

Strongly Abelian minimal varieties

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ABSTRACT

It is shown that every finite, simple, strongly Abelian algebra generating a minimal variety is term equivalent to a full matrix power of a 2-element unary algebra. The proof is based on a classification of reducts of matrix powers of unary algebras.

Introduction

The matrix power construction for algebras and varieties was first considered (under different names) by T. Evans [3], M. Saade [13], W. D. Neumann [11], and S. Fajtlowicz [4]. Since then, matrix powers have occurred in various topics; e.g. fine spectra of varieties ([20]), or bounded equational theories ([6]). Recently, important applications of reducts of matrix powers have been found in the structure theory of finite algebras ([9], [5]), and in the study of category equivalence of varieties ([10]).

The term operations of matrix powers of sets (or equivalently, the term operations of matrix powers of algebras whose operations are projections) — interpreted in a different form, and called “co-operations” — were investigated by B. Csákány [1], [2] from the point of view of completeness. A Rosenberg-type general completeness criterion for such operations was found by Z. Székely [14]. In an unpublished work [15] this result was extended to matrix powers of unary permutational algebras as well.

In this paper we extend Székely’s results as follows: among all reducts of matrix powers of unary algebras we characterize the algebras that are term equivalent to full matrix powers, by describing (in terms of a single kind of preservation property) the maximal possible clones of the remaining algebras (Theorem 1.5 and Corollary 1.6). Making use of this classification theorem we prove that there are no other locally finite, strongly Abelian minimal varieties, than the well-known minimal varieties generated by matrix powers of some 2-element algebras (Theorem 2.2). In fact, this application was the motivation for the investigation of reducts of matrix powers of arbitrary unary algebras. Theorem 1.5 allows a slight refinement in the main result of [16] as well (Theorem 3.4).

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1. A classification of reducts of matrix powers of unary algebras

If not stated otherwise, algebras are denoted by boldface capitals and their universes by the corresponding letters in italics. The clone of term operations of an algebra \mathbf{A} is denoted by $\text{Clo } \mathbf{A}$. For algebras $\mathbf{A} = (A; F)$ and $\mathbf{A}' = (A; F')$ having a common universe A , \mathbf{A}' is called a *reduct* of \mathbf{A} if $F' \subseteq \text{Clo } \mathbf{A}$ (or equivalently, $\text{Clo } \mathbf{A}' \subseteq \text{Clo } \mathbf{A}$), and \mathbf{A}' is said to be *term equivalent* to \mathbf{A} if $\text{Clo } \mathbf{A}' = \text{Clo } \mathbf{A}$.

We identify every natural number n with the set $n = \{0, \dots, n-1\}$. For a set U , let T_U , S_U , and C_U denote the full transformation monoid on U , the full symmetric group on U , and the set of (unary) constant functions on U , respectively. We denote by id the identity mapping of each set. The cardinality of a set A is denoted by $|A|$. For a mapping φ , the kernel of φ is denoted by $\ker \varphi$.

For the notion and the history of matrix powers of algebras the reader is referred to [20]; recent applications can be found in [5] and [10]. In this paper we need the concept for unary algebras only. Let $\mathbf{U} = (U; F)$ be a unary algebra and let m be a positive integer. For arbitrary mappings $\mu: m \rightarrow n$, $\sigma \in T_m$ and for arbitrary unary term operations g_0, \dots, g_{m-1} of \mathbf{U} let us define an operation $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ on U^m as follows: for $x_i = (x_i^0, \dots, x_i^{m-1}) \in U^m$ ($0 \leq i \leq m-1$),

$$h_\mu^\sigma[g_0, \dots, g_{m-1}](x_0, \dots, x_{m-1}) = (g_0(x_{0\mu}^{0\sigma}), \dots, g_{m-1}(x_{(m-1)\mu}^{(m-1)\sigma})).$$

The mappings μ and σ will be referred to as the *variable mapping* and the *coordinate mapping* of the operation, respectively, while g_0, \dots, g_{m-1} will be called the *sequence of coordinate functions* of the operation. If $g_0 = \dots = g_{m-1} = \text{id}$, then we will write h_μ^σ instead of $h_\mu^\sigma[\text{id}, \dots, \text{id}]$; furthermore, if the operation is unary (and hence μ is the unique mapping $m \rightarrow 1$), then the subscript μ will be omitted.

The m -th *matrix power* of \mathbf{U} , denoted $\mathbf{U}^{[m]}$, is the algebra with universe U^m and with all $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ described above as fundamental operations.

Lemma 1.1. *Let \mathbf{U} be a unary algebra, and let m be a positive integer. For an n -ary operation $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ and for k -ary operations $h_{\nu_l}^{\tau_l}[f_{0l}, \dots, f_{m-1,l}]$ ($l = 0, \dots, n-1$) of $\mathbf{U}^{[m]}$ we have*

$$\begin{aligned} & h_\mu^\sigma[g_0, \dots, g_{m-1}](h_{\nu_0}^{\tau_0}[f_{00}, \dots, f_{m-1,0}](x_0, \dots, x_{k-1}), \dots \\ & \quad \dots, h_{\nu_{n-1}}^{\tau_{n-1}}[f_{0,n-1}, \dots, f_{m-1,n-1}](x_0, \dots, x_{k-1})) \\ &= (g_0 f_{0\sigma, 0\mu}(x_{0\sigma\nu_0}^{0\sigma\tau_0\mu}), \dots, g_{m-1} f_{(m-1)\sigma, (m-1)\mu}(x_{(m-1)\sigma\nu_{(m-1)\mu}}^{(m-1)\sigma\tau_{(m-1)\mu}})); \end{aligned}$$

hence the composite operation is an operation of $\mathbf{U}^{[m]}$, and its variable mapping, coordinate mapping and sequence of coordinate functions are

$$i \mapsto i\sigma\nu_{i\mu}, \quad i \mapsto i\sigma\tau_{i\mu}, \quad \text{and} \quad i \mapsto g_i f_{i\sigma, i\mu} \quad (0 \leq i \leq m-1),$$

respectively.

Proof. Straightforward. \diamond

Since the i th n -ary projection on U^m is $h_{m \rightarrow \{i\}}^{\text{id}}$, it follows immediately from Lemma 1.1 that $\mathbf{U}^{[m]}$ has no other term operations than its fundamental operations; that is to say, $\text{Clo } \mathbf{U}^{[m]}$ consists of all operations of the form $h_{\mu}^{\sigma}[g_0, \dots, g_{m-1}]$ described above. Clearly, every (term) operation of $\mathbf{U}^{[m]}$ depends on at most m variables.

We will say that an operation $h_{\mu}^{\sigma}[g_0, \dots, g_{m-1}]$ of $\mathbf{U}^{[m]}$ is *surjective in the i -th coordinate* if g_i is surjective ($0 \leq i \leq m-1$). Clearly, if \mathbf{U} is finite, then g_i is surjective if and only if it is a permutation of U . For a subset $I \subseteq m$, an operation $h_{\mu}^{\sigma}[g_0, \dots, g_{m-1}]$ of $\mathbf{U}^{[m]}$ will be called *coordinate-wise I -surjective* if h is surjective in the i -th coordinate for every $i \in I$, and *I -gluing* if there exist distinct indices $i, i' \in I$ with $i\mu = i'\mu$, $i\sigma = i'\sigma$. Instead of coordinate-wise m -surjective or m -gluing we say briefly *coordinate-wise surjective* or *gluing*, respectively. Notice that an operation $h_{\mu}^{\sigma}[g_0, \dots, g_{m-1}]$ of $\mathbf{U}^{[m]}$ is gluing if and only if $\ker \mu \cap \ker \sigma$ is different from the equality relation.

Lemma 1.2. *Let \mathbf{U} be a unary algebra, and let $m \geq 1$.*

(1.2.1) *Every surjective operation of $\mathbf{U}^{[m]}$ is coordinate-wise surjective.*

(1.2.2) *For \mathbf{U} finite, an operation of $\mathbf{U}^{[m]}$ is surjective if and only if it is coordinate-wise surjective and not gluing. \diamond*

Let A be a set and π a permutation of A . For an n -ary operation f on A the *conjugate of f via π* is the operation

$$\pi f \pi^{-1}: A^n \rightarrow A, (a_0, \dots, a_{n-1}) \mapsto \pi f(\pi^{-1}(a_0), \dots, \pi^{-1}(a_{n-1})).$$

Clearly, $\pi f \pi^{-1}$ is exactly the operation for which $\pi: (A; f) \rightarrow (A; \pi f \pi^{-1})$ is an isomorphism. For any set F of operations on A , the *conjugate of F via π* is

$$\pi F \pi^{-1} = \{\pi f \pi^{-1}: f \in F\}.$$

Obviously, if \mathcal{C} is a clone on A , then so is $\pi \mathcal{C} \pi^{-1}$. For arbitrary algebras $\mathbf{A} = (A; F)$ and $\mathbf{A}' = (A; F')$ with common universe A and for any permutation π of A the equality $\text{Clo } \mathbf{A} = \pi(\text{Clo } \mathbf{A}')\pi^{-1}$ means that the isomorphic copy of \mathbf{A} under π is term equivalent to \mathbf{A}' .

Let \mathbf{U} be a unary algebra, m a positive integer, and consider a reduct \mathbf{U}' of \mathbf{U} . Obviously, $\text{Clo } (\mathbf{U}')^{[m]}$ is a subclone of $\text{Clo } \mathbf{U}^{[m]}$. However, the subclones of $\text{Clo } \mathbf{U}^{[m]}$ that are conjugate to $\text{Clo } (\mathbf{U}')^{[m]}$ — even via permutations of U^m of the form $\pi = h^{\text{id}}[\pi_0, \dots, \pi_{m-1}]$ ($\in \text{Clo } (U; S_U)^{[m]}$) — need not be equal to clones of m th matrix powers of some unary algebras on U ; to see this, take $m = 2$, $\mathbf{U} = (2; T_2)$, $\mathbf{U}' = (2; 0)$, and $\pi_0 = \text{id}$, $\pi_1 = (0 \ 1)$. The following proposition provides an internal characterization for the conjugates of the clones of full matrix powers of reducts of a unary algebra.

Proposition 1.3. *Let \mathbf{U} be a finite unary algebra, and let \mathcal{C} be a subclone of $\text{Clo } \mathbf{U}^{[m]}$. The following three conditions are equivalent for \mathcal{C} :*

(1.3.1) for every transformation σ of m , \mathcal{C} contains a surjective operation with coordinate mapping σ and variable mapping id ;

(1.3.2) \mathcal{C} contains the operation $h_{\text{id}}^{\text{id}}$ and at least one surjective unary operation with coordinate mapping $\gamma = (0 \ 1 \ \dots \ m-1)$;

(1.3.3) \mathbf{U} has a reduct \mathbf{U}' such that \mathcal{C} is conjugate to $\text{Clo}(\mathbf{U}')^{[m]}$ via some permutation $\pi = h^{\text{id}}[\pi_0, \dots, \pi_{m-1}]$ in $\text{Clo } \mathbf{U}^{[m]}$.

Proof. By means of Lemma 1.1 one can easily check that for arbitrary operation $h = h_{\nu}^{\tau}[f_0, \dots, f_{m-1}] \in \mathcal{C}$ the conjugate of h via $\pi = h^{\text{id}}[\pi_0, \dots, \pi_{m-1}]$ ($\in \text{Clo}(U; S_U)^{[m]}$) is

$$\pi h \pi^{-1} = h_{\nu}^{\tau}[\pi_0 f_0 \pi_{0\tau}^{-1}, \dots, \pi_{m-1} f_{m-1} \pi_{(m-1)\tau}^{-1}].$$

Thus the implication (1.3.3) \Rightarrow (1.3.1) is obvious.

Suppose now that (1.3.1) holds. By assumption, $h_{\text{id}}^{\text{id}}[g_0, \dots, g_{m-1}]$ and $h_{\text{id}}^{\gamma}[g'_0, \dots, g'_{m-1}]$ belong to \mathcal{C} for some permutations $g_0, \dots, g_{m-1}, g'_0, \dots, g'_{m-1}$ of U . Identifying variables we get also $h^{\text{id}}[g_0, \dots, g_{m-1}], h^{\gamma}[g'_0, \dots, g'_{m-1}] \in \mathcal{C}$. Since $h^{\text{id}}[g_0, \dots, g_{m-1}]$ acts coordinate-wise on U^m and U is finite, composing $h_{\text{id}}^{\text{id}}[g_0, \dots, g_{m-1}]$ with some power of $h^{\text{id}}[g_0, \dots, g_{m-1}]$ yields $h_{\text{id}}^{\text{id}} = h_{\text{id}}^{\text{id}}[\text{id}, \dots, \text{id}]$, whence $h_{\text{id}}^{\text{id}} \in \mathcal{C}$. This proves (1.3.2).

Assuming (1.3.2) observe first that in view of $h_{\text{id}}^{\text{id}} \in \mathcal{C}$, for every unary operation $h^{\gamma}[g'_0, \dots, g'_{m-1}] \in \mathcal{C}$ we have

$$h_{\text{id}}^{\gamma}[g'_0, \dots, g'_{m-1}] = h_{\text{id}}^{\text{id}}(h^{\gamma}[g'_0, \dots, g'_{m-1]}(x_0), \dots, h^{\gamma}[g'_0, \dots, g'_{m-1]}(x_{m-1})) \in \mathcal{C}.$$

Thus \mathcal{C} contains $h_{\text{id}}^{\gamma}[g'_0, \dots, g'_{m-1}]$ for some permutations g'_0, \dots, g'_{m-1} of U . Consider now the permutations $\pi_0 = \text{id}$ and $\pi_i = g'_0 g'_1 \dots g'_{i-1}$ ($i = 1, \dots, m-1$). Clearly, $\pi = h^{\text{id}}[\pi_0, \dots, \pi_{m-1}]$ belongs to $\text{Clo } \mathbf{U}^{[m]}$. By the remark made in the first paragraph of this proof, the conjugate of the operation $h_{\text{id}}^{\gamma}[g'_0, \dots, g'_{m-1}]$ via π is of the form $h_{\text{id}}^{\gamma}[\text{id}, \dots, \text{id}, g]$ for some permutation g in $\text{Clo } \mathbf{U}$. The conjugate of $h_{\text{id}}^{\text{id}}$ via π is obviously $h_{\text{id}}^{\text{id}}$ itself. Thus

$$h_{\text{id}}^{\text{id}}, h_{\text{id}}^{\gamma}[\text{id}, \dots, \text{id}, g] \in \pi \mathcal{C} \pi^{-1} (\subseteq \text{Clo } \mathbf{U}^{[m]}).$$

We show that h_{id}^{γ} also belongs to $\pi \mathcal{C} \pi^{-1}$, and hence so is h^{γ} . This will prove that $\pi \mathcal{C} \pi^{-1}$ is the clone of the m th matrix power of a reduct of \mathbf{U} (cf. Lemma 2.10 in [17]), as required in (1.3.3). Let $q = h^{\gamma}[\text{id}, \dots, \text{id}, g]$. Clearly, $q^m = h^{\text{id}}[g, \dots, g]$, so for some natural number k , $q^{mk} = h^{\text{id}}[g^{-1}, \dots, g^{-1}] \in \mathcal{C}$. Thus

$$h_{\text{id}}^{\gamma}(x_0, \dots, x_{m-1}) = h_{\text{id}}^{\gamma}[\text{id}, \dots, \text{id}, g](x_0, \dots, x_{m-2}, q^{mk}(x_{m-1})) \in \mathcal{C},$$

completing the proof. \diamond

Let \mathbf{U} be a unary algebra, and let m be a positive integer. Further, let \mathcal{I} be the set of blocks of a uniform partial equivalence of m (i.e., \mathcal{I} is a non-empty family of pairwise disjoint, non-empty subsets of m of the same cardinality). We will say that an operation h of $\mathbf{U}^{[m]}$ respects \mathcal{I} , if for each block I of \mathcal{I} one of the following conditions holds:

(Rsp1) h is not coordinate-wise I -surjective,

- (Rsp2) h is coordinate-wise I -surjective and I -gluing,
- (Rsp3) h is coordinate-wise I -surjective, moreover, the variable mapping of h is constant on I , and the coordinate mapping of h maps I bijectively onto a block in \mathcal{I} .

Obviously, the three conditions (Rsp1)–(Rsp3) mutually exclude one another.

It will not cause confusion if we do not distinguish a uniform partial equivalence from the family of its blocks. A subclone \mathcal{C} of $\text{Clo } \mathbf{U}^{[m]}$ will be said to respect a uniform partial equivalence \mathcal{I} (or the family \mathcal{I} of blocks of a uniform partial equivalence) on m if every member of \mathcal{C} respects \mathcal{I} . Similarly, a reduct \mathbf{A} of $\mathbf{U}^{[m]}$ will be said to respect \mathcal{I} if every fundamental operation (or equivalently, cf. Proposition 1.4 (1.4.1) below, every term operation) of \mathbf{A} respects \mathcal{I} .

Proposition 1.4. *Let \mathbf{U} be a unary algebra, let $m \geq 1$, and let \mathcal{I} be the set of blocks of a uniform partial equivalence on m .*

(1.4.1) *The set $\text{Clo}_{\mathcal{I}} \mathbf{U}^{[m]}$ of all operations in $\text{Clo } \mathbf{U}^{[m]}$ respecting \mathcal{I} forms a clone.*

(1.4.2) *$\text{Clo}_{\mathcal{I}} \mathbf{U}^{[m]}$ is a proper subclone of $\text{Clo } \mathbf{U}^{[m]}$ if and only if \mathcal{I} is distinct from the equality relation.*

(1.4.3) *If \mathcal{I} is distinct from the equality relation, then $\text{Clo}_{\mathcal{I}} \mathbf{U}^{[m]}$ is not conjugate to the clone of the m -th matrix power of any reduct of \mathbf{U} .*

Proof. The claim in (1.4.1) can be checked directly by making use of Lemma 1.1 and the definition of respectability; alternately, it follows immediately from Lemma 3.2.

In (1.4.2) the necessity is obvious, while the sufficiency is a trivial consequence of (1.4.3).

To prove (1.4.3) one can apply Proposition 1.3 and the following observation. If the blocks in \mathcal{I} have at least two elements, then the operation $h_{\text{id}}^{\text{id}} \in \text{Clo } \mathbf{U}^{[m]}$ is not respected by \mathcal{I} , while if \mathcal{I} consists of singletons, but $\bigcup \mathcal{I} \neq m$, then none of the surjective unary operations in $\text{Clo } \mathbf{U}^{[m]}$ with coordinate mapping $\gamma = (0 \ 1 \ \dots \ m-1)$ is respected by \mathcal{I} . \diamond

Now we are in a position to state the classification theorem for the subclones of the clones of matrix powers of finite unary algebras.

Theorem 1.5. *Let \mathbf{U} be a finite unary algebra, and let m be a positive integer. For every subclone \mathcal{C} of $\text{Clo } \mathbf{U}^{[m]}$ one of the following two conditions holds:*

(1.5.a) *\mathbf{U} has a reduct \mathbf{U}' such that \mathcal{C} is conjugate to $\text{Clo } (\mathbf{U}')^{[m]}$ via some permutation $\pi = h^{\text{id}}[\pi_0, \dots, \pi_{m-1}]$ in $\text{Clo } \mathbf{U}^{[m]}$;*

(1.5.b) *\mathcal{C} respects a uniform partial equivalence on m distinct from the equality relation.*

Restated for algebras in a slightly weaker form Theorem 1.5 yields the following corollary.

Corollary 1.6. *If \mathbf{A} is a reduct of the m -th matrix power of a finite unary algebra \mathbf{U} ($m \geq 1$), then either*

(1.6.a) *\mathbf{A} is isomorphic to an algebra term equivalent to $(\mathbf{U}')^{[m]}$ for an appropriate reduct \mathbf{U}' of \mathbf{U} , or*

(1.6.b) \mathbf{A} respects a uniform partial equivalence on m distinct from the equality relation. \diamond

We postpone the proof of Theorem 1.5 till the last section, and discuss first some of its applications.

2. Minimal varieties generated by finite simple strongly Abelian algebras

A variety V is called *minimal* if it has exactly two subvarieties: V itself and the trivial variety. Obviously, every locally finite minimal variety is generated by a finite simple algebra having no nontrivial proper subalgebra. The variety generated by an algebra \mathbf{A} will be denoted by $V(\mathbf{A})$.

In tame congruence theory (the structure theory of finite algebras and locally finite varieties, see [5]) simple algebras are divided into 5 types. A finite simple algebra \mathbf{S} turns out to be of type **1** if and only if it is *strongly Abelian*, that is, it satisfies the following *strong term condition*: for all $n \geq 1$, for every n -ary term operation f of \mathbf{S} and for arbitrary elements $u, v, a_i, b_i, c_i \in S$ ($1 \leq i \leq n-1$),

$$f(u, a_1, \dots, a_{n-1}) = f(v, b_1, \dots, b_{n-1}) \Rightarrow f(u, c_1, \dots, c_{n-1}) = f(v, c_1, \dots, c_{n-1});$$

furthermore, \mathbf{S} is of type **2** if and only if it fails to be strongly Abelian, however, it is *Abelian*, that is, it satisfies the following *term condition*: for all $n \geq 1$, for every n -ary term operation f of \mathbf{S} and for arbitrary elements $u, v, a_i, b_i \in S$ ($1 \leq i \leq n-1$),

$$f(u, a_1, \dots, a_{n-1}) = f(u, b_1, \dots, b_{n-1}) \Leftrightarrow f(v, a_1, \dots, a_{n-1}) = f(v, b_1, \dots, b_{n-1}).$$

It is not hard to see that every strongly Abelian algebra is Abelian.

For strongly Abelian algebras the most important examples are matrix powers of unary algebras, and for Abelian (but not strongly Abelian) algebras the most important examples are affine algebras (i.e., algebras polynomially equivalent to unitary modules over rings). Clearly, subalgebras and reducts of strongly Abelian, resp. Abelian, algebras are also such. In many cases, the property of being strongly Abelian, or Abelian leads to analogous results. The following representation theorems from tame congruence theory illustrate the analogy: Every finite simple algebra of type **1** can be represented as a subalgebra of a reduct of a matrix power of some unary algebra [5, Theorem 13.3], and every finite simple algebra of type **2** can be represented as a subalgebra of a reduct of an affine algebra [5, Theorem 13.5].

Recently, while investigating the problem which finite simple algebras of type **2** generate residually small varieties, K. Kearnes, E. W. Kiss, and M. Valeriote noticed the following interesting fact:

Theorem 2.1. [7] *Every finite simple algebra of type **2** that generates a minimal variety is affine.*

Their proof is based on the techniques of tame congruence theory. In [19] we present another proof which makes use of the representation theorem mentioned above and a classification theorem for reducts of affine algebras. Now we can prove the type **1** analogue of Theorem 2.1, using an approach analogous to the one in [19]. We note that K. Kearnes, E. W. Kiss, and M. Valeriote were also able to extend their methods to get the same result [8].

Theorem 2.2. *Every finite simple algebra of type **1** that generates a minimal variety is isomorphic to an algebra term equivalent to $(2; \text{id})^{[m]}$ or $(2; 0)^{[m]}$ for some $m \geq 1$.*

For the proof we need a variant of the representation theorem for finite simple algebras of type **1**, which was used earlier in [18].

Lemma 2.3. [18, Lemma 3.3] *For arbitrary finite simple algebra \mathbf{S} of type **1**, there exist an integer $m \geq 1$ and a finite set U such that \mathbf{S} is isomorphic to a subalgebra \mathbf{S}' of a reduct \mathbf{A} of $(U; S_U \cup C_U)^{[m]}$ with S' satisfying the following conditions:*

$$(2.3.1) \quad \{s^i: (s^0, \dots, s^{m-1}) \in S'\} = U \text{ for all } 0 \leq i \leq m-1,$$

$$(2.3.2) \quad |\{(s^i, s^j): (s^0, \dots, s^{m-1}) \in S'\}| > |U| \text{ for all } 0 \leq i < j \leq m-1.$$

A quadruple $(\mathbf{A}, U, m, \mathbf{S}')$ satisfying all requirements in Lemma 2.3 will be called a *condensed representation of \mathbf{S}* .

Lemma 2.4. *Let \mathbf{S} be a finite simple algebra of type **1**. If $(\mathbf{A}, U, m, \mathbf{S}')$ is a condensed representation of \mathbf{S} , then $V(\mathbf{A}) = V(\mathbf{S})$.*

Proof. Let $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ and $h_\nu^\tau[f_0, \dots, f_{m-1}]$ be arbitrary term operations of \mathbf{A} . Suppose they agree on \mathbf{S}' , that is

$$g_i(x_{i\mu}^{i\sigma}) = f_i(x_{i\nu}^{i\tau}) \quad \text{for all } i \ (0 \leq i \leq m-1) \text{ and all } x_0, \dots, x_{m-1} \in S'.$$

For each i ($0 \leq i \leq m-1$), condition (2.3.1) for S' ensures that either $g_i = f_i$ are constants, or both g_i, f_i are permutations and $i\mu = i\nu$. In the latter case condition (2.3.2) implies that $i\sigma = i\tau$ and $g_i = f_i$. Thus the two term operations of \mathbf{A} coincide. This shows that every identity satisfied in \mathbf{S} is satisfied in \mathbf{A} as well. The converse is trivial, so the proof is complete. \diamond

Lemma 2.5. *Let \mathbf{U} be a unary algebra, let $m \geq 1$, and let \mathcal{I} be the set of blocks of a uniform partial equivalence on m . Then the mapping*

$$\varphi_{\mathcal{I}}: \text{Clo}_{\mathcal{I}} \mathbf{U}^{[m]} \rightarrow \text{Clo}(2; 0)^{[|\mathcal{I}|]}, \quad h = h_\mu^\sigma[g_0, \dots, g_{m-1}] \mapsto h_{\bar{\mu}}^{\bar{\sigma}}[(f_I)_{I \in \mathcal{I}}]$$

with

$$I\bar{\mu} = \begin{cases} i\mu \ (i \in I) & \text{if (Rsp3) holds for } h \\ \text{arbitrary} & \text{otherwise} \end{cases},$$

$$I\bar{\sigma} = \begin{cases} I\sigma & \text{if (Rsp3) holds for } h \\ \text{arbitrary} & \text{otherwise} \end{cases},$$

$$f_I = \begin{cases} \text{id} & \text{if (Rsp3) holds for } h \\ 0 & \text{otherwise} \end{cases}$$

is a clone homomorphism.

Proof. Notice that $\varphi_{\mathcal{I}}$ is well defined, because if (Rsp1) or (Rsp2) holds for $h \in \text{Clo}_{\mathcal{I}}\mathbf{U}^{[m]}$ and $I \in \mathcal{I}$, then f_I is constant, and hence $h_{\bar{\mu}}^{\bar{\sigma}}[(f_I)_{I \in \mathcal{I}}]$ is independent of the choice of $I\bar{\mu}$ and $I\bar{\sigma}$. Clearly, $\varphi_{\mathcal{I}}$ assigns to the i th n -ary projection $h_{m \rightarrow \{i\}}^{\text{id}}$ in $\text{Clo}_{\mathcal{I}}\mathbf{U}^{[m]}$ the i th n -ary projection in $\text{Clo}(2; 0)^{[|\mathcal{I}|]}$. Making use of Lemma 1.1 one can easily check that $\varphi_{\mathcal{I}}$ commutes with composition; the details are left to the reader. \diamond

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. Let \mathbf{S} be a finite simple algebra of type **1** such that the variety $V = V(\mathbf{S})$ is minimal. By Lemmas 2.3 and 2.4 V contains an algebra \mathbf{A} which is a reduct of $\mathbf{U}^{[m]}$ for some $m \geq 1$ and some finite unary algebra \mathbf{U} . Select \mathbf{A} so that $|A|$ be minimal, and fix \mathbf{U} and m . If \mathcal{I} is a uniform partial equivalence on m respected by \mathbf{A} , then the image of $\text{Clo } \mathbf{A}$ under the clone homomorphism $\varphi_{\mathcal{I}}$ in Lemma 2.5 yields a reduct of $(2; 0)^{[|\mathcal{I}|]}$ which — when made into an algebra of the same similarity type as \mathbf{A} — belongs to V . Moreover, since V is a minimal variety, $\varphi_{\mathcal{I}}$ is injective.

Combining this observation with the minimality of $|A|$ we conclude that \mathbf{A} respects no uniform partial equivalence \mathcal{I} of m distinct from the equality relation, furthermore (since \mathbf{A} obviously respects the equality relation on m), \mathbf{U} is a two-element algebra. We may assume without loss of generality that $\mathbf{U} = (2; T_2)$. Now Corollary 1.6 implies that \mathbf{A} is isomorphic to an algebra term equivalent to the m th matrix power of a reduct of $(2; T_2)$. Applying again the observation in the preceding paragraph with \mathcal{I} the equality relation on m , and making use of the injectivity of $\varphi_{\mathcal{I}}$, we see that \mathbf{A} has to be isomorphic to the m th matrix power of a reduct \mathbf{U}_0 of $(2; 0)$. Clearly, $\mathbf{U}_0 = (2; \text{id})$ or $\mathbf{U}_0 = (2; 0)$.

Since V is a minimal variety, we have $V = V(\mathbf{A}) = V(\mathbf{U}_0^{[m]})$. It is well known (cf. e.g. [20]) that an algebra belongs to $V(\mathbf{U}_0^{[m]})$ if and only if it is isomorphic to the m th matrix power of an algebra in $V(\mathbf{U}_0)$. In particular, it follows that, up to isomorphism, $\mathbf{U}_0^{[m]}$ is the only simple algebra in $V(\mathbf{U}_0^{[m]})$. Since $\mathbf{S} \in V = V(\mathbf{U}_0^{[m]})$, we conclude that $\mathbf{S} \cong \mathbf{U}_0^{[m]}$. This completes the proof. \diamond

3. Further refinement of the Primal Algebra Characterization Theorem

In [16] the following strong version of Rosenberg's Primal Algebra Characterization Theorem was proved:

Theorem 3.1. [16] *Let \mathbf{A} be a finite simple algebra having no proper subalgebra. Then one of the following conditions holds:*

- (3.1.a) \mathbf{A} is quasiprimal;
- (3.1.b) \mathbf{A} is affine with respect to an elementary Abelian p -group (p prime);
- (3.1.c) \mathbf{A} is isomorphic to a reduct of $(2; T_2)^{[m]}$ for some integer $m \geq 1$;
- (3.1.d) \mathbf{A} has a compatible k -regular relation ($k \geq 3$);
- (3.1.e) \mathbf{A} has a compatible k -ary central relation ($k \geq 2$);
- (3.1.f) \mathbf{A} has a compatible bounded partial order.

Recall that for an algebra $\mathbf{A} = (A; F)$ and a k -ary relation ρ on A , ρ is called a *compatible relation of \mathbf{A}* if ρ is a subuniverse of \mathbf{A}^k . If, for an operation f on A , ρ is a compatible relation of $(A; f)$, we also say that f *preserves* ρ . A family $T = \{\Theta_0, \dots, \Theta_{m-1}\}$ ($m \geq 1$) of equivalence relations on A is called *k -regular* if each Θ_i ($0 \leq i \leq m-1$) has exactly k blocks and $\Theta_T = \Theta_0 \cap \dots \cap \Theta_{m-1}$ has exactly k^m blocks; the relation corresponding to T is defined as follows:

$$\lambda_T = \{(a_0, \dots, a_{k-1}) \in A^k: \text{for all } i (0 \leq i \leq m-1), \\ a_0, \dots, a_{k-1} \text{ are not pairwise incongruent modulo } \Theta_i\}.$$

A relation on A is called *k -regular* if it is of the form λ_T for a k -regular family T of equivalence relations on A with $k \geq 3$. (Note that for $k = 2$, $\lambda_T = \Theta_T$ is an equivalence relation.)

We will need a description of what it means for an operation to preserve a k -regular relation.

Lemma 3.2. [12, Lemma 7.3] *Let f be an n -ary operation on a finite set A , and let $T = \{\Theta_0, \dots, \Theta_{m-1}\}$ be a k -regular family of equivalence relations on A such that λ_T is preserved by f . For every i ($0 \leq i \leq m-1$) such that the range of f meets each block of Θ_i , there exist j, l ($0 \leq j \leq m-1$, $0 \leq l \leq n-1$) such that for all $x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \in A$ we have*

$$f(x_0, \dots, x_{n-1}) \Theta_i f(y_0, \dots, y_{n-1}) \Leftrightarrow x_l \Theta_j y_l.$$

The other notions appearing in Theorem 3.1 will not be required in the arguments later on, therefore they will not be defined here. For these concepts the reader is referred to [16].

Now let \mathbf{U} be a finite unary algebra, and let m be a positive integer. Furthermore, let \mathcal{I} be a uniform partial equivalence on m , and denote by r the common size of blocks of \mathcal{I} . First we give an alternate characterization for the property that an operation $h \in \text{Clo } \mathbf{U}^{[m]}$ respects \mathcal{I} .

For $0 \leq i \leq m-1$ let Φ_i denote the kernel of the projection of U^m onto its i th coordinate, and for $I \subseteq m$ put $\Phi_I = \bigcap_{i \in I} \Phi_i$. Clearly,

$$T(\mathcal{I}) = \{\Phi_I: I \text{ is block of } \mathcal{I}\}$$

is a $|U|^r$ -regular family of equivalence relations on U^m . Put $q = |U|^r$, and consider the following q -ary relation on U^m :

$$\lambda(\mathcal{I}) = \{(a_0, \dots, a_{q-1}) \in (U^m)^q: \text{for each } I \in \mathcal{I}, \\ \text{the } r\text{-tuples } (a_0^i)_{i \in I}, \dots, (a_{q-1}^i)_{i \in I} \text{ are not pairwise distinct}\}.$$

If $q = 2$ (i.e., $|U| = 2$ and $r = 1$), then $\lambda(\mathcal{I}) = \Phi_{\cup \mathcal{I}}$, while if $q \geq 3$, then $\lambda(\mathcal{I})$ is the q -regular relation corresponding to $T(\mathcal{I})$.

Lemma 3.3. *Let \mathbf{U} be a finite unary algebra, and let m be a positive integer. For arbitrary operation $h \in \text{Clo } \mathbf{U}^{[m]}$ and for the set \mathcal{I} of blocks of any uniform partial equivalence on m the following conditions are equivalent:*

(3.3.1) h respects \mathcal{I} ,

(3.3.2) h preserves the relation $\lambda(\mathcal{I})$.

Proof. Let $h = h_\mu^\sigma[g_0, \dots, g_{m-1}] \in \text{Clo } \mathbf{U}^{[m]}$ be n -ary.

(3.3.1) \Rightarrow (3.3.2). Assume h respects \mathcal{I} , and consider arbitrary q -tuples

$$(1) \quad (x_{00}, \dots, x_{0,q-1}), \dots, (x_{n-1,0}, \dots, x_{n-1,q-1}) \in \lambda(\mathcal{I}).$$

Applying h we get the q -tuple

$$((g_0(x_{0\mu,0}^{0\sigma}), \dots, g_{m-1}(x_{(m-1)\mu,0}^{(m-1)\sigma})), \dots, (g_0(x_{0\mu,q-1}^{0\sigma}), \dots, g_{m-1}(x_{(m-1)\mu,q-1}^{(m-1)\sigma}))).$$

To prove (3.3.2) we have to check that for each $I \in \mathcal{I}$, the q r -tuples

$$(2) \quad (g_i(x_{i\mu,0}^{i\sigma}))_{i \in I}, \dots, (g_i(x_{i\mu,q-1}^{i\sigma}))_{i \in I}$$

in U^r are not pairwise distinct; or equivalently (since $q = |U|^r$), that not all r -tuples in U^r appear among the q r -tuples in (2).

Let $I \in \mathcal{I}$, and make use of the assumption that h respects \mathcal{I} . If (Rsp1) holds, say g_j ($j \in I$) is non-surjective, then our claim in the preceding paragraph is obvious, for the j th coordinates in the r -tuples in (2) belong to the range of g_j , a proper subset of U . If (Rsp2) holds, and, say $k\mu = k'\mu$ and $k\sigma = k'\sigma$ for distinct $k, k' \in I$, then our claim is again trivial, since in this case in each r -tuple in (2) the k' th coordinate arises from the k th coordinate by applying the permutation $g_{k'}g_k^{-1}$ of U . Finally, if (Rsp3) holds, then with l denoting the constant value of μ on I we see that the r -tuples in (2) arise from the r -tuples

$$(3) \quad (x_{l,0}^j)_{j \in I\sigma}, \dots, (x_{l,q-1}^j)_{j \in I\sigma}$$

by applying the coordinate-wise permutation $(y^j)_{j \in I\sigma} \mapsto (g_{j\sigma^{-1}}(y^j))_{j \in I\sigma}$ of $U^{I\sigma}$. In view of (Rsp3) $I\sigma$ belongs to \mathcal{I} , therefore by (1) the r -tuples in (3) are not pairwise distinct, and hence the same holds for the r -tuples in (2).

(3.3.2) \Rightarrow (3.3.1). Assume now that h preserves $\lambda(\mathcal{I})$. If $q \geq 3$, then $\lambda(\mathcal{I}) = \lambda_{T(\mathcal{I})}$ is a q -regular relation on U^m , hence by Lemma 3.2

(4) for each $I \in \mathcal{I}$ such that the range of h meets each block of Φ_I , there exist $J \in \mathcal{I}$ and l ($0 \leq l \leq n-1$) such that

$$(g_i(x_{i\mu,0}^{i\sigma}))_{i \in I} = (g_i(y_{i\mu,0}^{i\sigma}))_{i \in I} \Leftrightarrow (x_l^j)_{j \in J} = (y_l^j)_{j \in J}.$$

Notice that (4) holds true also in the case $q = 2$, when \mathcal{I} consists of singletons, each Φ_i has exactly two blocks, and $\lambda(\mathcal{I}) = \Phi_{\cup \mathcal{I}}$.

To prove that h respects \mathcal{I} , let us consider a block $I \in \mathcal{I}$, and assume neither (Rsp1) nor (Rsp2) are satisfied by h . Then the range of h is easily seen to meet each block of Φ_I , so it follows from (4) that $i\mu = l$ and $i\sigma \in J$ for all $i \in I$. Since h is not I -gluing and $|I| = |J|$, therefore σ restricts to a mapping $I \rightarrow J$ which is bijective. Thus (Rsp3) holds for h . \diamond

Making use of Corollary 1.6 and Lemma 3.3 we get the following refinement of Theorem 3.1:

Theorem 3.4. *Let \mathbf{A} be a finite simple algebra having no proper subalgebra. Then one of the following conditions holds:*

- (3.4.a) \mathbf{A} is quasiprimal;
- (3.4.b) \mathbf{A} is affine with respect to an elementary Abelian p -group (p prime);
- (3.4.c) \mathbf{A} is isomorphic to an algebra term equivalent $\mathbf{U}^{[m]}$ for some 2-element unary algebra \mathbf{U} and some integer $m \geq 1$;
- (3.4.d) \mathbf{A} has a compatible k -regular relation ($k \geq 3$);
- (3.4.e) \mathbf{A} has a compatible k -ary central relation ($k \geq 2$);
- (3.4.f) \mathbf{A} has a compatible bounded partial order.

Proof. Let \mathbf{A} be a finite simple algebra with no proper subalgebra. By Theorem 3.1 the only case to be considered is when (3.1.c) holds and (3.4.c) fails for \mathbf{A} . Thus \mathbf{A} is isomorphic to a reduct \mathbf{A}' of $(2; T_2)^{[m]}$, and by Corollary 1.6 \mathbf{A}' respects a uniform partial equivalence \mathcal{I} on m distinct from the equality relation. Now Lemma 3.3 implies that $\lambda(\mathcal{I})$ is a compatible relation of \mathbf{A}' . Since \mathbf{A} — and hence \mathbf{A}' — is simple, \mathcal{I} cannot consist of singletons. Thus $\lambda(\mathcal{I})$ is a q -regular compatible relation of \mathbf{A}' for some $q \geq 3$, yielding that (3.4.d) holds for \mathbf{A} . \diamond

4. Proof of Theorem 1.5

Let \mathcal{C} be a subclone of $\text{Clo } \mathbf{U}^{[m]}$ for some finite unary algebra \mathbf{U} and some integer $m \geq 1$. We select a surjective operation $h_0 = h_\alpha^\zeta[q_0, \dots, q_{m-1}]$ from \mathcal{C} such that h_0 has maximal essential arity among all surjective operations in \mathcal{C} , and keep h_0 fixed until the end of this section; say h_0 is n -ary and depends on all of its variables. The blocks of the partition of m determined by the kernel $\ker \alpha$ of the variable mapping will be denoted by I_0, \dots, I_{n-1} so that $|I_0| = \dots = |I_{l-1}| > |I_l| \geq |I_{l+1}| \geq \dots \geq |I_{n-1}|$. Permuting the variables of h_0 if necessary we may assume that $I_j \alpha = \{j\}$ for $j = 0, \dots, n-1$. Note that since h_0 is surjective, Lemma 1.2 ensures that q_0, \dots, q_{m-1} are surjective, moreover, ζ is injective on each block I_0, \dots, I_{n-1} of $\ker \alpha$.

Lemma 4.1. *For every j with $0 \leq j \leq l-1$ the set I_j is mapped by ζ bijectively onto some I_t with $0 \leq t \leq l-1$.*

Proof. It suffices to carry out the proof for $j = 0$. Clearly, the operation

$$h_\alpha^\zeta[q_0, \dots, q_{m-1}](h_\alpha^\zeta[q_0, \dots, q_{m-1}](x_0, \dots, x_{n-1}), x_n, \dots, x_{2n-2})$$

in \mathcal{C} is surjective, and by Lemma 1.1 its variable mapping is

$$i \mapsto \begin{cases} i\zeta\alpha (\leq n-1) & \text{if } i \in I_0 \\ i+n-1 & \text{if } i \in I_j (1 \leq j \leq n-1) \end{cases}.$$

If I_0 contained elements i, i' such that $i\zeta$ and $i'\zeta$ belonged to different blocks of $\ker \alpha$, then the operation above would depend on at least $n + 1$ variables, contradicting the choice of h_0 .

Hence $I_0\zeta \subseteq I_t$ for some t ($0 \leq t \leq n - 1$). Since ζ is injective on I_0 and $|I_0| \geq |I_t|$, we conclude that $|I_0| = |I_t|$, $0 \leq t \leq l - 1$, and ζ maps I_0 bijectively onto I_t . \diamond

Now we define certain sets of blocks of $\ker \alpha$ which will play a crucial role in the sequel. For an integer k with $0 \leq k \leq l - 1$, a block I_j will be called I_k -reachable (or *reachable from I_k*) if $0 \leq j \leq l - 1$ and there exists a unary operation $h_1 = h^\sigma[g_0, \dots, g_{m-1}]$ in \mathcal{C} such that h_1 is coordinate-wise $I_k\zeta$ -surjective and its coordinate mapping σ maps $I_k\zeta$ bijectively onto I_j . Note that $I_k\zeta$ itself is I_k -reachable, as shown by the unary projection h^{id} . The family of I_k -reachable blocks of $\ker \alpha$ will be denoted by \mathcal{I}_k . Clearly, \mathcal{I}_k yields a uniform partial equivalence on m .

Our aim is to prove that \mathcal{C} respects \mathcal{I}_k . First we show a weaker property.

Lemma 4.2. *Let k be an integer, $0 \leq k \leq l - 1$. For arbitrary operation h in \mathcal{C} and for arbitrary I_k -reachable block $I_j \in \mathcal{I}_k$ one of the following conditions holds:*

- (4.2.1) *h is not coordinate-wise I_j -surjective,*
- (4.2.2) *h is coordinate-wise I_j -surjective and I_j -gluing,*
- (4.2.3) *h is coordinate-wise I_j -surjective, moreover, the variable mapping of h is constant, while the coordinate mapping of h is injective on I_j .*

Proof. Let $h = h_\nu^\tau[f_0, \dots, f_{m-1}] \in \mathcal{C}$ be a t -ary operation, let $I_j \in \mathcal{I}_k$, and suppose both (4.2.1) and (4.2.2) fail for them. Select a unary operation $h_1 = h^\sigma[g_0, \dots, g_{m-1}]$ witnessing the I_k -reachability of I_j , and consider the operation

$$(5) \quad h_\alpha^\zeta[g_0, \dots, g_{m-1}](x_t, \dots, x_{t+k-1}, \\ h^\sigma[g_0, \dots, g_{m-1}](h_\nu^\tau[f_0, \dots, f_{m-1}](x_0, \dots, x_{t-1})), x_{t+k}, \dots, x_{t+n-2})$$

in \mathcal{C} . By Lemma 1.1 the variable mapping, the coordinate mapping, and the sequence of coordinate functions of this operation are

$$i \mapsto \begin{cases} s + t & \text{if } i \in I_s, s = 0, \dots, k - 1 \\ s + t - 1 & \text{if } i \in I_s, s = k + 1, \dots, n - 1, \\ i\zeta\sigma\nu & \text{if } i \in I_k \end{cases}, \quad i \mapsto \begin{cases} i\zeta & \text{if } i \in I_s \text{ with } s \neq k \\ i\zeta\sigma\tau & \text{if } i \in I_k \end{cases},$$

and

$$i \mapsto \begin{cases} q_i & \text{if } i \in I_s \text{ with } s \neq k \\ q_i g_{i\zeta} f_{i\zeta\sigma} & \text{if } i \in I_k \end{cases}.$$

Making use of the facts that for $i \in I_k$ we have $i\zeta\sigma \in I_k\zeta\sigma = I_j$, moreover, by assumption, h_1 is coordinate-wise $I_k\zeta$ -surjective, and h is coordinate-wise I_j -surjective and not I_j -gluing, one can easily check by means of Lemma 1.2 that the operation (5) is surjective. Furthermore, it has more than n essential variables unless $I_k\zeta\sigma\nu = I_j\nu$ is a singleton. So, by the choice of h_0 we conclude that ν is constant on I_j . Since h is not I_j -gluing, it follows that τ is injective on I_j . \diamond

Now we are in a position to prove

Lemma 4.3. *For every integer k with $0 \leq k \leq l - 1$, the clone \mathcal{C} respects \mathcal{I}_k .*

Proof. Let k be an integer with $0 \leq k \leq l - 1$, let $h = h_\nu^\tau[f_0, \dots, f_{m-1}] \in \mathcal{C}$ be a t -ary operation, and let $I_j \in \mathcal{I}_k$. By Lemma 4.2 the only case to be considered is when condition (4.2.3) holds for h , and it remains to show that $I_j\tau \in \mathcal{I}_k$. Renaming the variables of the operation $h_\nu^\tau[f_0, \dots, f_{m-1}]$ if necessary we may assume that $I_j\nu = \{0\}$.

Now consider the operation

$$(6) \quad h_\nu^\tau[f_0, \dots, f_{m-1}](h_\alpha^\zeta[q_0, \dots, q_{m-1}](x_0, \dots, x_{n-1}), x_n, \dots, x_{n+t-2})$$

in \mathcal{C} . For every $i \in I_j$ the variable mapping and the coordinate mapping of (6) sends i to $i\tau\alpha$ and $i\tau\zeta$, respectively, and the i th coordinate function of (6) is $f_i q_{i\tau}$, which is surjective. This shows that (6) is coordinate-wise I_j -surjective and not I_j -gluing. As for the latter property, notice that τ is injective on I_j and in view of the surjectivity of h_0 , h_0 is not $I_j\tau$ -gluing. Thus, by Lemma 4.2, the variable mapping $i \mapsto i\tau\alpha$ on I_j is constant, implying that $I_j\tau \subseteq I_s$ for some s ($0 \leq s \leq n - 1$). Since τ is injective on I_j and $|I_0| = |I_j| \geq |I_s|$, we get that $|I_j| = |I_s|$, $0 \leq s \leq l - 1$, and τ maps I_j bijectively onto I_s .

Using the same notation for the unary operation witnessing the I_k -reachability of I_j as in the proof of Lemma 4.2, let us form the (unary) operation

$$(7) \quad h^\sigma[g_0, \dots, g_{m-1}](h_\nu^\tau[f_0, \dots, f_{m-1}](x, \dots, x))$$

in \mathcal{C} . The coordinate mapping of (7) is $i \mapsto i\sigma\tau$, and for each i the i th coordinate function of (7) is $g_i f_{i\sigma}$. Clearly, the mapping $\sigma\tau$ maps $I_k\zeta$ bijectively onto $I_j\tau = I_s$ via I_j . Further, for each $i \in I_k\zeta$ the coordinate function $g_i f_{i\sigma}$ is surjective, since $i\sigma \in I_j$ and hence both factors are surjective. Thus the unary operation (7) witnesses that $I_j\tau$ is I_k -reachable. \diamond

The claim of Lemma 4.3 establishes that (1.5.b) holds for \mathcal{C} unless $\mathcal{I}_k = \{\{0\}, \dots, \{m-1\}\}$ for all k with $0 \leq k \leq l - 1$. Therefore, to complete the proof of Theorem 1.5 it suffices to consider this case.

Lemma 4.4. *If $\mathcal{I}_k = \{\{0\}, \dots, \{m-1\}\}$ for all k with $0 \leq k \leq l - 1$, then $n = l = m$, $h_0 = h_{\text{id}}^\zeta[q_0, \dots, q_{m-1}]$, and for every transformation σ of m , \mathcal{C} contains a surjective operation with coordinate mapping σ and variable mapping id .*

Proof. Suppose $\mathcal{I}_k = \{\{0\}, \dots, \{m-1\}\}$ for all k with $0 \leq k \leq l - 1$. Obviously, this implies that $\ker \alpha$ is the equality relation, whence $n = l = m$, $I_j = \{j\}$ for all j ($0 \leq j \leq m - 1$), and $h_0 = h_{\text{id}}^\zeta[q_0, \dots, q_{m-1}]$.

Let σ be an arbitrary transformation of m . For each k ($0 \leq k \leq m - 1$) select a unary operation $h^{\sigma_k}[g_{0k}, \dots, g_{m-1,k}]$ in \mathcal{C} witnessing the I_k -reachability of $\{k\sigma\}$. By definition, σ_k maps $I_k\zeta = \{k\zeta\}$ onto $\{k\sigma\}$ — that is to say, $k\zeta\sigma_k = k\sigma$ — and $g_{k\zeta,k}$ is surjective.

Consider now the operation in \mathcal{C} arising from $h_{\text{id}}^\zeta[q_0, \dots, q_{m-1}]$ by substituting the essentially unary operations $h_{m \rightarrow \{k\}}^{\sigma_k}[g_{0k}, \dots, g_{m-1,k}]$ ($0 \leq k \leq m - 1$) for its variables. It

is easy to check by Lemma 1.1 that the variable mapping, the coordinate mapping, and the sequence of coordinate functions of this composite operation are

$$i \mapsto i\zeta(m \rightarrow \{i\}) = i, \quad i \mapsto i\zeta\sigma_i = i\sigma, \quad \text{and} \quad i \mapsto q_i g_{i\zeta, i}.$$

Hence, this operation satisfies the requirements of the lemma. \diamond

Now Proposition 1.3 yields that (1.5.a) holds for \mathcal{C} . This completes the proof of Theorem 1.5.

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