subuniverses of $\mathbf{A}^2$, and hence by Lemma 3.8 every m-element subalgebra of $\mathbf{A}$ is trivial. Since by (S2) all m-element subalgebras of $\mathbf{A}$ are isomorphic, we conclude that

$$\operatorname{Clo} \mathbf{A} \subseteq \mathcal{Q}_{\operatorname{Iso} \mathbf{A}} \cap \mathcal{E}_m.$$

This obviously holds for $m = 1$ as well.

To prove the reverse inclusion it suffices to show that every operation $f \in \mathcal{Q}_{\operatorname{Iso} \mathbf{A}} \cap \mathcal{E}_m$ preserves the reduced subuniverses of finite powers of $\mathbf{A}$. Let $B \preceq B_1 \times \ldots \times B_k$ be a reduced subuniverse of $\mathbf{A}^k$ $(k \geq 1)$. If $k = 1$, then B is a subuniverse of $\mathbf{A}$, hence the identity mapping $B \to B$ in $\operatorname{Iso} \mathbf{A}$ is preserved by f. Thus f preserves B as well. In case $k \geq 2$ we apply Theorem 2.2. In case (2.2.2) the subalgebra $\mathbf{B}_1$ of $\mathbf{A}$ has no nonsingleton proper subalgebras. Hence $|B_1| = 2$. For arbitrary internal isomorphisms $\pi_i \colon B_i \to B_1$ $(i = 1, \ldots, k)$ of $\mathbf{A}$, the description of $B[\pi_1, \ldots, \pi_k]$ immediately shows that all operations in $\mathcal{E}_1$ preserve $B[\pi_1, \ldots, \pi_k]$. (Note that $L_{a,b} = \{(x, y, x + y) \colon x, y \in \{a, b\}\}$ if $(\{a, b\}; +)$ is the group with neutral element a.) Thus f preserves B. Consider now the case (2.2.1). By the choice of m we have $m \geq 2$ and $|B_i| \leq m$ for all $i = 1, \ldots, k$. Let, say, $|B_1| \geq |B_2|, \ldots, |B_k|$, and using (S2) select internal isomorphisms $\pi_i \colon B_i \to C_i$ of $\mathbf{A}$ with $C_i \subseteq B_1$ $(i = 1, \ldots, k)$. Then $B[\pi_1, \ldots, \pi_k]$ is a subset of B_1^k, $|B_1| \leq m$, hence it is preserved by f. Thus f preserves B as well, concluding the proof.

4. Existence

Let A be a finite set. Now we look at the question: For which monoids T of partial bijective transformations of A do there exist the various types of symmetric algebras $\mathbf{A}$ such that $T = \operatorname{Iso}\mathbf{A}$. Obvious necessary conditions for T are the following. As the set of fixed points of each automorphism of an algebra is a subuniverse, we have to require that

(T0) $\mathrm{id}_B \in T$ whenever B is the set of fixed points of a permutation in T;

furthermore, we need the analogues of (S1) and (S2):

(T1) for every subset B of A with $\mathrm{id}_B \in T$ the permutations of B in T form a primitive permutation group on B,

(T2) for every subset B of A with $\mathrm{id}_B \in T$ and for every set $C \subseteq A$ with $|C| = |B|$, T contains a bijection $B \to C$.

Again, (T1)–(T2) imply that there exists a natural number ν_T, $0 \leq \nu_T \leq |A| - 1$, such that for a proper subset B of A, $\mathrm{id}_B \in T$ if and only if $|B| \leq \nu_T$. Moreover, by (T0), every nonidentity permutation in T has at most ν_T fixed points.

Clearly, every algebra $\mathbf{A}$ on A with $\operatorname{Iso}\mathbf{A} = T$ is symmetric, and we have $\nu(\mathbf{A}) = \nu_T$. For the exceptional types (i.e., (1.1.1)–(1.1.3) and (1.2.3)–(1.2.4)), the possible monoids $T = \operatorname{Iso}\mathbf{A}$ are determined by Proposition 1.1 and Theorem 1.2, respectively. In these cases the corresponding algebras do exist. For example, if $T = T(\widehat{A})$ as in Proposition 1.1, then

$$\operatorname{Iso}(A; \mathcal{Q}_T) = T,$$

hence the symmetric algebra $\mathbf{A}$ of type (1.1.1) with $\operatorname{Iso}\mathbf{A} = T$ exists; the other cases are similar and even more trivial.

So we can concentrate on types (1.2.1) and (1.2.2). Let T be a monoid of bijective partial transformations of A such that (T0)–(T2) hold and $\nu_T \geq 1$. The problem of existence now stands in determining those T and m ($1 \leq m \leq \nu_T$ or $m = |A|$) for which

$$\operatorname{Iso}(A; \mathcal{Q}_T \cap \mathcal{D}_m) = T, \quad \text{resp.,} \quad \operatorname{Iso}(A; \mathcal{Q}_T \cap \mathcal{E}_m) = T.$$

Since these equalities are trivial if $|A| = 2$, we will assume $|A| \geq 3$.

We construct some operations. Let $|A| = n$, $A = \{a_1, \ldots, a_n\}$; for $1 \leq k \leq n$ we set $A_k = \{a_1, \ldots, a_k\}$. For $B, C \subseteq A$, $|B| = |C|$, we denote by $T_{B,C}$ the set of mappings $B \to C$ in T; for brevity we write T_B instead of $T_{B,B}$.

For every natural number $k \geq 3$ with $k \leq \nu_T$ or $k = n$, we define

$$\Delta_{T,k}(x_1, \ldots, x_k) = \begin{cases} d(x_1, x_3, x_2) & \text{if } (x_1, \ldots, x_k) = (a_1\sigma, \ldots, a_k\sigma) \\ & \quad \text{for some } \sigma \in T_{A_k, C},\ |C| = k, \\ d(x_1, x_2, x_3) & \text{otherwise,} \end{cases}$$

$$\Lambda_{T,k}(x_1,\dots,x_k) = \begin{cases} x_2 & \text{if } (x_1,\dots,x_k) = (a_1\sigma,\dots,a_k\sigma) \\ & \quad \text{for some } \sigma \in T_{A_k,C},\ |C| = k, \\ x_1 & \text{otherwise} \end{cases}$$

$$(x_1,\dots,x_k \in A).$$

In case $2 \le \nu_T < n-1$, resp. $1 \le \nu_T < n-1$, we define two operations of arity $k = \nu_T+1$ as follows:

$$\bar{\Delta}_T(x_1,\dots,x_k) = \begin{cases} a_{k+1}\sigma & \text{if } (x_1,\dots,x_k) = (a_1\sigma,\dots,a_k\sigma) \text{ for some } \sigma \in T_A, \\ d(x_1,x_2,x_3) & \text{otherwise,} \end{cases}$$

$$\bar{\Lambda}_T(x_1,\dots,x_k) = \begin{cases} a_{k+1}\sigma & \text{if } (x_1,\dots,x_k) = (a_1\sigma,\dots,a_k\sigma) \text{ for some } \sigma \in T_A, \\ x_1 & \text{otherwise} \end{cases}$$

$$(x_1,\dots,x_k \in A).$$

Note that $\bar{\Delta}_T$ and $\bar{\Lambda}_T$ are well defined, since if $(a_1\sigma,\dots,a_k\sigma) = (a_1\sigma',\dots,a_k\sigma')$ for some $\sigma,\sigma' \in T_A$, then $\sigma'\sigma^{-1} \in T_A$ has at least $k > \nu_T$ fixed points, hence by our assumptions on T it must be the identity, implying $a_{k+1}\sigma = a_{k+1}\sigma'$.

We will also need the ternary discriminator operation and the so-called switching operation:

$$t(x,y,z) = \begin{cases} z & \text{if } x = y, \\ x & \text{otherwise,} \end{cases} \qquad s(x,y,z) = \begin{cases} z & \text{if } x = y, \\ y & \text{if } x = z, \\ x & \text{otherwise} \end{cases} \qquad (x,y,z \in A).$$

4.1. Lemma. *The operations defined above have the following properties:*

(4.1.1) *all of them belong to $\mathcal{Q}_T$;*

(4.1.2) *$\Delta_{T,k}$ and $\Lambda_{T,k}$ preserve none of the permutations of A_k outside T_{A_k};*

(4.1.3) *$\bar{\Delta}_T$ and $\bar{\Lambda}_T$ do not preserve the identity mapping of A_{ν_T+1};*

(4.1.4) *$\Delta_{T,k},\ \bar{\Delta}_T \in \mathcal{D}_l$ for $l = n$ and all $1 \le l \le \nu_T$;*

(4.1.5) *$\Lambda_{T,k} \in \mathcal{D}_{k-1} - \mathcal{D}_k$ if $k \le \nu_T$, and $\Lambda_{T,n}, \bar{\Lambda}_T \in \mathcal{D}_{\nu_T} - \mathcal{D}_n$;*

(4.1.6) *$\Lambda_{T,k} \in \mathcal{E}_{k-1} - \mathcal{E}_k$ if $k \le \nu_T$, and $\Lambda_{T,n}, \bar{\Lambda}_T \in \mathcal{E}_{\nu_T} - \mathcal{E}_n$ if $\nu_T \ge 2$;*

(4.1.7) *$t \in \mathcal{D}_1 - \mathcal{D}_2$ and $s \in \mathcal{E}_1 - \mathcal{E}_2$.*

The proof is straightforward, the details are left to the reader.

4.2. Proposition. *Let A be an n-element set ($n \ge 3$), and T a monoid of bijective partial transformations of A such that $\nu_T \ge 1$ and* (T0)–(T2) *hold. Then*

$$\operatorname{Iso}(A; \mathcal{Q}_T \cap \mathcal{D}_m) = T \tag{4.2.1}$$

if and only if either $m = \nu_T = 1$, or $\nu_T \ge 2$ and $m \in \{1,2,\dots,\nu_T,n\}$. Moreover, for distinct indices m the clones $\mathcal{Q}_T \cap \mathcal{D}_m$ are pairwise distinct.

Proof. In the first claim the necessity is obvious if $\nu_T \geq 2$, while if $\nu_T = 1$, then it follows by applying Lemma 3.3 for the algebra $\mathbf{A} = (A; \mathcal{Q}_T \cap \mathcal{D}_m)$. For the sufficiency assume T and m are as described in the proposition, and observe that $\supseteq$ obviously holds in (4.2.1). To prove the equality, we have to exhibit operations in $\mathcal{Q}_T \cap \mathcal{D}_m$ ensuring that no partial bijection of A outside T is an internal isomorphism. Since (T1)–(T2) are satisfied by T, it suffices to exclude the identity mapping of the set A_{ν_T+1} (if $\nu_T < n-1$) and, for every positive integer $k \geq 3$ with $k \leq \nu_T$ or $k = n$, every permutation of A_k outside T_{A_k}. (Note that for $k \leq 2$ such permutations cannot exist.) Using Lemma 4.1 one can easily see that for $\nu_T = m = 1$ the operations $\bar{\Lambda}_T$, $\Delta_{T,n}$, while in the remaining cases the operations $\bar{\Delta}_T$, $\Delta_{T,k}$ ($k \geq 3$, and $k \leq \nu_T$ or $k = n$) do the job.

For T and m as above with $\nu_T \geq 2$, the algebras $(A; \mathcal{Q}_T \cap \mathcal{D}_m)$ are symmetric, hence by Lemma 3.4 we have

$$\mathcal{Q}_T \cap \mathcal{D}_n \subseteq \mathcal{Q}_T \cap \mathcal{D}_{\nu_T} \subseteq \ldots \subseteq \mathcal{Q}_T \cap \mathcal{D}_{m+1} \subseteq \mathcal{Q}_T \cap \mathcal{D}_m \subseteq \ldots \subseteq \mathcal{Q}_T \cap \mathcal{D}_2 \subseteq \mathcal{Q}_T \cap \mathcal{D}_1.$$

By Lemma 4.1 the operations $\Lambda_{T,n}$, $\Lambda_{T,m+1}$ ($m = 2, \ldots, \nu_T - 1$), t, resp., show that the inclusions are proper, proving the second claim.

4.3. Proposition. *Let A be an n-element set ($n \geq 3$), and T a monoid of bijective partial transformations of A such that $\nu_T \geq 1$ and (T0)–(T2) hold. Then*

$$\operatorname{Iso}(A; \mathcal{Q}_T \cap \mathcal{E}_m) = T \tag{4.3.1}$$

if and only if $\nu_T \geq 2$, either $m = n = \nu_T + 1$ or $m \leq \nu_T$, and T_B is the full symmetric group on B whenever $|B| \leq m$ ($B \subseteq A$). Moreover, for distinct indices m the clones $\mathcal{Q}_T \cap \mathcal{E}_m$ are pairwise distinct.

Proof. In the first claim the necessity is trivial from the definition of $\mathcal{E}_m$ and from Lemma 3.3. For the sufficiency assume T and m satisfy the conditions described in the proposition. In case $m = n = \nu_T + 1$ (4.3.1) is obvious, therefore let $m \leq \nu_T$. As before, it suffices to exhibit operations in $\mathcal{Q}_T \cap \mathcal{E}_m$ ensuring that neither the identity mapping of A_{ν_T+1} (if $\nu_T < n-1$) nor any permutation of A_k outside T_{A_k} for $k \geq \max\{3, m+1\}$ with $k \leq \nu_T$ or $k = n$, is an internal isomorphism. (Note that by our assumptions on T such permutations cannot exist if $k \leq 2$ or $k \leq m$.) Thus, by Lemma 4.1, the operations $\bar{\Lambda}_T$ and $\Lambda_{T,k}$ ($k \geq \max\{3, m+1\}$, and $k \leq \nu_T$ or $k = n$) do the job.

By Lemma 4.1 the operations $\Lambda_{T,n}$, $\Lambda_{T,m+1}$ ($m = 2, \ldots, \nu_T - 1$), s, resp., show also that the obvious inclusions

$$\mathcal{Q}_T \cap \mathcal{E}_n \subseteq \mathcal{Q}_T \cap \mathcal{E}_{\nu_T} \subseteq \ldots \subseteq \mathcal{Q}_T \cap \mathcal{E}_{m+1} \subseteq \mathcal{Q}_T \cap \mathcal{E}_m \subseteq \ldots \subseteq \mathcal{Q}_T \cap \mathcal{E}_2 \subseteq \mathcal{Q}_T \cap \mathcal{E}_1$$

are proper, completing the proof.

5. Applications

5.1. Finite algebras having all even permutations as automorphisms [2], [4], [5]. Let A be a finite set with $|A| = n \geq 4$. It is straightforward to check that if $\mathbf{A}$ is an algebra on A such that the alternating group $\operatorname{Alt} A$ on A is contained in $\operatorname{Aut} \mathbf{A}$, then $\mathbf{A}$ is symmetric. Thus the results of the previous section yield that a monoid T of bijective partial transformations of A equals $\operatorname{Iso} \mathbf{A}$ for some algebra $\mathbf{A}$ of this kind if and only if T satisfies (T0)–(T2) and $\operatorname{Alt} A \subseteq T$. To determine these monoids T let us denote by $\operatorname{Bij}_k A$ the set of all bijections $B \to C$ with $B, C \subseteq A$, $|B| = |C| \leq k$ $(1 \leq k \leq n)$, and by $\operatorname{Sym} A$ the full symmetric group on A. Noticing that by (T0) and $\operatorname{Alt} A \subseteq T$ we have $\mathrm{id}_B \in T$ for every $B \subseteq A$ with $|B| \leq n-3$, we get that $\operatorname{Bij}_{n-3} A \subseteq T$. Thus, there are the following six possibilities for T:

$$T_1 = \operatorname{Bij}_{n-3} A \cup \operatorname{Alt} A, \quad T_2 = \operatorname{Bij}_{n-2} A \cup \operatorname{Alt} A,$$

$$T_3 = T_2 \cup \{\pi \in \operatorname{Bij}_{n-1} A - \operatorname{Bij}_{n-2} A\colon\ \pi \text{ is the restriction of a permutation in } \operatorname{Alt} A\},$$

$$T_4 = \operatorname{Bij}_{n-1} A \cup \operatorname{Alt} A, \quad T_5 = \operatorname{Bij}_{n-2} A \cup \operatorname{Sym} A, \quad T_6 = \operatorname{Bij}_n A.$$

The lattice they form is shown in Figure 2.

Figure 2

Obviously, $\nu_{T_1} = n-3$, $\nu_{T_2} = \nu_{T_5} = n-2$, and $\nu_{T_3} = \nu_{T_4} = \nu_{T_6} = n-1$.

Now, using Theorem 1.2 and the results of the previous section we can easily draw the lattice of clones of all algebras $\mathbf{A}$ on A such that $\operatorname{Alt} A \subseteq \operatorname{Aut} \mathbf{A}$. The diagrams are presented in Figures 3 and 4 for $n > 4$ and $n = 4$, respectively. Notice that for $n > 4$ no algebras of exceptional types exist, while for $n = 4$ both of types (1.2.3) and (1.2.4) occur.

Figure 4

5.2. Finite symmetric algebras in which every bijection between proper subsets is an internal isomorphism. Let A be a finite set with $|A| = n \geq 3$, and let $T_0 = \mathrm{Bij}_{n-1} A \cup G_0$ where G_0 is a primitive permutation group on A. As in the previous case, every algebra $\mathbf{A}$ on A with $T_0 \subseteq \mathrm{Iso}\,\mathbf{A}$ is symmetric. Moreover, a monoid T of bijective partial transformations of A equals $\mathrm{Iso}\,\mathbf{A}$ for an algebra $\mathbf{A}$ of this kind if and only if

$$T = \mathrm{Bij}_{n-1} A \cup G = T_0 \cup G \quad \text{for some permutation group } G \ \text{ with } G_0 \subseteq G \subseteq \mathrm{Sym}\,A.$$

Clearly, $\nu_T = n - 1$ for all such T.

Thus, by Theorem 1.2 and by the results in Section 4 the lattice of clones of algebras $\mathbf{A}$ on A with $T_0 \subseteq \mathrm{Iso}\,\mathbf{A}$ is as shown in Figure 5. The intervals $[\mathcal{Q}_{T_0 \cup \mathrm{Sym}\,A} \cap \mathcal{X},\ \mathcal{Q}_{T_0} \cap \mathcal{X}]$ for $\mathcal{X} = \mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}_1, \ldots, \mathcal{E}_{n-1}$ are all isomorphic to the dual of the interval $[G_0, \mathrm{Sym}\,A]$ in the subgroup lattice of $\mathrm{Sym}\,A$.

Figure 5

5.3. Some 3-element algebras admitting a cyclic automorphism [3]. Let $A = \{0,1,2\}$ and $\widehat{A} = (A;+)$ with $+$ denoting addition modulo 3. Furthermore, let γ be the translation $x+1$, i.e., $\gamma = (0\ 1\ 2)$. We consider those algebras $\mathbf{A}$ on A for which $\gamma \in \operatorname{Aut} A$. Clearly, if $\mathbf{A}$ has a 2-element subuniverse, then every 2-element subset of A is a subuniverse of $\mathbf{A}$, and all 2-element subalgebras of $\mathbf{A}$ are isomorphic.

Thus, $\mathbf{A}$ is a symmetric algebra if and only if either

(a) $\mathbf{A}$ has no 2-element subalgebras, or

(b) the transposition $(0\ 1)$ on $\{0,1\}$ is an internal isomorphism of $\mathbf{A}$.

In case (a), if $\nu(\mathbf{A}) = 0$, then by Proposition 1.1 $\operatorname{Iso} \mathbf{A} = T(\widehat{A})$ and $\operatorname{Clo} \mathbf{A}$ is one of the clones $\mathcal{Q}_{T(\widehat{A})}$, $\mathcal{Q}_{T(\widehat{A})} \cap \mathcal{P}(\widehat{A})$, and $[T(\widehat{A})]$ (the clone generated by $T(\widehat{A})$). If $\nu(\mathbf{A}) = 1$, then by Theorem 1.2 and by the results in Section 4 we have two possibilities for $\operatorname{Iso} A$: $\operatorname{Bij}_1 A \cup T(\widehat{A})$ and $\operatorname{Bij}_1 A \cup \operatorname{Sym} A$; hence $\operatorname{Clo} \mathbf{A}$ equals one of the clones $\mathcal{Q}_{T(\widehat{A})} \cap \mathcal{D}_1$, $\mathcal{Q}_{\operatorname{Sym} A} \cap \mathcal{D}_1$, and

$\mathcal{I}({}_K\widehat{A}) = \mathcal{I}(\widehat{A})$ (as $|K| = 3$). If (b) holds, then we have $\mathrm{Bij}_2\, A \cup T(\widehat{A}) \subseteq \mathrm{Iso}\,\mathbf{A}$, which is a special case of the problem settled in 5.2. Thus $\mathrm{Iso}\,\mathbf{A}$ equals one of the monoids $T_0 = \mathrm{Bij}_2\, A \cup T(\widehat{A})$, $T_1 = \mathrm{Bij}_2\, A \cup \mathrm{Sym}\, A$, and hence the clones of 3-element algebras $\mathbf{A}$ of this kind form a lattice like the one in Figure 5 with 3 "levels" and 11 elements. (See Figure 6 showing part of the lattice of all clones of algebras $\mathbf{A}$ on A with $\gamma \in \mathrm{Aut}\,\mathbf{A}$.)

Suppose now that $\mathbf{A}$ is not symmetric. Then every 2-element subset of A is a subuniverse of $\mathbf{A}$, any two 2-element subalgebras of $\mathbf{A}$ are isomorphic, and the 2-element subalgebras have no nontrivial automorphisms. This implies that $\mathrm{Aut}\,\mathbf{A} = T(\widehat{A})$, and that for arbitrary 2-element subsets B, C of A there is a *unique* internal isomorphism $B \to C$ of $\mathbf{A}$.

If $\mathbf{A}^2$ has no reduced subuniverse of size 3, then Theorem 2.2 and the latter property of the internal isomorphisms yields that for all $k \geq 1$, every reduced subuniverse of $\mathbf{A}^k$ can be uniquely written in the form $D[\pi_1, \ldots, \pi_k]$ where D is a subuniverse of $\mathbf{A}^k$ with $D \subseteq \{0,1\}^k$, and $\pi_1, \ldots, \pi_k$ are internal isomorphisms between 2-element subalgebras of $\mathbf{A}$. Thus, by Theorem 2.1, an operation belongs to $\mathrm{Clo}\,\mathbf{A}$ if and only if it preserves the internal isomorphisms of $\mathbf{A}$ and all such D. Consequently,

$$\mathrm{Clo}\,\mathbf{A} = \{f \in \mathcal{Q}_{T(\widehat{A})} \cap \mathcal{D}_1\colon\ f \text{ preserves } \{0,1\}, \text{ and } f|_{\{0,1\}} \in \mathcal{T}\}$$

for some idempotent clone $\mathcal{T}$ on $\{0,1\}$ (here $f|_{\{0,1\}}$ denotes the restriction of f to $\{0,1\}$). Our assumptions on $\mathbf{A}$ yield that $\mathcal{T}$ must contain an operation not preserving $(0\ 1)$. These clones are known from Post's description [8] (cf. [11]), hence the clones on A satisfying the above conditions are

$$\mathcal{C}_1 = \{f \in \mathcal{Q}_{T(\widehat{A})} \cap \mathcal{D}_1\colon\ f \text{ preserves } \{0,1\}\},$$

$$\mathcal{O}_1 = \{f \in \mathcal{C}_1\colon\ f \text{ preserves the natural order } 0 \leq 1 \text{ on } \{0,1\}\},$$

$$\mathcal{C}_k^i = \{f \in \mathcal{C}_1\colon\ f \text{ preserves } \{0,1\}^k - \{(1-i, \ldots, 1-i)\}\} \quad (i \in \{0,1\},\ k = 2,3,\ldots),$$

$$\mathcal{C}_\omega^i = \bigcap_{k=1}^{\infty} \mathcal{C}_k^i, \quad \mathcal{O}_k^i = \mathcal{O}_1 \cap \mathcal{C}_k^i \quad (i \in \{0,1\},\ k = 2,3,\ldots,\omega),$$

$$\mathcal{O}_{\omega+1}^i = \{f \in \mathcal{O}_\omega^i\colon\ f \text{ preserves } \{(0,0,0),(1-i,i,i),(i,1-i,i),(i,i,1-i),(1,1,1)\}\} \qquad (i \in \{0,1\}).$$

It is straightforward to check (by constructing operations) that these clones are pairwise distinct. Furthermore, since $\mathcal{Q}_{T_0} \cap \mathcal{E}_2$ is contained in all these clones and the square of the symmetric algebra $(A; \mathcal{Q}_{T_0} \cap \mathcal{E}_2)$ has no reduced subuniverse of size 3, therefore the same property holds for the larger clones as well. Thus the possibilities for $\mathrm{Clo}\,\mathbf{A}$ in this case are exactly the clones listed above.

Note that these clones, together with $\mathcal{Q}_{T_0} \cap \mathcal{D}_1$, $\mathcal{Q}_{T_0} \cap \mathcal{D}_2$, $\mathcal{Q}_{T_0} \cap \mathcal{E}_1$, $\mathcal{Q}_{T_0} \cap \mathcal{E}_2$ (which correspond to the symmetric algebras $\mathbf{A}$ with $\operatorname{Aut}\mathbf{A} = T(\widehat{A})$ such that $\mathbf{A}^2$ has no reduced subuniverse of size 3), form a lattice isomorphic to the lattice of all idempotent clones on $\{0,1\}$.

Finally, assume $\mathbf{A}^2$ has a reduced subuniverse of size 3. Then the argument in the proof of Lemma 3.3 can be repeated, with some simplifications due to $|A| = 3$, implying that there is a 2×2 cross among the subuniverses of $\mathbf{A}^2$. A slight modification of the proof of Lemma 3.2 yields also that there is a 3×2 cross among the subuniverses of $\mathbf{A}^2$. Indeed, using the notation of the proof of Lemma 3.2 we see that if for the subuniverse B of size 3 and for the elements a_1, a_2 selected as said we have $|B(x, a_2)| = 3$, then $B \cap (A \times \{a_1, a_2\})$ is already of the same form as C, while in the opposite case the role of a_1, a_2 is symmetric, and the construction of C works independently of the choice of the 2×2 cross X with $\operatorname{pr}_1 X = \{a_1, a_2\}$.

Taking into account the internal isomorphisms of $\mathbf{A}$ we conclude that one of the crosses

$$X_i = X(A, \{0,1\}, 0, i) \quad (i \in \{0,1\})$$

is a subuniverse of $\mathbf{A}^2$. If X_1 is a subuniverse of $\mathbf{A}^2$, then

$$X_1 \cap \{0,1\}^2 = \{(0,0),(0,1),(1,1)\} \quad \text{and} \quad X_1[\gamma, \mathrm{id}] \cap \{0,1\}^2 = \{(1,0),(1,1),(0,1)\}$$

are also subuniverses of $\mathbf{A}^2$, whence $\operatorname{Clo}\mathbf{A} \subseteq \mathcal{O}_2^1$. Similarly, if X_0 is a subuniverse of $\mathbf{A}^2$, then $\operatorname{Clo}\mathbf{A} \subseteq \mathcal{O}_2^0$. (Since all operations in $\mathcal{O}_2^0 \cap \mathcal{O}_2^1$ preserve the transposition $(0\ 1)$ on $\{0,1\}$, X_0 and X_1 cannot simultaneously be subuniverses of $\mathbf{A}^2$.) Thus $\operatorname{Clo}\mathbf{A}$ is contained in one of the clones

$$\mathcal{M}_k^i = \{f \in \mathcal{O}_k^i \colon\ f \text{ preserves } X_i\} \quad (i \in \{0,1\},\ k = 2,3,\ldots,\omega,\omega+1).$$

It is not hard to check (by constructing operations) that these clones are pairwise distinct. This implies that for each $i \in \{0,1\}$ and $k = 2,3,\ldots,\omega,\omega+1$, $\mathcal{O}_k^i$ covers $\mathcal{M}_k^i$ in the lattice of clones on A.

Applying Theorem 2.4 one can easily verify that both of $\mathcal{M}_2^0$ and $\mathcal{M}_2^1$ cover the clone $\mathcal{Q}_{T_0} \cap \mathcal{D}_3$. In addition, it is shown in [3] that for $i \in \{0,1\}$, every proper subclone of $\mathcal{M}_2^i$ not contained in $\mathcal{Q}_{T_0} \cap \mathcal{D}_3$ is contained in $\mathcal{M}_3^i$. This completes the description of the part, illustrated in Figure 6, of the lattice of subclones of $\mathcal{Q}_{T(\widehat{A})}$.

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Figure 2: T_1 T_2 T_3 T_4 T_5 T_6

Figure 3: $\mathcal{Q}_{T_1} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_2} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_3} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_4} \cap \mathcal{D}_1$
$\mathcal{Q}_{T_5} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_6} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_1} \cap \mathcal{D}_2$ $\mathcal{Q}_{T_1} \cap \mathcal{D}_3$
$\mathcal{Q}_{T_1} \cap \mathcal{D}_{n-3}$ $\mathcal{Q}_{T_2} \cap \mathcal{D}_{n-2}$ $\mathcal{Q}_{T_3} \cap \mathcal{D}_{n-1}$ $\mathcal{Q}_{T_1} \cap \mathcal{D}_n$
$\mathcal{Q}_{T_1} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_2} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_3} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_4} \cap \mathcal{E}_1$
$\mathcal{Q}_{T_5} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_6} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_1} \cap \mathcal{E}_2$ $\mathcal{Q}_{T_1} \cap \mathcal{E}_3$

$\mathcal{Q}_{T_1} \cap \mathcal{E}_{n-3}$ $\mathcal{Q}_{T_2} \cap \mathcal{E}_{n-2}$ $\mathcal{Q}_{T_4} \cap \mathcal{E}_{n-1}$ $\mathcal{Q}_{T_6} \cap \mathcal{E}_n$

Figure 4: $\mathcal{Q}_{T_1} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_2} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_3} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_4} \cap \mathcal{D}_1$
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$\mathcal{Q}_{T_6} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_2} \cap \mathcal{E}_2$ $\mathcal{Q}_{T_4} \cap \mathcal{E}_3$ $\mathcal{I}({}_K\widehat{A})$ $\mathcal{I}(\widehat{A})$

Figure 5: $\mathcal{Q}_{T_0} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_0} \cap \mathcal{D}_2$ $\mathcal{Q}_{T_0} \cap \mathcal{D}_{n-1}$ $\mathcal{Q}_{T_0} \cap \mathcal{D}_n$

$\mathcal{Q}_{T_0 \cup G} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_0 \cup \mathrm{Sym}\, A} \cap \mathcal{D}_1$ $\mathcal{Q}_{T_0} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_0} \cap \mathcal{E}_2$
$\mathcal{Q}_{T_0} \cap \mathcal{E}_{n-1}$ $\mathcal{Q}_{T_0 \cup G} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_0 \cup \mathrm{Sym}\, A} \cap \mathcal{E}_1$ $\mathcal{Q}_{T_0 \cup \mathrm{Sym}\, A} \cap \mathcal{E}_n$

Figure 6: $\mathcal{Q}_{T(\widehat{A})}$ $\mathcal{Q}_{T(\widehat{A})} \cap \mathcal{P}(\widehat{A})$ $[T(\widehat{A})]$ $\mathcal{Q}_{T(\widehat{A})} \cap \mathcal{D}_1$
$\mathcal{Q}_{\mathrm{Sym}\, A} \cap \mathcal{D}_1$ $\mathcal{I}(\widehat{A})$ $\mathcal{C}_1$ $\mathcal{O}_1$ $\mathcal{C}_2^1$ $\mathcal{C}_3^1$ $\mathcal{C}_4^1$ $\mathcal{C}_\omega^1$ $\mathcal{O}_2^1$ $\mathcal{O}_3^1$ $\mathcal{O}_4^1$ $\mathcal{O}_\omega^1$
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