Expansions of minimal varieties

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Dedicated to Professor K. Tandori on his 70th birthday and
to Professor L. Leindler on his 60th birthday

Abstract

Let $\mathbf{A}$ be a strictly simple algebra generating a locally finite minimal variety, and let us expand $\mathbf{A}$ arbitrarily with new operations to get an algebra $\mathbf{A}^\bullet$. We investigate the question under what conditions $\mathbf{A}^\bullet$ generates a minimal variety. Our result shows that if the tame congruence theoretic type label of $\mathbf{A}$ is distinct from $\mathbf{5}$ or if $\mathbf{A}$ has a trivial automorphism group, then $\mathbf{A}^\bullet$ generates a minimal variety if and only if $\mathbf{A}^\bullet$ is nonabelian or has a trivial subalgebra.

A variety $\mathcal{V}$ is called *minimal* if $\mathcal{V}$ has exactly two subvarieties: $\mathcal{V}$ itself and the variety of trivial (i.e. one-element) algebras. We restrict our attention to *locally finite* varieties, that is, varieties in which every finitely generated algebra is finite. If $\mathcal{V}$ is a locally finite variety and $\mathbf{A}$ is a nontrivial member of $\mathcal{V}$ of minimum cardinality, then $\mathbf{A}$ is a *strictly simple algebra*, that is, $\mathbf{A}$ is finite, simple, and has no nontrivial proper subalgebras. Moreover, if $\mathcal{V}$ is minimal, then $\mathcal{V} = \mathcal{V}(\mathbf{A})$, the variety generated by $\mathbf{A}$. Thus every locally finite minimal variety is generated by a strictly simple algebra. In fact, it was proved recently in [8] that the strictly simple generator of a minimal variety is uniquely determined, up to isomorphism.

Now let $\mathcal{V}(\mathbf{A})$ be a minimal variety with strictly simple generator $\mathbf{A}$, and consider an algebra $\mathbf{A}^\bullet$ arising from $\mathbf{A}$ by adding new operations to it; such an algebra $\mathbf{A}^\bullet$ will be called an *expansion* of $\mathbf{A}$. Clearly, $\mathbf{A}^\bullet$ is also strictly simple. For many familiar minimal varieties $\mathcal{V}(\mathbf{A})$ the varieties $\mathcal{V}(\mathbf{A}^\bullet)$ generated by expansions $\mathbf{A}^\bullet$ of $\mathbf{A}$ turn out almost exclusively to be also minimal. For example, if $\mathcal{V}(\mathbf{A})$ is congruence distributive, then

Research supported by a fellowship from the Alexander von Humboldt Stiftung and by the Hungarian National Foundation for Scientific Research grant no. 1903.

AMS Subject Classification: 08B05, 08A05, 08A40

$\mathcal{V}(\mathbf{A}^\bullet)$, too, is congruence distributive, as the existence of Jónsson term operations is inherited by expansions. Hence, by Jónsson's Lemma [6], $\mathcal{V}(\mathbf{A}^\bullet)$ is a minimal variety for every expansion $\mathbf{A}^\bullet$ of $\mathbf{A}$.

If congruence distributivity is replaced by congruence modularity, we have already some exceptions $\mathbf{A}^\bullet$ to this conclusion. Recall that an algebra $\mathbf{C}$ is said to be *affine* if there exists an abelian group $\widehat{C} = (C; +)$ on its base set such that $x - y + z$ is a term operation of $\mathbf{C}$ and every fundamental operation of $\mathbf{C}$ is a polynomial operation of the module ${}_{\mathrm{End}\,\widehat{C}}\widehat{C}$ (i.e. $\widehat{C}$ considered as a module over its endomorphism ring). By C. Herrmann's results ([3], [4], cf. also [2]), if $\mathbf{S}$ is a strictly simple algebra such that the variety $\mathcal{V}(\mathbf{S})$ is congruence modular, then either $\mathcal{V}(\mathbf{S})$ is congruence distributive or $\mathbf{S}$ is an affine algebra; hence $\mathcal{V}(\mathbf{S})$ is minimal if and only if $\mathbf{S}$ is not an affine algebra without trivial subalgebras. Therefore it follows that if $\mathcal{V}(\mathbf{A})$ is a congruence modular minimal variety with strictly simple generator $\mathbf{A}$ and $\mathbf{A}^\bullet$ is an expansion of $\mathbf{A}$, then $\mathcal{V}(\mathbf{A}^\bullet)$ is a minimal variety except for the case when $\mathbf{A}^\bullet$ is affine and has no trivial subalgebras.

In congruence modular varieties affine algebras are exactly those algebras which are abelian in the sense used in commutator theory ([4]). The definition of 'abelian', which is not restricted to the congruence modular case, is the following: an algebra $\mathbf{C}$ is *abelian* if for all $n \geq 1$, for every n-ary term operation t of $\mathbf{C}$ and for arbitrary elements $u, v \in C$ and $\bar{c}, \bar{d} \in C^{n-1}$ we have

$$f(u, \bar{c}) = f(u, \bar{d}) \;\Leftrightarrow\; f(v, \bar{c}) = f(v, \bar{d}).$$

Our aim in this paper is to look at the question in general: Given a strictly simple algebra $\mathbf{A}$ generating a minimal variety, under what conditions is it true that $\mathcal{V}(\mathbf{A}^\bullet)$ is a minimal variety for 'almost all' expansions $\mathbf{A}^\bullet$ of $\mathbf{A}$. We show that under mild restrictions on $\mathbf{A}$, the same conclusion holds as in the congruence modular case: for any expansion $\mathbf{A}^\bullet$ of $\mathbf{A}$, the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is minimal except when $\mathbf{A}^\bullet$ is abelian and has no trivial subalgebras. For example, to get this conclusion, it is enough to require that the tame congruence theoretic type label of $\mathbf{A}$ is not **5**, or $\mathbf{A}$ has a trivial automorphism group, or the number of trivial subalgebras of $\mathbf{A}$ is distinct from 1. (See Corollary 8 and Theorem 11.)

Our arguments are based on the characterization of locally finite minimal varieties in [8] and the tools applied there. It will be convenient to use the tame congruence theoretic type labels **1**–**5** for strictly simple algebras; in particular, this makes it easy to distinguish between abelian and nonabelian algebras: a simple algebra is abelian if and only if it is of type **1** or **2** (see [5]). Besides this fact, however, no techniques or results from tame congruence theory will be explicitly used in the paper.

Algebras are denoted by boldface capitals and their base sets by the corresponding letters in italics. Since we study expansions, we will often deal with distinct algebras defined on the same base set. These algebras will be distinguished by superscripts, and the superscript will be omitted when referring to the base set.

If there is no danger of confusion, we will use the same notation for a term and for the corresponding term operation; in fact, even in words, we may sometimes talk about "a term of an algebra $\mathbf{A}$ with specific properties" as an abbreviation for "a term in the language of $\mathbf{A}$ such that the term operation of $\mathbf{A}$ corresponding to t has those specific properties".

The clone of term operations of an algebra $\mathbf{A}$ is denoted by $\operatorname{Clo}\mathbf{A}$. Two algebras $\mathbf{A}$, $\mathbf{A}'$ on the same base set are called term equivalent if $\operatorname{Clo}\mathbf{A} = \operatorname{Clo}\mathbf{A}'$. We will find it convenient to extend this notion to algebras $\mathbf{A}$, $\mathbf{B}$ on arbitrary bases sets as follows: $\mathbf{A}$, $\mathbf{B}$ are called term equivalent if $\mathbf{A}$ is isomorphic to an algebra $\mathbf{B}'$ on B such that $\mathbf{B}'$ is term equivalent (in the narrower sense) to $\mathbf{B}$.

We now introduce the concepts that are needed for the statement of the characterization theorem on strictly simple algebras generating minimal varieties. Let $\mathbf{A}$ be an algebra, and let e be a unary term in the language of $\mathbf{A}$. We will say that e is an *idempotent term (operation) of* $\mathbf{A}$ if $\mathbf{A} \models e^2 = e$ (or equivalently, if the term operation e acts identically on its range $e(A)$).

An idempotent term e of $\mathbf{A}$ is called a *minimal idempotent term of* $\mathbf{A}$ (or briefly: a *minimal idempotent of* $\mathbf{A}$) if the term operation e is not constant and its range $e(A)$ is minimal (with respect to inclusion) among the ranges of all non-constant idempotent term operations of $\mathbf{A}$. Clearly, every finite algebra has a minimal idempotent. It is not hard to see (cf. [13]) that if $\mathbf{A}$ is a strictly simple algebra, then for any minimal idempotents e, e' of $\mathbf{A}$ there exist unary terms g, h such that

$$(*) \qquad \mathbf{A} \models ghe' = e', hge = e, ge = e'ge, he' = ehe',$$

that is, the term operations g, h restrict to mutually inverse bijective mappings $e(A) \to e'(A)$ and $e'(A) \to e(A)$, respectively.

For a variety $\mathcal{V}$ and for a term e in the language of $\mathcal{V}$ the condition occurring in [8] that

$$\text{if } \mathbf{C} \in \mathcal{V} \text{ and } \mathbf{C} \models e(x) = e(y), \text{ then } |C| = 1$$

will be referred to as follows: *e is nonconstant throughout $\mathcal{V}$.*

Theorem 1. ([8]) *Let $\mathbf{A}$ be a strictly simple algebra and let e be a minimal idempotent of $\mathbf{A}$. The following conditions are equivalent:*

(1) *$\mathcal{V}(\mathbf{A})$ is a minimal variety;*

(2) *e is nonconstant throughout $\mathcal{V}(\mathbf{A})$ and $\mathbf{A}$ has a trivial subalgebra provided it is abelian.*

If $\mathbf{A}$ is a strictly simple algebra and e, e' are minimal idempotents of $\mathbf{A}$, then the existence of unary terms g, h with $(*)$ immediately implies that e is nonconstant throughout $\mathcal{V}(\mathbf{A})$ if and only if e' is nonconstant throughout $\mathcal{V}(\mathbf{A})$. Thus the fact whether a minimal idempotent term is nonconstant throughout $\mathcal{V}(\mathbf{A})$ does not depend on the choice of the minimal idempotent.

The proof of Theorem 1 in [8] is based on a construction that will also be useful in the application we are going to discuss. To describe this construction let $\mathcal{V}$ be a variety and let e be a unary term in the language of $\mathcal{V}$ such that $\mathcal{V} \models e^2 = e$. Clearly, for every term t in the language of $\mathcal{V}$, et is a term of the same arity as t, and for each algebra $\mathbf{C} \in \mathcal{V}$ the term operations of $\mathbf{C}$ of the form et are exactly those term operations whose range is contained in $e(C)$. We want to define, for all algebras $\mathbf{C} \in \mathcal{V}$, algebras $e(\mathbf{C})$ — all of the same similarity type, as $\mathbf{C}$ runs over $\mathcal{V}$ — such that the base set of $e(\mathbf{C})$ is $e(C)$ and the *set* of fundamental operations of $e(\mathbf{C})$ consists of the restrictions $et|_{e(C)}$ of all term operations et of $\mathbf{C}$ to $e(C)$. In [8] this was achieved by fixing the set of fundamental operation symbols to be

$$\{et\colon\ t \text{ is a term in the language of } \mathcal{V}\}$$

and defining the interpretation of et in the algebra $e(\mathbf{C})$ to be $et|_{e(C)}$. In this paper it will be more convenient to allow a richer set of fundamental operation symbols, namely

$$\{\bar{t}\colon\ \bar{t} \text{ is a term in the language of } \mathcal{V} \text{ and } \mathcal{V} \models e\bar{t} = \bar{t}\},$$

and, as before, define the interpretation of $\bar{t}$ in the algebra $e(\mathbf{C})$ to be $\bar{t}|_{e(C)}$. Obviously, what happens by this modification is merely that instead of each operation symbol we may take several distinct operation symbols with the same interpretation in all algebras in question.

For $\mathcal{V}$ and e as in the previous paragraph we denote the class $\{e(\mathbf{C})\colon\ \mathbf{C} \in \mathcal{V}\}$ of algebras by $e(\mathcal{V})$. It can be proved that $e(\mathcal{V})$ is a variety (cf. [11; Remark 6]). Moreover, for every algebra $\mathbf{A}$ and for every idempotent term e of $\mathbf{A}$ we have $e\big(\mathcal{V}(\mathbf{A})\big) = \mathcal{V}\big(e(\mathbf{A})\big)$.

The construction $\mathbf{A} \mapsto e(\mathbf{A})$ has particularly useful properties if $\mathbf{A}$ is a strictly simple algebra and e is a minimal idempotent term of $\mathbf{A}$. The elementary properties (1)–(3) stated in the next theorem were established along with the first applications of this construction in [1] and [12], while (4) was verified in [8].

Theorem 2. ([1], [12], [8]) *Let $\mathbf{A}$ be a strictly simple algebra and let e be a minimal idempotent of $\mathbf{A}$. Then*

(1) *$e(\mathbf{A})$ is a strictly simple algebra;*

(2) *$e(\mathbf{A})$ has the same trivial subalgebras as $\mathbf{A}$ (i.e., for $u \in A$, $\{u\}$ is a trivial subalgebra of $\mathbf{A}$ if and only if $u \in e(A)$ and $\{u\}$ is a trivial subalgebra of $e(\mathbf{A})$);*

(3) *$e(\mathbf{A})$ has the same tame congruence theoretic type label as $\mathbf{A}$;*

(4) *the automorphism group of $e(\mathbf{A})$ is isomorphic (via restriction) to the automorphism group of $\mathbf{A}$.*

It is worth noting that if $\mathbf{A}$ is a strictly simple algebra and e, e' are minimal idempotents of $\mathbf{A}$, then the fact mentioned before Theorem 1 that there exist unary terms inducing mutually inverse bijections between $e(A)$ and $e'(A)$, yields that the algebras $e(\mathbf{A})$ and $e'(\mathbf{A})$ are term equivalent. Hence in this case the algebra $e(\mathbf{A})$ is essentially independent of the choice of e.

The algebras of the form $e(\mathbf{A})$ where $\mathbf{A}$ is strictly simple and e is a minimal idempotent of $\mathbf{A}$ are completely described, up to term equivalence, see [13]. Here we will not need this description in full details; its consequence stated in Theorem 3 below will suffice.

For a group G, G^0 will denote the extension of G with a new zero element 0, and we let L_G $[R_G]$ denote the group of all permutations on G^0 determined by multiplication on the left [right] with elements of G. An algebra $\mathbf{B}$ will be called a *G^0-algebra* if its base set is $B = G^0$, $\{0\}$ is a subalgebra of $\mathbf{B}$, the constant with value 0 and the permutations in L_G are unary operations of $\mathbf{B}$, and the permutations in R_G are automorphisms of $\mathbf{B}$. It is easy to show that under these conditions the set of unary term operations of $\mathbf{B}$ is $\{0\} \cup L_G$, while the automorphism group of $\mathbf{B}$ is R_G.

Theorem 3. ([13]) *Let $\mathbf{A}$ be a strictly simple algebra of type $\mathbf{i} \in \{\mathbf{1}, \ldots, \mathbf{5}\}$, and let e be a minimal idempotent of $\mathbf{A}$. Then the following condition in ($\mathbf{i}$) holds for the algebra $e(\mathbf{A})$:*

- ($\mathbf{1}$) *$e(\mathbf{A})$ is term equivalent to a strictly simple unary algebra in which every fundamental operation is either constant or a permutation;*
- ($\mathbf{2}$) *$e(\mathbf{A})$ is a strictly simple affine algebra;*
- ($\mathbf{3}$)–($\mathbf{4}$) *$e(\mathbf{A})$ is a strictly simple algebra that generates a congruence distributive variety;*
- ($\mathbf{5}$) *either $e(\mathbf{A})$ is term equivalent to a two-element semilattice, possibly with some bounds added, or $e(\mathbf{A})$ is a strictly simple G^0-algebra of type $\mathbf{5}$ for some group G, $|G| > 1$.*

In the problem we are going to investigate we are given a strictly simple algebra $\mathbf{A}$ generating a minimal variety, and we study the expansions $\mathbf{A}^\bullet$ of $\mathbf{A}$. Clearly, $\mathbf{A}^\bullet$ is also strictly simple, however, a minimal idempotent term of $\mathbf{A}$ may not be a minimal idempotent term of $\mathbf{A}^\bullet$. The first claim of the next lemma shows that $\mathbf{A}^\bullet$ has a minimal idempotent term that is convenient to work with. The second claim will not be needed until Lemma 16 near the end of the paper.

Lemma 4. *Let $\mathbf{A}$ be a strictly simple algebra and let e be a minimal idempotent of $\mathbf{A}$.*

(1) *Every expansion $\mathbf{A}^\bullet$ of $\mathbf{A}$ has a minimal idempotent f such that $\mathbf{A}^\bullet \models ef = fe = f$.*

(2) *If f is a minimal idempotent of an expansion of $\mathbf{A}$ satisfying the identities in (1), then*

- (i) *the expansion $\mathbf{A}^f$ of $\mathbf{A}$ with f is the least expansion of $\mathbf{A}$ (up to term equivalence) which has f as a minimal idempotent, and*
- (ii) *$e(\mathbf{A}^f)$ is term equivalent to the expansion $e(\mathbf{A})^f$ of $e(\mathbf{A})$ with f.*

Proof. (1) Since e is an idempotent term of $\mathbf{A}^\bullet$ with $|e(A)| > 1$, $\mathbf{A}^\bullet$ has a minimal idempotent term f' such that $f'(A) \subseteq e(A)$, $|f'(A)| > 1$. Clearly, $\mathbf{A}^\bullet \models ef' = f'$. Let $f = f'e$. It is straightforward to check that $\mathbf{A}^\bullet \models ef = fe = f = f^2$. Moreover, $f(A) = f'(A)$, because $f'(A) = f'\big(f'(A)\big) \subseteq f'\big(e(A)\big) \subseteq f'(A)$ and here $f'\big(e(A)\big) = f(A)$. Thus f, too, is a minimal idempotent of $\mathbf{A}^\bullet$.

(2) If f is a minimal idempotent of $\mathbf{A}^\bullet$, then obviously $\operatorname{Clo}\mathbf{A}^f \subseteq \operatorname{Clo}\mathbf{A}^\bullet$. In particular, this implies the minimality of f for $\mathbf{A}^f$, proving (i).

In (ii) the inclusion $\operatorname{Clo} e(\mathbf{A})^f \subseteq \operatorname{Clo} e(\mathbf{A}^f)$ follows immediately from the identity $\mathbf{A}^f \models ef = f$ and from the definition of $e(\mathbf{A}^f)$. To prove the reverse inclusion it suffices to show that every fundamental operation of $e(\mathbf{A}^f)$ belongs to $\operatorname{Clo} e(\mathbf{A})^f$. We proceed by induction on the number of occurrences of f in the terms of $\mathbf{A}^f$, and use the assumption $\mathbf{A}^f \models ef = fe = f$. By definition every fundamental operation of $e(\mathbf{A}^f)$ is of the form $eT|_{e(A)}$ for some term T of $\mathbf{A}^f$. If f does not occur in T, then T is a term of $\mathbf{A}$ and $eT|_{e(A)}$ is an operation of $e(\mathbf{A})$ (and hence of $e(\mathbf{A})^f$). If f occurs in T, then T has the form $T = t\big(f(T_0), T_1, \ldots, T_{k-1}\big)$ for some term t of $\mathbf{A}$ and some terms $T_0, T_1, \ldots, T_{k-1}$ of $\mathbf{A}^f$. We have

$$\mathbf{A}^f \models eT = et\big(ef(eT_0), T_1, \ldots, T_{k-1}\big),$$

therefore $eT|_{e(A)}$ is a composition of the operations $et(\cdot, T_1, \ldots, T_{k-1})|_{e(A)}$, $ef|_{e(A)} = f|_{e(A)}$, and $eT_0|_{e(A)}$. By the induction hypothesis the first and third ones are term operations of $e(\mathbf{A})^f$, so the same is true for $eT|_{e(A)}$, as required. $\diamond$

Let $\mathbf{A}$ be a strictly simple algebra generating a minimal variety. Our aim is to show that under mild restrictions on $\mathbf{A}$, for arbitrary expansion $\mathbf{A}^\bullet$ of $\mathbf{A}$ the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is minimal if and only if $\mathbf{A}^\bullet$ is not an abelian algebra without trivial subalgebras. In view of Theorem 1 we have to check whether a minimal idempotent term of $\mathbf{A}^\bullet$ is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$. The idea of the proof is the following. Let us fix a pair e, f of minimal idempotent terms of $\mathbf{A}$ and $\mathbf{A}^\bullet$, respectively, such that the condition in Lemma 4 (1) is satisfied. First we observe that the condition that f is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$ can be verified via the 'intermediate variety' $e\big(\mathcal{V}(\mathbf{A}^\bullet)\big) = \mathcal{V}\big(e(\mathbf{A}^\bullet)\big)$ by showing that f is nonconstant throughout $\mathcal{V}\big(e(\mathbf{A}^\bullet)\big)$ and e is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$. Then we check these two conditions separately. For the latter one we can make use of the facts that $\mathbf{A}^\bullet$ is an expansion of $\mathbf{A}$ and that e is nonconstant throughout $\mathcal{V}(\mathbf{A})$, while for the first one, where $e(\mathbf{A}^\bullet)$ is obviously an expansion of $e(\mathbf{A})$, we can apply Theorem 3. The details of these arguments can be found in Lemmas 5–7.

Lemma 5. *Let $\mathcal{V}$ be a variety and let e, f be unary terms in the language of $\mathcal{V}$ such that $\mathcal{V} \models e^2 = e, f^2 = ef = f$. The following conditions are equivalent:*

(1) *f is nonconstant throughout $\mathcal{V}$;*

(2) *f is nonconstant throughout $e(\mathcal{V})$ and e is nonconstant throughout $\mathcal{V}$.*

Proof. Notice that under the hypotheses of the lemma f is a fundamental operation symbol in the language of $e(\mathcal{V})$, so (2) does indeed make sense.

To prove the equivalence assume first that (1) holds. If $\mathbf{C} \models e(x) = e(y)$ for some algebra $\mathbf{C}$ in $\mathcal{V}$, then $\mathbf{C} \models f(x) = ef(x) = ef(y) = f(y)$, hence in view of (1) $|C| = 1$. This shows that e is nonconstant throughout $\mathcal{V}$. Now let $e(\mathbf{C}) \in e(\mathcal{V})$ ($\mathbf{C} \in \mathcal{V}$), and suppose $e(\mathbf{C}) \models f(x) = f(y)$. For arbitrary elements $c, d \in C$, the elements $f(c) = ef(c)$ and $f(d) = ef(d)$ belong to $e(\mathbf{C})$, therefore we have $f(c) = f\big(f(c)\big) = f\big(f(d)\big) = f(d)$. Thus $\mathbf{C} \models f(x) = f(y)$, whence by (1) we get that $|C| = 1$. Hence $|e(C)| = 1$, proving that f is nonconstant throughout $e(\mathcal{V})$.

Conversely, assume now that (2) holds, and let $\mathbf{C}$ be any algebra in $\mathcal{V}$ such that $\mathbf{C} \models f(x) = f(y)$. Clearly, $e(\mathbf{C}) \models f(x) = f(y)$. Since f is nonconstant throughout $e(\mathcal{V})$, we must have $|e(C)| = 1$. In other words, $\mathbf{C} \models e(x) = e(y)$. However, as e is nonconstant throughout $\mathcal{V}$, we get that $|C| = 1$. Thus (1) holds. $\diamond$

Lemma 6. *Let $\mathbf{A}$ be an algebra, $\mathbf{A}^\bullet$ an expansion of $\mathbf{A}$, and let e be an idempotent term of $\mathbf{A}$ (and hence of $\mathbf{A}^\bullet$). If e is nonconstant throughout $\mathcal{V}(\mathbf{A})$, then e is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$, too.*

Proof. Assume e is nonconstant throughout $\mathcal{V}(\mathbf{A})$, and consider an algebra $\mathbf{D}$ in $\mathcal{V}(\mathbf{A}^\bullet)$ such that $\mathbf{D} \models e(x) = e(y)$. Omitting the newly added operations we get a reduct $\mathbf{D}^\circ$ of $\mathbf{D}$ of the same similarity type as $\mathbf{A}$. Clearly, $\mathbf{D} \in \mathcal{V}(\mathbf{A}^\bullet)$ implies that $\mathbf{D}^\circ \in \mathcal{V}(\mathbf{A})$. Moreover, we have $\mathbf{D}^\circ \models e(x) = e(y)$. As e is nonconstant throughout $\mathcal{V}(\mathbf{A})$, we conclude that $|D| = 1$, completing the proof. $\diamond$

Lemma 7. *Let* $\mathbf{A}$ *be a strictly simple algebra generating a minimal variety and let* e *be a minimal idempotent of* $\mathbf{A}$*. Further, let* $\mathbf{A}^\bullet$ *be an expansion of* $\mathbf{A}$ *and* f *a minimal idempotent of* $\mathbf{A}^\bullet$*. Then* f *is nonconstant throughout* $\mathcal{V}(\mathbf{A}^\bullet)$ *unless*

(†) $\quad e(\mathbf{A})$ *is a* G^0*-algebra of type* $\mathbf{5}$ *for some group* G *with* $|G| > 1$*.*

Proof. Without loss of generality we may select f as described in Lemma 4 (1). The assumption that $\mathcal{V}(\mathbf{A})$ is minimal implies by Theorem 1 that e is nonconstant throughout $\mathcal{V}(\mathbf{A})$. Hence, by Lemma 6, e is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$. Now Lemma 5 shows that f is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$ if and only if f is nonconstant throughout $e\big(\mathcal{V}(\mathbf{A}^\bullet)\big)$. According to our remark before Theorem 2, here $e\big(\mathcal{V}(\mathbf{A}^\bullet)\big) = \mathcal{V}\big(e(\mathbf{A}^\bullet)\big)$.

We show that f is a minimal idempotent of $e(\mathbf{A}^\bullet)$. In fact, it is easy to see that every term operation of $e(\mathbf{A}^\bullet)$ (not only every fundamental operation) is of the form $eg|_{e(A)}$ where g is a term of $\mathbf{A}^\bullet$. In particular, if $eg|_{e(A)}$ is a minimal idempotent of $e(\mathbf{A}^\bullet)$ such that its range $eg\big(e(A)\big)$ is contained in the range $f\big(e(A)\big) = f(A)$ of $f|_{e(A)}$, then ege is an idempotent term of $\mathbf{A}^\bullet$ with the same range as $eg|_{e(A)}$. So the minimality of f for $\mathbf{A}^\bullet$ implies that $ege(A) = f(A)$. That is, the ranges of $eg|_{e(A)}$ and $f|_{e(A)}$ coincide, as required.

Let $\mathbf{B} = e(\mathbf{A})$ and $\mathbf{B}^\bullet = e(\mathbf{A}^\bullet)$. In view of Theorem 2 $\mathbf{B}$ is strictly simple and of the same type as $\mathbf{A}$. Moreover, as $\mathbf{A}$ generates a minimal variety, $\mathbf{A}$ has a trivial subalgebra provided it is abelian (cf. Theorem 1). Therefore $\mathbf{B}$, too, has a trivial subalgebra provided it is abelian.

Clearly, $\mathbf{B}^\bullet$ is an expansion of $\mathbf{B}$, so it is also strictly simple. In the first two paragraphs of this proof it was shown that f is a minimal idempotent of $\mathbf{B}^\bullet$; furthermore, the claim of the lemma to be proved was reduced to the following: *f is nonconstant throughout* $\mathcal{V}(\mathbf{B}^\bullet)$ *unless* $\mathbf{B}$ *is a* G^0*-algebra of type* $\mathbf{5}$ *for a group* G *with* $|G| > 1$*.*

To verify this assertion we make use of Theorem 3. If $\mathbf{B}$ is of type $\mathbf{1}$, then $\mathbf{B}$ is abelian, so it must have a trivial subalgebra. Thus Theorem 3 yields that $|B| = 2$. Now the identity mapping is the only possible choice for the term operation f of $\mathbf{B}^\bullet$, hence $\mathbf{B}^\bullet \models f(x) = x$. It is obvious that in this case f is nonconstant throughout $\mathcal{V}(\mathbf{B}^\bullet)$. If $\mathbf{B}$ is of type $\mathbf{5}$ and is not a G^0-algebra with $|G| > 1$, then again we get from Theorem 3 that $|B| = 2$. So the argument can be finished as in the previous case. Finally, assume $\mathbf{B}$ is of type $\mathbf{2}$, $\mathbf{3}$, or $\mathbf{4}$. We see from Theorem 3 that $\mathbf{B}$ is either affine or generates a congruence distributive variety. Applying in the affine case McKenzie's theorem [9] (cf. [10; Theorem 22]) on strictly simple algebras in congruence permutable varieties, we see that the expansion $\mathbf{B}^\bullet$ of $\mathbf{B}$ is also either affine or generates a congruence distributive variety. Hence either $\mathbf{B}^\bullet$ is an affine algebra having no trivial subalgebras, or $\mathcal{V}(\mathbf{B}^\bullet)$ is a minimal variety. In the latter case Theorem 1 implies that f is nonconstant throughout $\mathcal{V}(\mathbf{B}^\bullet)$. If $\mathbf{B}^\bullet$ is affine with no trivial subalgebras, then for any fixed element $o \in f(B)$ the algebra

$$\mathbf{B}^* = \big(B; \{t \in \operatorname{Clo} \mathbf{B}^\bullet \colon\ t(o, \dots, o) = o\}\big)$$

is a reduct of $\mathbf{B}^\bullet$ such that $\mathbf{B}^*$ is strictly simple, affine, and has a trivial subalgebra; hence $\mathbf{B}^*$ generates a minimal variety. Clearly, f is a minimal idempotent of $\mathbf{B}^*$. Thus a combination of Theorem 1 and Lemma 6 yields that f is nonconstant throughout $\mathcal{V}(\mathbf{B}^\bullet)$. $\diamond$

Corollary 8. *Let* $\mathbf{A}$ *be a strictly simple algebra generating a minimal variety. If*
— $\mathbf{A}$ *is of type* $\neq \mathbf{5}$, *or*
— $\mathbf{A}$ *has a trivial automorphism group, or*
— *the number of trivial subalgebras of* $\mathbf{A}$ *is* $\neq 1$,
then for every expansion $\mathbf{A}^\bullet$ *of* $\mathbf{A}$, $\mathbf{A}^\bullet$ *generates a minimal variety if and only if* $\mathbf{A}^\bullet$ *is nonabelian or has a trivial subalgebra.*

Proof. Let $\mathbf{A}$ be a strictly simple algebra and let e be a minimal idempotent of $\mathbf{A}$. By the definition of G^0-algebras a G^0-algebra has exactly $|G|$ automorphisms. Therefore, in view of Theorem 3, condition (†) in Lemma 7 holds if and only if $e(\mathbf{A})$ has the following three properties (the first two of which imply the third): $e(\mathbf{A})$ has type $\mathbf{5}$, $e(\mathbf{A})$ has a nontrivial automorphism group, and $e(\mathbf{A})$ has a unique trivial subalgebra. By Theorem 2 all these properties are invariant under the construction $\mathbf{A} \mapsto e(\mathbf{A})$, hence (†) is equivalent to the following condition:

(‡) $\mathbf{A}$ is of type $\mathbf{5}$ and $\mathbf{A}$ has a nontrivial automorphism group (whence also $\mathbf{A}$ has a unique trivial subalgebra).

Now let $\mathbf{A}^\bullet$ be an expansion of $\mathbf{A}$ and let f be a minimal idempotent of $\mathbf{A}^\bullet$. If (‡) fails, then by Lemma 7 f is nonconstant throughout $\mathcal{V}(\mathbf{A}^\bullet)$, so our claim is an immediate consequence of Theorem 1. $\diamond$

The claim of Corollary 8 can be slightly refined if we take into account that abelian strictly simple algebras generating minimal varieties have a very special structure. The structure theorem stated below was proved independently in [7] and in [14]–[15] (cf. also [8] where the same results are deduced directly from Theorem 1).

Theorem 9. ([7], [14], [15]) *Let* $\mathbf{A}$ *be an abelian strictly simple algebra.*

(1) *If* $\mathbf{A}$ *is of type* $\mathbf{1}$, *then* $\mathbf{A}$ *generates a minimal variety if and only if* $\mathbf{A}$ *is term equivalent to* $\mathbf{B}^{[m]}$ *for some integer* $m \geq 1$ *and for some two-element unary algebra* $\mathbf{B}$ *that has a trivial subalgebra.*

(2) *If* $\mathbf{A}$ *is of type* $\mathbf{2}$, *then* $\mathbf{A}$ *generates a minimal variety if and only if* $\mathbf{A}$ *is affine and* $\mathbf{A}$ *has a trivial subalgebra.*

Recall that for an algebra $\mathbf{B}$ and for an integer $m \geq 1$ the *mth matrix power* $\mathbf{B}^{[m]}$ *of* $\mathbf{B}$ is the algebra arising from the mth direct power $\mathbf{B}^m$ of $\mathbf{B}$ by expanding it with an m-ary operation d and a unary operation p defined as follows: for all $x_i = (x_i^j)_{j<m} \in B^m$ $(i = 0, \ldots, m-1)$ and for all $y = (y^j)_{j<m} \in B^m$

$$\begin{aligned} d(x_0, x_1, \ldots, x_{m-1}) &= (x_0^0, x_1^1 \ldots, x_{m-1}^{m-1}), \\ p(y) &= (y^1, \ldots, y^{m-1}, y^0). \end{aligned}$$

For more details on this construction and on its applications the reader is referred to [16], [5] and [11].

We will need the following easy fact on expansions of matrix powers.

Lemma 10. *Let* $\mathbf{B}$ *be any algebra and let* m *be a positive integer. Every expansion of* $\mathbf{B}^{[m]}$ *is term equivalent to* $(\mathbf{B}^\bullet)^{[m]}$ *for some expansion* $\mathbf{B}^\bullet$ *of* $\mathbf{B}$.

Sketch of proof. The coordinates of an m-tuple x will be denoted by $x^0, \ldots, x^{m-1}$. For every positive integer n an n-ary operation $g = g(x_0, \ldots, x_{n-1})$ on B^m can be written in the form $g = \big(g_0(\bar{x}), \ldots, g_{m-1}(\bar{x})\big)$ where $\bar{x} = (x_i^j)_{0 \le i < n,\ 0 \le j < m}$ and $g_0, \ldots, g_{m-1}$ are mn-ary operations on B. In this notation, if h is an n-ary operation on B, then the coordinatewise action of h on B^m is

$$\hat{h}(x_0, \ldots, x_{n-1}) = \big(h((x_i^0)_{0 \le i < n}), \ldots, h((x_i^{m-1})_{0 \le i < n})\big).$$

It is not hard to show that for every operation $g = \big(g_0(\bar{x}), \ldots, g_{m-1}(\bar{x})\big)$ on B^m, g is contained in the clone generated by the operations d, p, $\hat{g}_i$ $(0 \le i \le m-1)$, and conversely, each $\hat{g}_i$ $(0 \le i \le m-1)$ is contained in the clone generated by the operations d, p, g. The proof is based on the observation that d and p allow to form from any given m-tuples a new m-tuple by selecting for each coordinate any of the coordinates of the given ones.

Now let $\mathbf{C}$ be an expansion of $\mathbf{B}^{[m]}$ (hence $C = B^m$), and consider the algebra $\mathbf{B}^\bullet$ on B whose set of fundamental operations is

$$\bigcup\Big(\{g_0(\bar{x}), \ldots, g_{m-1}(\bar{x})\}\colon g(x_0, \ldots, x_{n-1}) = \big(g_0(\bar{x}), \ldots, g_{m-1}(\bar{x})\big) \\ \text{is a fundamental operation of } \mathbf{C}\Big).$$

The facts in the preceding paragraph imply that $(\mathbf{B}^\bullet)^{[m]}$ is term equivalent to $\mathbf{C}$, and (up to adding fictitious variables to the fundamental operations) $\mathbf{B}^\bullet$ is an expansion of $\mathbf{B}$. $\diamond$

Now we are in a position to prove the main result of the paper.

Theorem 11. *Let* $\mathbf{A}$ *be a strictly simple algebra generating a minimal variety, and let* $\mathbf{A}^\bullet$ *be an expansion of* $\mathbf{A}$. *Then* $\mathbf{A}^\bullet$ *also generates a minimal variety unless one of the following conditions holds:*

(1) $\mathbf{A}$ *is term equivalent to* $\mathbf{B}^{[m]}$ *for some* $m \ge 1$ *and for some two-element unary algebra* $\mathbf{B}$ *with at least one trivial subalgebra, while its expansion* $\mathbf{A}^\bullet$ *is term equivalent to* $(\mathbf{B}^\bullet)^{[m]}$ *where* $\mathbf{B}^\bullet$ *is an expansion of* $\mathbf{B}$ *which is either*

- *a unary algebra without trivial subalgebras, or*
- *an affine algebra without trivial subalgebras (in the latter case* $\mathbf{A}^\bullet$*, too, is an affine algebra without trivial subalgebras);*

(2) $\mathbf{A}$ *and* $\mathbf{A}^\bullet$ *are affine,* $\mathbf{A}$ *has at least one trivial subalgebra, while* $\mathbf{A}^\bullet$ *has no trivial subalgebras;*

(3) $\mathbf{A}$ *is of type* **5** *and* $\mathbf{A}$ *has a nontrivial automorphism group (hence* $\mathbf{A}$ *has a unique trivial subalgebra).*

Proof. Let e be a minimal idempotent of $\mathbf{A}$ and f a minimal idempotent of $\mathbf{A}^\bullet$. If $\mathbf{A}$ is nonabelian, then $\mathbf{A}^\bullet$ is also nonabelian, and hence by Corollary 8 the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is minimal unless (3) holds.

Assume now that $\mathbf{A}$ is abelian, and apply Theorem 9 to $\mathbf{A}$. First we consider the case when $\mathbf{A}$ is affine and has a trivial subalgebra. By Corollary 8 the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is not minimal if and only if $\mathbf{A}^\bullet$ is abelian and has no trivial subalgebras. Clearly, the expansion $\mathbf{A}^\bullet$ of $\mathbf{A}$ generates a congruence permutable variety, and it is well known that an abelian algebra in a congruence permutable variety is affine ([9]; cf. [5; Exercise 3.2 (3)]). Therefore the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is not minimal if and only if (2) holds.

In the remaining case $\mathbf{A}$ is term equivalent to $\mathbf{B}^{[m]}$ for some $m \geq 1$ and some two-element unary algebra $\mathbf{B}$ with at least one trivial subalgebra. Again by Corollary 8 the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is not minimal if and only if $\mathbf{A}^\bullet$ is abelian and has no trivial subalgebras. The expansion $\mathbf{A}^\bullet$ of $\mathbf{A}$ is term equivalent to an expansion of $\mathbf{B}^{[m]}$. Thus by Lemma 10 $\mathbf{A}^\bullet$ is term equivalent to $(\mathbf{B}^\bullet)^{[m]}$ for some expansion $\mathbf{B}^\bullet$ of $\mathbf{B}$. It is well known that $\mathbf{A}^\bullet$ is abelian and has no trivial subalgebras if and only if $\mathbf{B}^\bullet$ has the same properties (cf. [5; Exercise 3.12 (5)]). Since every two-element abelian algebra is either affine or term equivalent to a unary algebra, it follows that the variety $\mathcal{V}(\mathbf{A}^\bullet)$ is not minimal if and only if (1) holds. ◇

Let us look at the 'exceptions' listed in Theorem 11, that is, at the pairs $\mathbf{A}, \mathbf{A}^\bullet$ where $\mathbf{A}$ is a strictly simple algebra and $\mathbf{A}^\bullet$ is an expansion of $\mathbf{A}$ such that $\mathbf{A}$ generates a minimal variety, while $\mathbf{A}^\bullet$ does not. In cases (1) and (2), where $\mathcal{V}(\mathbf{A}^\bullet)$ is not minimal because $\mathbf{A}^\bullet$ violates the condition that "it has a trivial subalgebra provided it is abelian", $\mathbf{A}$ and $\mathbf{A}^\bullet$ are explicitly described. Moreover, it is known that in these cases the variety $\mathcal{V}(\mathbf{A}^\bullet)$ has a unique non-trivial proper subvariety.

Case (3), where the non-minimality of $\mathcal{V}(\mathbf{A}^\bullet)$ is caused by the failure of the condition that "a minimal idempotent is nonconstant throughout the variety", is much less understood. We have no explicit description for the exceptional pairs $\mathbf{A}, \mathbf{A}^\bullet$. However, in Corollary 17 we will prove that every algebra $\mathbf{A}$ satisfying condition (3) in Theorem 11 does indeed occur in an exceptional pair $\mathbf{A}, \mathbf{A}^\bullet$. Furthermore, in Examples 13 and 14 we will construct algebras showing that in the exceptional pairs $\mathbf{A}, \mathbf{A}^\bullet$ such that $\mathbf{A}$ satisfies condition (3) in Theorem 11,

— there is no restriction on the type of the nonabelian algebra $\mathbf{A}^\bullet$, and

— there is no finite bound on the number of subvarieties of $\mathcal{V}(\mathbf{A}^\bullet)$.

In the examples it will be necessary to check for certain algebras $\mathbf{A}$ whether an idempotent term is nonconstant throughout $\mathcal{V}(\mathbf{A})$. For this purpose we will apply a result from [8] stating that this condition is equivalent to a Mal'cev-like condition.

Theorem 12. ([8]) *Let* $\mathbf{A}$ *be an arbitrary algebra and let* e *be an idempotent term of* $\mathbf{A}$*. The following conditions are equivalent:*

(1) *e is nonconstant throughout $\mathcal{V}(\mathbf{A})$;*

(2) *for some $n \geq 1$, there exist binary terms t_i and unary terms g_i, h_i $(0 \leq i \leq n)$*

such that

$$\begin{aligned} \mathcal{V}(\mathbf{A}) \quad \models \quad & x = t_0\bigl(x, eg_0(x)\bigr), \\ & t_i\bigl(x, eh_i(x)\bigr) = t_{i+1}\bigl(x, eg_{i+1}(x)\bigr) \quad (0 \le i \le n-1), \\ & t_n\bigl(x, eh_n(x)\bigr) = e(x). \end{aligned}$$

All examples we consider are G^0-algebras for some group G with $|G| > 1$. The unit element of G will be denoted by 1. Let us recall from [13] some facts on strictly simple G^0-algebras of type **5** that will be needed later on.

— Every nonabelian strictly simple G^0-algebra has the following semilattice operation $\wedge$ as a term operation:

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in G^0).$$

— The term operations of every strictly simple G^0-algebra of type **5** are monotone with respect to the natural order $\le$ of the semilattice $(G^0; \wedge)$; moreover, they also preserve the quaternary relation

$$\mu_0 = \{(a,a,a,a),\ (a,a,0,0),\ (a,0,a,0),\ (a,0,0,0)\colon\ a \in G^0\}.$$

— Every nonabelian strictly simple G^0-algebra generates a minimal variety.

— $(G^0; \wedge, L_G, 0)$ is a strictly simple G^0-algebra of type **5**; in fact, up to term equivalence, this is the least nonabelian strictly simple G^0-algebra.

— The clones of strictly simple G^0-algebras of type **5** form an interval in the clone lattice; hence, up to term equivalence, there exists a largest strictly simple G^0-algebra of type **5**. The clone of this algebra consists exactly of those operations on G^0 which preserve μ_0 and admit all permutations in R_G as automorphisms (in view of $|G| > 1$ the latter condition implies that $\{0\}$ is a subalgebra).

Example 13. Let $\mathbf{B} = (G^0; \wedge, L_G, 0)$ (hence $B = G^0$) where G is a group with $|G| > 1$, and let W be a nonempty set of operations on B such that each member of W has range $\{0,1\}$ or $\{1\}$. Then the expansion $\mathbf{B}^\bullet$ of $\mathbf{B}$ with all operations in W fails to generate a minimal variety.

Proof. Notice first that $\mathbf{B}^\bullet$ has a minimal idempotent f with range $\{0,1\}$. In fact, for any fundamental operation $w \in W$ (say, w is k-ary) we have $w(b_0, \ldots, b_{k-1}) = 1$ for some $b_0, \ldots, b_{k-1} \in B$. If $w(0, \ldots, 0) = 0$, then allowing the notation l_0 for the constant operation 0 we get that $f(x) = w\bigl(l_{b_0}(x), \ldots, l_{b_{k-1}}(x)\bigr)$ is an idempotent term operation of $\mathbf{B}^\bullet$ with range $\{0,1\}$. Otherwise, $w(0, \ldots, 0) = 1$ is a constant term operation of $\mathbf{B}^\bullet$, whence $f(x) = x \wedge 1$ is an idempotent term operation of $\mathbf{B}^\bullet$ with range $\{0,1\}$. In both cases f is clearly minimal.

Claim A. If t_0 is a binary term of $\mathbf{B}^\bullet$ such that $\mathbf{B}^\bullet \models x = t_0\bigl(x, fg_0(x)\bigr)$ for some unary term g_0, then $\mathbf{B}^\bullet \models x = t_0(x, y)$.

Proof of Claim A. Clearly, $t_0(B,B) = B$ (i.e., the range of t_0 is B). However, it is straightforward to check by induction on the length of the terms that for every term t of $\mathbf{B}^\bullet$, either t is term of $\mathbf{B}$ or $t(B,\dots,B) \subseteq \{0,g\}$ for some $g \in G$. Thus t_0 is a term of $\mathbf{B}$. Hence

$$\begin{aligned} \mathbf{B} \models t_0(x,y) &= l_a(x), \text{ or} \\ t_0(x,y) &= l_b(y), \text{ or} \\ t_0(x,y) &= l_a(x) \wedge l_b(y) \end{aligned}$$

for some $a, b \in G$, and the same identity holds in $\mathbf{B}^\bullet$ as well. Since $\mathbf{B}^\bullet \models x = t_0\big(x, fg_0(x)\big)$, the last two cases cannot occur, while in the first case we get $a = 1$ and hence $\mathbf{B}^\bullet \models x = t_0(x,y)$.

Now assume that, in contrary to our assertion, $\mathbf{B}^\bullet$ generates a minimal variety. Then by Theorems 1 and 12 there exist terms t_i, g_i, h_i $(0 \le i \le n)$ such that $\mathbf{B}^\bullet$ satisfies the identities in Theorem 12 (2) for f in place of e. Suppose these terms are selected so that the length n of the chain is as small as possible. By Claim A, $\mathbf{B}^\bullet \models x = t_0(x,y)$. Thus t_0, g_0, h_0 can be omitted from the Mal'cev chain, contradicting the minimality of n. This completes the proof. $\diamond$

Obviously, in Example 13 the two-element algebra $f(\mathbf{B}^\bullet)$ has all $w|_{f(B)}$ $(w \in W)$ as fundamental operations. Furthermore, for each of the relations $\rho = \le$ and $\rho = \mu_0$, if ρ is preserved by all operations in W, then $\rho|_{f(B)}$ is an invariant relation of $f(\mathbf{B}^\bullet)$. Therefore one can easily construct sets W of operations such that $\mathbf{B}^\bullet$ has type **3**, **4** or **5**.

Example 14. Let $\mathbf{B}$ be as in the previous example, and let $\{1\} = H_0 \subset H_1 \subset \dots \subset H_{k-1} \subset H_k = G$ be an ascending chain of subgroups of G. For $i = 0, 1, \dots, k-1, k$ define a unary operation f_i on B by

$$f_i(x) = \begin{cases} x & \text{if } x \in H_i \\ 0 & \text{otherwise} \end{cases} \qquad (x \in B),$$

and let $\mathbf{B}^\bullet$ be the expansion of $\mathbf{B}$ with the operations f_i $(0 \le i \le k)$. Then, for $i = 0, 1, \dots, k-1, k$, the identities $f_i(x) = f_i(y)$ define pairwise distinct proper subvarieties of $\mathcal{V}(\mathbf{B}^\bullet)$.

Proof. For $0 \le i \le k$ let $\mathcal{V}_i$ denote the subvariety of $\mathcal{V}(\mathbf{B}^\bullet)$ defined by the identity $f_i(x) = f_i(y)$. Since $\mathbf{B}^\bullet \models f_i f_j = f_j$ for $j < i$, we have $\mathcal{V}_k \subseteq \mathcal{V}_{k-1} \subseteq \dots \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_0 \subseteq \mathcal{V}(\mathbf{B}^\bullet)$. It remains to show that the inclusions are proper. This is obvious for the last inclusion, because $\mathbf{B}^\bullet \not\models f_0(x) = f_0(y)$.

Now let $0 \le i \le k-1$, consider the subset K_i of B^G consisting of all $|G|$-tuples in which all coordinates belong to the set $\{0\} \cup cH_i$ for some left cosets cH_i $(c \in G)$ of G, and let ρ_i denote the equivalence relation on B^G whose only nonsingleton block is K_i. It is straightforward to check that ρ_i is a congruence of $(\mathbf{B}^\bullet)^G$. Furthermore, we have $(\mathbf{B}^\bullet)^G/\rho_i \in \mathcal{V}_i \setminus \mathcal{V}_{i+1}$, completing the proof. $\diamond$

Our last example is an essential step toward proving the fact that every algebra $\mathbf{A}$ satisfying condition (3) in Theorem 11 occurs in an exceptional pair $\mathbf{A}, \mathbf{A}^\bullet$.

Example 15. Let G be a group with $|G| > 1$, and let $\mathbf{B}$ be the largest strictly simple G^0-algebra of type **5**. Define a unary operation f on B by

$$f(x) = \begin{cases} x & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (x \in B).$$

The expansion $\mathbf{B}^f$ of $\mathbf{B}$ with f fails to generate a minimal variety.

Proof. Throughout the proof $\leq$ denotes the natural order of the semilattice $(G^0; \wedge)$. We will repeatedly need the following three facts stated in Claims B, C, and D.

Claim B. The term operations of $\mathbf{B}^f$ are monotone with respect to $\leq$, and for every term operation $\bar{h}(x_0, \ldots, x_{n-1})$ of $\mathbf{B}^f$ there exists a term operation $\bar{q}(x_0, \ldots, x_{n-1})$ of $\mathbf{B}$ such that $\bar{h}(x_0, \ldots, x_{n-1}) \leq \bar{q}(x_0, \ldots, x_{n-1})$ for all $x_0, \ldots, x_{n-1} \in B$.

Proof of Claim B. The operations of $\mathbf{B}$ as well as f are monotone with respect to $\leq$, and $f(x) \leq x$ for all $x \in B$. Therefore both parts of the claim follow by induction on the complexity of terms.

Claim C. If

(i) $\bar{t}(x_0, \ldots, x_{k-1})$ is a term operation of $\mathbf{B}$ such that $\bar{t}(c, 0, \ldots, 0) = c$ for some $c \in G = B \setminus \{0\}$, or

(ii) $\bar{t}(x_0, \ldots, x_{k-1})$ is a term operation of $\mathbf{B}^f$ such that $\bar{t}(x, 0, \ldots, 0) = x$ for all $x \in B$,

then $\bar{t}(x_0, \ldots, x_{k-1}) = x_0$ for all $x_0, \ldots, x_{k-1} \in B$.

Proof of Claim C. First assume (i) holds. Then $\bar{t}_0(x) = \bar{t}(x, 0, \ldots, 0)$ is a unary term operation of $\mathbf{B}$, and by the definition of a G^0-algebra the only unary term operations of $\mathbf{B}$ are 0 and l_a $(a \in G)$. Therefore $\bar{t}_0(c) = c$ implies that $\bar{t}(x, 0, \ldots, 0) = \bar{t}_0(x) = x$ for all $x \in B$.

So it remains to consider the case (ii). We use the monotonicity of the term operations of $\mathbf{B}^\bullet$ (see Claim B). The inequality $\bar{t}(x_0, x_1, \ldots, x_{k-1}) \geq \bar{t}(x_0, 0, \ldots, 0) = x_0$ yields that $\bar{t}(x_0, x_1, \ldots, x_{k-1}) = x_0$ for all $x_0 \in G = B \setminus \{0\}$ and all $x_1, \ldots, x_{k-1} \in B$. Furthermore, since $|G| > 1$ and $\bar{t}(0, x_1, \ldots, x_{k-1}) \leq \bar{t}(x_0, x_1, \ldots, x_{k-1}) = x_0$ if $x_0 \in G$, we conclude that $\bar{t}(0, x_1, \ldots, x_{k-1}) = 0$ holds for all $x_1, \ldots, x_{k-1} \in B$. This completes the proof of the claim.

Claim D. If $\bar{s}$ is a ternary term operation of $\mathbf{B}$ such that $\bar{s}(c, c, 0) = c$ and $\bar{s}(d, 0, d) = d$ for some elements $c, d \in G = B \setminus \{0\}$, then $\bar{s}(x, y, z) = x$ for all $x, y, z \in B$.

Proof of Claim D. Under the assumptions on $\bar{s}$ Claim C (i) implies that $\bar{s}(x, x, y) = \bar{s}(x, y, x) = x$ for all $x, y \in B$. Therefore we have $\bar{s}(1, 1, 1) \geq \bar{s}(1, 1, 0) = 1$, whence $\bar{s}(1, 1, 1) = 1$. Since $\bar{s}$ preserves μ_0, $\bar{s}$ applied to the quadruples $(1, 1, 1, 1)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ from μ_0 yields that $\big(1, 1, 1, \bar{s}(1, 0, 0)\big) \in \mu_0$. Thus $\bar{s}(1, 0, 0) = 1$, and hence Claim C (i) implies the required conclusion.

Analogously to Example 13, we want to refute the minimality of the variety $\mathcal{V}(\mathbf{B}^f)$ via showing that Claim A is true for $\mathbf{B}^f$ in place of $\mathbf{B}^\bullet$. We work toward this goal by determining first the unary term operations of $\mathbf{B}^f$.

Claim E. The unary term operations of $\mathbf{B}^f$ are exactly the following:

$$0, \quad l_a \ (a \in G), \quad \text{and} \quad l_b f l_c \ (b, c \in G).$$

Proof of Claim E. It suffices to verify that for every fundamental operation t' of $\mathbf{B}^f$ and for all elements $a_i, b_j, c_j \in G$ the unary operation

$$g(x) = t'\big(0, l_{a_0}(x), \ldots, l_{a_{l-1}}(x), l_{b_0} f l_{c_0}(x), \ldots, l_{b_{m-1}} f l_{c_{m-1}}(x)\big)$$

is one of those listed in Claim E. This is obvious if $t' = f$, so assume t' is a fundamental operation of $\mathbf{B}$. Using the term operation

$$t(x, y_0, \ldots, y_{m-1}) = t'\big(0, l_{a_0}(x), \ldots, l_{a_{l-1}}(x), l_{b_0}(y_0), \ldots, l_{b_{m-1}}(y_{m-1})\big)$$

of $\mathbf{B}$ we see that

$$g(x) = t\big(x, f l_{c_0}(x), \ldots, f l_{c_{m-1}}(x)\big).$$

By identifying variables in t and by adding fictitious variables to t if necessary we may assume that $m = |G|$ and $G = \{c_0, \ldots, c_{m-1}\}$.

The list in Claim E contains all operations assuming 0 for all but one element of G, therefore suppose $g(x) \neq 0$ for at least two distinct elements $x \in G$, say $x = c_0$ and $x = c_1$. Our aim is to show that $g \in L_G$. By Claim B there exists an element $d \in G$ such that $g(x) \leq l_d(x)$ for all $x \in B$. Thus, replacing g by $l_{d^{-1}} g$, we may assume without loss of generality that $g(x) \leq x$ for all $x \in B$. In particular, this implies that

$$t(c_0^{-1}, 1, 0, \ldots, 0) = g(c_0^{-1}) = c_0^{-1} \quad \text{and} \quad t(c_1^{-1}, 0, 1, 0, \ldots, 0) = g(c_1^{-1}) = c_1^{-1}.$$

Then for the ternary term operation $s(x, y, z) = t\big(x, l_{c_0}(y), l_{c_1}(z), 0, \ldots, 0\big)$ of $\mathbf{B}$ we have $s(c_0^{-1}, c_0^{-1}, 0) = c_0^{-1}$ and $s(c_1^{-1}, 0, c_1^{-1}) = c_1^{-1}$. Therefore applying Claim D we get that $s(1, 0, 0) = 1$, and hence $t(1, 0, \ldots, 0) = 1$, so by Claim C (i) t is the $(m+1)$-ary projection onto its first variable. Thus $g = l_1$, and the proof of Claim E is complete.

In case $|G| = 2$ knowledge of the unary term operations will not be enough to get the analogue of Claim A, therefore in this case we have to determine the binary term operations of $\mathbf{B}^f$ as well. Let $G = \{1, 2\}$, and consider the following binary operation $\circ$ and quaternary operation τ on B:

$$x \circ y = \begin{cases} x & \text{if } x, y \in G \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in B),$$

$$\tau(x, y, z, u) = \begin{cases} x & \text{if } x = y \text{ or } x = l_2(y) = z = u \\ 0 & \text{otherwise} \end{cases} \qquad (x, y, z, u \in B).$$

It is straightforward to check that $\circ$ and τ are term operations of $\mathbf{B}$. Furthermore, the following is a complete list of the binary term operations of $\mathbf{B}$:

$$0,\ l_a(x),\ l_a(y),\ l_a(x) \wedge l_b(y),\ l_a(x) \circ y,\ \text{and } l_a(y) \circ x \quad (a, b \in G).$$

We will call a binary operation h on B *absorptive* if $h(x, 0) = 0 = h(0, x)$ for all $x \in B$, and *essentially unary* if it depends on at most one of its variables.

Claim F. For $|G| = 2$ and for a binary operation h on B the following conditions are equivalent:

(i) h is a term operation of $\mathbf{B}^f$;
(ii) h has the following two properties:
   (a) there exists a binary term operation q of $\mathbf{B}$ such that $h(x,y) \leq q(x,y)$ for all $x, y \in B$, and
   (b) h is either absorptive or essentially unary;
(iii) h is one of the following operations:

$$q(x,y), \quad q\big(fl_a(x),y\big), \quad q\big(x,fl_b(y)\big) \quad \text{or} \quad q\big(fl_a(x),fl_b(y)\big)$$

for some $a, b \in G$ and some binary term operation q of $\mathbf{B}$; or

$$l_a\tau\big(l_b(x), l_c(y), fl_b(x), fl_{2\cdot c}(y)\big)$$

for some $a, b, c \in G$.

Proof of Claim F. (iii)⇒(i) is obvious and (i)⇒(ii)(a) is established in Claim B. The implication (iii)⇒(ii)(b) follows easily by inspecting the definition of τ and the list of binary term operations of $\mathbf{B}$ (which are all either absorptive or essentially unary). Thus (iii)⇒(ii).

We show that (ii)⇒(iii). Let us call two binary operations h and h' L_G*-equivalent* if $h'(x,y) = l_a h\big(l_b(x), l_c(y)\big)$ for some $a, b, c \in G$. Clearly, this yields an equivalence relation $\approx$ on the set of binary term operations of $\mathbf{B}^f$. Moreover, each family of operations described in (ii), resp. (iii) is a union of full $\approx$-blocks. So it suffices to select a representative from each $\approx$-block of operations h satisfying condition (ii), and exhibit it in a form described in (iii). If h is essentially unary, then this is easily done, using Claim E. If h is absorptive, then $h(x,y)$ or $h(y,x)$ is $\approx$-equivalent to one of the absorptive operations $\chi(x,y)$ indicated below by the 2×2 matrices $\big(\chi(i,j)\big)_{i,j\in G}$ of their 'essential' values:

$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 2 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 0 & 2 \end{matrix}$
$f(x) \wedge f(y)$	$f(x) \circ y$	$x \circ f(y)$	$x \wedge y$	$\tau\big(x, y, f(x), fl_2(y)\big)$

Below each matrix we present a corresponding term operation form the list in (iii).

So far we have shown that (i)⇐(ii)⇔(iii), therefore what remains to prove is that $\mathbf{B}^f$ has no other binary term operations than those described by the equivalent conditions (ii) and (iii). That is, we have to verify that for every fundamental operation t' of $\mathbf{B}^f$ and for all operations $h_i(x,y)$ $(i = 0, 1, \ldots)$ satisfying the equivalent conditions (ii), (iii), the operation

$$h(x,y) = t'\big(h_0(x,y), h_1(x,y), \ldots\big)$$

also satisfies these conditions. If $t' = f$, then this is obvious by (ii). Now assume that t' is a fundamental operation of $\mathbf{B}$. Using the description in (iii) for the operations $h_i(x,y)$ we see that $\mathbf{B}$ has a term operation t such that

$$h(x,y) = t\big(x, y, fl_1(x), fl_2(x), fl_1(y), fl_2(y)\big).$$

By Claim B, h satisfies (ii)(a). Hence, by looking at the list of binary term operations of $\mathbf{B}$ we get that at least one of the unary operations $h(x,0)$, $h(0,x)$ is the constant 0. Let, say, $h(0,y)=0$ for all $y\in B$. If h is absorptive, then (ii) holds, so we are done.

Suppose from now on that h is not absorptive, that is, the unary term operation $h(x,0)$ of $\mathbf{B}^f$ is not the constant 0. We want to prove that h does not depend on its second variable. Without loss of generality we can replace h with an operation $\approx$-equivalent to it, therefore we may assume that $h(1,0)=1$, whence from the inequality $h(1,y)\geq h(1,0)$ it follows that $h(1,y)=1$ for all $y\in B$. If $h(2,0)=h(2,1)=h(2,2)=0$, then h does not depend on y, so we are done. Otherwise, in view of the inequality $h(2,0)\leq h(2,y)$ for all $y\in B$, at least one of $h(2,1)$, $h(2,2)$ is distinct from 0. Replacing $h(x,y)$ by $h(x,l_2(y))$ if necessary we may assume that $h(2,1)\neq 0$. Notice that the only possible binary term operation q of $\mathbf{B}$ such that $h(x,y)\leq q(x,y)$ for all $x,y\in B$ (cf. Claim B) is the first projection, therefore we have $h(x,y)\leq x$ for all $x,y\in B$. Thus $h(2,1)=2$. Hence for the ternary term operation $s(x,y,z)=t\big(x,l_2(y),z,l_2(y),l_2(y),0\big)$ of $\mathbf{B}$ we have $s(1,0,1)=t(1,0,1,0,0,0)=h(1,0)=1$ and $s(2,2,0)=t(2,1,0,1,1,0)=h(2,1)=2$. Now Claim D implies that $s(1,0,0)=1$, that is, $t(1,0,0,0,0,0)=1$, and hence by Claim C (i) t is the 6-ary projection onto its first variable. Thus $h(x,y)=x$ for all $x,y\in B$. This completes the proof of Claim F.

Now we finish the proof of the assertion in Example 15 by showing that the analogue of Claim A is true for $\mathbf{B}^f$. As in Example 13, this implies that $\mathbf{B}^f$ does not generate a minimal variety. Let t_0 be a binary term of $\mathbf{B}^f$ such that $\mathbf{B}^f\models x=t_0\big(x,fg_0(x)\big)$ for some unary term g_0. Since the term operation fg_0 occurs in the list in Claim E, there is an element $b\in G$ such that $fg_0(x)=0$ for all $x\in G\setminus\{b\}$. Hence $x=t_0(x,0)$ for all $x\in G\setminus\{b\}$. However, $t_0(x,0)$ is a unary term operation of $\mathbf{B}^f$. In case $|G|>2$ the only unary term operation of $\mathbf{B}^f$ with these properties is l_1, that is, we have $t_0(x,0)=x$ for all $x\in B$. By Claim C (ii) it follows that $\mathbf{B}^f\models x=t_0(x,y)$, as required. Finally, if $|G|=2$, then the description of binary term operations in Claim F (ii) immediately implies that the only possibility for t_0 is that $\mathbf{B}^f\models x=t_0(x,y)$. $\diamond$

Now we are in a position to settle the question whether every strictly simple algebra $\mathbf{A}$ generating a minimal variety $\mathcal{V}(\mathbf{A})$ and satisfying the conditions in Theorem 11 (3) has an expansion $\mathbf{A}^\bullet$ such that $\mathcal{V}(\mathbf{A}^\bullet)$ is not minimal. First, in Lemma 16, we reduce the general question to an analogous question on the corresponding G^0-algebras $e(\mathbf{A})$ (cf. Theorem 3), and then combine this observation with the result in Example 15 to get an affirmative answer.

Lemma 16. *Let $\mathbf{A}$ be nonabelian a strictly simple algebra generating a minimal variety, and let e be a minimal idempotent of $\mathbf{A}$. Furthermore, let f be a unary operation on A such that f is a minimal idempotent of the expansion $\mathbf{A}^f$ of $\mathbf{A}$ and $\mathbf{A}^f\models ef=fe=f$. Then f is a minimal idempotent of $e(\mathbf{A})^f$, and $\mathbf{A}^f$ generates a minimal variety if and only if $e(\mathbf{A})^f$ generates a minimal variety.*

Proof. The arguments in the first two paragraphs of the proof of Lemma 7 show that f is a minimal idempotent of $e(\mathbf{A}^f)$, and f is nonconstant throughout $\mathcal{V}(\mathbf{A}^f)$ if and only if f is nonconstant throughout $\mathcal{V}\big(e(\mathbf{A}^f)\big)$. Hence by Theorem 1 $\mathbf{A}^f$ generates a minimal variety if and only if $e(\mathbf{A}^f)$ does. In view of Lemma 4 (2)(ii) the algebra $e(\mathbf{A}^f)$ is term equivalent to $e(\mathbf{A})^f$, therefore the claims of the lemma follow. $\diamond$

Corollary 17. *Let* $\mathbf{A}$ *be strictly simple algebra generating a minimal variety. If* $\mathbf{A}$ *is of type* $\mathbf{5}$ *and has a nontrivial automorphism group, then* $\mathbf{A}$ *has an expansion* $\mathbf{A}^\bullet$ *which does not generate a minimal variety.*

Proof. By Theorem 3 the assumptions on $\mathbf{A}$ yield that $e(\mathbf{A})$ is a G^0-algebra for some group G such that $|G| > 1$. Let us expand $e(\mathbf{A})$ to get an algebra $\mathbf{B}$ which is, up to term equivalence, the largest strictly simple G^0-algebra of type $\mathbf{5}$ on the set $e(A) = G^0 = B$. By Example 15 there exists a unary operation f on B with two-element range such that f is a minimal idempotent of $\mathbf{B}^f$ and $\mathbf{B}^f$ does not generate a minimal variety. Clearly, f is a minimal idempotent of $e(\mathbf{A})^f$ as well, moreover, by Lemma 6 and Theorem 1 $e(\mathbf{A})^f$ does not generate a minimal variety.

It is easy to see that efe is a (well-defined!) unary operation on A with $efe|_{e(A)} = f$ and with the same range as f, so it can be considered as the operation on A denoted by the operation symbol f. Clearly, f is a minimal idempotent of $\mathbf{A}^f$ and $\mathbf{A}^f \models ef = fe = f$. Hence our claim follows from Lemma 16. $\diamond$

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