Nonfinitely based finite groupoids generating minimal varieties

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 80th birthday

Introduction

The first example of a finite algebra which has no finite basis for its identities was discovered by R. Lyndon [3] in 1954. Since that time a number of other nonfinitely based finite algebras have been constructed. (A short summary of these algebras can be found e.g. in [2] or [1].) More recently, V. L. Murskiĭ [5] and P. Perkins [6] observed that some of the finite algebras that are nonfinitely based 'spread' this property to all finite algebras in the variety they generate.

Following Perkins [6] we call a variety V *inherently nonfinitely based* if V is locally finite and V is contained in no finitely based, locally finite variety. An algebra $\mathbf{A}$ ($\mathbf{A}$ is not necessarily finite) is said to be *inherently nonfinitely based* if the variety $V(\mathbf{A})$ it generates has this property. Since every finite algebra generates a locally finite variety, it is clear that, if $\mathbf{A}$ is finite and inherently nonfinitely based, then $\mathbf{A}$ has no finite basis for its identities.

In 1976 R. McKenzie [4] asked whether there exists a nonfinitely based finite algebra which generates a minimal variety. An affirmative answer was given by D. Pigozzi [7], and to this day his algebras have been the only such examples. Pigozzi's construction yields algebras that have a large number of binary operations and (almost) all constants as nullary operations. Thus the following problem remains open in [7]:

Problem. [7] Does there exist a finite groupoid generating a nonfinitely based minimal variety?

The aim of this paper is to show that for every integer $n > 3$ there exists an inherently nonfinitely based n-element groupoid which generates a minimal variety (Corollary 3). These groupoids are constructed from a more general class of inherently nonfinitely based algebras generating minimal varieties (Theorem 1, Corollary 2). In contrast with Pigozzi's examples, in these algebras there is only one constant term operation and all other unary term operations form a permutation group. Nevertheless, this has a similar effect to the varieties the algebras generate as in Pigozzi's case: every nontrivial algebra in the variety has a subalgebra isomorphic to the given algebra, implying that the variety is minimal. To prove that the algebras are inherently nonfinitely based we make use of the 'shift automorphism method' due to K. A. Baker, G. F. McNulty, and H. Werner [2].

Preliminaries and main results

For an algebra $\mathbf{A}$ we denote by $\mathrm{Aut}\,\mathbf{A}$ the automorphism group of $\mathbf{A}$, and by $V(\mathbf{A})$ the variety generated by $\mathbf{A}$. A variety V is said to be *minimal* if it has exactly two subvarieties: V itself and the trivial variety consisting of one-element algebras only. A variety V is said to be *finitely based* (or to have a finite basis for its identities) if there exists a finite set Σ of identities valid in V such that every identity valid in V can be deduced from Σ; an algebra is finitely based if the variety $V(\mathbf{A})$ has this property. An algebra is called *locally finite* if all its finitely generated subalgebras are finite; a variety is locally finite if every algebra in it is such.

Let $\mathbf{A} = (A; F)$ be an algebra. An operation on A is a *term operation* [*polynomial operation*] of $\mathbf{A}$ if it can be constructed form the fundamental operations of $\mathbf{A}$ and from projections [resp., form the fundamental operations of $\mathbf{A}$, from projections, and from constant operations] via composition. The set of unary term operations [polynomial operations] of $\mathbf{A}$ will be denoted by $\mathrm{Clo}_1\,\mathbf{A}$ [$\mathrm{Pol}_1\,\mathbf{A}$, respectively].

Two algebras are called *term equivalent* if they have the same base sets and the same term operations. It is well known that the algebraic properties of term equivalent algebras are very similar (see e.g. [9]). In particular, the varieties they generate are isomorphic as categories; furthermore, if both algebras have finitely many fundamental operations, then one of them is finitely based [nonfinitely based, inherently nonfinitely based] if and only if the other one has the same property. In what follows, we will use these facts without further reference.

Let G be a group, and consider its extension with a zero element: $G^0 = \{0\} \cup G$

$(0 \notin G)$. For any element $g \in G$ define permutations $l_g, r_g \colon G^0 \to G^0$ as follows:

$$l_g(x) = \begin{cases} 0 & \text{if } x = 0 \\ gx & \text{if } x \in G \end{cases}, \quad r_g(x) = \begin{cases} 0 & \text{if } x = 0 \\ xg & \text{if } x \in G \end{cases},$$

and put

$$L_G = \{l_g \colon\ g \in G\}, \quad R_G = \{r_g \colon\ g \in G\}.$$

Clearly, L_G and R_G are permutation groups on G^0, and both of them act regularly on G, i.e. for all $a, b \in G$ there exists a unique permutation in L_G [resp. R_G] sending a to b. Making use of this fact one can easily check that for an algebra $\mathbf{B}$ with base set G^0 the following two conditions are equivalent:

(∗) $\mathrm{Clo}_1\,\mathbf{B} = \{0\} \cup L_G$;

(∗∗) $\{0\} \cup L_G \subseteq \mathrm{Clo}_1\,\mathbf{B}$, $\{0\}$ is a subalgebra of $\mathbf{B}$, and $R_G \subseteq \mathrm{Aut}\,\mathbf{B}$.

The algebras on G^0 satisfying these conditions will be called G^0-*algebras*.

Simple G^0-algebras were studied in detail in [8]. In particular, the following theorem was shown.

Theorem A. [8] *For every finite group G with $|G| > 1$ every simple G^0-algebra is either term equivalent to a one-dimensional vector space, or else it has the operation*

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in G^0)$$

among its term operations.

Clearly, $\wedge$ is a semilattice operation: 0 is the least element, and any two distinct elements $x, y \in G$ are incomparable.

Obviously, every algebra term equivalent to a one-dimensional vector space is simple and generates a minimal variety. The same holds true also for G^0-algebras having $\wedge$ among their term operations.

Proposition B. (cf. [8]) *Let $\mathbf{B}$ be a G^0-algebra such that $\wedge$ is a term operation of $\mathbf{B}$. Then $\mathbf{B}$ is simple, and $V(\mathbf{B})$ is a minimal variety.*

Proof. Let ρ be a congruence of $\mathbf{B}$. If $a \,\rho\, b$ for some elements $a, b \in G^0$, $a \neq b$ — say $a \neq 0$ —, then $a = a \wedge a \,\rho\, a \wedge b = 0$, whence for arbitrary element $g \in G$ we have $g = l_{ga^{-1}}(a) \,\rho\, l_{ga^{-1}}(0) = 0$. Thus $\mathbf{B}$ is simple.

The set $\mathrm{Clo}_1\,\mathbf{B}$ of unary term operations of $\mathbf{B}$ can be considered, in the usual way, as an algebra of the same type as $\mathbf{B}$. It is well known that this algebra is the free algebra of rank 1 in the variety $V(\mathbf{B})$. Further, it is clear, that the natural homomorphism

$$\mathrm{Clo}_1\,\mathbf{B} \to \mathbf{B}, \quad f \mapsto f(1)$$

is surjective; however, $|\mathbf{B}| = |\mathrm{Clo}_1\,\mathbf{B}|$, therefore it is an isomorphism. Consequently, $\mathbf{B}$ is the free algebra in $V(\mathbf{B})$ freely generated by $\{1\}$.

Let now $\mathbf{A}$ be a nontrivial algebra in $V(\mathbf{B})$. Obviously, all unary terms in the language of $V(\mathbf{B})$ that induce on $\mathbf{B}$ the constant operation 0, induce a unique constant operation on $\mathbf{A}$ as well; the element in their range will also be denoted by 0. Selecting $a \in A - \{0\}$ we conclude from the freeness of $\mathbf{B}$ that there exists a homomorphism $\varphi\colon \mathbf{B} \to \mathbf{A}$ such that $1\varphi = a$; moreover, clearly $0\varphi = 0$. Hence the simplicity of $\mathbf{B}$ ensures that φ is injective. Thus $\mathbf{B}$ can be embedded in each nontrivial algebra in $V(\mathbf{B})$, whence the minimality of $V(\mathbf{B})$ follows immediately.

The main result of the paper is that for every finite group G ($|G| > 2$) there exist simple G^0-algebras that are inherently nonfinitely based.

Theorem 1. *For every finite group G with at least 3 elements, the algebra $\mathbf{B} = (G^0;\ L_G, \star)$ with*

$$x \star y = \begin{cases} 0 & \text{if } 0 \in \{x,y\} \text{ or } x = y \\ x & \text{otherwise} \end{cases} \qquad (x, y \in G^0),$$

is inherently nonfinitely based.

It is easy to check that $\mathbf{B}$ is a G^0-algebra. Furthermore, $\wedge$ is a term operation of $\mathbf{B}$; ideed, if $|G| = n$ and, say, $G = \{1 = a_0, a_1, \dots, a_{n-1}\}$, then

$$x \wedge y = (\dots((x \star l_{a_1}(y)) \star l_{a_2}(y)) \dots) \star l_{a_{n-1}}(y) \quad (x, y \in G^0).$$

Thus by Proposition B we get the following corollary.

Corollary 2. *For every finite group G with at least 3 elements, the algebra $\mathbf{B} = (G^0;\ L_G, \star)$ generates an inherently nonfinitely based minimal variety.*

Consider now the special case when G is a cyclic group with n elements ($n \geq 3$); say, $a \in G$ generates G. Then the algebra $\mathbf{B} = (G^0;\ L_G, \star)$ has a term operation

$$x \bullet y = l_a(x \star l_a(y)) \quad (x, y \in G^0)$$

satisfying the following identities:

$$l_a(x) = x \bullet x, \quad l_{a^{-1}}(x) = (l_a)^{n-1}(x), \quad x \star y = l_{a^{-1}}(x \bullet l_{a^{-1}}(y)).$$

Hence the groupoid $(B; \bullet)$ is term equivalent to the algebra $\mathbf{B}$. Thus we get

Corollary 3. *For every finite cyclic group $G = \{1, a, a^2, \ldots, a^{n-1}\}$ with at least 3 elements, the groupoid $(G^0; \bullet)$ with*

$$x \bullet y = l_a(x \star l_a(y)) \quad (x, y \in G^0)$$

generates an inherently nonfinitely based minimal variety.

Proof of Theorem 1

We will use the 'shift automorphism method' due to K. A. Baker, G. F. McNulty, and H. Werner [2]. This method applies to algebras with an *absorbing element* 0. Recall that 0 is called an *absorbing element* of an algebra if every fundamental operation f of the algebra takes on the value 0 whenever at least one of its arguments is 0.

Let $\mathbf{A}$ be an arbitrary algebra with absorbing element 0, and let $\sigma \in \operatorname{Aut} \mathbf{A}$. The elements of $\mathbf{A}$ distinct from 0 will be called *proper elements*. An n-tulpe will be called *proper* if none of its components is 0. Clearly, if $\mathbf{A}$ is not unary, then 0 is the only absorbing element of $\mathbf{A}$, hence $0\sigma = 0$. Let f be an n-ary fundamental operation of $\mathbf{A}$. Considering f as an $(n+1)$-ary relation we see that f is invariant under (the componentwise action of) σ. Moreover, if $0\sigma = 0$, then each orbit of σ consists either entirely of proper or entirely of improper $(n+1)$-tuples.

The theorem underlying the method provides a sufficient condition for an infinite algebra with an absorbing element to be inherently nonfinitely based.

Theorem C. [2] *Let $\mathbf{A}$ be an infinite algebra with finitely many fundamental operations such that $\mathbf{A}$ is locally finite and has an absorbing element 0. If $\mathbf{A}$ has an automorphism σ such that*

(a) *$\{0\}$ is the only σ-orbit of $\mathbf{A}$ that is finite,*
(b) *for every fundamental operation f of $\mathbf{A}$, the proper part of f is partitioned by σ into only finitely many orbits, and*

(c) $a\sigma = p(a)$ *for some proper element* $a \in A$ *and some nonconstant polynomial operation* $p \in \mathrm{Pol}_1\,\mathbf{A}$,

then $\mathbf{A}$ *is inherently nonfinitely based.*

Now let $\mathbf{B} = (B; F)$ be an arbitrary finite algebra with finitely many fundamental operations, and with an absorbing element 0. Form its direct power $\mathbf{B}^{\mathbf{Z}}$ with $\mathbf{Z}$ the set of integers; the elements of $\mathbf{B}^{\mathbf{Z}}$ are doubly infinite sequences

$$\nu = \ldots\ \nu(-2)\ \nu(-1)\ \nu(0)\ \nu(1)\ \nu(2)\ \ldots$$

with $\nu(j) \in \mathbf{B}$ for all $j \in \mathbf{Z}$. For $i \in \mathbf{Z}$ the *translate* $\nu^{(i)}$ of $\nu \in B^{\mathbf{Z}}$ is the sequence ν shifted i positions (to the right if $i > 0$, or to the left if $i < 0$, or not at all if $i = 0$); in symbols, $\nu^{(i)}(j) = \nu(j - i)$ for all $i, j \in \mathbf{Z}$. It is clear that for any fixed integer i, shifting all sequences of $\mathbf{B}^{\mathbf{Z}}$ i positions is an automorphism of $\mathbf{B}^{\mathbf{Z}}$.

The shift automorphism method can now be described as follows. Suppose we are given a finite number of non-periodic sequences in $\mathbf{B}^{\mathbf{Z}}$ such that

($\dagger$) the translates of these sequences together with all improper sequences in $\mathbf{B}^{\mathbf{Z}}$ form a subalgebra $\mathbf{C}$ in $\mathbf{B}^{\mathbf{Z}}$.

Then the equivalence relation Θ on $\mathbf{C}$ such that one block of Θ consists of all improper sequences and all other blocks are singletons is a congruence of the algebra $\mathbf{C}$. Moreover, in the quotient algebra $\mathbf{A} = \mathbf{C}/\Theta$ the element corresponding to the nontrivial block is an absorbing element; we will denote it by 0. The right shift (i.e. shifting one position to the right) induces an automorphism σ of $\mathbf{A}$ with $0\sigma = 0$. Since we selected non-periodical sequences, condition (a) holds for $\mathbf{A}$ (and hence $\mathbf{A}$ is infinite). If we can select the starting sequences in such a way that

($\ddagger$) the algebra $\mathbf{A}$ described above is locally finite and satisfies conditions (b) and (c) in Theorem C,

then Theorem C yields that $\mathbf{A}$ is inherently nonfinitely based. However, every variety containing $\mathbf{B}$ contains $\mathbf{A}$ as well, so $\mathbf{B}$ is also inherently nonfinitely based.

Proof of Theorem 1. Let G be a fixed finite group with $|G| = n \geq 3$, say $G = \{1 = a_0, a_1, \ldots, a_{n-1}\}$, and let $B = G^0$. Consider the periodic sequence

$$\gamma = a_1 \ldots a_{n-1} a_1 \ldots a_{n-1} \ldots \in G^{\mathbf{N}},$$

and for $i \in \mathbf{N}$ let $[\gamma]_i$ denote the prefix of length i of γ. (Hence the period of γ is $[\gamma]_{n-1}$.)

We will call a sequence $\nu \in B^{\mathbf{Z}}$ *symmetric* if $\nu(-i) = \nu(i)$ for all $i \in \mathbf{N}$. Let $\alpha \in B^{\mathbf{Z}}$ be the symmetric sequence whose 'nonnegative half' is

$$\alpha|_{\mathbf{N}_0} = 1\ [\gamma]_{n-1}\ 1\ [\gamma]_n\ 1\ [\gamma]_{n+1}\ 1\ [\gamma]_{n+2}\ 1\ \dots\ ,$$

and for arbitrary $g \in G$ let $\alpha_g = l_g(\alpha)$. Clearly, these sequences are non-periodic, and for $g = 1$ we have $\alpha_1 = \alpha$. We will show that if we start with the sequences α_g $(g \in G)$ and apply the shift automorphism method sketched above, conditions (†)–(‡) hold true.

Observe that for arbitrary $\nu, \nu' \in \mathbf{B}^{\mathbf{Z}}$ the sequence $\nu \star \nu'$ is either improper or equals ν. Further, the set of translates of α_g $(g \in G)$ is closed under all operations in L_G. Thus (†) obviously holds. In the algebra $\mathbf{A}$, 0 is an absorbing element, and for any elements ν, ν' distinct from 0 we have

$$\nu \star \nu' = \begin{cases} \nu & \text{if } \nu(l) \neq \nu'(l) \text{ for all } l \in \mathbf{Z} \\ 0 & \text{otherwise} \end{cases} .$$

In particular, $\alpha^{(1)} = \alpha^{(1)} \star \alpha$, since $n \geq 3$ ensures that any two neighbouring components of α are distinct. Hence (c) is satisfied for the polynomial function $p(x) = \alpha^{(1)} \star x$ of $\mathbf{A}$. It is also clear from the operations of $\mathbf{A}$ that $\mathbf{A}$ is locally finite.

It remains to verify (b) for the fundamental operations of $\mathbf{A}$. For the unary operations this is trivial, while for $\star$ it will be derived from the following claim.

Claim. *For arbitrary elements $g, h \in G$ and for every integer $j \geq n$ there exists an index $l \geq j$ such that $\alpha_g(l) = \alpha_h^{(j)}(l)$.*

Let I denote the set of all indices $i \in \mathbf{N}$ such that $\alpha(i) = 1$, and let $0 \leq k \leq n-1$ be the (unique) index with $g = ha_k$. Then

$$\begin{aligned} \alpha_g(i) &= g \quad \text{for all} \quad i \in I, \\ \alpha_h(i) &= g \quad \text{for all} \quad i \in I + k, \\ \alpha_h^{(j)}(i) &= g \quad \text{for all} \quad i \in I + k + j. \end{aligned}$$

In view of $k + j \geq n$ it is clear from the construction of α that I contains two integers at a distance $k + j$. Therefore $I \cap (I + k + j) \neq \emptyset$, so any $l \in I \cap (I + k + j)$ is appropriate, proving the claim.

Now, making use of the fact that α is symmetric, we get from the Claim that for arbitrary elements $g, h \in G$ and for every integer $j \in \mathbf{Z}$

$$\alpha_g \star \alpha_h^{(j)} = 0, \quad \text{if} \quad |j| \geq n.$$

This implies that condition (b) holds for $\star$. The proof of Theorem 1 is complete.

References

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