Modules in General Algebra

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Abstract

We discuss structure theorems for abelian algebras, which state that the abelian algebras in question are close to modules. Each structure theorem reflects a major development in commutator theory. The highlights are Herrmann's Theorem stating that abelian algebras are affine in congruence modular varieties, and Quackenbush's Theorem characterizing quasi-affine algebras. We conclude by summarizing some recent results from a joint paper with K. A. Kearnes. Among others, we show that Herrmann's Theorem can be extended to any variety which satisfies a nontrivial lattice identity as a congruence equation.

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1 Introduction

Classical algebraic structures — such as groups, rings, modules, lattices — play a significant role in studying general algebraic systems. This paper focuses on the question how modules over rings occur in general algebra. Modules have inspired the development of general algebra various ways, the most profound influence being, to this day, the discovery that algebras very similar to modules may be present as 'building blocks' in arbitrary algebras just as abelian groups are in arbitrary groups. This phenomenon was revealed by *commutator theory* — a deep structure theory whose fundamental idea is a generalization of the group theoretic commutator to arbitrary algebras.

For simplicity we won't discuss general commutator theory, except for a short section at the end of the paper. Instead, we will restrict our attention to algebras which are "abelian", that is, which play the same role as abelian groups do among groups.

It is a well-known fact that a group $\mathbf{A}$ is abelian if and only if the **diagonal** subset

$$D = \{(a, a) \colon a \in A\}$$

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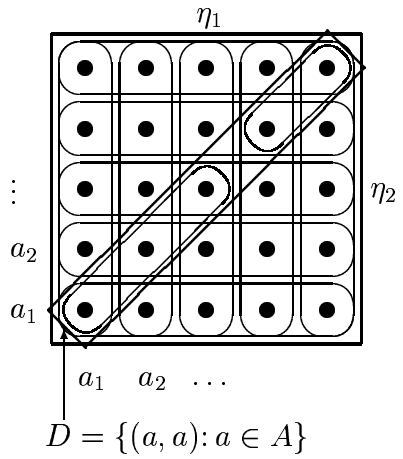

Figure 1: The algebra $\mathbf{A} \times \mathbf{A}$

of $A \times A$ is a normal subgroup of the group $\mathbf{A} \times \mathbf{A}$. Normal subgroups are exactly the subgroups which are congruence blocks. Thus the condition characterizing abelian groups among groups can be restated to say that

$$D \text{ is a block of a congruence of } \mathbf{A} \times \mathbf{A}, \tag{1}$$

which makes sense for arbitrary algebras $\mathbf{A}$.

Three congruences of the algebra $\mathbf{A} \times \mathbf{A}$ will play an important role throughout these discussions. The kernel of the two projections $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ will be denoted η_1 and η_2. To define the third congruence let us consider those congruences δ of $\mathbf{A} \times \mathbf{A}$ which have the property that D is a union of δ-blocks; or equivalently, a diagonal element of $A \times A$ is δ-related to diagonal elements only. Clearly, the congruence $\delta = 0$ (the equality relation) has this propety, and for any family of congruences with this property the transitive closure of their union also has this property. Therefore $\mathbf{A} \times \mathbf{A}$ has a largest congruence with the property that D is a union of congruence blocks. This congruence will be denoted by Δ, or if necessary, by $\Delta_{\mathbf{A}}$. Now it is clear that condition (1) is equivalent to the following:

$$D \text{ is a block of the congruence } \Delta \text{ of } \mathbf{A} \times \mathbf{A}. \tag{2}$$

Property (1) is not the only condition characterizing abelian groups within the class of groups. It requires justification that this characterization is "the right one" if one wants to capture the property of being abelian in general. The importance of condition (1) in studying general algebras was realized around the year 1976 when several authors — following different approaches — were led to investigating simple algebras in congruence permutable varieties. J. D. H. Smith [11] and R. McKenzie [8] independently discovered an interesting phenomenon concerning these algebras,

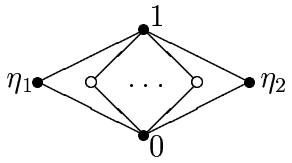

Figure 2: The lattice $\mathrm{Con}\,(\mathbf{A} \times \mathbf{A})$

which involved condition (1), and several other authors came close to finding the same results. We will now discuss these discoveries in more detail.

Recall first that a variety is called **congruence permutable** if $\alpha \circ \beta = \beta \circ \alpha$ holds for any two congruences α, β of each algebra in the variety. Classical examples of such varieties are all varieties of groups, rings, and R-modules for any ring R.

Let $\mathbf{A}$ be a simple algebra such that the variety $\mathcal{V}(\mathbf{A})$ it generates is congruence permutable. In this case the congruence lattice $\mathrm{Con}\,(\mathbf{A} \times \mathbf{A})$ of $\mathbf{A} \times \mathbf{A}$ is easily seen to be a lattice of height 2 whose least element is the equality relation 0, and whose greatest element is the full relation 1. Clearly, besides the trivial congruences 0 and 1 the projection kernels η_1, η_2 are also congruences of $\mathbf{A} \times \mathbf{A}$. Therefore $|\mathrm{Con}\,(\mathbf{A} \times \mathbf{A})| \geq 4$. The surprising fact found in 1976 was that $\mathbf{A}$ can have two sharply different behaviors according to whether $\mathbf{A} \times \mathbf{A}$ has exactly four, or more than four congruences.

Let us consider first the case when $|\mathrm{Con}\,(\mathbf{A} \times \mathbf{A})| = 4$. It can be proved that under the assumptions on $\mathbf{A}$ this implies that $|\mathrm{Con}\,(\mathbf{A}^k)| = 2^k$ for all $k \geq 1$. In particular, this has the following consequences.

- If $\mathbf{A}$ is finite and has no nontrivial subalgebras, then the variety $\mathcal{V}(\mathbf{A})$ is congruence distributive. (A variety is said to be **congruence distributive** if all algebras in the variety have a distributive congruence lattice.)

- If $\mathbf{A}$ is finite and $\mathbf{A}^+$ denotes the algebra arising from $\mathbf{A}$ by adding all constants as fundamental operations, then the variety $\mathcal{V}(\mathbf{A}^+)$ is congruence distributive.

In both cases $\mathbf{A}$ is a **functionally complete** algebra; that is, every operation on the base set of $\mathbf{A}$ is a polynomial operation of $\mathbf{A}$. An analogous conclusion holds even if $\mathbf{A}$ is infinite: every operation on the base set of $\mathbf{A}$ can be "interpolated" on any finite subset of its domain by a polynomial operation of $\mathbf{A}$. Recall that a **polynomial operation** of an algebra is an operation which arises from fundamental operations and from constant operations via composition. In contrast, an operation which arises from the fundamental operations only via composition, is called a **term operation** of the algebra.

Now let $|\mathrm{Con}\,(\mathbf{A} \times \mathbf{A})| > 4$. This means that $\mathbf{A} \times \mathbf{A}$ has a congruence Θ such that $\eta_i \cap \Theta = 0$ and $\eta_i \circ \Theta = 1$ for $i = 1, 2$. One can show that $\mathbf{A}$ has this property if and only if condition (2) — or the equivalent condition (1) — holds for $\mathbf{A}$. Moreover,

it follows also that under the assumptions on **A** condition (2) is equivalent to the following:

$$\left.\begin{array}{c}\text{There exist an abelian group } \widehat{A} = (A;+) \text{ and} \\ \text{a subring } R \text{ of its endomorphism ring End } \widehat{A} \text{ such that} \\ \mathbf{A} \text{ is polynomially equivalent to } {}_R\widehat{A}.\end{array}\right\} \tag{3}$$

Two algebras are said to be **polynomially equivalent** if they have the same polynomial operations.

2 Abelian Versus Affine

Properties (2) and (3) will play a central role throughout this paper. An algebra **A** is said to be **abelian** if (2) [or, equivalently, (1)] holds for **A**, and **affine** if (3) holds for **A**. Obviously, not all abelian algebras are affine: a set with no operations clearly forms an abelian algebra which is not affine. However, all affine algebras are abelian. To see this it suffices to notice that for any module ${}_R\widehat{A}$ the congruence Δ of ${}_R\widehat{A} \times {}_R\widehat{A}$ can be described as follows:

$$(a,b)\,\Delta\,(c,d) \quad \Longleftrightarrow \quad a-b=c-d.$$

Hence the same holds for any affine algebra **A** which is polynomially equivalent to ${}_R\widehat{A}$.

As we mentioned above, in congruence permutable varieties simple abelian algebras must be affine. In fact, simplicity is irrelevant here; in other words, in congruence permutable varieties 'being abelian' and 'being affine' are equivalent properties for *any* algebra. The proof that in congruence permutable varieties abelian algebras are affine is fairly straightforward. It makes use of the fact that by congruence permutability **A** has a ternary term operation p satisfying the identites $p(x,y,y) = x = p(y,y,x)$, and then using the abelian property proceeds to show that for any element 0 in **A** the operation $x+y = p(x,0,y)$ yields an abelian group $\widehat{A} = (A;+)$ such that every polynomial operation of **A** is a polynomial operation of the module ${}_{\mathrm{End}\,\widehat{A}}\widehat{A}$. This is enough to conclude that **A** is affine.

This result on abelian algebras in congruence permutable varieties was soon generalized to **congruence modular varieties**, that is, to varieties in which all algebras have a modular congruence lattice (see [3]). This is the first deep theorem on how modules occur in general algebra.

THEOREM 2.1 (C. Herrmann) *In congruence modular varieties every abelian algebra is affine.*

Idea of proof. Let $\mathcal{V}$ be a congruence modular variety, and let **A** be an algebra from $\mathcal{V}$. There are two important facts, both depending heavily on congruence modularity, which are crucial to the proof:

- If $\mathbf{A}$ is abelian, then so is the algebra $(\mathbf{A} \times \mathbf{A})/\Delta$.
- Let $\mathbf{A}$ be abelian. For any element $0 \in \mathbf{A}$ the mapping

$$\iota\colon \mathbf{A} \to (\mathbf{A} \times \mathbf{A})/\Delta,\ a \mapsto (a, 0)/\Delta$$

  is injective. Moreover, if $\{0\}$ is a trivial subalgebra of $\mathbf{A}$, then ι is an embedding.

From now on the proof splits into two cases.

Assume first that $\mathbf{A}$ has a trivial subalgebra, say $\{0\}$. The facts mentioned above allow us to define an infinite sequence of abelian algebras as follows:

$$\mathbf{A}_0 = \mathbf{A} \qquad \text{and} \qquad \mathbf{A}_{i+1} = (\mathbf{A}_i \times \mathbf{A}_i)/\Delta_{\mathbf{A}_i} \quad \text{for } i \geq 0.$$

In this sequence each algebra is embedded in the next one by the injective homomorphism described above, so we can form the direct union of this infinite family as shown in the diagram below:

$$\mathbf{A}_0 \hookrightarrow \mathbf{A}_1 \hookrightarrow \ldots \hookrightarrow \mathbf{A}_i \hookrightarrow \mathbf{A}_{i+1} \hookrightarrow \ldots \hookrightarrow \bigcup_{i=0}^{\infty} \mathbf{A}_i = \overline{\mathbf{A}}.$$

It turns out that there is a natural way for defining an abelian group operation $+$ on the universe of $\overline{\mathbf{A}}$ so that the algebra $(\overline{\mathbf{A}}; +)$, which arises from $\overline{\mathbf{A}}$ by adding $+$ as a new operation, becomes an affine algebra; namely, $+$ is defined in such a way that subtraction on $\overline{\mathbf{A}}$ is the union of the natural homomorphisms $\mathbf{A}_i \times \mathbf{A}_i \to \mathbf{A}_{i+1}$. This shows that

$$\mathbf{A} \text{ embeds in a reduct of an affine algebra.} \tag{4}$$

Making use of the assumption that $\mathcal{V}$ is congruence modular, one can derive from property (4) that $\mathbf{A}$ must be affine, as claimed.

Now assume that the abelian algebra $\mathbf{A}$ has no trivial subalgebras, and form the algebra $\mathbf{A}^\circ = (\mathbf{A} \times \mathbf{A})/\Delta$. Since the diagonal D is a block of Δ, this algebra has a trivial subalgebra. It is also abelian, therefore by the case settled above we conclude that the algebra $\mathbf{A}^\circ$ is affine. One can show now that $\mathbf{A}$, too, must be affine. This completes the proof of Herrmann's Theorem. $\square$

3 Abelian Versus Quasi-affine

The property displayed in (4) defines an important class of algebras. The members of this class are usually not affine, but they are still 'close enough' to modules so that their operations are tractable. The algebras satisfying condition (4) are said to be **quasi-affine**.

It is not hard to see that all quasi-affine algebras are abelian. Indeed, if $\mathbf{A}$ is quasi-affine, that is, it embeds in a reduct of an affine algebra $\mathbf{A}^*$, then the congruence $\Delta_{\mathbf{A}^*}$

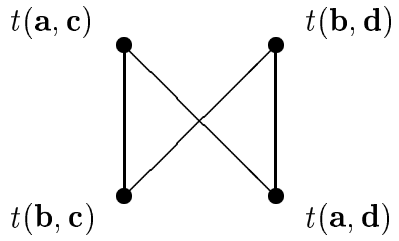

Figure 3: Square labelled with a twisted matrix in $\mathbf{A}$

of $\mathbf{A}^* \times \mathbf{A}^*$ restricts to $A \times A$ as a congruence of $\mathbf{A} \times \mathbf{A}$; moreover, this congruence of $\mathbf{A} \times \mathbf{A}$ has the diagonal as a congruence class, because $\Delta_{\mathbf{A}^*}$ has the same property with respect to $\mathbf{A}^* \times \mathbf{A}^*$. Thus $\mathbf{A}$ satisfies condition (1), and hence it is abelian.

It will be useful to present yet another condition which is equivalent to conditions (1)–(2) defining the abelian property.

CLAIM 3.1 *An algebra* $\mathbf{A}$ *is abelian if and only if* $\mathbf{A}$ *satisfies the following* **term condition***: for all polynomial operations* t *of* $\mathbf{A}$ *and for all tuples* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *of appropriate lengths in* $\mathbf{A}$ *we have*

$$t(\mathbf{a}, \mathbf{c}) = t(\mathbf{a}, \mathbf{d}) \quad \Longleftrightarrow \quad t(\mathbf{b}, \mathbf{c}) = t(\mathbf{b}, \mathbf{d}).$$

It will be convenient to visualize this condition as shown in Figure 3. We have a square whose vertices can be labelled with 2×2 matrices of the form $\begin{bmatrix} t(\mathbf{a}, \mathbf{c}) & t(\mathbf{b}, \mathbf{d}) \\ t(\mathbf{b}, \mathbf{c}) & t(\mathbf{a}, \mathbf{d}) \end{bmatrix}$ where t is a polynomial operation of $\mathbf{A}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are tuples of appropriate lengths in $\mathbf{A}$. Such matrices will be called **twisted matrices** in $\mathbf{A}$. The term condition requires that whenever the labelling is such that the two vertices of a diagonal have the same label, then the two vertices of the other diagonal also have the same label.

Claim 3.1 provides a new way for us to verify that quasi-affine algebras are abelian. Suppose that the algebra $\mathbf{A}$ is quasi-affine, that is, it embeds in a reduct of an affine algebra $\mathbf{A}^*$. Let ${}_R\widehat{A}^*$ be a module with underlying abelian group $\widehat{A}^* = (A^*; +)$ such that $\mathbf{A}^*$ is polynomially equivalent to ${}_R\widehat{A}^*$. Every polynomial operation $t(\mathbf{x}, \mathbf{y})$ of $\mathbf{A}^*$ is of the form

$$t(\mathbf{x}, \mathbf{y}) = \sum r_i x_i + \sum s_j y_j + u$$

for some $r_i, s_j \in R$ and $u \in A^*$. Therefore for any twisted matrix in $\mathbf{A}$ as displayed in Figure 3 the equality

$$t(\mathbf{a}, \mathbf{c}) + t(\mathbf{b}, \mathbf{d}) = t(\mathbf{b}, \mathbf{c}) + t(\mathbf{a}, \mathbf{d})$$

holds in the algebra $\mathbf{A}^*$, that is, the sum of the two top and the two bottom elements of the square — computed in the abelian group $\widehat{A}^*$ — are equal. Thus, if the elements

along one of the diagonals are equal, then the elements along the other diagonal must also be equal. This shows that the term condition holds for $\mathbf{A}$, hence $\mathbf{A}$ is abelian.

Again, one can construct examples to show that not all abelian algebras are quasi-affine.

In the literature there are quite a few (but not many) results stating that under certain conditions an abelian algebra must be quasi-affine (or affine). Herrmann's Theorem discussed in the preceding section is one of them. Further sufficient conditions are listed in the theorem below, where — unlike in Herrmann's Theorem — no assumption is made on the congruence properties of the variety the algebra comes from.

THEOREMS 3.2 *An abelian algebra* $\mathbf{A}$ *is quasi-affine provided that it satisfies one of the following conditions:*

(i) $\mathbf{A}$ *is finite and simple (or more generally: tame)* (**D. Hobby – R. McKenzie** [4]).

(ii) $\mathbf{A}$ *is finite and all its minimal congruences are of type* **2** (**R. McKenzie** [9]).

(iii) $\mathbf{A}$ *is simple and idempotent* (**K. A. Kearnes** [5]).

(iv) $\mathbf{A}$ *has a central, cancellative binary term* (**K. A. Kearnes** [6]).

The proof of (iii) uses Herrmann's method, while (i)–(ii) use the techniques of *tame congruence theory* — a structure theory of finite algebras and locally finite varieties —, for which the reader is referred to the book [4]. The same deep theory made it possible to prove in [4] another sufficient condition, which is more in the vein of Herrmann's Theorem.

THEOREM 3.3 (D. Hobby – R. McKenzie) *If* $\mathcal{V}$ *is a locally finite variety which satisfies a nontrivial lattice identity as a congruence equation, then in* $\mathcal{V}$ *every abelian algebra is affine.*

For more than ten years an outstanding open problem has been whether the same conclusion holds true if the assumption that $\mathcal{V}$ is locally finite is omitted.

PROBLEM 3.4 Is it true that if $\mathcal{V}$ is *any* variety which satisfies a nontrivial lattice identity as a congruence equation, then in $\mathcal{V}$ every abelian algebra is affine?

We will discuss this problem shortly. Before that we will look at the question how quasi-affine algebras can be characterized within the class of abelian algebras. Let $\mathbf{A}$ be a quasi-affine algebra, and let $\mathbf{A}^*$ be an affine algebra such that $\mathbf{A}$ embeds in a reduct of $\mathbf{A}^*$. Let ${}_R\widehat{A}^*$ be a module with underlying abelian group $\widehat{A}^* = (A^*; +)$ such that $\mathbf{A}^*$ is polynomially equivalent to ${}_R\widehat{A}^*$. As we have pointed out earlier, the reason for the term condition to hold in $\mathbf{A}$ is that for any labelling of the square

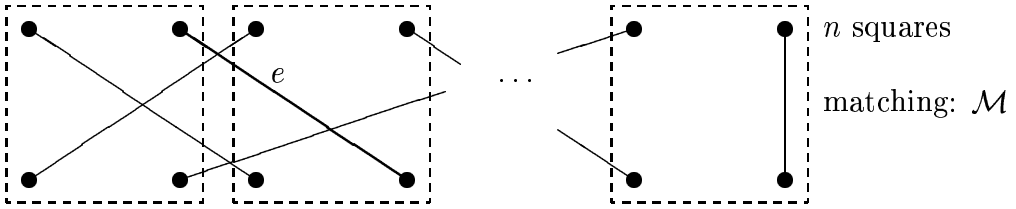

Figure 4: A hierarchy of "term conditions"

with a twisted matrix in $\mathbf{A}$, the sum — computed in $\widehat{A}^*$ — of the two top and the two bottom elements are equal.

A similar argument works for any finite number of squares instead of only one square. Suppose that we have n squares whose vertices are labelled with twisted matrices in $\mathbf{A}$ (see Figure 4). Since the two top and the two bottom elements of each square sum up in $\widehat{A}^*$ to the same element, therefore the sum of all $2n$ top elements is the same as the sum of all $2n$ bottom elements. Consequently, if the labelling is such that there exists a matching $\mathcal{M}$ between the $2n$ top vertices and the $2n$ bottom vertices which has the property that the two vertices on each edge from $\mathcal{M}$, except on a fixed edge $e \in \mathcal{M}$, have the same label, then the two vertices on e must also have the same label.

For each choice of $n, \mathcal{M}, e$ this is a kind of "term condition". The special case when $n = 1$ and $\mathcal{M}$ is the matching determined by the diagonals of the square, coincides with the term condition introduced in Claim 3.1. The other matching in the case $n = 1$ yields essentially the same condition, only the role of $\mathbf{x}$ and $\mathbf{y}$ is switched in the polynomial operations $t(\mathbf{x}, \mathbf{y})$. Thus we have found an infinite family of "term conditions", including the original term condition, which must hold in every quasi-affine algebra. R. Quackenbush [10] proved that these term conditions are not only necessary, but also sufficient for an algebra to be quasi-affine.

THEOREM 3.5 (R. Quackenbush) *An algebra* $\mathbf{A}$ *is quasi-affine if and only if for any positive integer* n*, for each matching* $\mathcal{M}$ *between the* $2n$ *top vertices and the* $2n$ *bottom vertices of* n *squares and for each edge* $e \in \mathcal{M}$*, if the vertices of the* n *squares are labelled with twisted matrices from* $\mathbf{A}$ *in such a way that the two vertices on each edge from* $\mathcal{M} - \{e\}$ *have the same label, then the two vertices on* e *also have the same label.*

This characterization of quasi-affine algebras shows that the class of quasi-affine algebras can be axiomatized by an infinite set of universal Horn sentences, and therefore quasi-affine algebras form a quasivariety. R. Quackenbush proved also that this quasivariety is not finitely axiomatizable, hence one cannot expect to find a much simpler axiom system than the one in Theorem 3.5.

Quackenbush's characterization of quasi-affine algebras was generally believed to be too complicated to be applicable in proving quasi-affineness. In fact, until

recently it has never been used to actually prove any result of the kind which are listed in Theorems 3.2.

4 New Results

This section surveys some results from the paper [7] where, among other things an affirmative solution is given to Problem 3.4. The main result in [7] on the relationship between abelian and quasi-affine algebras is the following sufficient condition for an abelian algebra to be quasi-affine.

THEOREM 4.1 *If* $\mathbf{A}$ *is an abelian algebra such that* $\Delta \cap \eta_2 = 0$ *holds in the congruence lattice of* $\mathbf{A} \times \mathbf{A}$*, then the algebra* $\mathbf{A}$ *is quasi-affine.*

Most earlier sufficient conditions on quasi-affineness are special cases of this theorem. Among the results listed in Theorems 3.2, (i) for simple (tame) algebras of type **2** and (ii)–(iii) are consequences of Theorem 4.1. However, Theorem 4.1 differs from the earlier results in that its proof makes use of Quackenbush's Theorem. To show how Quackenbush's characterization is applied we sketch the proof below.

Idea of proof of Theorem 4.1. Suppose that $\mathbf{A}$ is an abelian algebra which is not quasi-affine. We have to prove that the congruences Δ and η_2 of $\mathbf{A} \times \mathbf{A}$ intersect nontrivially.

Since $\mathbf{A}$ is not quasi-affine, there exist a positive integer n, a matching $\mathcal{M}$ between the $2n$ top vertices and the $2n$ bottom vertices of n squares, and an edge $e \in \mathcal{M}$ such that Quackenbush's condition fails; this failure means that the n squares can be labelled with twisted matrices from $\mathbf{A}$ in such a way that the two vertices on each edge in $\mathcal{M} - \{e\}$ have the same label, but the two vertices of e have different labels.

Let us fix $n, \mathcal{M}, e$ so that n be minimal. Since $\mathbf{A}$ is abelian, we must have $n \geq 2$. The square which contains the top vertex of e will be called the **critical square**. We will select another edge, namely an edge whose bottom vertex is in the critical square, and will denote it e'. (Notice that usually we have two choices for e'.)

Now let us define a binary relation ρ on $A \times A$ as follows:

$$(p, q)\, \rho\, (r, s) \quad \Longleftrightarrow \quad \begin{array}{l} \text{the partial labelling of the } n \text{ squares with } p, q, r, s \\ \text{(as shown in Figure 5) can be extended to a full} \\ \text{labelling of the } n \text{ squares with twisted matrices in } \mathbf{A} \\ \text{so that the two vertices of each edge in } \mathcal{M} - \{e, e'\} \\ \text{have the same label.} \end{array}$$

The assumption that Quackenbush's condition fails for $n, \mathcal{M}, e$ means that there exist pairs (p, q) and (r, s) in $A \times A$ such that $(p, q)\, \rho\, (r, s)$ and $s = q$, but $p \neq r$. Consequently

$$\rho \cap \eta_2 \neq 0. \tag{5}$$

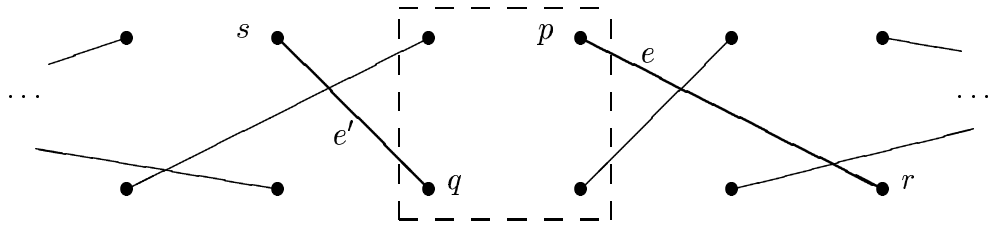

Figure 5: Choice of e' for the definition of ρ

It is easy to see that ρ is a compatible binary relation of the algebra $\mathbf{A} \times \mathbf{A}$. The crucial steps in the proof are to show that

(a) for any ρ-related pairs $(p, q)\, \rho\, (r, s)$ we have $p = q$ if and only if $r = s$,

and

(b) the relation ρ is reflexive (for appropriate choice of e').

(a) can be proved by making use of the minimality of n. If $(p, q)\, \rho\, (r, s)$, then by the definition of ρ the n squares can be labelled with twisted matrices in $\mathbf{A}$ such that the elements p, q, r, s occur at the appropriate vertices, and the two vertices of each edge in $\mathcal{M} - \{e, e'\}$ get the same label. It turns out that if either one of the equalities $p = q$, resp. $r = s$ holds, then we can omit the critical square together with all edges starting there, and then add (at most two) new edges so that the same labelling as before is a labelling of the remaining $n - 1$ squares with twisted matrices from $\mathbf{A}$ in such a way that all edges but one have the same label on their two vertices; moreover the exceptional edge has labels r, s, resp. p, q on its two vertices. Since Quackenbush's conditions must hold for all labellings of less than n squares, we can derive the missing equality $r = s$, resp. $p = q$.

The proof of (b) depends on a simple graph theoretical argument which allows us to show that for at least one choice of the edge e' the n squares can be labelled with the matrices

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix}, \quad \begin{bmatrix} b & b \\ b & b \end{bmatrix}, \quad \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \quad \begin{bmatrix} b & a \\ b & a \end{bmatrix}, \quad \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad \begin{bmatrix} b & a \\ a & b \end{bmatrix},$$

where a, b are any two fixed elements of $\mathbf{A}$, in such a way that the two vertices of each edge in $\mathcal{M}$ get the same label, moreover, e is labelled with a's and e' is labelled with b's. These matrices are obviously twisted matrices in $\mathbf{A}$, therefore it follows that $(a, b)\, \rho\, (a, b)$.

Once (a)–(b) are verified for ρ, we can easily complete the proof of the theorem. Firstly, (b) immediately implies that the symmetric, transitive closure $\widehat{\rho}$ of ρ is

a congruence of $\mathbf{A}$ and we have $\rho \subseteq \widehat{\rho}$. Secondly, (a) has the consequence that the congruence $\widehat{\rho}$ inherits from ρ the property that in the algebra $\mathbf{A} \times \mathbf{A}$ diagonal elements are $\widehat{\rho}$-related to diagonal elements only. Since Δ is the largest congruence of $\mathbf{A} \times \mathbf{A}$ with this property, therefore $\widehat{\rho} \subseteq \Delta$. Hence $\rho \subseteq \Delta$. Combining this with (5) we see that $\Delta \cap \eta_2 \neq 0$, what was to be proved. $\square$

One can show that the hypotheses of Theorem 4.1 always hold for an abelian algebra $\mathbf{A}$ which belongs to a variety satisfying a nontrivial idempotent Mal'cev condition. The property that a variety **satisfies an idempotent Mal'cev condition** means that the variety satisfies a condition of the form "there exist idempotent terms in the language of the variety for which certain identities hold", and such a condition is called **nontrivial** if it fails in at least one variety. It is worth noting that a variety satisfies a nontrivial idempotent Mal'cev condition if and only if a nontrivial congruence condition holds for all congruences of all members of the variety. Here, by a **congruence condition** we mean an equation expressible in terms of $\wedge$ (intersection of relations), $\vee$ (congruence generated by the union of the two relations), and $\circ$ (composition of relations).

Thus we can derive the following important corollary to Theorem 4.1.

COROLLARY 4.2 *If $\mathcal{V}$ is a variety which satisfies a nontrivial idempotent Mal'cev condition (or, equivalently, a nontrivial congruence condition), then in $\mathcal{V}$ every abelian algebra is quasi-affine.*

Under slightly stronger assumptions we can also conclude that abelian algebras must be affine.

THEOREM 4.3 *If $\mathcal{V}$ is a variety which satisfies an idempotent Mal'cev condition that fails in the variety of semilattices, then in $\mathcal{V}$ every abelian algebra is affine.*

It is a well-known fact that every variety in which a nontrivial lattice identity holds as a congruence equation, satisfies the hypotheses of the preceding theorem. Thus the corollary below is an immediate consequence of Theorem 4.3.

COROLLARY 4.4 *If $\mathcal{V}$ is a variety which satisfies a nontrivial lattice identity as a congruence equation, then in $\mathcal{V}$ every abelian algebra is affine.*

This solves Problem 3.4.

5 Concluding Remarks

As we mentioned at the beginning of the paper, the investigation of abelian algebras is part of *general commutator theory*, which is an important tool in studying the structure of algebras in general. A **commutator** is a binary operation defined on the congruence lattices of algebras. The notion generalizes the well-known commutator operation for groups. As in groups, the commutator of two congruences α, β is

denoted $[\alpha, \beta]$, and an algebra is called abelian exactly when $[1, 1] = 0$ holds for its congruences 1 (full relation) and 0 (equality relation).

The results on abelian algebras which are surveyed in this paper all reflect an important development in commutator theory. A general commutator theory was first worked out for congruence permutable varieties by J. D. H. Smith [11]. This was soon extended to congruence modular varieties by J. Hagemann and C. Herrmann [2]. The notion of the congruence modular commutator is defined in terms of the *centrality relation*, which is based on a relativized version of the term condition from Claim 3.1; if α, β are congruences of an algebra $\mathbf{A}$ then $[\alpha, \beta]$ is defined to be the least congruence γ of $\mathbf{A}$ such that α centralizes β modulo γ.

In congruence modular varieties the commutator has nice algebraic and categorical properties. These properties have made commutator theory a very effective technique in studying algebras in congruence modular varieties. For the details of these developments the reader is referred to the book [1].

The definition of the commutator based on the centrality relation makes sense for arbitrary algebras, not only for those in congruence modular varieties. And, in fact, the theory of this commutator produced very significant results, especially for finite algebras and locally finite varieties; see e.g. [4]. However, some of the nice properties that are true in the congruence modular case remain no longer valid in a more general setting. For instance, the commutator is no longer symmetric — that is, $[\alpha, \beta] = [\beta, \alpha]$ is not true in general —, and we don't understand the structure of abelian algebras.

R. Quackenbush hinted in [10] that these shortcomings could be avoided if one defined a commutator using his infinite family of "term conditions". This approach would yield a commutator $[\alpha, \beta]_\ell$ — called the **linear commutator** — which is symmetric and such that the abelian algebras with respect to this commutator are exactly the quasi-affine algebras.

Until recently, the main problem with this commutator was that it was not known whether the linear commutator is a 'true commutator' in the sense that it agrees with the usual commutator in congruence modular varieties. In the paper [7] we prove that the answer to this question is 'yes'. In fact, much more is true:

If $\mathcal{V}$ is a variety which satisfies a nontrivial idempotent Mal'cev condition (or, equivalently, a nontrivial congruence condition), then the linear commutator coincides with the symmetric commutator throughout $\mathcal{V}$.

The **symmetric commutator** is the symmetric version of the commutator based on the centrality relation; in more detail, if α, β are congruences of an algebra $\mathbf{A}$ then their symmetric commutator $[\alpha, \beta]_s$ is defined to be the least congruence γ of $\mathbf{A}$ such that α centralizes β modulo γ and also β centralizes α modulo γ.

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