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Simple Abelian algebras

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ABSTRACT

An algebra is called Abelian if all its term operations satisfy the so-called 'term condition'. By a recent result of M. Valeriote [22], a finite simple Abelian algebra has no nontrivial proper subalgebra. Using this fact we prove that every finite simple Abelian algebra whose fundamental operations are surjective is either polynomially equivalent to a simple unitary module, or term equivalent to a matrix power of a unary permutational algebra.

The term condition (see Section 1) defining Abelian algebras has played an important role in studying the representation problem for congruence lattices, and in commutator theory for congruence modular varieties (see [9], [5] for the history and references). It features also in tame congruence theory ([7]), a new structure theory for finite algebras. The name 'Abelian' is justified by the fact that a group has this property if and only if it is commutative. Typical examples of Abelian algebras are unitary modules and unary algebras. Commutator theory and tame congruence theory yield two sufficient conditions for an Abelian algebra **A** to be almost a unitary module: if **A** belongs to a congruence modular variety, or if **A** is finite and belongs to a variety satisfying a nontrivial congruence identity, then **A** is a so-called affine algebra, i.e. **A** is polynomially equivalent to a unitary module ([6], [7; 9.20 (2), 9.8, 9.18]).

The main results of this paper concern finite simple Abelian algebras. By a recent theorem of M. Valeriote [22], these algebras have no nontrivial proper subalgebras. We make no assumption on the congruence properties of the varieties the algebras generate, however, we assume that all their fundamental operations are surjective (briefly we say that the algebras are surjective). Thus, this family includes every finite simple Abelian algebra with a single nonconstant fundamental operation.

We present a complete description for finite, simple, surjective Abelian algebras: if **A** is a finite, simple, surjective Abelian algebra, then either **A** is affine, or **A** is isomorphic to an algebra term equivalent to a matrix power $(N;G)^{[m]}$ of an algebra

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$(N;G)$ where G is a permutation group acting primitively on N provided $|N| > 2$ (Corollary 4.2). Most considerations are devoted to the case when $\mathbf{A}$ has a trivial subalgebra (Theorems 2.1, 4.1), as the opposite case was settled in [18].

Simple algebras having no nontrivial proper subalgebras (briefly called strictly simple algebras) are closely related to minimal varieties. It is well known and easy to see that every locally finite minimal variety is generated by a finite strictly simple algebra. However, it is an open problem (Problem 10 in [7]), which finite strictly simple algebras generate minimal varieties. In [18] we described, up to term equivalence, all finite, strictly simple, surjective algebras without trivial subalgebras. It turned out that such an algebra generates a minimal variety if and only if it is quasiprimal. In contrast, every finite strictly simple algebra which is idempotent (that is, every singleton is a trivial subalgebra), generates a minimal variety ([17]). The structure of finite strictly simple algebras having several trivial subalgebras seems to be more complicated even if we assume that they are surjective; in fact, no complete description (up to term equivalence) can be expected for them, except in the Abelian case discussed here (cf. [20]). From the theorem above it follows that a finite, simple, surjective Abelian algebra generates a minimal variety if and only if it has a trivial subalgebra (Corollary 4.3).

1. Preliminaries

If not stated otherwise, algebras are denoted by boldface capitals and their universes by the corresponding letters in italics. Two algebras are called *term equivalent* [*polynomially equivalent*], if they have the same clone of term [polynomial] operations. The clone of term operations [the set of n-ary term operations] of an algebra $\mathbf{A}$ is denoted by $\mathrm{Clo}\,\mathbf{A}$ [resp., $\mathrm{Clo}_n\,\mathbf{A}$]. Similarly, the clone of polynomial operations [the set of n-ary polynomial operations] of $\mathbf{A}$ is denoted by $\mathrm{Pol}\,\mathbf{A}$ [resp., $\mathrm{Pol}_n\,\mathbf{A}$].

For a set N, let T_N, S_N, and C_N denote the full transformation monoid on N, the full symmetric group on N and the set of (unary) constant operations on N, respectively. It will cause no confusion if we denote the unary constant operation on N with value a simply by a. The identity mapping on N is denoted by id_N (or id if N is clear from the context). For convenience we identify every natural number n with the set $n = \{0, 1, \dots, n-1\}$.

Recall that an algebra $\mathbf{A}$ is said to be *strictly simple* if $|A| \geq 2$, $\mathbf{A}$ is simple and $\mathbf{A}$ has no nontrivial proper subalgebra. By a trivial algebra we always mean a one-element algebra. For an algebra $\mathbf{A}$, the set of all elements $u \in A$ such that $\{u\}$ is a subalgebra of $\mathbf{A}$ will be denoted by $U_{\mathbf{A}}$. An algebra $\mathbf{A}$ is called *idempotent* if $U_{\mathbf{A}} = A$, or equivalently, $\mathrm{Clo}_1\,\mathbf{A} = \{\mathrm{id}\}$.

For a set A and for $k \geq 1$, the nonvoid subsets of A^k will also be called *k-ary relations* (on A), and for an algebra $\mathbf{A}$ the universes of subalgebras of $\mathbf{A}^k$ will be called *compatible relations* of $\mathbf{A}$.

For $k \geq 1$ and for a subset $I = \{i_0, \dots, i_{l-1}\}$ of k with $i_0 < \dots < i_{l-1}$, we denote the projection mapping $A^k \to A^I$, $(x_0, \dots, x_{k-1}) \mapsto (x_{i_0}, \dots, x_{i_{l-1}})$ by pr_I.

A finite algebra $\mathbf{A}$ is called *quasiprimal* ([12], [13]) if every operation on A preserving the internal isomorphisms (i.e. isomorphisms between subalgebras) of $\mathbf{A}$ is a term operation of $\mathbf{A}$. An algebra $\mathbf{A}$ is said to be *affine with respect to an*

Abelian group $\widehat{A}$ if $\mathbf{A}$ and $\widehat{A}$ have the same universe,

$$Q_{\widehat{A}} = \{(a,b,c,d) \in A^4\colon\ a-b+c=d\}$$

is a compatible relation of $\mathbf{A}$ (or equivalently, the operations of $\mathbf{A}$ commute with $x-y+z$), and $x-y+z$ is a term operation of $\mathbf{A}$. It is well known that an algebra $\mathbf{A}$ is affine with respect to an Abelian group $\widehat{A}$ on its universe if and only if $\mathbf{A}$ is polynomially equivalent to a module ${}_R\widehat{A}$ for some subring R of $\operatorname{End}\widehat{A}$. For simple affine algebras, an explicit description, up to term equivalence, can be found in [4] (cf. also [15]).

Let $\mathbf{C} = (C;F)$ be a unary algebra and let $m \geq 1$. For arbitrary mappings $\sigma\colon m \to m$, $\mu\colon m \to n$ and $g_0, \dots, g_{m-1} \in \operatorname{Clo}_1 \mathbf{C}$ let us define an operation $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ on C^m as follows: for $x_i = (x_i^0, \dots, x_i^{m-1}) \in C^m$ $(0 \leq i \leq n-1)$,

$$h_\mu^\sigma[g_0, \dots, g_{m-1}](x_0, \dots, x_{n-1}) = (g_0(x_{0\mu}^{0\sigma}), \dots, g_{m-1}(x_{(m-1)\mu}^{(m-1)\sigma})).$$

The *m-th matrix power* of $\mathbf{C}$, denoted $\mathbf{C}^{[m]}$, is the algebra with universe C^m and with all $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ as fundamental operations. It is easy to see that $\mathbf{C}^{[m]}$ has no other term operations than its fundamental operations; that is to say, $\operatorname{Clo}\mathbf{C}^{[m]}$ consists of all operations of the form $h_\mu^\sigma[g_0, \dots, g_{m-1}]$ as above. Clearly, every term operation of $\mathbf{C}^{[m]}$ depends on at most m variables.

Following D. Hobby and R. McKenzie [7] we call an algebra $\mathbf{A}$ *Abelian* if it satisfies the so-called *term condition* (or *TC*): for all $n \geq 1$, for every n-ary term operation f of $\mathbf{A}$ and for arbitrary elements $u, v, a_i, b_i \in A$ $(1 \leq i \leq n-1)$,

$$f(u, a_1, \dots, a_{n-1}) = f(u, b_1, \dots, b_{n-1}) \;\Leftrightarrow\; f(v, a_1, \dots, a_{n-1}) = f(v, b_1, \dots, b_{n-1}).$$

Furthermore, $\mathbf{A}$ is *strongly Abelian* if it satisfies the *strong term condition* (or *TC**): for all $n \geq 1$, for every n-ary term operation f of $\mathbf{A}$ and for arbitrary elements $u, v, a_i, b_i, c_i \in A$ $(1 \leq i \leq n-1)$,

$$f(u, a_1, \dots, a_{n-1}) = f(v, b_1, \dots, b_{n-1}) \;\Rightarrow\; f(u, c_1, \dots, c_{n-1}) = f(v, c_1, \dots, c_{n-1}).$$

It is not hard to see that every strongly Abelian algebra is Abelian. By the basics of tame congruence theory ([7]), a finite simple algebra is of type **1** if and only if it is strongly Abelian, and it is of type **2** if and only if it is Abelian but not strongly Abelian. A remarkable result in tame congruence theory is that every finite simple algebra of type **2** is representable as a subalgebra of a reduct of a finite simple affine algebra ([7; Theorem 13.5]), and, analogously, every finite simple algebra of type **1** is representable as a subalgebra of a reduct of a matrix power of a finite unary algebra ([7; Theorem 13.3]).

We will use also the following result, which was proved using the techniques of tame congruence theory:

Theorem 1.1. (Valeriote [22]) *Every finite simple Abelian algebra is strictly simple.*

A k-ary relation B on A is called *totally reflexive* if it contains each k-tuple from A^k whose components are not pairwise distinct. Further, B is called *totally* *symmetric* if it is closed under permuting the components. (As a rule, "totally" is omitted if $k = 2$.) A totally reflexive, totally symmetric relation $B \subseteq A^k$ is called *central* if $B \neq A^k$ and there exists a $c \in A$ such that $(c, a_1, \dots, a_{k-1}) \in B$ for all $a_1, \dots, a_{k-1} \in A$. The set of all such elements c is called the *center* of B. Observe that every unary relation is totally reflexive and symmetric, hence the unary central relations are exactly the nonvoid proper subsets of A. For a fixed subset U of A, a central relation will be called *U-central* if U is contained in its center.

For an element $a \in A$ we set

$$X^a = (A \times \{a\}) \cup (\{a\} \times A).$$

As usual, a binary relation on A is called *irreflexive*, if none of the pairs (a, a), $a \in A$, belongs to it. For a fixed subset U of A, the binary relations of the form

$$X^u \cup Y \quad \text{with } u \in U \text{ and } Y \text{ an irreflexive binary relation on } A - U$$

will be called *U-crosses* on A.

In [19] we proved a theorem, which essentially determines the maximal possible clones for finite strictly simple algebras with trivial subalgebras not generating congruence permutable varieties:

Theorem 1.2. ([19]) *For every finite strictly simple algebra* $\mathbf{A}$ *with at least one trivial subalgebra, one of the following conditions holds:*

(a) $\mathbf{A}$ *is quasiprimal;*

(b) $\mathbf{A}$ *is affine;*

(c) $\mathbf{A}$ *has a k-ary compatible $U_{\mathbf{A}}$-central relation for some $k \geq 2$;*

(d) $\mathbf{A}$ *has a compatible bounded partial order such that every element of $U_{\mathbf{A}}$ is a bound (consequently $|U_{\mathbf{A}}| \leq 2$);*

(e) $\mathbf{A}$ *has a compatible symmetric $U_{\mathbf{A}}$-cross.*

2. A classification of simple Abelian algebras

Our aim in this section is to specialize Theorem 1.2 to Abelian algebras. In Lemmas 2.3 and 2.9 below we show that if $\mathbf{A}$ is a finite simple Abelian algebra such that (d) or (e) holds, then $\mathbf{A}$ is strongly Abelian. Since a quasiprimal algebra is never Abelian, this will prove the main result of the section:

Theorem 2.1. *For every finite simple Abelian algebra* $\mathbf{A}$ *with at least one trivial subalgebra, one of the following conditions holds:*

(a) $\mathbf{A}$ *is affine;*

(b) $\mathbf{A}$ *has a k-ary compatible $U_{\mathbf{A}}$-central relation for some $k \geq 2$;*

(c) $\mathbf{A}$ *is strongly Abelian.*

We present examples satisfying the assumptions of Theorem 2.1 and exactly one of conditions (a)–(c).

Examples 2.2. (1) The affine algebras $\mathbf{A}$ satisfying the assumptions of Theorem 2.1 are 'almost' simple modules; a precise description, up to term equivalence, is given in [4] (cf. also [15]). It is easy to see that none of (b) or (c) holds for them.

(2) Let ${}_K\widehat{A} = (A; +, K)$ be a finite vector space of dimension $d \geq 2$, and let $\mathbf{A}$ be the algebra with base set A whose operations are all nonsurjective term operations of the module ${}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A}$. Then $\mathbf{A}$ is a simple Abelian algebra which is neither affine, nor strongly Abelian. The unique trivial subalgebra of $\mathbf{A}$ is $\{0\}$, and for $k = |K|^{d-1}$, a compatible k-ary $\{0\}$-central relation is

$$\{(a_0, \ldots, a_{k-1}) \in A^k\colon a_0, \ldots, a_{k-1} \text{ are not pairwise distinct,}$$
$$\text{or one of them is } 0\}.$$

Another algebra with similar properties is $(A; \mathcal{B}_0)$, where A is an arbitrary finite set containing 0, and $\mathcal{B}_0$ is the clone consisting of all operations in Burle's clone ([2]) that preserve the singleton $\{0\}$.

(3) The matrix power $\mathbf{A} = (2; \mathrm{id})^{[m]}$ is a simple strongly Abelian algebra with exactly two trivial subalgebras, and none of conditions (a) or (b) in Theorem 2.1 holds. To see the failure of (b), make use of the fact that $\mathbf{A}$ is term equivalent to an algebra with surjective fundamental operations ([21]), and apply Lemma 4.4.

For the proof of Theorem 2.1 we recall some basic notions and facts from tame congruence theory. Let $\mathbf{A}$ be an algebra, and B a subset of A of the form $B = e(A)$ for some $e \in \mathrm{Pol}_1\,\mathbf{A}$ with $e^2 = e$. Since e acts identically on B, for every polynomial operation $g \in \mathrm{Pol}\,\mathbf{A}$, if g can be restricted to B (that is if $g(B, \ldots, B) \subseteq B$), then its restriction $g|_B$ to B coincides with the restriction $eg|_B$ of the polynomial operation eg to B. The *induced algebra* of $\mathbf{A}$ on B is defined as follows:

$$\mathbf{A}|_B = (B;\ \{eg|_B\colon\ g \in \mathrm{Pol}\,\mathbf{A}\}).$$

For $B \subseteq A$ and for a k-ary relation ρ on A we set $\rho|_B = \rho \cap B^k$.

Now assume $\mathbf{A}$ is a finite simple algebra. A set $N \subseteq A$ is called a *minimal set* for $\mathbf{A}$ if N is of the form $N = f(A)$ for some nonconstant unary polynomial operation $f \in \mathrm{Pol}_1\,\mathbf{A}$, and it is minimal (with respect to inclusion) among the sets of this form. It is shown in the theory (cf. [7; 2.10]) that for a minimal set N, there always exists a unary polynomial $e \in \mathrm{Pol}_1\,\mathbf{A}$ with $e^2 = e$ such that $N = e(A)$. Furthermore, $\mathbf{A}$ is strongly Abelian (or of type **1**) if and only if $\mathbf{A}|_N$ is essentially unary, and $\mathbf{A}$ is Abelian but not strongly Abelian (or of type **2**) if and only if $\mathbf{A}|_N$ is polynomially equivalent to a vector space.

Lemma 2.3. *Let $\mathbf{A}$ be a finite simple Abelian algebra. If $\mathbf{A}$ has a compatible bounded partial order, then $\mathbf{A}$ is strongly Abelian.*

Proof. Let $\leq$ be a compatible partial order of $\mathbf{A}$ with least element 0 and greatest element 1, and let N be a minimal set of $\mathbf{A}$ with $N = e(A)$, $e \in \mathrm{Pol}_1\,\mathbf{A}$, $e^2 = e$. Since e is monotone with respect to $\leq$ and acts identically on N, therefore $e(\leq) = \leq|_N$. Further, this relation is a compatible partial order of $\mathbf{A}|_N$ with least element $e(0)$ and greatest element $e(1)$. Since an algebra polynomially equivalent to a vector space cannot have a compatible bounded partial order (e.g. because the Mal'tsev polynomial $x - y + z$ does not preserve any bounded partial order), $\mathbf{A}$ must be strongly Abelian.

As the induced minimal algebra is formed by using all polynomial operations, it is crucial in the argument in the proof of Lemma 2.3 that partial orders are reflexive. For handling crosses we use a similar construction with term operations instead of polynomial operations. This idea was applied earlier by C. Bergman and R. McKenzie [1].

For an algebra $\mathbf{A}$ and a subset B of A of the form $B = e(A)$ for some $e \in \mathrm{Clo}_1 \mathbf{A}$ with $e^2 = e$, we introduce the *induced term algebra* of $\mathbf{A}$ on B as follows:

$$\mathbf{A}||_B = (B;\ \{eg|_B\colon\ g \in \mathrm{Clo}\,\mathbf{A}\}).$$

Lemma 2.4. *Let $\mathbf{A}$ be an algebra, and let $e^2 = e \in \mathrm{Clo}_1 \mathbf{A}$, $B = e(A)$, $|B| > 1$.*

(i) *If $\mathbf{A}$ is Abelian, then so is $\mathbf{A}||_B$.*

(ii) *If ρ is a compatible relation of $\mathbf{A}$, then $e(\rho) = \rho|_B$ is a compatible relation of $\mathbf{A}||_B$.*

(iii) *If $\mathbf{A}$ is generated by B, then $\mathbf{A}||_B$ is polynomially equivalent to $\mathbf{A}|_B$.*

(iv) *If $\mathbf{A}$ is strictly simple, then so is $\mathbf{A}||_B$.*

Proof. (i) is trivial.

(ii) The equality in the claim follows from the facts that ρ is preserved by e, and that e acts identically on B. Obviously, $\rho|_B$ is a compatible relation of $\mathbf{A}||_B$.

(iii) The proof of this claim is implicit in [1]. Clearly, $\mathrm{Pol}\,\mathbf{A}||_B \subseteq \mathrm{Pol}\,\mathbf{A}|_B$. To prove the reverse inclusion, let $eg|_B$ ($g \in \mathrm{Pol}_k \mathbf{A}$, $k \geq 1$) be an operation of $\mathbf{A}|_B$, say

$$g(x_0, \ldots, x_{k-1}) = f(x_0, \ldots, x_{k-1}, a_0, \ldots, a_{l-1})$$

for some $l \geq 0$, $f \in \mathrm{Clo}_{k+l} \mathbf{A}$ and $a_0, \ldots, a_{l-1} \in A$. Since B generates $\mathbf{A}$, there exist an integer t, some elements $b_0, \ldots, b_{t-1} \in B$ and operations $h_0, \ldots, h_{l-1} \in \mathrm{Clo}_t \mathbf{A}$ such that $a_i = h_i(b_0, \ldots, b_{t-1})$ for $0 \leq i \leq l-1$. Thus, for the term operation

$$\begin{aligned}&\bar{f}(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{t-1})\\ &\qquad = f(x_0, \ldots, x_{k-1}, h_0(y_0, \ldots, y_{t-1}), \ldots, h_{l-1}(y_0, \ldots, y_{t-1}))\end{aligned}$$

of $\mathbf{A}$,

$$g(x_0, \ldots, x_{k-1}) = \bar{f}(x_0, \ldots, x_{k-1}, b_0, \ldots, b_{t-1}).$$

Hence $eg|_B$ is a polynomial operation of $\mathbf{A}||_B$.

(iv) Assume $\mathbf{A}$ is strictly simple. Let $\mathbf{C}$ be a nontrivial subalgebra of $\mathbf{A}||_B$. Since C generates $\mathbf{A}$, therefore every element $a \in A$ is contained in $f(C, \ldots, C)$ for some $f \in \mathrm{Clo}\,\mathbf{A}$. Thus $B = e(A)$ is in the subalgebra of $\mathbf{A}||_B$ generated by C, implying $C = B$. Thus $\mathbf{A}||_B$ has no nontrivial proper subalgebra.

In view of (iii), to show that $\mathbf{A}||_B$ is simple, we can consider $\mathbf{A}|_B$ instead of $\mathbf{A}||_B$. By an observation of P. P. Pálfy and P. Pudlák [11] (see also [7; 2.3]), for $B = e(A)$ with $e^2 = e \in \mathrm{Pol}_1 \mathbf{A}$ and $|B| > 1$, "restriction to B" is a surjective homomorphism of the congruence lattice of $\mathbf{A}$ onto the congruence lattice of $\mathbf{A}|_B$. This completes the proof.

Lemma 2.5. *Let $\mathbf{A}$ be a finite algebra, and let $e^2 = e \in \mathrm{Clo}_1 \mathbf{A}$, $B = e(A)$, $|B| > 1$. If B is minimal (with respect to inclusion) among all subsets of A of this form, then the monoid $T = \mathrm{Clo}_1 \mathbf{A}||_B$ satisfies the following condition:*

(2.5) *every element $h \in T$ with $h^2 = h$ is either the identity or constant.*

Proof. Let $h \in T$ be such that $h^2 = h$. By the definition of $\mathbf{A}||_B$, $h = g|_B$ for some $g \in \mathrm{Clo}_1\,\mathbf{A}$ with $eg = g$. Let $C = h(B)(\subseteq B)$. Since $h^2 = h$, therefore $h|_C = \mathrm{id}_C$, implying $g^k|_C = \mathrm{id}_C$ for all $k \geq 1$. However, $g^2(A) \subseteq g(B) = h(B) = C$, hence the range of each power g^k $(k \geq 2)$ of g is C. By the finiteness some power g^k $(k \geq 2)$ satisfies $(g^k)^2 = g^k$, so the minimality of B yields that $C = B$ or $|C| = 1$. Accordingly, $h = \mathrm{id}_B$ or h is constant.

Lemma 2.6. *Let $\mathbf{A}$ be a finite algebra such that every proper subalgebra of $\mathbf{A}$ is trivial, and let $e^2 = e \in \mathrm{Clo}_1\,\mathbf{A}$, $B = e(A)$, $|B| > 1$. Then $U_{\mathbf{A}} \subseteq B$, and the following conditions hold for $U = U_{\mathbf{A}}$ and the monoid $T = \mathrm{Clo}_1\,\mathbf{A}||_B$:*

(2.6) *$f(u) = u$ for all $f \in T$ and $u \in U$, and*

(2.6)$'$ *$\{f(b)\colon\ f \in T\} = B$ for all $b \in B - U$.*

Consequently, $U_{\mathbf{A}||_B} = U_{\mathbf{A}}$.

Proof. We write U for $U_{\mathbf{A}}$. Since $g(u) = u$ for all $g \in \mathrm{Clo}_1\,\mathbf{A}$ and $u \in U$, therefore $U \subseteq B$ and (2.6) is obvious. For $b \in B - U$, we have $\{g(b)\colon\ g \in \mathrm{Clo}_1\,\mathbf{A}\} = A$, as b generates $\mathbf{A}$. Hence (2.6)$'$ follows.

Lemma 2.7. *Let B be a finite set, $|B| > 1$, let U be a subset of B, and let T be a transformation monoid on B satisfying conditions (2.5), (2.6), and (2.6)$'$.*

(i) *If $|U| > 1$, then $U = B$ and $T = \{\mathrm{id}\}$.*

(ii) *If $|U| = 1$, say $U = \{0\}$, then $\{0\} \subset T \subseteq S_B \cup \{0\}$.*

Proof. (i) Let $|U| > 1$. By (2.5) and (2.6) every element $e \in T$ with $e^2 = e$ is the identity. However, by the finiteness of B, each $f \in T$ has some power $e = f^k$ with $e^2 = e$, implying that f is a permutation. Now by (2.6) $f(b) \in B - U$ for all $b \in B - U$, whence (2.6)$'$ yields that $U = B$. Thus by (2.6) $T = \{\mathrm{id}\}$.

(ii) Letting $U = \{0\}$ we have $f(0) = 0$ for all $f \in T$. Thus T contains at most one constant, namely 0. Suppose T contains a transformation f which is neither a permutation nor the constant 0. Let $b \in B$ be such that $f(b) \neq 0$. In view of (2.6)$'$ there exists an $h \in T$ such that $hf(b) = b$. Thus $hf \in T$ is not a permutation, however, $hf(0) = 0$ and $hf(b) = b$ $(b \neq 0)$. Some power $(hf)^i$ of hf satisfies $((hf)^i)^2 = (hf)^i$, however, $(hf)^i$ is neither constant nor the identity. This contradiction to (2.5) shows that $T \subseteq S_B \cup \{0\}$. The inclusion $T \subseteq S_B$ cannot hold, since then we would get $U = B$, $T = \{\mathrm{id}\}$ as in case (i), contradicting $|B| > 1 = |U|$. Thus $0 \in T$.

Lemma 2.8. *Let $\mathbf{B}$ be a finite Abelian algebra with an element $0 \in B$ such that the constant 0 is a unary term operation of $\mathbf{B}$ and $(\mathrm{Clo}_1\,\mathbf{B}) - \{0\}$ is a permutation group. Then $\mathbf{B}$ is either essentially unary or term equivalent to a vector space.*

Proof. We denote the permutation group $(\mathrm{Clo}_1\,\mathbf{B}) - \{0\}$ by G. First we show that for $k \geq 1$,

(2.8) for every $f \in \mathrm{Clo}_k\,\mathbf{B}$ depending on its first variable and for arbitrary elements $b_1, \ldots, b_{k-1} \in B$,

$$f(x, b_1, \ldots, b_{k-1}) \in S_B.$$

By TC, for any elements $a, a' \in B$ and $b_1, \ldots, b_{k-1} \in B$,

$$f(a, 0, \ldots, 0) = f(a', 0, \ldots, 0) \quad \Leftrightarrow \quad f(a, b_1, \ldots, b_{k-1}) = f(a', b_1, \ldots, b_{k-1}).$$

Since f depends on its first variable, we conclude that $f(x, 0, \ldots, 0)$ is not constant, whence $f(x, 0, \ldots, 0) \in G$. By a repeated application of this equivalence we get that $f(x, b_1, \ldots, b_{k-1}) \in S_B$.

Now by symmetry it follows that every unary polynomial operation of $\mathbf{B}$ is either constant or a permutation. If $|B| > 2$, then P. P. Pálfy's theorem [10] yields that $\mathbf{B}$ is essentially unary or polynomially equivalent to a vector space. If $|B| = 2$, then we can get the same conclusion from E. L. Post's description [14] of the two-element algebras (up to term equivalence) and the assumption that $\mathbf{B}$ is Abelian. Finally, if $\mathbf{B}$ is polynomially equivalent to a vector space, then using that $\{0\}$ is a subalgebra of $\mathbf{B}$, we can get that $\mathbf{B}$ is in fact term equivalent to a vector space. (For a direct proof of the conclusion from (2.8), cf. [15].)

Lemma 2.9. *Let $\mathbf{A}$ be a finite simple Abelian algebra having at least one trivial subalgebra. If $\mathbf{A}$ has a compatible symmetric $U_{\mathbf{A}}$-cross, then $\mathbf{A}$ is strongly Abelian.*

Proof. For convenience, we write U instead of $U_{\mathbf{A}}$. Let e be a unary term operation of $\mathbf{A}$ such that $e^2 = e$, $|e(A)| > 1$, and $e(A)$ is minimal among the subsets of A of this form. Set $B = e(A)$, and form the induced term algebra $\mathbf{B} = \mathbf{A}||_B$. By Theorem 1.1 and Lemma 2.4 $\mathbf{B}$ is (strictly) simple and Abelian. Let $0 \in U$ and let ρ be a compatible symmetric U-cross of $\mathbf{A}$ with $X^0 \subseteq \rho$. Since $e(0) = 0$, we have $0 \in B$, implying by Lemma 2.4 (ii) that $\rho|_B$ is a compatible symmetric U-cross of $\mathbf{B}$. (Note that by Lemma 2.6 $U_{\mathbf{B}} = U$.) Applying Lemmas 2.5, 2.6, and 2.7 for the monoid $T = \mathrm{Clo}_1\,\mathbf{B}$ of all unary term operations of $\mathbf{B}$, we see that the following two cases are to be considered:

Case I. $U = B$ and $T = \{\mathrm{id}\}$.

Case II. $U = \{0\}$ and $\{0\} \subset T \subseteq S_B \cup \{0\}$.

In Case I $\mathbf{B}$ is an idempotent algebra. Taking into account the description [16] of finite, idempotent, strictly simple algebras (up to term equivalence), and the fact that $\mathbf{B}$ is Abelian, we see that there are only two possibilities:

(a) $\mathbf{B}$ is affine, or

(b) $\mathbf{B}$ is a two-element essentially unary algebra.

It is easy to verify that an affine algebra cannot have a compatible U-cross (again, the Mal'tsev operation $x - y + z$ does not preserve any U-crosses). Therefore (a) is impossible. Hence by (b) and Lemma 2.4 (iii) the induced algebra $\mathbf{A}|_B$ is essentially unary. Clearly, B is a minimal set, so $\mathbf{A}$ is strongly Abelian.

In Case II $\mathrm{Clo}_1\,\mathbf{B} = G \cup \{0\}$ for some permutation group $G \subseteq S_B$ such that $g(0) = 0$ for all $g \in G$. By Lemma 2.8 and by the simplicity of $\mathbf{B}$,

(a)$'$ $\mathbf{B}$ is term equivalent to a one-dimensional vector space (and hence is affine), or

(b) $\mathbf{B}$ is a two-element essentially unary algebra.

Hence we can conclude the proof as in Case I.

3. Simple, surjective, strongly Abelian algebras

An algebra $\mathbf{A}$ will be called *surjective* if all its fundamental operations are surjective. A permutation group G acting on a set N is said to be *primitive*, if the unary algebra $(A; G)$ is simple and $|G| > 1$ (if $|N| = 2$). The main result of this section is a strong version of the representation theorem for finite simple algebras of type **1** for the case when the algebra is surjective.

Theorem 3.1. *Every finite simple surjective algebra of type* **1** *is isomorphic to an algebra term equivalent to* $(N;G)^{[m]}$ *for some finite set* N *(*$|N| \geq 2$*), some integer* $m \geq 1$*, and for some permutation group* G *on* N *such that either* G *is primitive or* $|N| = 2$ *and* $|G| = 1$*.*

The special case of Theorem 3.1 when the algebra has a single nonconstant fundamental operation is a consequence of a result of R. McKenzie [9].

In [18; Theorem 2.2] Theorem 3.1 was proved for simple surjective algebras of type **1** that are reducts of $(N;S_N)^{[m]}$ for some finite set N ($|N| \geq 2$). In the sequel we will need a slightly stronger version of this result, which is implicit in the proof given in [18]. For a reduct $\mathbf{U}$ of $(N;S_N)^{[m]}$, set

$$T_{\mathbf{U}} = \{\sigma \in T_m\colon\ h^{\sigma}_{\mu}[g_0,\ldots,g_{m-1}] \in \operatorname{Clo}\mathbf{U} \text{ for some } \mu \text{ and } g_0,\ldots,g_{m-1}\}.$$

It is easy to check that $T_{\mathbf{U}}$ is a submonoid of T_m. Recall that a submonoid T of T_m is called *transitive* if the unary algebra $(m;T)$ has no proper subalgebras.

Theorem 3.2. (cf. [18; Theorem 2.2]) *Let* $\mathbf{A}$ *be a surjective algebra that is isomorphic to a reduct* $\mathbf{U}$ *of* $(N;S_N)^{[m]}$ *for some finite set* N ($|N| \geq 2$) *and for some* $m \geq 1$ *such that* $T_{\mathbf{U}}$ *is a transitive submonoid of* T_m*. If* m *is chosen minimal with respect to the existence of such an isomorphism, then* $\mathbf{U}$ *(*$\cong \mathbf{A}$*) is term equivalent to* $(N;G)^{[m]}$ *for some subgroup* G *of* S_N*.*

In [18] $\mathbf{A}$ was assumed to be simple, however, one can check that the proof works also if simplicity is replaced by the weaker condition (see [18; Lemma 2.6]) that $T_{\mathbf{U}}$ is a transitive submonoid of T_m. Of course, in this case $(N;G)^{[m]}$ is not necessarily simple, so we cannot conclude that G is primitive unless $|N| = 2$, $|G| = 1$.

We start the proof of Theorem 3.1 with a variant of the representation theorem for finite simple algebras of type **1** ([7; 13.3]).

Lemma 3.3. *For arbitrary finite simple surjective algebra* $\mathbf{A}$ *of type* **1***, there exist an integer* $m \geq 1$ *and a finite set* N *such that* $\mathbf{A}$ *is isomorphic to a subalgebra* $\mathbf{W}$ *of a surjective reduct* $\mathbf{U} = (N^m;\ h^{\sigma}_{\mu}[g_0,\ldots,g_{m-1}],\ldots)$ *of* $(N;S_N)^{[m]}$ *with* W *satisfying the following conditions:*

(i) $\operatorname{pr}_{\{i\}} W = N$ *for all* $0 \leq i \leq m-1$,

(ii) $|\operatorname{pr}_{\{i,j\}} W| > |N|$ *for all* $0 \leq i < j \leq m-1$.

Proof. The proof is a modification of the proof of [7; 13.3]. Let N be a minimal set in $\mathbf{A}$, and e a unary polynomial of $\mathbf{A}$ with $e^2 = e$ and $e(A) = N$. Let $F = \{f_0 = e, f_1,\ldots,f_{k-1}\}$ be the family of all unary polynomial operations of $\mathbf{A}$ with range N. Let us define a relation $\approx$ on F as follows: $f_i \approx f_j$ if and only if there exists an $h \in S_N$ such that $f_j = hf_i$. Clearly, $\approx$ is an equivalence relation. We can assume without loss of generality that $\{f_0,\ldots,f_{m-1}\}$ contains exactly one element from each block of $\approx$. By the basics of tame congruence theory ([7; 2.8.4]), for any distinct elements $x, y \in A$ there exists an $f_i \in F$ such that $f_i(x) \neq f_i(y)$. Clearly, such an i with $0 \leq i \leq m-1$ also exists. Thus the assignment

$$x \mapsto (f_0(x),\ldots,f_{m-1}(x)) \quad (x \in A)$$

defines a bijective mapping of A onto a subset W of N^m having properties (i) and (ii). Let us denote this mapping by φ.

Consider now any, say n-ary, fundamental operation g of $\mathbf{A}$. Since $\mathbf{A}$ is strongly Abelian, by tame congruence theory ([7; Claim (3) in 5.6]), the polynomial operations $f_i g$ $(0 \le i \le m-1)$ of $\mathbf{A}$ depend on at most one variable. Since g is surjective, each $f_i g$ maps onto N. Thus there exist mappings $\sigma: m \to m$, $\mu: m \to n$ and permutations $g_0 \dots, g_{m-1} \in S_N$ such that

$$f_i g(x_0, \dots, x_{n-1}) = g_i(f_{i\sigma}(x_{i\mu})) \quad \text{for all} \quad 0 \le i \le m-1.$$

Hence for arbitrary elements $(f_0(x_j), \dots, f_{m-1}(x_j)) \in W$ $(x_j \in A,\ 0 \le j \le n-1)$ we have

$$\begin{aligned} & g((f_0(x_0), \dots, f_{m-1}(x_0))\varphi^{-1}, \dots, (f_0(x_{n-1}), \dots, f_{m-1}(x_{n-1}))\varphi^{-1}) \\ &\qquad = g(x_0, \dots, x_{n-1}) = (g(x_0, \dots, x_{n-1})\varphi)\varphi^{-1} \\ &\qquad = (g_0(f_{0\sigma}(x_{0\mu})), \dots, g_{m-1}(f_{(m-1)\sigma}(x_{(m-1)\mu})))\varphi^{-1} \\ &= h^\sigma_\mu[g_0, \dots, g_{m-1}]((f_0(x_0), \dots, f_{m-1}(x_0)), \dots, (f_0(x_{n-1}), \dots, f_{m-1}(x_{n-1})))\varphi^{-1}. \end{aligned}$$

This shows that if we make correspond to every fundamental operation g of $\mathbf{A}$ the operation $h^\sigma_\mu[g_0, \dots, g_{m-1}]$ with σ, μ and $g_0, \dots, g_{m-1}$ as described above, then φ is an isomorphism between $\mathbf{A} = (A; g, \dots)$ and the subalgebra $\mathbf{W} = (W; h^\sigma_\mu[g_0, \dots, g_{m-1}], \dots)$ of the reduct $\mathbf{U} = (N^m;\ h^\sigma_\mu[g_0, \dots, g_{m-1}], \dots)$ of $(N; S_N)^{[m]}$.

It remains to show that $\mathbf{U}$ is surjective. Consider a fundamental operation $h^\sigma_\mu[g_0, \dots, g_{m-1}]$ of $\mathbf{U}$. Its range is

$$R = \{(g_0(x^{0\sigma}_{0\mu}), \dots, g_{m-1}(x^{(m-1)\sigma}_{(m-1)\mu})):\ x_i = (x^0_i, \dots, x^{m-1}_i) \in N^m,\ 0 \le i \le n-1\}.$$

Thus $h^\sigma_\mu[g_0, \dots, g_{m-1}]$ is surjective unless there exist indices $0 \le i < j \le m-1$ such that $i\mu = j\mu$ and $i\sigma = j\sigma$. In the latter case $\mathrm{pr}_{\{i,j\}} R$ is a permutation of N (considered as a binary relation), hence by (ii) $R \cap W \subset W$, contradicting the surjectivity of the fundamental operations of $\mathbf{W}$ $(\cong \mathbf{A})$.

Lemma 3.4. *Let $\mathbf{A}$ be a finite simple surjective algebra which is isomorphic to a subalgebra $\mathbf{W}$ of a surjective reduct $\mathbf{U} = (N^m;\ h^\sigma_\mu[g_0, \dots, g_{m-1}], \dots)$ of $(N; S_N)^{[m]}$ for some finite set N ($|N| \ge 2$) and some integer $m \ge 2$. If m is chosen minimal with respect to the existence of such an isomorphism, then $T_{\mathbf{U}}$ is a transitive submonoid of T_m.*

Proof. Let I be a nonvoid subset of m closed under all transformations in $T_{\mathbf{U}}$. Consider an arbitrary fundamental operation $h^\sigma_\mu[g_0, \dots, g_{m-1}]$ of $\mathbf{U}$, say it is n-ary. Clearly, σ and μ can be restricted to I to yield mappings $\sigma|_I: I \to I$ and $\mu|_I: I \to n$. We show that the projection mapping $\mathrm{pr}_I: N^m \to N^I$ is a homomorphism of $\mathbf{U} = (N^m; h^\sigma_\mu[g_0, \dots, g_{m-1}], \dots)$ onto a reduct $\mathbf{U}' = (N^I; h^{\sigma|_I}_{\mu|_I}[g_0, \dots, g_{m-1}], \dots)$ of $(N; S_N)^{[|I|]}$. For convenience, we carry out the computation for $I = k$ $(0 < k \le m-1)$: for arbitrary elements $x_i = (x^0_i, \dots, x^{m-1}_i) \in N^m$ $(0 \le i \le n-1)$,

$$\begin{aligned} \mathrm{pr}_k\,(h^\sigma_\mu[g_0, \dots, g_{m-1}](x_0, \dots, x_{n-1})) &= \mathrm{pr}_k\,((g_0(x^{0\sigma}_{0\mu}), \dots, g_{m-1}(x^{(m-1)\sigma}_{(m-1)\mu}))) \\ &= (g_0(x^{0\sigma}_{0\mu}), \dots, g_{k-1}(x^{(k-1)\sigma}_{(k-1)\mu})) \\ &= h^{\sigma|_k}_{\mu|_k}[g_0, \dots, g_{m-1}](\mathrm{pr}_k\, x_0, \dots, \mathrm{pr}_k\, x_{n-1}). \end{aligned}$$

Clearly, pr_I is onto N^I, hence $\mathbf{U}'$ is surjective. Composing pr_I with the fixed embedding of $\mathbf{A}$ into $\mathbf{U}$, we get a homomorphism $\mathbf{A} \to \mathbf{U}'$.

Suppose the image of $\mathbf{A}$ under this homomorphism is a one-element algebra. Then $\mathrm{pr}_{\{i\}} W$ is a one-element set for all $i \in I$. Since $\mathbf{W}$ is a surjective subalgebra of $\mathbf{U}$, every fundamental operation $h^\sigma_\mu[g_0, \ldots, g_{m-1}]$ of $\mathbf{U}$ maps onto W when restricted to W, implying that $m - I$ is closed under σ. Now it is easy to check that $m - I$ is also closed under all transformations in $T_{\mathbf{U}}$. Replacing I with $m - I$ in the previous paragraph and noting that in this case pr_{m-I} is one-to-one, we get that $\mathbf{A}$ is isomorphic to a subalgebra of a surjective reduct of $(N; S_N)^{[m-|I|]}$, contradicting the minimality of m.

Hence the image of $\mathbf{A}$ under the homomorphism $\mathbf{A} \to \mathbf{U}'$ has at least two elements. By the simplicity of $\mathbf{A}$ it follows that $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{U}'$. Thus the minimality of m yields $I = m$, proving that $T_{\mathbf{U}}$ is transitive.

Proof of Theorem 3.1. Let $\mathbf{A}$ be a finite simple surjective algebra of type $\mathbf{1}$, and using Lemma 3.3 fix a representation of $\mathbf{A}$ as an isomorphic copy of a subalgebra $\mathbf{W}$ of a surjective reduct $\mathbf{U} = (N^m;\ h^\sigma_\mu[g_0, \ldots, g_{m-1}], \ldots)$ of $(N; S_N)^{[m]}$ for some finite set N ($|N| \geq 2$) and some integer $m \geq 1$. Assume this representation is chosen so that m be minimal. By Lemma 3.4 $T_{\mathbf{U}}$ is a transitive submonoid of T_m, so the assumptions of Theorem 3.2 (including the minimality of m) hold for $\mathbf{U}$. Hence $\mathbf{U}$ is term equivalent to $(N; G)^{[m]}$ for some subgroup G of S_N. It is well known (cf. [21]) and easy to check that every subalgebra of $(N; G)^{[m]}$ is of the form $(N'; G')^{[m]}$ with $G' = G|_{N'}$ for some subset N' of N. Thus $\mathbf{A}$ is isomorphic to an algebra term equivalent to $(N'; G')^{[m]}$ for some finite set N' and some permutation group G' on N'. Since $\mathbf{A}$ is simple, $(N'; G')$ must be simple, yielding that G' acts primitively on N' unless $|N'| = 2$, $|G'| = 1$.

4. Simple, surjective, Abelian algebras

In this section we study how Theorem 2.1 specializes to surjective algebras. Our aim is to prove the following theorem.

Theorem 4.1. *For every finite, simple, surjective Abelian algebra* $\mathbf{A}$ *having a trivial subalgebra, one of the following conditions holds:*

(a) $\mathbf{A}$ *is affine;*

(c)′ $\mathbf{A}$ *is isomorphic to an algebra term equivalent to* $(2; \mathrm{id})^{[m]}$ *for some integer* $m \geq 1$.

From [18; Theorem 3.4] and from Theorem 1.1 an analogous result follows for finite, simple, surjective Abelian algebras with no trivial subalgebras. Thus we have

Corollary 4.2. *Every finite, simple, surjective Abelian algebra is either affine or isomorphic to an algebra term equivalent to* $(N; G)^{[m]}$ *for some finite set* N ($|N| \geq 2$)*, some integer* $m \geq 1$*, and for some permutation group* G *on* N *such that either* G *is primitive or* $|N| = 2$ *and* $|G| = 1$.

Combining this with some well-known results on simple affine algebras ([8], [3]) and the matrix powers of finite unary algebras ([21]), we get

Corollary 4.3. *Let* $\mathbf{A}$ *be a finite, simple, surjective Abelian algebra. The variety* $V(\mathbf{A})$ *generated by* $\mathbf{A}$ *is minimal if and only if* $\mathbf{A}$ *has a trivial subalgebra;* $V(\mathbf{A})$ *is finitely based provided it is of finite type.*

For the proof of Theorem 4.1 we have to eliminate case (b) in Theorem 2.1 provided $\mathbf{A}$ is surjective.

Lemma 4.4. *Let* $\mathbf{A}$ *be a finite surjective algebra. If* B *is an* n*-ary compatible relation of* $\mathbf{A}$ $(n \geq 1)$*, then for arbitrary* k $(1 \leq k \leq n-1)$*,*

$$(B)_k = \{(x_0, \ldots, x_{k-1}) \in A^k \colon\ (x_0, \ldots, x_{n-1}) \in B \text{ for all } x_k, \ldots, x_{n-1} \in A\}$$

is a k*-ary compatible relation of* $\mathbf{A}$ *provided it is not empty.*

In particular, if B *is a compatible central relation of* $\mathbf{A}$*, then the center* $(B)_1$ *of* B *is a proper subalgebra of* $\mathbf{A}$*.*

The proof is straightforward.

We show that the center of a compatible $U_{\mathbf{A}}$-central relation of an algebra $\mathbf{A}$ of type $\mathbf{2}$ satisfying the assumptions of Theorem 4.1 has more than one element even if $|U_{\mathbf{A}}| = 1$. The following claim is a slight modification of the representation theorem for finite simple algebras of type $\mathbf{2}$ for the case when the algebra has a trivial subalgebra. It follows immediately from the proof given in [7; 13.5].

Lemma 4.5. *If* $\mathbf{A}$ *is a finite simple algebra of type* $\mathbf{2}$ *with a trivial subalgebra* $\{0\}$*, then there exist a finite field* K *and a finite vector space* ${}_K\widehat{V} = (V; +, 0, K)$ *such that* $\mathbf{A}$ *is a subalgebra of a reduct* $\mathbf{V}$ *of the module* ${}_{(\mathrm{End}\,{}_K\widehat{V})}\widehat{V}$*, and* ${}_K\widehat{V}$ *is spanned by* A *(as a vector space).*

The *essential arity* of an operation f is the number of variables of f on which f depends.

Lemma 4.6. *If* $\mathbf{A}$ *is a finite, simple, surjective algebra of type* $\mathbf{2}$ *having a trivial subalgebra, then there is no bound on the essential arities of surjective term operations of* $\mathbf{A}$*.*

Proof. Since $\mathbf{A}$ is surjective, every term operation of $\mathbf{A}$ arises from a surjective term operation by identification of variables. Hence it suffices to verify that there is no bound on the essential arities of (arbitrary) term operations of $\mathbf{A}$.

As we have seen in the proof of Lemma 2.9, the algebra $\mathbf{B} = \mathbf{A}||_B$ with B as described there is either affine or a two-element essentially unary algebra. Since $\mathbf{A}$ is of type $\mathbf{2}$, we have the first case. Thus there is no bound on the essential arities of term operations of $\mathbf{A}||_B$, which implies the same property for $\mathbf{A}$.

Lemma 4.7. *If* $\mathbf{A}$ *is a finite, simple, surjective algebra of type* $\mathbf{2}$ *having a trivial subalgebra* $\{0\}$*, then the center of every, at least binary, compatible* $\{0\}$*-central relation of* $\mathbf{A}$ *has more than one element.*

Proof. Consider a representation of $\mathbf{A}$ described in Lemma 4.5, and using Lemma 4.6, take a surjective m-ary term operation h of $\mathbf{A}$ such that h depends on all of its variables and $m > |A|$. Denote the corresponding term operation of $\mathbf{V}$ by $\bar{h}$. Clearly, $\bar{h}$ has the form

$$\bar{h} = \sum_{j=0}^{m-1} s_j x_j \quad (s_j \in \mathrm{End}\,{}_K\widehat{V},\ 0 \leq j \leq m-1)$$

with $s_j \neq 0$ for all $0 \leq j \leq m-1$. By the surjectivity of h we have $A = \sum_{j=0}^{m-1} s_j A$; moreover, since A spans ${}_K\widehat{V}$, therefore $s_j A \supset \{0\}$ for all $0 \leq j \leq m-1$. Clearly,

$$\{0\} \subset s_0 A \subseteq s_0 A + s_1 A \subseteq \ldots \subseteq s_0 A + \ldots + s_{m-1} A = A.$$

Since $m > |A|$, equality holds somewhere, that is,

$$s_0 A + \ldots + s_{i-1} A = s_0 A + \ldots + s_{i-1} A + s_i A \quad \text{for some} \quad 1 \leq i \leq m-1,$$

whence

$$A = s_0 A + \ldots + s_{i-1} A + s_{i+1} A + \ldots + s_{m-1} A. \tag{4.7}$$

Suppose B is a k-ary compatible $\{0\}$-central relation of $\mathbf{A}$ ($k \geq 2$). Then $(B)_2$ is a binary $\{0\}$-central relation, which by Lemma 4.4 is a compatible relation of $(A; h)$. Hence for arbitrary elements $a_j \in A$ ($0 \leq j \leq m-1$) we have $(a_j, 0) \in (B)_2$ for $j \neq i$ and $(0, a_i) \in (B)_2$, implying by an application of h that

$$(\sum_{\substack{j=0 \\ j \neq i}}^{m-1} s_j a_j, s_i a_i) \in (B)_2.$$

Thus, by (4.7), $s_i A$ belongs to the center of $(B)_2$, and hence to the center of B as well.

Proof of Theorem 4.1. Apply Theorem 2.1 for $\mathbf{A}$. In case (a) we have nothing to prove, while in case (c) Theorem 3.1 yields (c)$'$, as the algebras $(N; G)^{[m]}$ with G a primitive permutation group on N have no trivial subalgebras. Finally, assume that, according to case (b), $\mathbf{A}$ has an n-ary compatible $U_{\mathbf{A}}$-central relation B with $n \geq 2$. By Lemma 4.4 the center $(B)_1$ of B is a proper subalgebra of $\mathbf{A}$. However, by Lemma 4.7, $|(B)_1| > 1$, contradicting the fact (cf. Theorem 1.1) that $\mathbf{A}$ is strictly simple. This completes the proof.

References

[1] C. Bergman, R. McKenzie, Minimal varieties and quasivarieties, *J. Austral. Math. Soc.* **48** (1990), 133–147.

[2] G. A. Burle, The classes of k-valued logics containing all one-variable functions, *Diskret. Analiz* **10** (1967), 3–7. (Russian)

[3] D. M. Clark, P. H. Krauss, Para primal algebras, *Algebra Universalis* **6** (1976), 165–192.

[4] D. M. Clark, P. H. Krauss, Plain para primal algebras, *Algebra Universalis* **11** (1980), 365–388.

[5] R. Freese, R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, London Math. Soc. Lecture Note Series 125, Cambridge Univ. Press, Cambridge, 1987.

[6] C. Herrmann, Affine algebras in congruence modular varieties, *Acta Sci. Math. (Szeged)* **41** (1979), 119–125.

[7] D. Hobby, R. McKenzie, *The Structure of Finite Algebras (Tame Congruence Theory)*, Contemporary Mathematics, vol. 76, Amer. Math. Soc., Providence, R. I., 1988.

[8] R. McKenzie, On minimal, locally finite varieties with permuting congruence relations, Preprint, 1976.

[9] R. McKenzie, Finite forbidden lattices, in: *Universal Algebra and Lattice Theory* (Proc. Conf. Puebla, 1982), Lecture Notes in Math. 1004, Springer-Verlag, 1983; pp. 176–205.

[10] P. P. Pálfy, Unary polynomials in algebras I, *Algebra Universalis* **18** (1984), 262–273.

[11] P. P. Pálfy, P. Pudlák, Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups, *Algebra Universalis* **11** (1980), 22–27.

[12] A. F. Pixley, Functionally complete algebras generating distributive and permutable classes, *Math. Z.* **114** (1970), 361–372.

[13] A. F. Pixley, The ternary discriminator function in universal algebra, *Math. Ann.* **191** (1971), 167–180.

[14] E. L. Post, *The two-valued iterative systems of mathematical logic,* Ann. Math. Studies 5, Princeton Univ. Press, 1941.

[15] Á. Szendrei, *Clones in Universal Algebra,* Séminaire de Mathématiques Supérieures, vol. 99, Les Presses de l'Université de Montréal, Montréal, 1986.

[16] Á. Szendrei, Idempotent algebras with restrictions on subalgebras, *Acta Sci. Math. (Szeged)* **51** (1987), 251–268.

[17] Á. Szendrei, Every idempotent plain algebra generates a minimal variety, *Algebra Universalis* **25** (1988), 36–39.

[18] Á. Szendrei, Simple surjective algebras having no proper subalgebras, *J. Austral. Math. Soc.* **48** (1990), 434–454.

[19] Ágnes Szendrei, A classification of strictly simple algebras with trivial subalgebras, *Demonstr. Math.* (submitted).

[20] Ágnes Szendrei, Term minimal algebras, manuscript, 1990.

[21] W. Taylor, The fine spectrum of a variety, *Algebra Universalis* **5** (1975), 262–303.

[22] M. Valeriote, Finite simple Abelian algebras are strictly simple, *Proc. Amer. Math. Soc.* **108** (1990), 49–57.

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