A SURVEY ON STRICTLY SIMPLE ALGEBRAS AND MINIMAL VARIETIES

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The investigation of minimal varieties (or equationally complete varieties) started in the fifties with the works of Kalicki and Scott [40], [41], [74], and it became clear already at the very beginning that while in a number of classical varieties the minimal subvarieties are locally finite (hence generated by a finite algebra) and can be explicitly determined, this is far from being the case in general (i.e. in similarity classes). During the decade 1970–1980, great progress was made in understanding the structure of locally finite minimal varieties. Around 1976, a number of authors (Caine [9], Clark and Krauss [10], Quackenbush [68], Smith [75]) came close to revealing the structure of finite strictly simple algebras in congruence permutable varieties; the most transparent description was given in the unpublished paper by McKenzie [46], yielding also explicitly all locally finite, minimal, congruence permutable varieties. That the locally finite, minimal, congruence distributive varieties are exactly the congruence distributive varieties generated by a finite strictly simple algebra, has been known from Jónsson's lemma [38]. With the rapid development of commutator theory for congruence modular varieties it became possible to combine these two results and get a full description for locally finite, minimal, congruence modular varieties; the result is due to Herrmann (cf. [24]). In contrast, Pigozzi [61] constructed a finite strictly simple algebra generating a non-finitely based minimal variety, and in [62] he proved among other things that there exists a locally finite minimal variety which is residually large, showing how 'bad' locally finite minimal varieties can behave. Tame congruence theory, developed by McKenzie and his students opened up new perspectives in the study of locally finite varieties; its efficient tools may lead to the complete solution of the most fundamental problem on

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locally finite minimal varieties (Problem 10 in the book by Hobby and McKenzie [25]): describe all finite, strictly simple algebras that generate minimal varieties.

The aim of this paper is to give a fairly broad survey on *locally finite* minimal varieties, and is not intended to include all known results on minimal varieties. As the emphasis is put on locally finite minimal varieties and their strictly simple generators, we do not deal with their equational theories beyond the question whether they are finitely based. Furthermore, the results stating that some variety has 'a lot of' minimal subvarieties are touched upon in Section 1 only in order to justify the restriction to the locally finite case.

Following Section 2 which is a short introduction into the main problems on locally finite minimal varieties, in Section 3 we collect a number of theorems describing all minimal varieties in classical varieties (e.g. groups, rings, semigroups, relation algebras) and some generalizations thereof (e.g. n-groups, medial quasigroups, semirings, etc.). Section 4 is, in part, a preparation for the last two sections, listing some examples of finite strictly simple algebras generating minimal varieties, that are already typically of universal algebraic character, and include as special cases many of the algebras appearing in the preceding section. Section 5 is rather short, although it contains the deepest results achieved so far in the topic: the characterization of locally finite, minimal, congruence modular varieties. Finally, in Section 6 some recent results of the author are presented, that concern finite strictly simple algebras in which all fundamental operations are surjective. These algebras include all finite strictly simple algebras with a single nonconstant fundamental operation.

Except for some references to the types **1** up to **5** assigned by tame congruence theory to finite simple algebras, and some other results in the theory, for which the reader is referred to [25], only some basic notions and facts of universal algebra are used, which can be found in [22], [8] or [51]. The reader not familiar with the elements of tame congruence theory can skip those parts, without loosing much of the content.

We remark that the expressions *term operation*, *polynomial operation* are used as in [51]. Throughout the paper, by a *trivial algebra* we mean a one-element algebra.

For simplicity of notation we identify every natural number n with the set $n = \{0, \ldots, n-1\}$. The full symmetric group on n is denoted by S_n. The identity mapping on each set is denoted by id.

1. General facts

For a type τ (that is a sequence of operation symbols with given arities), the class of all algebras of type τ will be denoted by V_τ. Clearly, V_τ is a variety, defined by the empty set of identities; it is often called the *similarity class* of type τ. As is well known, the subvarieties of

V_τ form a lattice which is dually isomorphic to the lattice of equational theories of type τ (or, what is essentially the same, the lattice of fully invariant congruences of the free algebra in V_τ on $\aleph_0$ free generators). The least element of this lattice is the *trivial variety* consisting of trivial algebras only. Clearly, it is defined by the identity $x = y$. A variety V is called *minimal*, if it has exactly two subvarieties: V and the trivial variety. Since the trivial variety is a dually compact element in the lattice of subvarieties of V_τ, therefore it follows

Theorem 1.1. *Every nontrivial subvariety of V_τ contains a minimal variety.*

The first result giving the exact number of minimal varieties in a similarity class concerns groupoids.

Theorem 1.2. (Kalicki [40]) *There are $2^{\aleph_0}$ minimal varieties of groupoids.*

The proof is a nice application of Theorem 1.1: $2^{\aleph_0}$ varieties W_i $(i < 2^{\aleph_0})$ are defined in terms of identities so that $W_i \cap W_j$ is the trivial variety for all distinct indices i, j, and hence they contain pairwise distinct minimal varieties. In this way the proof yields no minimal variety of groupoids explicitly. Later Bol'bot [6] and Quackenbush [69] constructed $2^{\aleph_0}$ countable groupoids generating pairwise distinct minimal varieties. Bol'bot's groupoids satisfy the identities $xy = yx$, $(xy)y = x$, and hence are in fact totally symmetric quasigroups (see Section 3), while Quackenbush's groupoids have the discriminator function (see Section 4) among their term operations.

Applying Kalicki's method Burris [7] and Ježek [28] proved independently the analogue of Theorem 1.2 for almost all types.

Theorem 1.3. (Burris [7], Ježek [28]) *Let τ be a type.*

(i) *If τ contains at least one operation symbol of arity ≥ 2, then V_τ has $2^{\aleph_0+|\tau|}$ minimal subvarieties, where $|\tau|$ is the number of operation symbols in τ.*

(ii) *If τ consists of operation symbols of arity ≤ 1 and contains at least two unary operation symbols, then V_τ has $2^{\aleph_0+m_\tau}$ minimal subvarieties, where m_τ is the number of unary operation symbols in τ.*

It remains to consider the types containing a single (or no) unary operation symbol and a number of nullary operation symbols. For these types all varieties can be explicitly described; for the type consisting of a single unary operation symbol this was done by Jacobs and Schwabauer [26], and the general case was solved by Ježek [27]. In particular, for minimal varieties we have

Theorem 1.4. (Kalicki–Scott [41]) *For the type τ consisting of a single unary operation symbol f, V_τ has two minimal subvarieties, one defined by the identity $f(x) = x$, and the other one defined by the identity $f(x) = f(y)$.*

Clearly, a minimal variety is generated by any of its nontrivial members. The two minimal varieties in Theorem 1.4 can also be given as follows: they are the varieties generated by the unary algebras

(1.4) $(2; \mathrm{id})$, and

(1.4)′ $(2; f)$ with $f(0) = f(1) = 0$,

respectively.

Theorem 1.5. (Ježek [27]) *Let τ be a type consisting of at most one unary operation symbol and some nullary operation symbols c_i $(i \in I)$.*

(i) *If τ contains a unary operation symbol f, then V_τ has two minimal subvarieties, one defined by the identities $f(x) = x$, $c_i = c_j$ $(i, j \in I)$, and the other one defined by the identities $f(x) = f(y) = c_i$ $(i \in I)$.*

(ii) *If τ contains nullary operation symbols only, then V_τ has a unique minimal subvariety, the one defined by the identities $c_i = c_j$ $(i, j \in I)$.*

The reader will easily find two-element algebras, analogous to (1.4), (1.4)′, generating these minimal varieties.

The similarity classes V_τ for τ as in Theorem 1.3 (i) have "a lot of" minimal subvarieties not only as regards their numbers, but also in the sense that V_τ has no proper subvariety containing all minimal varieties in V_τ.

Theorem 1.6. (Bol'bot [5], Ježek [28]) *Let τ be a type.*

(i) *If τ contains at least one operation symbol of arity ≥ 2, then the least subvariety of V_τ containing all minimal subvarieties of V_τ is V_τ itself.*

(ii) *If τ consists of operation symbols of arity ≤ 1 and contains at least two unary operation symbols, then the least subvariety of V_τ containing all minimal subvarieties of V_τ is the variety defined by all constant identities (i.e. all identities $c_i = c_j$ with c_i, c_j nullary operation symbols in τ).*

We shall see later that in many classical varieties V the situation is different: V has only "a few" minimal subvarieties, they can be explicitly described, and the least subvariety of V containing all of them is much smaller than V.

From the algebraic point of view, the most important would be to know more about the structure of minimal varieties. However, without imposing further restrictions, this is a hopeless task. Pigozzi [62] has some results which, loosely speaking, state that the structure of a minimal variety can be as complicated as that of almost any variety, unless it is locally finite and satisfies further special conditions like for instance congruence distributivity or congruence permutability (see Section 5). Therefore we restrict our attention to locally finite minimal varieties.

2. Locally finite minimal varieties

A variety V is said to be *locally finite* if every finitely generated algebra in V is finite. For an algebra $\mathbf{A}$ we denote by $V(\mathbf{A})$ the variety generated by $\mathbf{A}$. Following [46], [19], [25] we call an algebra $\mathbf{A}$ *strictly simple* * if $|A| \geq 2$, $\mathbf{A}$ is simple, and $\mathbf{A}$ has no proper, nontrivial subalgebra. It is easy to see that every nontrivial locally finite variety contains a finite strictly simple algebra; in fact, every nontrivial algebra of minimal cardinality in the variety is strictly simple. Thus every locally finite minimal variety is generated by a finite strictly simple algebra. However, not every finite strictly simple algebra generates a minimal variety. The most trivial example is the 2-element mono-unary algebra whose operation is the transposition. So the following problem arises naturally:

Problem 2.1. Determine all finite strictly simple algebras $\mathbf{A}$ such that the variety $V(\mathbf{A})$ is minimal.

Most results known so far, which are related to Problem 2.1, are full lists of (locally finite) minimal subvarieties of certain varieties (see Sections 3 and 5). For finitely generated varieties we have

Theorem 2.2. (Scott [74]) *For any finite algebra $\mathbf{A}$, the variety $V(\mathbf{A})$ has only finitely many minimal subvarieties.*

Moreover, as it is pointed out in [74], the proof yields an algorithm deciding for every finite algebra $\mathbf{A}$ whether the variety $V(\mathbf{A})$ is minimal (or more generally, determining for every finite algebra $\mathbf{A}$ the minimal subvarieties of $V(\mathbf{A})$). Of course, from the algebraic point of view, this is not a satisfactory solution to Problem 2.1.

Since 1954, when Lyndon [45] constructed the first example of a finite algebra $\mathbf{A}$ (in fact, a groupoid with constant) which is not finitely based (i. e. $V(\mathbf{A})$ has no finite basis for its identities), much effort has been devoted to investigating under what conditions a finite algebra of finite similarity type is finitely based. The question is not settled even for strictly simple algebras generating minimal varieties. So we can supplement Problem 2.1 with

Problem 2.3. For which finite, strictly simple algebras $\mathbf{A}$ of finite type such that $V(\mathbf{A})$ is a minimal variety is $\mathbf{A}$ finitely based?

To this day, the algebras constructed by Pigozzi [61] are the only known strictly simple algebras of finite type that generate minimal varieties and are not finitely based. Recall that an identity is called *variable uniform* if every variable that occurs on the left-hand side of the identity

* In [75] and in a number of subsequent papers, the name 'plain' is used.

occurs also on the on the right-hand side, and vice-versa. An algebra is *variable uniform* if each of its identities is such. **

Example 2.4. (Pigozzi [61]) Let $\mathbf{G} = (G; \circ)$ be a variable-uniform groupoid that fails to be finitely based (e.g. $\mathbf{G}$ can be chosen to be Murskiĭ's 3-element groupoid [52]). Let $A(G)$ denote the set obtained from G by adjoining 4 new elements b, l, α, ∞, and define binary operations $\circ$ and Q_a ($a \in A(G) - \{\infty\}$) on $A(G)$ as follows: $\circ$ is the extension of the operation of $\mathbf{G}$ indicated by the table

** In the early sixties Płonka initiated a general theory for these algebras and varieties (see [65]), which was developed further by him and many of his students. His school uses the term 'regular' instead of 'variable uniform'. Another name appearing in the literature (especially in semigroup theory) is 'homogeneous'.

$$\begin{array}{c|ccc|cccc}
\circ & & G & & b & l & \alpha & \infty \\ \hline
 & & & & l & l & \alpha & \infty \\
 & & & & & & & \\
G & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & \\
 & & & & l & l & \alpha & \infty \\ \hline
b & l & \cdots & l & b & l & \alpha & \infty \\
l & l & \cdots & l & l & l & \alpha & \infty \\
\alpha & \alpha & \cdots & \alpha & \alpha & \alpha & \alpha & \infty \\
\infty & \infty & \cdots & \infty & \infty & \infty & \infty & \infty
\end{array},$$

and for $a \in A(G) - \{\infty\}$,

$$Q_a(x, y) = \begin{cases} y & \text{if } x = a \\ \infty & \text{otherwise} \end{cases} \qquad (x, y \in A(G)).$$

Pigozzi's result is the following.

If $\mathbf{G} = (G; \circ)$ is a variable-uniform groupoid that fails to be finitely based, then the algebra

$$\mathbf{A}(\mathbf{G}) = (A(G);\ \circ, Q_a, a)_{a \in A(G) - \{\infty\}}$$

generates a minimal variety and is not finitely based.

For later comparison several properties of $\mathbf{A}(\mathbf{G})$ are worth mentioning.

(2.4) All constants $a \in A(G) - \{\infty\}$ are fundamental operations of $\mathbf{A}(\mathbf{G})$, and the constant ∞ is a term operation of $\mathbf{A}(\mathbf{G})$.

$(2.4)'$ $\mathbf{A}(\mathbf{G})$ is simple and has no proper subalgebra.

$(2.4)''$ $V(\mathbf{A}(\mathbf{G}))$ satisfies no nontrivial congruence identity.

The last assertion follows from the fact that of the five types assigned by tame congruence theory [25] to finite simple algebras, $\mathbf{A}(\mathbf{G})$ has type **5**. (For the definition and basic properties of types, the reader is referred to [25].) To show this, call ∞ an *absorbing element* for an operation f on $A(G)$ if the value of f is ∞ whenever at least one of the elements substituted in f is ∞. Now, the range $\{\alpha, \infty\}$ of the unary polynomial operation $x \circ \alpha$ of $\mathbf{A}(\mathbf{G})$ is a minimal set, and is of type **5** since ∞ is an absorbing element for every nonconstant fundamental operation of $\mathbf{A}(\mathbf{G})$, and hence also for every nonconstant polynomial operation of $\mathbf{A}(\mathbf{G})$ depending on all of its variables.

3. Minimal varieties of classical algebras and some generalizations

Unlike in similarity classes, in most varieties of classical algebras (such as groups, rings, lattices, semigroups) it turns out that there are only countably many minimal subvarieties, all of them locally finite; moreover, the minimal varieties can be explicitly described. As a rule, the main steps of the proof are the following:

— showing that every nontrivial subvariety of the given variety V contains a nontrivial finite algebra,

— finding all finite strictly simple algebras in V, and

— selecting those generating minimal varieties.

Each locally finite minimal variety presented below is finitely based, and hence could also be described via a finite set of defining identities rather than by a generating algebra. However, these identities will not be discussed here; in most cases they can be found in the papers cited.

GROUPS. Groups form a variety if inversion is considered a unary fundamental operation. The three steps in the argument sketched above are now easy to carry out (cf. [41]).

Theorem 3.1. *The minimal varieties of groups are exactly the varieties generated by*

(3.1) *the cyclic groups of prime order.*

RINGS. A *zero ring* is a ring in which the product of any two elements is 0.

Theorem 3.2. (Tarski [84]) *The minimal varieties of rings are exactly the varieties generated by*

(3.2) *the finite fields of prime order, and*

(3.2)′ *the zero rings of prime order.*

MODULES. It is easy to see that for a fixed ring R with 1, the subvarieties of the variety of unitary R-modules are in one-to-one correspondence with the ideals of R. This immediately implies

Theorem 3.3. *For a fixed ring R with 1, the minimal varieties of unitary R-modules are exactly the varieties generated by the modules*

(3.3) ${}_R(R/I)$ *(i.e. the ring R/I considered as an R-module), where I runs over the maximal ideals of R.*

MEDIAL QUASIGROUPS. Quasigroups, considered as algebras $(A; \cdot, \backslash, /)$ with $\cdot, \backslash, /$ being the multiplication, left division, and right division, respectively, form a variety. Since both divisions are uniquely determined by the multiplication, when presenting a quasigroup, it suffices to describe its multiplication.

A quasigroup is called *totally symmetric* if its operations $\cdot, \backslash, /$ coincide and $\cdot$ is commutative. Bol'bot [4], [6] proved that the variety of totally symmetric quasigroups has $2^{\aleph_0}$ minimal subvarieties, and hence the variety of commutative quasigroups and the variety of all quasigroups have the same property. However, there is a nice variety of quasigroups, whose minimal subvarieties can be explicitly described. A quasigroup is called *medial* *** if it satisfies the identity

$$\text{(MED)} \qquad (xy)(zu) = (xz)(yu).$$

Theorem 3.4. (Ježek–Kepka [32]) *The minimal varieties of medial quasigroups are exactly the varieties generated by the quasigroups*

(3.4) $(K;\ ax+by)$ *where K is a finite field and $a, b \in K$ are nonzero elements generating K.*

The same result was proved also by Smith [76], under the assumption that the minimal varieties are locally finite. The varieties of idempotent medial quasigroups (that is, in which $xx = x$ also holds), and in particular the minimal ones, were described earlier in [13].

Corollary 3.5. (Csákány–Megyesi [13]) *The minimal varieties of idempotent medial quasigroups are exactly the varieties generated by the quasigroups*

(3.5) $(K;\ ax+(1-a)y)$ *where K is a finite field and $a \in K - \{0,1\}$ generates K.*

In [33] Ježek and Kepka investigated various properties of the quasigroup varieties defined by so-called *strictly balanced* identities of length ≤ 6, that is, by identities of length ≤ 6 containing the operation symbol $\cdot$ only and such that every variable occurs exactly once on each side. There are 11 such varieties; the poset they form under inclusion is shown in Figure 1.

*** This property is called by several authors 'Abelian' (e.g. [32]) or 'entropic' (e.g. [76]); in [76] furhter names are also listed. Idempotent entropic algebras (not necessarily groupoids) have been investigated in detail by Romanowska and Smith [71].

Figure 1

Here Q, C, G, A denote the variety of all quasigroups, the variety of commutative quasigroups, the variety of groups, and the variety of Abelian groups, respectively; D_1 $[D_2]$ is the variety of *left [right] permutable* quasigroups, defined by the identity

(PER) $$x(yz) = y(xz) \qquad [(xy)z = (xz)y];$$

E_1 $[E_2]$ is the variety of *left [right] modular* quasigroups, defined by the identity

(MOD) $$x(yz) = z(yx) \qquad [(xy)z = (zy)x];$$

furthermore, $F_1 = D_1 \cap E_2$, $F_2 = D_2 \cap E_1$, $H = E_1 \cap E_2$.

It is shown in [33] that D_1, D_2 have $2^{\aleph_0}$ minimal subvarieties. Thus all four varieties above the dotted line in the diagram have uncountably many minimal subvarieties. The remaining ones, however, turn out to contain only countably many minimal subvarieties. In fact, it follows that E_1, E_2 are subvarieties of the variety of medial quasigroups, so an application of Theorem 3.4 yields

Corollary 3.6. *The minimal subvarieties of E_1, F_2, and H, respectively, are exactly the varieties generated by the following quasigroups:*

$(3.6)_{E_1}$ $(K;\ a^2x + ay)$ *where K is a finite field and $a \in K$ is a nonzero element generating K;*

$(3.6)_{F_2}$ $(K;\ x + ay)$ *where K is a finite field, $a \in K$ generates K, and $a^2 = 1$;*

$(3.6)_{H}$ $(K;\ a^2x + ay)$ *where K is a finite field, $a \in K$ generates K, and $a^3 = 1$.*

n-GROUPS. An *n-group* is an algebra $(A;\ f, f_1, \dots, f_n)$ with $n+1$ n-ary operations $f, f_1, \dots, f_n$ such that f is *associative*, i.e.

$$\text{(n-ASS)} \qquad \begin{aligned} &f(f(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) \\ &= f(x_1, \dots, x_{i-1}, f(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}) \\ &\qquad\qquad \text{for all } i\ (2 \le i \le n), \end{aligned}$$

and the following identities are satisfied:

$$\text{(n-INV)} \qquad \begin{aligned} &f(x_1, \dots, x_{i-1}, f_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) = x_i, \\ &f_i(x_1, \dots, x_{i-1}, f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) = x_i \\ &\qquad\qquad \text{for all } i\ (1 \le i \le n). \end{aligned}$$

An n-group is *commutative* if the identity

$$f(x_1, \dots, x_n) = f(x_{1\pi}, \dots, x_{n\pi})$$

holds in it for all permutations $\pi \in S_n$.

Clearly, 2-groups are exactly classical groups, and commutative 2-groups are exactly Abelian groups. Like in groups or quasigroups, for an n-group $(A;\ f, f_1, \dots, f_n)$, the operations $f_1, \dots, f_n$ are uniquely determined by f, so an n-group can be given merely by describing its operation f.

Theorem 3.7. (Artamonov [1]) *The minimal varieties of n-groups are exactly the varieties generated by the n-groups*

(3.7) $(K;\ x_1 + ax_2 + \ldots + a^{n-1}x_n)$ *where K is a finite field, $a \in K$ generates K, and* $a^{n-1} = 1$.

Special cases of Theorem 3.7, in particular for commutative n-groups, were obtained earlier in Page–Butson [56] and Page [55].

(m,n)-RINGS. An *(m,n)-ring* is an algebra $(A; f, f_0, \dots, f_n, g)$ such that $(A; f, f_0, \dots, f_n)$ is a commutative $(m+1)$-group, g is an $(n+1)$-ary associative operation, and the following distributive laws hold:

$$\text{((m,n)-DISTR)} \qquad \begin{aligned} &g(x_0, \dots, x_{i-1}, f(y_0, \dots, y_m), x_{i+1}, \dots, x_n) \\ &= f(g(x_0, \dots, x_{i-1}, y_0, x_{i+1}, \dots, x_n), \dots, \\ &\qquad\qquad g(x_0, \dots, x_{i-1}, y_m, x_{i+1}, \dots, x_n)) \\ &\qquad\qquad \text{for all } i\ (0 \le i \le n). \end{aligned}$$

Again, it is sufficient to give an (m,n)-ring via its addition f and multiplication g.

Theorem 3.8. (Artamonov [1]) *The minimal varieties of (m,n)-rings are exactly the varieties generated by the (m,n)-rings*

(3.8) $(K; x_0+\ldots+x_m, ax_0\ldots x_n)$ *where K is a finite field and $a\in K$ generates K over its prime field,*

(3.8)′ $(K; x_0+\ldots+x_m, x_0+ax_1+a^2x_2+\ldots+a^nx_n)$ *where K is a finite field whose characteristic divides m, $a\in K$ generates K, and $a^n=1$, and*

(3.8)″ $(K; x_0+\ldots+x_m, g_i)$ *with $g_i(x_0,\ldots,x_n)=x_i$ for all $x_0,\ldots,x_n\in K$, where $i\in\{0,n\}$ and K is a finite prime field whose characteristic divides m.*

MEDIAL n-QUASIGROUPS. An *n-quasigroup* is an algebra $(A; f, f_1,\ldots,f_n)$ with $n+1$ n-ary operations satisfying the identities (n-INV). Clearly, 2-quasigroups are exactly quasigroups. Again, $f_1,\ldots,f_n$ are uniquely determined by f, so for an n-quasigroup it suffices to describe its multiplication f. An n-quasigroup is called *medial* if it satisfies the identity

$$\begin{aligned}&f(f(x_{11},x_{12},\ldots,x_{1n}),f(x_{21},x_{22},\ldots,x_{2n}),\ldots,f(x_{n1},x_{n2},\ldots,x_{nn}))\\ =\ &f(f(x_{11},x_{21},\ldots,x_{n1}),f(x_{12},x_{22},\ldots,x_{n2}),\ldots,f(x_{1n},x_{2n},\ldots,x_{nn})),\end{aligned}$$

and *idempotent* if it satisfies the identity

$$f(x,\ldots,x)=x.$$

Theorem 3.9. (Csákány–Megyesi [14]) *The minimal varieties of idempotent medial n-quasigroups are exactly the varieties generated by the n-quasigroups*

(3.9) $(K; a_1x_1+\ldots+a_nx_n)$ *where K is a finite field, $a_1,\ldots,a_n\in K$ are nonzero elements generating K, and $a_1+\ldots+a_n=1$.*

It seems very plausible that the analogue of Theorem 3.4 is also true for n-quasigroups, though I have not found it in the literature. However, since n-quasigroups form a congruence permutable variety (see Section 5), therefore applying Theorem 5.2 one can show that the locally finite minimal varieties of medial n-quasigroups are exactly the varieties generated by the n-quasigroups

$(K;\ a_1x_1+\ldots+a_nx_n)$ *where K is a finite field and $a_1,\ldots,a_n$ are nonzero elements generating K.*

So what remains to verify is that every minimal variety of medial n-quasigroups is locally finite.

LATTICES AND BOOLEAN ALGEBRAS. Consider the following operations on the set $2 = \{0, 1\}$:

$$\begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}, \qquad \begin{array}{c|cc} \wedge & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

and let $^-$ be the unary operation defined by $\bar{0} = 1$, $\bar{1} = 0$. The following facts are well known (cf. [41]).

Theorem 3.10. *The only minimal variety of lattices is the variety of distributive lattices, which is generated by*

(3.10) *the two-element lattice* $(2; \vee, \wedge)$.

Theorem 3.11. *The variety of Boolean algebras, which is generated by*

(3.11) *the two-element Boolean algebra* $(2; \vee, \wedge, ^-, 0, 1)$,

is minimal.

DISTRIBUTIVE PSEUDOCOMPLEMENTED LATTICES. A *distributive pseudocomplemented lattice* is an algebra $(A; \vee, \wedge, ^*, 0, 1)$ such that $(A; \vee, \wedge, 0, 1)$ is a distributive lattice with least and greatest elements $0, 1$, respectively, and for every $a \in A$, a^* is a *pseudocomplement* of a; that is,

$$c \leq a^* \quad \text{if and only if} \quad c \wedge a = 0 \quad \text{for any } c \in A.$$

Lee [44] proved that distributive pseudocomplemented lattices form a variety, namely, in addition to the identities for bounded distributive lattices the following identities are required:

$$x \wedge x^* = 0, \quad x \vee x^{**} = x^{**}, \quad 0^* = 1,$$

$$(x \vee y)^* = x^* \wedge y^*, \quad (x \wedge y)^{**} = x^{**} \wedge y^{**}.$$

It is easy to see that in each distributive pseudocomplemented lattice $(A; \vee, \wedge, ^*, 0, 1)$, $\{0, 1\}$ forms a subalgebra. This implies

Theorem 3.12. *The only minimal variety of distributive pseudocomplemented lattices is the variety of Boolean algebras, generated by (3.11).*

Lee [44] proved that the varieties of distributive pseudocomplemented lattices form an $(\omega+1)$-chain.

RELATION ALGEBRAS. A *relation algebra* is an algebra $(A; \vee, \wedge, ^-, 0, 1, \circ, 1', ^\vee)$ such that $(A; \vee, \wedge, ^-, 0, 1)$ is a Boolean algebra, $(A; \circ, 1')$ is a monoid, $^\vee$ is an involution (i.e. $(x^\vee)^\vee = x$

for all $x \in A$) which is an automorphism of $(A; \vee, \wedge, ^-, 0, 1)$ and an antiautomorphism of $(A; \circ)$, moreover, the following condition is satisfied:

$$(x \circ y) \wedge z \leq x \circ (y \wedge (x^\vee \circ z)) \quad \text{for all} \quad x, y, z \in A,$$

where $\leq$ is the natural order of the underlying Boolean algebra (see [84], [39]). Clearly, the last condition is equivalent to an identity, so relation algebras form a variety.

Typical examples are the relation algebras of binary relations on a fixed set, where the Boolean operations are the set theoretic operations, $\circ$ is the composition, and $^\vee$ is the inversion of binary relations. For a set N, let Δ_N, ∇_N denote the diagonal relation and the full relation on N, respectively, and put $\sigma_N = \nabla_N - \Delta_N$.

Theorem 3.13. (Tarski [84]) *The minimal varieties of relation algebras are exactly the three varieties generated by the following relation algebras of binary relations:*

(3.13) $(\{\emptyset, \Delta_N, \sigma_N, \nabla_N\};\ \cup, \cap, ^-, \emptyset, \nabla_N, \circ, \Delta_N, ^\vee)$ *where* $N = 1, 2,$ *or* 3.

Note that for $N = 1$ this is a 2-element algebra, since $\Delta_N = \nabla_N$ and $\emptyset = \sigma_N$.

SEMIGROUPS. In addition to $\vee$, introduced earlier, consider the following operations on the set $2 = \{0, 1\}$:

$$\begin{array}{c|cc} \circ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}, \qquad \begin{array}{c|cc} *_l & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}, \qquad \begin{array}{c|cc} *_r & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}.$$

The semigroups $(2; \vee)$, $(2; \circ)$, $(2; *_l)$, $(2; *_r)$ are called *the two-element semilattice, zero semigroup, left zero semigroup, right zero semigroup,* respectively.

Theorem 3.14. (Kalicki–Scott [41]) *The minimal varieties of semigroups are exactly the varieties generated by*

(3.14) *the groups in (3.1) (considered as semigroups), and*

(3.14)′ $(2; \vee)$, $(2; \circ)$, $(2; *_l)$, $(2; *_r)$.

Corollary 3.15. *The minimal varieties of commutative semigroups are exactly the varieties generated by*

(3.15) *the groups in (3.1) (considered as semigroups), and*

(3.15)′ $(2; \vee)$, $(2; \circ)$.

n-SEMIGROUPS. An *n-semigroup* is an algebra $(A; f)$ with a single n-ary operation f satisfying the identities (n-ASS).

Theorem 3.16. (Artamonov [1]) *The minimal varieties of n-semigroups are exactly the varieties generated by*

(3.16) *the n-groups in (3.7) (considered as n-semigroups), and*

(3.16)′ $(2; x_1 \vee \ldots \vee x_n)$, $(2; x_1 \circ \ldots \circ x_n)$, $(2; x_1 *_l \ldots *_l x_n)$, $(2; x_1 *_r \ldots *_r x_n)$.

Some precursors of this theorem can be found in [56], [55] (cf. the section on n-groups).

SEMIRINGS. A *semiring* is an algebra $(A; +, \cdot)$ such that $(A; +)$ is a commutative semigroup, $(A; \cdot)$ is a semigroup, and $\cdot$ is distributive over $+$. We use the operations $\vee$, $\wedge$, $\circ$, $*_l$, $*_r$ defined earlier.

Theorem 3.17. (Polin [66]) *The minimal varieties of semirings are exactly the varieties generated by*

(3.17) *the rings in (3.2), (3.2)′ (considered as semirings), and*

(3.17)′ $(2; \vee, *_l)$, $(2; \vee, *_r)$, $(2; \vee, \circ)$, $(2; \wedge, \circ)$, $(2; \vee, \wedge)$, $(2; \vee, \vee)$, $(2; \circ, \circ)$, $(2; \circ, \wedge)$.

In their results on semirings, Pastijn and Romanowska do not assume the commutativity of $(A; +)$, however, they assume that both semigroups $(A; +)$ and $(A; \cdot)$ are idempotent. They have a complete description for all varieties, and hence all minimal varieties, of such semirings

— with both $(A; +)$ and $(A; \cdot)$ rectangular ([60]),
— with $(A; +)$ a semilattice ([70]), or
— with both $(A; +)$ and $(A; \cdot)$ normal ([59]).

MEDIAL GROUPOIDS. Similarly to quasigroups, a groupoid is called *medial* if it satisfies the identity (MED). Unlike in the case of quasigroups, *not all* minimal varieties of medial groupoids are known. In [34] the authors give a full list of minimal varieties of commutative medial groupoids, and using their description of finite simple medial groupoids, one can determine also all locally finite minimal varieties of medial groupoids.

Theorem 3.18 (Ježek–Kepka [34]) *The locally finite minimal varieties of medial groupoids are exactly the varieties generated by*

(3.18) *the quasigroups in (3.4) (considered as groupoids),*

(3.18)′ *the semigroups in (3.14)′, and*

(3.18)″ *the groupoids* $(\{0, a_0, \ldots, a_{n-1}\}; \cdot)$ $(n \geq 2)$ *with*

$$xy = \begin{cases} a_{i+1} & \text{if } x = y = a_i \\ 0 & \text{otherwise} \end{cases}$$

(in the subscript addition is modulo n).

Theorem 3.19 (Ježek–Kepka [34]) *The minimal varieties of commutative medial groupoids are exactly the varieties generated by the following groupoids:*

(3.19) $(K; ax + ay)$ *where* K *is a finite field and* $a \in K$ *is a nonzero element generating* K,

(3.19)′ *the semigroups in (3.15)′, and*

(3.19)″ *the groupoids in (3.18)″.*

DISTRIBUTIVE, PERMUTABLE, AND MODULAR GROUPOIDS. In some other varieties of groupoids all minimal varieties are known.

A groupoid $(A; \cdot)$ is called *distributive* if it satisfies the identities

$$x(yz) = (xy)(xz) \quad \text{and} \quad (xy)z = (xz)(yz),$$

that is, $\cdot$ is distributive over itself.

Theorem 3.20. (Ježek–Kepka [31]) *The minimal varieties of distributive groupoids are exactly the varieties generated by*

(3.20) *the quasigroups in (3.5) (considered as groupoids), and*

(3.20)′ *the semigroups in (3.14)′.*

Similarly to quasigroups, a groupoid $(A; \cdot)$ is called *left [right] permutable* if it satisfies the respective identity in (PER), and *bi-permutable* if both of them are satisfied.

Theorem 3.21. (Ježek–Kepka [35]) *The minimal varieties of bi-permutable groupoids are exactly the minimal varieties of commutative semigroups* (see Corollary 3.15).

Note that for quasigroups the corresponding result is trivial, as $D_1 \cap D_2 = A$. In [35] it remains open whether there are uncountably many minimal varieties of left permutable groupoids.

A groupoid $(A; \cdot)$ is called *left [right] modular* if it satisfies the respective identity in (MOD), and *bi-modular* if both of them are satisfied. It is not hard to show that the variety of left modular groupoids is a subvariety of the variety of medial groupoids.

Theorem 3.22. (Ježek–Kepka [36]) *The minimal varieties of left modular groupoids are exactly the varieties generated by*

(3.22) *the quasigroups in* $(3.6)_{E_1}$ *(considered as groupoids), and*

(3.22)′ *the semigroups in (3.15)′.*

Corollary 3.23. *The minimal varieties of bi-modular groupoids are exactly the varieties generated by*

(3.23) *the quasigroups in* $(3.6)_H$ *(considered as groupoids), and*

(3.23)′ *the semigroups in (3.15)′.*

SEMIGROUPS WITH INVOLUTION. An algebra $(A; \cdot, f)$ is called a *semigroup with involution* if $(A; \cdot)$ is a semigroup and f is an involutive antiautomorphism of $(A; \cdot)$, that is, the identities

$$f^2(x) = x, \quad f(xy) = f(y)f(x)$$

are satisfied.

Theorem 3.24. (Fajtlowicz [17]) *The minimal varieties of semigroups with involution are the varieties generated by*

(3.24) *the semigroups in (3.15), (3.15)′ augmented with the identity mapping as involution,*

(3.24)′ $(3; \cdot, f)$ *where* $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, *and*

$$xy = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

(3.24)″ $(2 \times 2; \cdot, f)$ *where* $(a, b)(c, d) = (a, d)$ *and* $f((a, b)) = (b, a)$ *for all* $a, b, c, d \in 2$.

SEMILATTICES WITH TWO COMMUTING ENDOMORPHISMS. The last variety we consider in this section differs interestingly from the pattern we got used to up till now: though it has countably many minimal subvarieties, each generated by a strictly simple algebra, some of the minimal varieties fail to be locally finite.

A *semilattice with two commuting endomorphisms* is an algebra $(A; \wedge, f, g)$ where $(A; \wedge)$ is a semilattice (i.e. a commutative idempotent semigroup) and f, g are endomorphisms of $(A; \wedge)$ such that $fg = gf$. Clearly, these algebras form a variety. In [29] Ježek described all simple algebras in this variety. From this description it is not hard to see that if a semilattice $\mathbf{A} = (A; \wedge, f, g)$ with two commuting endomorphisms is simple, then

(i) up to isomorphism, $\mathbf{A}$ is the only simple algebra in $V(\mathbf{A})$, provided $(A; \wedge)$ has a least element, and

(ii) $V(\mathbf{A})$ contains the minimal variety generated by $(2; \wedge, \mathrm{id}, \mathrm{id})$, provided $(A; \wedge)$ has no least element.

This yields all minimal varieties of semilattices with two commuting endomorphisms. For the explicit statement of the result we need some notation. In the theorem below each algebra has 0 in its base set, and $\wedge$ is defined as follows:

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

For arbitrary set A and element $a \in A$, we denote by c_a the unary constant operation on A with value a. Finally, for a group G (which is always assumed to be disjoint from $\{0\}$) and for $a \in G$, let l_a denote the unary operation on $\{0\} \cup G$ defined by

$$l_a(x) = \begin{cases} 0 & \text{if } x = 0 \\ ax & \text{if } x \in G \end{cases}.$$

Theorem 3.25 (Ježek [29]) *The minimal varieties of semilattices with two commuting endomorphisms are exactly the varieties generated by the following algebras:*

(3.25) $(2; \wedge, \mathrm{id}, c_1)$, $(2; \wedge, c_1, \mathrm{id})$, $(2; \wedge, c_0, c_0)$, $(2; \wedge, c_1, c_1)$;

(3.25)′ $(\{0\} \cup G;\ \wedge, c_0, l_a)$ *and* $(\{0\} \cup G;\ \wedge, l_a, c_0)$ *where* G *is a finite cyclic group with generator* a;

(3.25)″ $(\{0\} \cup G;\ \wedge, l_a, l_b)$ *where* G *is the (finite) Abelian group given by the defining relations* $G = [a, b|\ ab = ba,\ a^{rp} = b^{rq} = 1,\ a^p = b^{sq}]$, *and the integers* p, q, r, s *satisfy the conditions* $p, q, r \geq 1$, $0 \leq s < r$, $\gcd(r, s) = 1$;

(3.25)‴ $(\{0\} \cup G;\ \wedge, l_a, l_b)$ *where* G *is the (infinite) Abelian group given by the defining relations* $G = [a, b|\ ab = ba,\ a^n b^m = 1]$, *and* $n, m \geq 1$.

4. Further examples of minimal varieties

Recall that two algebras are said to be *term equivalent (polynomially equivalent)* if they have the same base set and the same set of term (polynomial) operations. Having the same term operations is equivalent to requiring that the fundamental operations of the algebras are mutually expressible from each other. A well-known, classical example is the two-element Boolean algebra $(2; \vee, \wedge, ^-, 0, 1)$ and the two-element Boolean ring $(2; +, \cdot, 0, 1)$ with unit.

Let $\tau = (f_i : i \in I)$ and $\sigma = (g_j : j \in J)$ be any types with f_i n_i-ary and g_j m_j-ary ($i \in I$, $j \in J$). A variety V of type τ is said to be *equivalent* to a variety W of type σ, if for every $i \in I$ and $j \in J$ there exist an n_i-ary term F_i of type σ and an m_i-ary term G_i of type τ such that for arbitrary algebras $\mathbf{A} = (A; (f_i : i \in I)) \in V$, $\mathbf{B} = (B; (g_j : j \in J)) \in W$, the algebra $\mathbf{A}^\sigma = (A; (G_j : j \in J))$ belongs to W, the algebra $\mathbf{B}^\tau = (B; (F_i : i \in I))$ belongs to V, and the mappings

$$V \to W,\ \mathbf{A} \mapsto \mathbf{A}^\sigma \quad \text{and} \quad W \to V,\ \mathbf{B} \mapsto \mathbf{B}^\tau$$

are inverses of each other. For example, the variety of Boolean algebras and the variety of Boolean rings with unit are equivalent. More generally, it is not hard to see that for arbitrary term equivalent algebras $\mathbf{A}$ and $\mathbf{A}'$, the varieties $V(\mathbf{A})$ and $V(\mathbf{A}')$ are equivalent.

A finite family $V_0, \ldots, V_{n-1}$ ($n \geq 2$) of varieties of the same type τ are called *independent* if there exists an n-ary term T of type τ in the variables $x_0, \ldots, x_{n-1}$ such that for every i ($0 \leq i \leq n-1$), the identity $T = x_i$ holds in V_i. (This implies also the more general property that for arbitrary terms $T_0, \ldots, T_{n-1}$ of type τ, there exists a term $\bar{T}$ of type τ such that for every i ($0 \leq i \leq n-1$), the identity $\bar{T} = T_i$ holds in V_i.)

Primal algebras, introduced and investigated by Foster [18], constitute the first set of examples for finite algebras generating minimal varieties, that are typically of universal algebraic character. A finite algebra $\mathbf{A}$ is called *primal* if every operation on A is a term operation of $\mathbf{A}$. Thus primal algebras are the most natural generalizations of the two-element Boolean algebra, and the structure of the varieties they generate is very similar to that of the variety of Boolean algebras. (For more details on this similarity, see e.g. [22; Appendix 5], and the references given there.) There is, however, one important difference. While in the two-element Boolean algebra the operations are more or less fixed: $\vee, \wedge, ^-, 0, 1$ (though some of them can be omitted), in primal algebras there is a large freedom in choosing the fundamental operations. By the preceding remarks on equivalent varieties it is clear that the choice of the fundamental operations of a primal algebra is irrelevant if we are interested only in the variety it generates.

However, by a result of O'Keefe [54], for arbitrary $n \geq 2$, and for arbitrary pairwise nonisomorphic primal algebras $\mathbf{A}_0, \ldots, \mathbf{A}_{n-1}$ of the same type with a single fundamental operation, the varieties $V(\mathbf{A}_0), \ldots, V(\mathbf{A}_{n-1})$ are independent. This shows that the equational theories of term equivalent algebras can be quite different. Nevertheless, there are properties of equational theories that are invariant under equivalence, the most important being the property of having a finite basis provided both similarity types are finite.

Since equational theories are not a central topic in this survey, we will not distinguish between term equivalent algebras; that is to say, whenever we consider algebras that are determined up to term equivalence only, we leave it implicit how their fundamental operations are selected.

QUASIPRIMAL ALGEBRAS. A finite algebra $\mathbf{A}$ is called *quasiprimal* if every operation on A preserving the internal isomorphisms (i.e. isomorphisms between subalgebras) of $\mathbf{A}$ is a term operation of $\mathbf{A}$. The concept as well as the following characterization of quasiprimal algebras is due to Pixley [63], [64].

A finite algebra $\mathbf{A}$ is quasiprimal if and only if the ternary discriminator

$$t(x,y,z) = \begin{cases} z & \text{if } x = y \\ x & \text{otherwise} \end{cases} \qquad (x,y,z \in A)$$

on A is a term operation of $\mathbf{A}$.

Strictly simple quasiprimal algebras are easy to describe up to term equivalence. Indeed, let $\mathbf{A}$ be a strictly simple quasiprimal algebra, and let U denote the set of those elements $u \in A$ for which $\{u\}$ is a trivial subalgebra of $\mathbf{A}$. Then the term operations of A are exactly the operations preserving the trivial subalgebras and the automorphisms of $\mathbf{A}$ (as $\mathbf{A}$ has no nontrivial proper subalgebra). Since the fixed points of each automorphism of $\mathbf{A}$ form a subalgebra, we conclude that every nonidentity automorphism of $\mathbf{A}$ has at most one fixed point, and the fixed point (if any) belongs to U.

For a subset U of A and a permutation group G acting on A let $\mathcal{R}_U(G)$ denote the clone of all operations f on A such that $f(u,\ldots,u) = u$ for all $u \in U$, and f admits each member of

G as an automorphism. Combining the facts mentioned above we get the necessity part of the following claim. The verification of the sufficiency is straightforward.

A finite algebra $\mathbf{A}$ *is a strictly simple quasiprimal algebra if and only if it is term equivalent to* $(A; \mathcal{R}_U(G))$ *for a subset* U *of* A *and for some permutation group* G *on* A *such that every nonidentity permutation in* G *has at most one fixed point, and the fixed point (if any) belongs to* U.

It is easy to see that in a quasiprimal algebra $\mathbf{A}$, $t(x, t(x, y, z), z)$ is a majority term operation. Therefore every quasiprimal algebra can be defined by finitely many fundamental operations, and as a special case of Theorem 5.1 discussed in the next section we have

Theorem 4.1. *If* $\mathbf{A}$ *is a strictly simple quasiprimal algebra, then* $\mathbf{A}$ *generates a minimal variety, and* $\mathbf{A}$ *has a finite basis for its identities provided it is of finite type.*

AFFINE ALGEBRAS. With a terminology introduced in [46], an algebra $\mathbf{A}$ is said to be *affine with respect to an Abelian group* $\widehat{A} = (A; +, -, 0)$ if $\mathbf{A}$ and $\widehat{A}$ have the same universe, $x - y + z$ is a term operation of $\mathbf{A}$, and the quaternary relation

$$Q_{\widehat{A}} = \{(a, b, c, d) \in A^4\colon\ a - b + c = d\}$$

is a subuniverse of $\mathbf{A}^4$ (i.e., the fundamental operations of $\mathbf{A}$ commute with $x - y + z$). The algebras that are affine with respect to $\widehat{A}$ are well known to be related to the module ${}_{(\mathrm{End}\,\widehat{A})}\widehat{A}$ (i.e. $\widehat{A}$ considered as a module over its endomorphism ring $\mathrm{End}\,\widehat{A}$).

Let $\mathbf{A}$ *be an algebra and* $\widehat{A}$ *an Abelian group on its universe. The algebra* $\mathbf{A}$ *is affine with respect to* $\widehat{A}$ *if and only if* $\mathbf{A}$ *is polynomially equivalent to a module* ${}_R\widehat{A}$ *for some subring* R *of* $\mathrm{End}\,\widehat{A}$.

For brevity, we shall call an algebra *affine* if it is affine with respect to some Abelian group. It is not hard to see that for an affine algebra the corresponding Abelian group is uniquely determined, up to the choice of its neutral element 0.

There is a more explicit description, up to term equivalence, for finite simple affine algebras, due to Clark and Krauss [12] (cf. [77; Chapter 2] for a direct proof). For a vector space ${}_K\widehat{A}$, let $T({}_K\widehat{A})$ denote the group $\{x + a\colon\ a \in A\}$ of translations of ${}_K\widehat{A}$, and $C({}_K\widehat{A})$ the set of all binary operations $rx + (1 - r)y$ with $r \in \mathrm{End}\,{}_K\widehat{A}$.

A finite algebra $\mathbf{A}$ *is a simple affine algebra if and only if there exist a finite field* K, *a vector space* ${}_K\widehat{A} = (A; +, K)$, *and an endomorphism* e *of* ${}_K\widehat{A}$ *with* $e^2 = e$ *such that* $\mathbf{A}$ *is term equivalent to*

$$(A;\ x - y + z,\ C({}_K\widehat{A}),\ e) \quad \text{or} \quad (A;\ x - y + z,\ C({}_K\widehat{A}),\ e,\ T({}_K\widehat{A})).$$

Clearly, the second algebra has no proper subalgebra, while in the first one the proper subalgebras are exactly the singletons $\{u\}$ with $u \in e(A)$. Thus every finite simple affine algebra is strictly simple.

Theorem 4.2. (McKenzie [46], Clark–Krauss [10]) *If* $\mathbf{A}$ *is a finite simple affine algebra, then*

(i) $\mathbf{A}$ *generates a minimal variety if and only if* $\mathbf{A}$ *has a trivial subalgebra, and*

(ii) $\mathbf{A}$ *has a finite basis for its identities provided it is of finite type.*

We note that a common generalization of quasiprimal algebras and affine algebras (called paraprimal algebras), and the varieties they generate were investigated by Clark and Krauss [10], [11], [12] and McKenzie [47].

PREPRIMAL ALGEBRAS. A finite algebra $\mathbf{A}$ is called *preprimal* if its clone is a maximal (proper) subclone of the clone $\mathcal{O}_A$ of all operations on A. The maximal subclones of $\mathcal{O}_A$ were described by Rosenberg [72], in terms of invariant relations. For an n-ary operation f on A and a k-ary relation B on A (i.e. $B \subseteq A^k$), f is said to *preserve* B if B is a subuniverse of the algebra $(A; f)^k$. For a k-ary relation B on A we denote by $\mathcal{P}_B$ the clone consisting of all operations on A preserving B.

A k-ary relation B on A is called *totally reflexive* if it contains each k-tuple from A^k whose components are not pairwise distinct. Further, B is called *totally symmetric* if it is closed under permuting the components. A totally reflexive, totally symmetric relation $B \subseteq A^k$ is called *central* if $B \neq A^k$ and there exists an element $c \in A$ such that $(c, a_1, \ldots, a_{k-1}) \in B$ for all $a_1, \ldots, a_{k-1} \in A$. The set of these elements c is called the *center* of B. Observe that every unary relation is totally reflexive and symmetric, hence the unary central relations are exactly the nonvoid proper subsets of A. Now let $k \geq 3$. A family $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$ $(m \geq 1)$ of equivalence relations on A is called *k-regular* if each Θ_i $(0 \leq i \leq m-1)$ has exactly k blocks and $\Theta_T = \Theta_0 \cap \ldots \cap \Theta_{m-1}$ has exactly k^m blocks. The relation determined by T is

$$\lambda_T = \{(a_0, \ldots, a_{k-1}) \in A^k\colon \text{ for all } i \ (0 \leq i \leq m-1), \text{ the elements } a_0, \ldots, a_{k-1} \text{ are not pairwise incongruent modulo } \Theta_i\}.$$

These relations λ_T are called *k-regular relations.*

Now we can state Rosenberg's theorem ([72]):

For a finite set A ($|A| \geq 2$), the maximal subclones of $\mathcal{O}_A$ are exactly the clones $\mathcal{P}_B$ where B is one of the following relations:

(i) *bounded partial orders on A,*

(ii) *permutations of A (considered as binary relations) having cycles of equal prime length and no fixed points,*

(iii) *equivalence relations on A distinct from the full relation and the diagonal relation,*

(iv) *quaternary relations of the form*

$$Q_{\widehat{A}} = \{(a, b, c, d) \in A^4\colon\ a - b + c = d\}$$

for some elementary Abelian p-group $\widehat{A} = (A; +)$ (p prime),

(v) *k-ary central relations on A ($k \geq 1$),*

(vi) *k-regular relations on A ($k \geq 3$).*

This description determines preprimal algebras up to term equivalence. Note that the preprimal algebras of type (ii) or of type (v) with $k = 1$ are quasiprimal, while those of type (iv) are affine.

Theorem 4.3. (Knoebel [42]) *A preprimal algebra generates a minimal variety if and only if it is of one of the types (i), (ii), or of type (v) with $k \geq 2$.*

It is worth noting that the preprimal algebras of types (ii)–(vi) are all term equivalent to algebras with finitely many fundamental operations, or equivalently, their clones are finitely generated (see Lau [43]). However, this is not the case for type (i), though many of them (e.g. those corresponding to a lattice order) have this property; in fact, the only preprimal algebra whose clone is proved to be not finitely generated is the one corresponding to the partial order

(Tardos [83]).

Another striking difference between type (i) and the remaining types is that for preprimal algebras of types (ii)–(vi) the generated varieties are residually finite (see [42]), while, for instance, for the partial order displayed above the corresponding preprimal algebra of type (i) generates a residually large variety (cf. [25; Exercise 10.5]). Recently, McKenzie [50] proved that the variety generated by a preprimal algebra of type (i) is residually small if and only if it is congruence distributive.

MATRIX POWERS OF ALGEBRAS. Let $\mathbf{A} = (A; F)$ be an arbitrary algebra, and m an integer, $m \geq 1$. The *mth matrix power* of $\mathbf{A}$, denoted $\mathbf{A}^{[m]}$, is the algebra $(A^m;\ F, d, p)$ where $(A^m; F)$ is the mth direct power of $\mathbf{A}$, and d is an m-ary while p is a unary operation on A^m defined as follows: for $a_i = (a_i^0, \ldots, a_i^{m-1}) \in A^m$ $(0 \leq i \leq m-1)$ and $a = (a^0, \ldots, a^{m-1}) \in A^m$,

$$d(a_0, a_1, \ldots, a_{m-1}) = (a_0^0, a_1^1, \ldots, a_{m-1}^{m-1}),$$

$$p(a) = (a^1, \ldots, a^{m-1}, a^0).$$

d is often called the *diagonal operation.*

Note that the algebra in (3.24)$''$ is $(2; \emptyset)^{[2]}$, which is term equivalent to $(2; \mathrm{id})^{[2]}$.

One can check directly that for algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type, a mapping $\bar{\varphi}\colon A^m \to B^m$ is a homomorphism $\mathbf{A}^{[m]} \to \mathbf{B}^{[m]}$ if and only if there exists a homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ such that $(a^0, \ldots, a^{m-1})\bar{\varphi} = (a^0\varphi, \ldots, a^{m-1}\varphi)$ for all $(a^0, \ldots, a^{m-1}) \in A^m$. Now it follows that for any variety V and for any fixed integer $m \geq 1$, the algebras isomorphic to $\mathbf{A}^{[m]}$ with $\mathbf{A} \in V$ again form a variety, which is called the mth matrix power of V, and is denoted by $V^{[m]}$. Furthermore, it follows

Theorem 4.4. *For any algebra $\mathbf{A}$ and integer $m \geq 1$, $\mathbf{A}^{[m]}$ generates a minimal variety if and only if $\mathbf{A}$ does.*

For any variety V and integer $m \geq 1$, one can extend every basis for the identities of V with some additional identities involving d and p to get a basis for the identities of $V^{[m]}$ (cf. [85]). The explicit form of this extension immediately yields

Theorem 4.5. *If V is a finitely based variety of finite type and $m \geq 1$, then $V^{[m]}$ is also finitely based.*

Matrix powers of algebras or varieties (under different names) were considered e.g. by Evans [15], Saade [73], Neumann [53], and Fajtlowicz [16]. The present name is justified by the fact that for arbitrary unitary R-module ${}_R\widehat{A}$ and integer $m \geq 1$, $({}_R\widehat{A})^{[m]}$ is term equivalent to the module ${}_{R_{m\times m}}(\widehat{A}^m)$, that is, $\widehat{A}^m$ considered in a natural way as a module over the ring $R_{m\times m}$ of $m \times m$ matrices with entries in R. For a more detailed history of the notion, consult Taylor [85].

Later on we will primarily be interested in matrix powers of unary algebras, or algebras term equivalent to them, so it is useful to describe their term operations. Let C be a set and let $m, n \geq 1$. For arbitrary mappings $\mu\colon m \to n$, $\sigma\colon m \to m$ and for arbitrary transformations $g_0, \ldots, g_{m-1}$ of C let us define an operation $h_\mu^\sigma[g_0, \ldots, g_{m-1}]$ on C^m as follows: for $x_i = (x_i^0, \ldots, x_i^{m-1}) \in C^m$ $(0 \leq i \leq n-1)$,

$$h_\mu^\sigma[g_0, \ldots, g_{m-1}](x_0, \ldots, x_{n-1}) = (g_0(x_{0\mu}^{0\sigma}), \ldots, g_{m-1}(x_{(m-1)\mu}^{(m-1)\sigma})).$$

For a unary algebra $\mathbf{C} = (C; F)$ *and for an integer* $m \geq 1$, *the term operations of* $\mathbf{C}^{[m]}$ *are exactly the operations* $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ *where* $n \geq 1$, $\mu: m \to n$, $\sigma: m \to m$ *are arbitrary mappings, and* $g_0, \dots, g_{m-1}$ *are unary term operations of* $\mathbf{C}$.

Clearly, every term operation of $\mathbf{C}^{[m]}$ depends on at most m variables. We will need the following fact which is an easy application of Theorem 4.4.

Corollary 4.6. *Let* N *be a finite set,* G *a permutation group on* N, *and let* $m \geq 1$. *The algebra* $(N; G)^{[m]}$ *generates a minimal variety if and only if* $|G| = 1$.

BOUNDED VARIETIES. A variety V of finite similarity type is said to be *bounded*, if there exists a natural number m such that every term in the language of V is equivalent, with respect to V, to a term of length at most m. The equational theories of bounded varieties were studied in detail by Ježek and McNulty [37]. In particular, they determined all equational theories corresponding to minimal bounded varieties whose type contains a single fundamental operation. The varieties themselves were described by Ježek [30]. A mapping $\mu: m \to n$ will be called *reducible* if there is a divisor d of m such that $i\mu = j\mu$ whenever $i \equiv j \pmod{d}$ $(i, j \in m)$; otherwise, μ is called *irreducible*.

Theorem 4.7. (Ježek [30]) *The minimal bounded varieties whose type contains a single* k*-ary fundamental operation* ($k \geq 2$), *are exactly the varieties generated by*

(4.7) *the algebras* $(2^m; h_\mu^\gamma[\mathrm{id}, \dots, \mathrm{id}])$ *where* $m \geq 1$, γ *is the cyclic permutation* $(0\ 1\ \dots\ m-1)$, *and* $\mu: m \to k$ *is an arbitrary irreducible mapping, and*

$(4.7)'$ *the two-element algebra with constant operation.*

The algebras in (4.7) are easily seen to be term equivalent to $(2; \mathrm{id})^{[m]}$. However, similarly to the result of O'Keefe on primal algebras mentioned at the beginning of this section, it is shown in [30] that for every fixed k (≥ 2) and for arbitrary finite family of pairwise distinct irreducible mappings μ_i: $m_i \to k$ $(i = 0, \dots, n-1)$, the varieties generated by the algebras $(2^{m_i}; h_{\mu_i}^\gamma[\mathrm{id}, \dots, \mathrm{id}])$ $(i = 0, \dots, n-1)$ are independent.

ω-CATEGORICAL VARIETIES. Let λ be a cardinal. A class of algebras is called λ*-categorical* if it contains exactly one algebra of cardinality λ, up to isomorphism. Varieties and quasivarieties λ-categorical for some infinite $\lambda \geq$ the cardinality of the similarity type were determined independently by Givant [20], [21] and Palyutin [57], [58]. Using tame congruence theory, McKenzie [49] found a nice algebraic proof for the case $\lambda = \omega$. It is relatively easy to see (cf. [49; Theorem 1.3]) that an ω-categorical variety of countable type has to be a locally finite minimal variety.

Theorem 4.8 (Givant [20], [21], Palyutin [57], [58], McKenzie [49]) *A variety of countable type is* ω*-categorical if and only if it is generated by one of the following algebras:*

(4.8) *a finite simple affine algebra having a trivial subalgebra,*

(4.8)′ *an algebra term equivalent to* $(2; \mathrm{id})^{[m]}$ *or* $(2; \mathrm{id}, 0)^{[m]}$ *for some* $m \geq 1$.

5. The congruence modular case

The deepest and most general result concerning locally finite minimal varieties solves Problems 2.1 and 2.3 under the assumption that $\mathbf{A}$ belongs to a congruence modular variety. Recall that a variety is said to be *congruence modular* [*congruence distributive*] if every algebra in the variety has a modular [distributive] congruence lattice. Similarly, a variety is called *congruence permutable* if any two congruences of each algebra in the variety permute.

Theorem 5.1. (Jónsson [38], Baker [2]) *Let* $\mathbf{A}$ *be a finite strictly simple algebra. If* $\mathbf{A}$ *belongs to a congruence distributive variety, then* $V(\mathbf{A})$ *is a minimal variety, and* $\mathbf{A}$ *has a finite basis for its identities provided it is of finite type.*

Theorem 5.2. (McKenzie [46], Smith [75], Gumm [23]) *Let* $\mathbf{A}$ *be a finite strictly simple algebra. If* $\mathbf{A}$ *belongs to a congruence permutable variety, then* $\mathbf{A}$ *is either quasiprimal (and hence generates a congruence distributive variety) or affine; the variety* $V(\mathbf{A})$ *is minimal if and only if* $\mathbf{A}$ *is not an affine algebra without any trivial subalgebra; furthermore,* $\mathbf{A}$ *is finitely based if it is of finite type.*

Theorem 5.3. (C. Herrmann, cf. [24, Corollary 4.2]) *Let* $\mathbf{A}$ *be a finite strictly simple algebra. If* $\mathbf{A}$ *belongs to a congruence modular variety, then either* $V(\mathbf{A})$ *is congruence distributive or* $\mathbf{A}$ *is an affine algebra; hence Theorem 5.1 or 5.2 applies.*

The proof of Theorem 5.3 depends heavily on commutator theory for congruence modular varieties (see Freese–McKenzie [19]).

Theorems 3.1–3.13 provide a large collection of illustrations to the above results, although those theorems are not special cases of Theorem 5.3 as they involve also the claim that every minimal subvariety of the classical variety in question is locally finite. Only Artamonov's proof of Theorem 3.8 and Smith' version of Theorem 3.4 makes use of Theorem 5.2, the other results were found either earlier than, or independently of the corresponding theorems in this section.

We mention here a theorem of Quackenbush [68] which is closely related to Theorem 5.2.

Theorem 5.4 (Quackenbush [68]) *Let* $\mathbf{A}$ *be a finite algebra of cardinality* k. *If every subalgebra of a finite power of* $\mathbf{A}$ *has cardinality a power of* k, *then* $\mathbf{A}$ *is a strictly simple algebra generating a congruence permutable variety.*

The case $k = 2$ was considered earlier by Taylor [85]. In [48] McKenzie gives an elegant and relatively simple proof for Theorem 5.4.

It is well known that for every finite strictly simple algebra $\mathbf{A}$ generating a minimal, congruence modular variety, $\mathbf{A}$ is the only subdirectly irreducible algebra in $V(\mathbf{A})$. Recently Bergman and McKenzie [3] proved the following stronger result.

Theorem 5.5 (Bergman–McKenzie [3]) *Every locally finite, minimal, congruence modular variety is minimal as a quasivariety.*

6. Surjective strictly simple algebras

To step beyond the congruence modular case, D. Hobby and R. McKenzie [25] list Problem 2.1 in their book among the questions they suggest for investigation by making use of tame congruence theory.

In this section we present several recent results on finite strictly simple algebras, which were obtained by a combination of the techniques of clone theory and tame congruence theory. We do not make any assumption on the generated variety, however, we assume that the algebras are *surjective*, that is, all their fundamental operations are surjective. As we shall see, in some cases surjective strictly simple algebras have much nicer structure than strictly simple algebras in general. On the other hand, it is easy to see that every strictly simple algebra with a single fundamental operation is surjective unless it is a two-element algebra whose operation is constant. Thus the class of surjective algebras is wide enough to include many algebras for which Problems 2.1 and 2.3 are still open.

Clearly, surjectivity of algebras is not invariant under term equivalence, however, every result on surjective algebras carries over in a natural way to algebras that are term equivalent to surjective algebras. In this way each result discussed below can be considered a generalization of the corresponding known result for the congruence permutable case. Indeed, every algebra generating a congruence permutable variety is term equivalent to a surjective algebra: if the algebra $(A; \{f_i : i \in I\})$ with f_i n_i-ary $(i \in I)$ has a Mal'tsev operation p among its term operations, then it is term equivalent to the algebra $(A; p, \{g_i : i \in I\})$ with

$$g_i(x_0, \ldots, x_{n_i+1}) = p(f_i(x_0, \ldots, x_{n_i-1}), x_{n_i}, x_{n_i+1}) \quad (i \in I),$$

whose fundamental operations are surjective. The analogous claim fails for the algebras generating congruence distributive varieties.

IDEMPOTENT ALGEBRAS. I became interested in Problems 2.1 and 2.3 while investigating idempotent algebras. Recall that an operation f is said to be *idempotent* if it satisfies the identity

$f(x, \ldots, x) = x$, while an algebra $\mathbf{A}$ is *idempotent* if every fundamental operation (and hence every term operation) of $\mathbf{A}$ is idempotent. In other words, $\mathbf{A}$ is idempotent if and only if every singleton $\{a\}$ $(a \in A)$ is a subalgebra of $\mathbf{A}$. It turns out that for a finite idempotent algebra, having no nontrivial proper subalgebra (or equivalently, being strictly simple) is a rather strong constraint; there are only a few such algebras, up to term equivalence, and they can be described explicitly.

To state the result we need some preparation. For $a \in A$ and for an integer $k \geq 2$ we define a k-ary relation on A as follows:

$$X_k^a = \{(a_0, \ldots, a_{k-1}) \in A^k \colon\ a_i = a \text{ for at least one } i,\ 0 \leq i \leq k-1\}.$$

For $a \in A$ and $k \geq 2$ let $\mathcal{F}_k^a$ denote the clone of all operations f on A such that X_k^a is a subuniverse of $(A; f)^k$. Furthermore, we put $\mathcal{F}_\omega^a = \bigcap_{k=2}^\infty \mathcal{F}_k^a$. For a permutation group G acting on A we will write $\mathcal{I}(G)$ for $\mathcal{R}_A(G)$ (see Section 4).

Theorem 6.1. [78] *Let $\mathbf{A}$ be a finite, idempotent, strictly simple algebra with at least 3 elements. Then one of the following conditions holds:*

(a) *$\mathbf{A}$ is quasiprimal;*

(b) *$\mathbf{A}$ is affine;*

(c) *$\mathbf{A}$ is term equivalent to the algebra $(A;\ \mathcal{F}_k^a \cap \mathcal{I}(G))$ for some k $(2 \leq k \leq \omega)$, some element $a \in A$, and for some permutation group G acting on A such that a is the unique fixed point of each nonidentity permutation in G.*

The proof of Theorem 6.1 yields that the algebras $\mathbf{A}$ in (c) are functionally complete (i.e. every operation on A is a polynomial operation of $\mathbf{A}$), and the same is well known to hold for quasiprimal algebras as well. Hence in cases (a) and (c) $\mathbf{A}$ is of type **3**, while in case (b) it is of type **2**. Thus in the context of tame congruence theory Theorem 6.1 means the following:

*Let $\mathbf{A}$ be a finite, idempotent, strictly simple algebra with at least 3 elements. If $\mathbf{A}$ is of type **2**, then $\mathbf{A}$ is affine, while if $\mathbf{A}$ is of type **3**, then $\mathbf{A}$ is either quasiprimal or term equivalent to the algebra $(A;\ \mathcal{F}_k^a \cap \mathcal{I}(G))$ for some k $(2 \leq k \leq \omega)$, some element $a \in A$, and for some permutation group G acting on A such that a is the unique fixed point of each nonidentity permutation in G. $\mathbf{A}$ cannot be of type **1**, **4** or **5**.*

The 2-element algebras were determined, up to term equivalence, by E. Post [67]. Combining this with Theorem 6.1 and the observation that all algebras in (c) generate congruence distributive varieties, we get

Corollary 6.2. [79] *Every finite, idempotent, strictly simple algebra generates a minimal variety, and is finitely based provided it is of finite type.*

ALGEBRAS WITH NO PROPER SUBALGEBRAS. We have a definitive answer to Problems 2.1 and 2.3 also for surjective strictly simple algebras that have no trivial subalgebras. Recall that a permutation group G on a set N is said to *act primitively* on N if the unary algebra $(N; G)$ is simple and $|G| > 1$ (if $|N| = 2$).

Theorem 6.3. [80] *Let* $\mathbf{A}$ *be a finite, simple, surjective algebra having no proper subalgebra. Then one of the following conditions holds:*

(a) $\mathbf{A}$ *is quasiprimal;*

(b) $\mathbf{A}$ *is affine;*

(c) $\mathbf{A}$ *is isomorphic to an algebra term equivalent to* $(N; G)^{[m]}$ *for some finite set* N $(|N| \geq 2)$, *some* $m \geq 1$, *and for some permutation group* G *on* N *which acts primitively on* N.

In the context of tame congruence theory, Theorem 6.3 can be restated as follows:

Let $\mathbf{A}$ *be a finite, simple, surjective algebra having no proper subalgebra. If* $\mathbf{A}$ *is of type* $\mathbf{1}$, *then* $\mathbf{A}$ *is isomorphic to an algebra term equivalent to* $(N; G)^{[m]}$ *for some primitive permutation group* G *on* N; *if* $\mathbf{A}$ *is of type* $\mathbf{2}$, *then* $\mathbf{A}$ *is affine, while if* $\mathbf{A}$ *is of type* $\mathbf{3}$, *then* $\mathbf{A}$ *is quasiprimal.* $\mathbf{A}$ *cannot be of type* $\mathbf{4}$ *or* $\mathbf{5}$.

As was mentioned in Theorem 4.2, a simple affine algebra with no proper subalgebra does not generate a minimal variety. Furthermore, by Corollary 4.6, none of the algebras in Theorem 6.3 (c) generates a minimal variety either. Thus we have

Corollary 6.4. [80] *A finite, simple, surjective algebra* $\mathbf{A}$ *having no proper subalgebra generates a minimal variety if and only if it is quasiprimal. If so,* $\mathbf{A}$ *is finitely based provided it is of finite type.*

It is interesting to compare this result with the algebras in Example 2.4. Clearly, if the operation $\circ$ of $\mathbf{G}$ is surjective (this is the case, for instance, if $\mathbf{G}$ is Murskiĭ's groupoid), then $\mathbf{A}(\mathbf{G})$ fails to be surjective *only* because of the constants.

In [61] Pigozzi raises the question whether there exists a finite, strictly simple groupoid generating a minimal variety, which is not finitely based. Corollary 6.4 shows that such a groupoid with no proper subgroupoids cannot exist. In fact, since every finite simple groupoid without proper subgroupoids is surjective, if it generates a minimal variety, then it is quasiprimal, and hence is finitely based.

The special case of Theorem 6.3 for algebras $\mathbf{A}$ in which the unary term operations form a permutation group, combined with some deep results in tame congruence theory, makes it possible to prove a slight extension of Theorem 5.5.

Theorem 6.5. [80] *A locally finite minimal variety omitting type* $\mathbf{1}$ *is minimal as a quasivariety if and only if it has a unique subdirectly irreducible algebra.*

ABELIAN ALGEBRAS. The case of surjective strictly simple algebras having several trivial subalgebras seems to be more complicated than the two extremes settled above. However, if we restrict our attention to Abelian algebras, then we can get a result analogous to Theorem 6.3.

Following [25] we call an algebra $\mathbf{A}$ *Abelian* if it satisfies the so-called *term condition* (or *TC*): for all $n \geq 1$, for every n-ary term operation f of $\mathbf{A}$ and for arbitrary elements $u, v, a_i, b_i \in A$ $(1 \leq i \leq n-1)$,

$$f(u, a_1, \ldots, a_{n-1}) = f(u, b_1, \ldots, b_{n-1}) \quad \text{iff} \quad f(v, a_1, \ldots, a_{n-1}) = f(v, b_1, \ldots, b_{n-1}).$$

It is known from tame congruence theory that a finite simple algebra is Abelian if and only if it is of type **1** or **2**. Recently, Valeriote [86] proved that every finite simple Abelian algebra is strictly simple.

Theorem 6.6. [81] *Let $\mathbf{A}$ be a finite, surjective, simple Abelian algebra having a trivial subalgebra. Then one of the following conditions holds:*

(a) *$\mathbf{A}$ is affine;*

(b) *$\mathbf{A}$ is isomorphic to an algebra term equivalent to $(2; \mathrm{id})^{[m]}$ for some $m \geq 1$.*

Again, rephrased in the context of tame congruence theory, this result mean the following:

*Let $\mathbf{A}$ be a finite, surjective, simple Abelian algebra having a trivial subalgebra. If $\mathbf{A}$ is of type **1**, then $\mathbf{A}$ is isomorphic to an algebra term equivalent to $(2; \mathrm{id})^{[m]}$ for some $m \geq 1$, while if $\mathbf{A}$ is of type **2**, then $\mathbf{A}$ is affine.*

In view of Theorems 4.4 and 4.5, every algebra $(2; \mathrm{id})^{[m]}$ $(m \geq 1)$ generates a finitely based minimal vartiety. Combining this with the analogous result for strictly simple affine algebras having trivial subalgebras (cf. Theorem 4.2), we get

Corollary 6.7. [81] *Every finite, surjective, simple Abelian algebra having a trivial subalgebra generates a minimal variety, and is finitely based provided it is of finite type.*

ALGEBRAS WITH A UNIQUE TRIVIAL SUBALGEBRA. Finally we mention a result showing that on each finite set with at least three elements, there exist $2^{\aleph_0}$ pairwise inequivalent surjective, strictly simple algebras. For a group G and an element $0 \notin G$ we use the operations $\wedge$ and l_a $(a \in G)$ on $\{0\} \cup G$ introduced preceding Theorem 3.25. Further, let $\mathbf{N}$ denote the set of positive integers.

Theorem 6.8. [82] *For arbitrary finite group G with at least two elements, there exist operations f_n $(n \in \mathbf{N})$ on $\{0\} \cup G$ $(0 \notin G)$ such that the algebras*

$$\mathbf{A}_I = (\{0\} \cup G;\ \wedge, \{l_a \colon a \in G\}, \{f_n \colon n \in I\}), \quad I \subseteq \mathbf{N},$$

have the following properties:

(i) $\mathbf{A}_I$ *is strictly simple and surjective, and* $\{0\}$ *is the unique trivial subalgebra of* $\mathbf{A}_I$*;*

(ii) $\mathbf{A}_I$ *generates a minimal variety;*

(iii) *for distinct sets* $I, J \subseteq \mathbf{N}$*, the algebras* $\mathbf{A}_I$ *and* $\mathbf{A}_J$ *are not term equivalent.*

We note that the operations f_n ($n \in \mathbf{N}$) are constructed in such a way that in the classification by tame congruence theory, the algebras $\mathbf{A}_I$ are of type **5**. However, by adding a single ternary or binary operation, respectively, one can get a similar family of algebras of type **4** or **3** as well.

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