A Completeness Criterion for Semi-Affine Algebras

Ágnes Szendrei

Bolyai Institute, József Attila University
Szeged, Hungary

Abstract

A Rosenberg-type completeness criterion is proved for a semi-affine algebra to be a simple affine algebra.

Introduction

Affine algebras, i.e. algebras polynomially equivalent to modules, and their reducts, called semi-affine algebras, are of interest in universal algebra as well as in multiple-valued logic: in universal algebra they play an important role in the study of congruence modular varieties and in the structure theory of finite algebras, while in multiple-valued logic they come up most naturally as algebras whose clones are contained in one of the maximal clones of linear type in Rosenberg's Theorem [3].

In both of these areas it is a basic question: what operations can be constructed from a given set F of operations by composition, or, alternatively, given a clone (a composition closed set) $\mathcal{C}$ of operations, under what conditions a subset F of $\mathcal{C}$ generates $\mathcal{C}$ (via composition); if it does, then F is said to be *complete* in $\mathcal{C}$.

In a more algebraic setting, the question is: under what conditions a reduct $(A; F)$ of an algebra $(A; \mathcal{C})$ is term equivalent to $(A; \mathcal{C})$. The most important result of this type is Rosenberg's Theorem [3] solving the problem for primal algebras (i.e., for $\mathcal{C}$ the clone of all operations on a finite set A), and it is typical, too, in that completeness is characterized in terms of exluded compatible relations of $(A; F)$. It is clear that the structure of the algebra $(A; \mathcal{C})$ and its reducts $(A; F)$ might be essential in these considerations, therefore an algebraic approach proves often useful.

This research is partially supported by the Hungarian National Foundation for Scientific Research, grant no. 1903.

In this paper we consider a semi-affine algebra 'complete' if it is a simple affine algebra, and investigate the question under what conditions a semi-affine algebra is complete. We get the following Rosenberg-type completeness criterion (Theorem 2.1): a finite algebra $\mathbf{A}$ that is semi-affine with respect to an elementary Abelian group $\widehat{A}$ is complete if and only if $\mathbf{A}$ admits no nontrivial congruence of $\widehat{A}$ and no q-regular relation corresponding to a q-regular family of congruences of $\widehat{A}$, and $\mathbf{A}$ is not isomorphic to a matrix power of a unary semi-affine algebra.

We note that Słupecki-type completeness criteria for reducts of certain simple affine algebras were proved earlier in [8].

Preliminaries

An *algebra* is a pair $\mathbf{A} = (A; F)$ with A a nonvoid set called the *universe* of $\mathbf{A}$, and F a set of finitary operations on A called the set of *fundamental operations* of $\mathbf{A}$. An operation f on A is a *term operation* [*polynomial operation*] of $\mathbf{A}$ if f can be constructed from the fundamental operations of $\mathbf{A}$ and from projection operations [from the fundamental operations of $\mathbf{A}$, from projections, and from constant operations] via composition.

A set $\mathcal{C}$ of operations on A is called a *clone* if it contains the projections and is closed under composition. Obviously, the term operations [polynomial operations] of any algebra form a clone.

If not stated otherwise, algebras are denoted by boldface capitals and their universes by the corresponding letters in italics. The clone of term operations [the set of n-ary term operations] of an algebra $\mathbf{A}$ is denoted by $\operatorname{Clo}\mathbf{A}$ [resp., $\operatorname{Clo}_n \mathbf{A}$]. Similarly, the clone of polynomial operations [the set of n-ary polynomial operations] of $\mathbf{A}$ is denoted by $\operatorname{Pol}\mathbf{A}$ [resp., $\operatorname{Pol}_n \mathbf{A}$].

We will call an algebra $\mathbf{A}$ *surjective* if every fundamental operation of $\mathbf{A}$ is surjective. For algebras $\mathbf{A} = (A; F)$ and $\mathbf{A}' = (A'; F')$, we say that $\mathbf{A}$ is a *reduct* [*polynomial reduct*] of $\mathbf{A}'$ if $A = A'$ and $F \subseteq \operatorname{Clo} \mathbf{A}'$ [$F \subseteq \operatorname{Pol} \mathbf{A}'$]. The algebras $\mathbf{A} = (A; F)$ and $\mathbf{A}' = (A'; F')$ are called *term equivalent* [*polynomially equivalent*] if $A = A'$ and $\operatorname{Clo} \mathbf{A} = \operatorname{Clo} \mathbf{A}'$ [$\operatorname{Pol} \mathbf{A} = \operatorname{Pol} \mathbf{A}'$].

For a set N, let T_N, S_N, and C_N denote the full transformation monoid on N, the full symmetric group on N and the set of (unary) constant operations on N, respectively. The identity mapping and the equality relation on N are denoted by id and Δ, respectively (N will be clear from the context). For convenience we identify every natural number n with the set $n = \{0, 1, \ldots, n-1\}$.

For a set A and for $k \geq 1$, the nonvoid subsets of A^k will also be called *k-ary relations* (on A), and for an algebra $\mathbf{A}$ the universes of subalgebras of $\mathbf{A}^k$ will be called *compatible relations* of $\mathbf{A}$. An operation f on A is said to *preserve* a relation ρ if ρ is a compatible relation of the algebra $(A; f)$.

We say that an algebra $\mathbf{A}$ is *semi-affine with respect to an Abelian group* $\widehat{A} = (A; +)$ if $\mathbf{A}$ and $\widehat{A}$ have the same universe and

$$Q_{\widehat{A}} = \{(a, b, c, d) \in A^4 \colon\ a - b + c = d\}$$

is a compatible relation of $\mathbf{A}$ (or equivalently, the operations of $\mathbf{A}$ commute with $x - y + z$). Furthermore, $\mathbf{A}$ is said to be *affine with respect to* $\widehat{A}$ if it is semi-affine with respect to $\widehat{A}$ and, in addition, $x - y + z$ is a term operation of $\mathbf{A}$. It is well known (cf. [10; 2.1, 2.7– 2.8]) that

— an algebra $\mathbf{A}$ is semi-affine with respect to an Abelian group $\widehat{A}$ if and only if $\mathbf{A}$ is a polynomial reduct of the module ${}_{(\operatorname{End} \widehat{A})}\widehat{A}$ (i.e. $\widehat{A}$ considered as a module over its endomorphism ring $\operatorname{End} \widehat{A}$), and

— $\mathbf{A}$ is affine with respect to $\widehat{A}$ if and only if $\mathbf{A}$ is polynomially equivalent to a module ${}_R\widehat{A}$ for some subring R of $\operatorname{End} \widehat{A}$.

Let $q \geq 3$. A family $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$ ($m \geq 1$) of equivalence relations on A is called *q-regular* if each Θ_i ($0 \leq i \leq m-1$) has exactly q blocks and $\Theta_T = \Theta_0 \cap \ldots \cap \Theta_{m-1}$ has exactly q^m blocks. A relation on A is called *q-regular* if it is of the form

$$\lambda_T = \{(a_0, \ldots, a_{q-1}) \in A^q \colon \text{ for all } i\ (0 \leq i \leq m-1),\ a_0, \ldots, a_{q-1} \text{ are not pairwise incongruent modulo } \Theta_i\}$$

for a q-regular family T of equivalence relations on A.

Let U be a q-element set and $m \geq 1$. The kernels of the m distinct projections $U^m \to U$ form a q-regular family of equivalences on U^m, which will be called the *standard q-regular family of equivalences on U^m*; the corresponding q-regular relation is called the *standard q-regular relation on U^m*. It is well known that the *mth matrix power* $\mathbf{U}^{[m]}$ of any unary algebra $\mathbf{U} = (U; F)$ admits the standard q-regular relation as a compatible relation. We recall that the universe of $\mathbf{U}^{[m]}$ is U^m, and its operations are exactly all operations $h^\sigma_\mu[g_0, \ldots, g_{m-1}]$ defined for arbitrary mappings $\sigma: m \to m$, $\mu: m \to n$ and $g_0, \ldots, g_{m-1} \in \mathrm{Clo}_1\, \mathbf{U}$ as follows: for $x_i = (x_i^0, \ldots, x_i^{m-1}) \in U^m$ $(0 \leq i \leq n-1)$,

$$h^\sigma_\mu[g_0, \ldots, g_{m-1}](x_0, \ldots, x_{n-1}) \\ = (g_0(x^{0\sigma}_{0\mu}), \ldots, g_{m-1}(x^{(m-1)\sigma}_{(m-1)\mu})).$$

The mappings σ, μ will be called the *component mapping* and the *variable mapping* of $h^\sigma_\mu[g_0, \ldots, g_{m-1}]$, respectively. For unary operations the subscript indicating the variable mapping $m \to 1$ will be omitted.

In the lemma below we collect some well-known facts on finite algebras admitting q-regular compatible relations.

Lemma 1.1. *Let $\mathbf{A} = (A; F)$ be a finite algebra, and let $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$ be a q-regular family of equivalence relations on A such that λ_T is a compatible relation of $\mathbf{A}$.*

(1.1.i) *$T/\Theta_T = \{\Theta_0/\Theta_T, \ldots, \Theta_{m-1}/\Theta_T\}$ is a q-regular family of equivalences on A/Θ_T, and there exists a bijection $\varphi: A/\Theta_T \to q^m$ carrying T/Θ_T into the standard q-regular family of equivalences on q^m.*

(1.1.ii) *If $f \in F$ is an n-ary operation whose range meets each block of some Θ_i, then there exist j, l $(0 \leq j \leq m-1,\ 0 \leq l \leq n-1)$ such that for $x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1} \in A$ we have*

$$f(x_0, \ldots, x_{m-1})\, \Theta_i\, f(y_0, \ldots, y_{m-1}) \text{ whenever } x_l\, \Theta_j\, y_l.$$

(1.1.iii) *If $\mathbf{A}$ is a surjective algebra, then*

(1) *Θ_T is a congruence of $\mathbf{A}$,*

(2) *the relation λ_{T/Θ_T} is a compatible relation of $\mathbf{A}/\Theta_T$, and*

(3) *the bijection φ yields an isomorphism between $\mathbf{A}/\Theta_T$ and a reduct of the matrix power $(q; S_q)^{[m]}$.*

The proof of (1.1.ii) can be found, e.g. in [5; Lemma 7.3]. The claims in (1.1.iii) are well-known consequences of (1.1.i) and (1.1.ii); see [6], [4]. We note that Rousseau [6] (cf. also [4]) proved (1.1.iii)(3) for the case $\Theta_T = \Delta$, however, in view of (1.1.iii)(1)–(2) the more general claim follows immediately from this special case.

Our basic tool in proving the main result of this paper is a strong version of Rosenberg's primal algebra characterization theorem [3]. Recall that a finite algebra $\mathbf{A}$ is called *quasiprimal* ([1], [2]) if every operation on A preserving the internal isomorphisms (i.e. isomorphisms between subalgebras) of $\mathbf{A}$ is a term operation of $\mathbf{A}$. Further, a k-ary relation ρ on A is said to be *central* if $\rho \neq A^k$, ρ is totally reflexive, totally symmetric, and there exists a $c \in A$ such that $(c, a_1, \ldots, a_{k-1}) \in \rho$ for all $a_1, \ldots, a_{k-1} \in A$.

Theorem 1.2. [11] *Let $\mathbf{A}$ be a finite simple algebra having no proper subalgebra. Then one of the following conditions holds:*

(1.2.a) *$\mathbf{A}$ is quasiprimal;*

(1.2.b) *$\mathbf{A}$ is affine with respect to an elementary Abelian p-group (p prime);*

(1.2.c) *$\mathbf{A}$ is isomorphic to a reduct of $(2; T_2)^{[m]}$ for some integer $m \geq 1$;*

(1.2.d) *$\mathbf{A}$ has a compatible q-regular relation for some $q \geq 3$;*

(1.2.e) *$\mathbf{A}$ has a compatible k-ary central relation for some $k \geq 2$;*

(1.2.f) *$\mathbf{A}$ has a compatible bounded partial order.*

Main results

On the base set p (p prime), $+$ and $\cdot$ will always denote addition, resp. multiplication modulo p. Further, we let L_p denote the set of all unary linear operations on p, i.e.

$$L_p = \{cx + a\colon\ 0 \leq c, a \leq p - 1\}.$$

Our main result is

Theorem 2.1. *For arbitrary finite algebra $\mathbf{A}$ that is semi-affine with respect to an elementary Abelian p-group $\widehat{A} = (A; +)$ (p prime), one of the following conditions holds:*

(2.1.a) *$\mathbf{A}$ is affine with respect to $\widehat{A}$;*

(2.1.b) *$\mathbf{A}$ has a nontrivial congruence which is a congruence of $\widehat{A}$;*

(2.1.c) *there is a group isomorphism $\widehat{A} \to (p; +)^m$ which is simultaneously an isomorphism between $\mathbf{A}$ and a reduct of $(p; L_p)^{[m]}$;*

(2.1.d) *$\mathbf{A}$ has a compatible relation λ_T for some q-regular family T of congruences of $\widehat{A}$ with $q > p$.*

Clearly, if for an algebra $\mathbf{A}$ as in Theorem 2.1 condition (2.1.c) or (2.1.d) holds, then $\mathbf{A}$ cannot be affine. Thus Theorem 2.1 yields a necessary and sufficient condition for simple semi-affine algebras to be affine.

Corollary 2.2. *Let* $\mathbf{A}$ *be a finite simple algebra that is semi-affine with respect to an elementary Abelian* p*-group* $\widehat{A} = (A; +)$ *(*p *prime). Then* $\mathbf{A}$ *is affine with respect to* $\widehat{A}$ *if and only if both of conditions (2.1.c) and (2.1.d) fail for* $\mathbf{A}$*.*

The rest of this section is devoted to the proof of Theorem 2.1.

Let $\widehat{A} = (A; +)$ be an Abelian group. The group $\{x + a\colon\ a \in A\}$ of all translations of $\widehat{A}$ will be denoted by $T(\widehat{A})$. For an algebra $\mathbf{A} = (A; F)$ that is semi-affine with respect to $\widehat{A}$, $\mathbf{A}^*$ will stand for the algebra $(A; F, T(\widehat{A}))$ arising from $\mathbf{A}$ by adding all translations of $\widehat{A}$ as unary operations.

Lemma 2.3. *For an algebra* $\mathbf{A}$ *that is semi-affine with respect to an Abelian group* $\widehat{A} = (A; +)$*,* $\mathbf{A}^*$ *is affine with respect to* $\widehat{A}$ *if and only if* $\mathbf{A}$ *is such.*

Proof. It is straightforward to check that the clone of $\mathbf{A}^*$ is

$$\begin{aligned}\operatorname{Clo}\mathbf{A}^* = \{ & \sum_{i=0}^{n-1} r_i x_i + a\colon\ n \geq 1,\ a \in A, \text{ and} \\ & \sum_{i=0}^{n-1} r_i x_i + a_0 \in \operatorname{Clo}\mathbf{A} \text{ for some } a_0 \in A\}.\end{aligned}$$

This implies the claim of the lemma. ◇

In view of this lemma, when we want to prove Theorem 2.1 via applying Theorem 1.2 for semi-affine algebras $\mathbf{A}$, we can always replace $\mathbf{A}$ by $\mathbf{A}^*$, i.e. we may assume that the translations in $T(\widehat{A})$ are operations of $\mathbf{A}$. Thus, in what follows, we look more closely at the relations preserved by all translations of an Abelian group.

For equivalence relations the following fact is easy and well-known.

Lemma 2.4. *For an Abelian group* $\widehat{A} = (A; +)$*, if* Θ *is an equivalence relation on* A *such that* Θ *is preserved by all translations in* $T(\widehat{A})$*, then* Θ *is a congruence of* $\widehat{A}$*.*

For studying q-regular relations we shall need a group theoretical result. First we recall some notions and notation. Let $G \subseteq S_N$ be a permutation group acting on a set N. The *orbits* of G are the minimal nonvoid subsets of N that are closed under all permutations in G. Clearly, the orbits of G yield a partition of N. We say that G is *transitive* on N if N is an orbit of G, and G acts *regularly* on N if it is transitive and no non-identity permutation in G has fixed points.

Let k and m be arbitrary positive integers, and let P be a subgroup of S_m. Clearly, the unary term operations $h^\sigma[g_0,\ldots,g_{m-1}]$ of $(k;S_k)^{[m]}$ with $\sigma \in P$ form a permutation group acting on the set k^m. In group theory this group is called the *general wreath product* of S_k and P, and is denoted by $S_k \operatorname{Wr} P$ (cf. [7; p. 272]). In $S_k \operatorname{Wr} P$ the elements $h^{\mathrm{id}}[g_0,\ldots,g_{m-1}]$ form a normal subgroup (isomorphic to the mth direct power of S_k), which will be denoted by $(S_k)^m$, while the elements $h^\pi[\mathrm{id},\ldots,\mathrm{id}]$ form a subgroup (isomorphic to P), which will be denoted by $\tilde{P}$. Obviuosly, $\tilde{P}$ is a *complement* of $(S_k)^m$ in $S_k \operatorname{Wr} P$ in the sense that $(S_k)^m \cap \tilde{P} = \{\mathrm{id}\}$ and $(S_k)^m \tilde{P} = S_k \operatorname{Wr} P$.

If P is a regular permutation group on m, then $S_k \operatorname{Wr} P$ essentially coincides with the so-called *complete wreath product* of S_k and P (cf. [7; pp. 270, 272]).

Lemma 2.5. *Let G be a subgroup of the permutation group $S_q \operatorname{Wr} S_m$ where q is a power of a prime number p and m is an arbitrary positive integer. If G is an elementary Abelian p-group which acts regularly on q^m, then G is a subgroup of $(S_q)^m$.*

Proof. Let G be a subgroup of $S_q \operatorname{Wr} S_m$ satisfying the assumptions of the lemma, and let P denote the group of component mappings of permutations in G. Thus G is an elementary Abelian p-subgroup of $S_q \operatorname{Wr} P$ acting regularly on q^m. Let $I_0,\ldots,I_{t-1}$ denote the orbits of P. Then each member $h^\sigma[g_0,\ldots,g_{m-1}]$ of G acts componentwise, via $h^{\sigma|_{I_l}}[g_i\colon i \in I_l]$ $(l = 0,\ldots,t-1)$ on the set $q^m = q^{I_0} \times \ldots \times q^{I_{t-1}}$. By the well-known fact that every commutative, transitive permutation group is regular, it follows that in each component we have a regular permutation group. Consequently, for cardinality reasons, G splits into a direct product of t regular, elementary Abelian p-subgroups of $S_q \operatorname{Wr} S_{I_l}$ $(l = 0,\ldots,t-1)$, respectively. Hence it suffices to prove that if P is transitive, then $m = 1$.

Assume that P is transitive. Since P is a homomorphic image of G, therefore P is an elementary Abelian p-group. From the transitivity and commutativity of P it follows that P is regular as well.

Consider the subgroup $G_0 = G \cap (S_q)^m$ of G. Since G is finite and Abelian, it has a subgroup P_0 that is a complement of G_0 in G (that is, $G_0 \cap P_0 = \{\mathrm{id}\}$ and $G_0 P_0 = G$). Clearly, for each $\sigma \in P$, P_0 contains exactly one permutation with component mapping σ. Thus P_0 is a complement of $(S_q)^m$ in the complete wreath product $S_q \operatorname{Wr} P$. It is known (cf. [7; 10.7 in Chapter 2]) that any two complements of $(S_q)^m$ in $S_q \operatorname{Wr} P$ — specifically $\tilde{P}$ and P_0 — are conjugate. Since all assumptions on G and the required conclusion as well are invariant under conjugation, we may assume without loss of generality that $\tilde{P} \subseteq G$. However, as G is Abelian, G is contained in the centralizer of $\tilde{P}$ in $S_q \operatorname{Wr} P$, which is easily seen to be equal to

$$\{h^\sigma[g, \ldots, g]\colon\ g \in S_q,\ \sigma \in P\}$$

(cf. [7; Exercise 2 on p. 277]). Obviously, this group is transitive only if $m = 1$, completing the proof. ⋄

Lemma 2.6. *Let $\widehat{A}$ be a finite elementary Abelian p-group (p prime), and let $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$ be a q-regular family of equivalences on A such that λ_T is preserved by all translations in $T(\widehat{A})$. Then*

(2.6.i) *$\Theta_0, \ldots, \Theta_{m-1}$, and hence their intersection Θ_T as well, are congruences of $\widehat{A}$, and*

(2.6.ii) *for any elementary Abelian p-group $(q; +)$, there exists an isomorphism $\widehat{A}/\Theta_T \to (q; +)^m$ carrying T/Θ_T into the standard q-regular family of equivalences on q^m.*

Proof. Consider the unary algebra $\mathbf{A} = (A; T(\widehat{A}))$. By our assumption λ_T is a compatible relation of $\mathbf{A}$. Since $\mathbf{A}$ is surjective, we get from Lemma 1.1 (1.1.iii)(1) that Θ_T is a congruence of $\mathbf{A}$. So by Lemma 2.4 Θ_T is a congruence of $\widehat{A}$. Applying Lemma 1.1 (1.1.i) and (1.1.iii)(3) we get also that there exists an isomorphism φ between the algebra $\mathbf{A}/\Theta_T = (A/\Theta; T(\widehat{A}/\Theta))$ and a reduct of the matrix power $(q; S_q)^{[m]}$ such that φ carries T/Θ_T into the standard q-regular family $\{\Phi_0, \ldots, \Phi_{m-1}\}$ of equivalences on q^m. Let G denote the subgroup of S_{q^m} corresponding to the group $T(\widehat{A}/\Theta)$ under φ. Clearly, G is a subgroup of $S_q \operatorname{Wr} S_m$. Furthermore, by construction, G is an elementary Abelian p-group, which acts transitively on q^m. Now Lemma 2.5 states that $G \subseteq (S_q)^m$, whence it follows that $\Phi_0, \ldots, \Phi_{m-1}$ are congruences of $(q^m; G)$. Via the isomorphism φ we get that $\Theta_0/\Theta_T, \ldots, \Theta_{m-1}/\Theta_T$ are congruences of $\mathbf{A}/\Theta_T$, and hence $\Theta_0, \ldots, \Theta_{m-1}$ are congruences of $\mathbf{A}$. Now by Lemma 2.4 we conclude that (2.6.i) holds.

Since the family T of congruences of $\widehat{A}$ is q-regular, the natural embedding

$$\widehat{A}/\Theta_T \to \widehat{A}/\Theta_0 \times \ldots \times \widehat{A}/\Theta_{m-1}$$

is an isomorphism, and all quotient groups on the right are elementary Abelian p-groups with q elements. Up to isomorphism, we can replace them with the given group $(q; +)$, and the requirements in (2.6.ii) obviously hold. ⋄

Lemma 2.7. *Let $\mathbf{A}$ be a finite algebra that is semiaffine with respect to an elementary Abelian p-group $\widehat{A} = (A; +)$ (p prime), and let T be a p-regular family of congruences of $\widehat{A}$ such that λ_T is a compatible relation of $\mathbf{A}^*$. Then*

(2.7.i) Θ_T *is a congruence of* $\mathbf{A}$*, and*

(2.7.ii) *if* $\Theta_T = \Delta$*, then there is a group isomorphism* $\widehat{A} \to (p;+)^m$ *which is simultaneously an isomorphism between* $\mathbf{A}$ *and a reduct of* $(p; L_p)^{[m]}$.

Proof. Let $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$. By the previous lemma these equivalences are congruences of $\widehat{A}$, and so is their intersection Θ_T.

To prove (2.7.i) let f be an n-ary operation of $\mathbf{A}$, and let $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1} \in A$ be arbitrary elements of $\mathbf{A}$ such that $x_k \,\Theta_T\, y_k$ for all $0 \leq k \leq n-1$. Let $0 \leq i \leq m-1$. Assume first that the range of f meets at least two blocks of Θ_i. Since $\widehat{A}/\Theta_i$ is a p-element cyclic group and $\mathbf{A}$ is semi-affine with respect to $\widehat{A}$, it is clear that the range of f meets each block of Θ_i. Thus we get from Lemma 1.1 (1.1.ii) that $f(x_0, \ldots, x_{n-1}) \,\Theta_i\, f(y_0, \ldots, y_{n-1})$. The same conclusion is obvious, if the range of f meets only one block of Θ_i. Since i was arbitrary, we conclude that $f(x_0, \ldots, x_{n-1}) \,\Theta_T\, f(y_0, \ldots, y_{n-1})$, as required.

Now let $\Theta_T = \Delta$. By Lemma 2.6 (2.6.ii) there exists an isomorphism $\widehat{A} \to (p;+)^m$ carrying T into the standard p-regular family of equivalences on p^m. Let $\mathbf{B} = (p^m; F)$ be the algebra corresponding to $\mathbf{A}$ under this isomorphism. Notice that the standard p-regular relation on p^m is a compatible relation of $\mathbf{B}$, and apply Lemma 1.1 (1.1.ii) to each operation f of $\mathbf{B}$. Let, say, f be n-ary. For $b \in p^m$ the components of b will be denoted by $b^0, \ldots, b^{m-1}$. Let $0 \leq i \leq m-1$ be arbitrary. As in the previous paragraph, we see that the set of ith components of $f(b_0, \ldots, b_{n-1})$ as the arguments run over all elements of p^m is either p or a one-element set. In the first case we get from (1.1.ii) that there exist indices j_i, l_i $(0 \leq j_i \leq m-1,\ 0 \leq l_i \leq n-1)$ and a permutation $g_i \in S_p$ such that the ith component of $f(b_0, \ldots, b_{n-1})$ equals $g_i(b_{l_i}^{j_i})$ for all $b_0, \ldots, b_{n-1} \in p^m$. In the second case the same holds with g_i constant (and j_i, l_i arbitrary). Thus $f = h_\mu^\sigma[g_0, \ldots, g_{m-1}]$ where σ and μ are the mappings $\sigma\colon m \to m,\ i \mapsto j_i$ and $\mu\colon m \to n,\ i \mapsto l_i$. Hence $\mathbf{B}$ is a reduct of $(p; S_p \cup C_p)^{[m]}$. Taking into consideration that $\mathbf{B}$ is semi-affine with respect to $(p;+)^m$, one can easily derive that $\mathbf{B}$ is a reduct of $(p; L_p)^{[m]}$, completing the proof. ◇

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathbf{A}$ be a finite algebra that is semi-affine with respect to an elementary Abelian p-group $\widehat{A} = (A;+)$ (p prime), and consider the algebra $\mathbf{A}^*$. Because of the translations, $\mathbf{A}^*$ has no proper subalgebra, no compatible bounded partial order and no compatible central relation. If $\mathbf{A}^*$ is not simple, then by Lemma 2.4 (2.1.b) trivially holds, so assume $\mathbf{A}^*$ is simple. Now we can apply Theorem 1.2

for $\mathbf{A}^*$. Since a semi-affine algebra cannot be quasiprimal, condition (1.2.b), (1.2.c) or (1.2.d) in Theorem 1.2 holds for $\mathbf{A}^*$.

Assume first that (1.2.b) holds for $\mathbf{A}^*$. It is well known that if an algebra is affine with respect to an Abelian group, then (because of the term operation $x - y + z$) this group is uniquely determined up to the choice of the element 0. Thus (2.1.a) holds for $\mathbf{A}^*$ and hence for $\mathbf{A}$ as well.

Now let us consider the case when (1.2.c) holds for $\mathbf{A}^*$, that is, there exists an isomorphism φ between $\mathbf{A}^*$ and a reduct of the matrix power $(2; T_2)^{[m]}$ (hence $p = 2$). Let G denote the subgroup of S_{2^m} corresponding to the group $T(\widehat{A})$ under φ. Clearly, G is a subgroup of $S_2 \operatorname{Wr} S_m$, and G is an elementary Abelian 2-group acting transitively on 2^m. By Lemma 2.5 we have $G \subseteq (S_2)^m$, so for cardinality reasons $G = (S_2)^m$. Let ω be the image of $0 \in A$ under φ, and let τ be the translation $x + \omega$ of the Abelian group $(2; +)^m$. It is straightforward to check that the mapping $\varphi\tau$ is a group isomorphism $\widehat{A} \to (2; +)^m$ which is simultaneously an isomorphism between $\mathbf{A}$ and a reduct of $(2; T_2)^{[m]}$. Obviously, $T_2 = L_2$, hence (2.1.c) holds with $p = 2$.

Finally, suppose condition (1.2.d) holds for $\mathbf{A}^*$, and let T be a q-regular family of equivalences on A such that λ_T is a compatible relation of $\mathbf{A}^*$. Obviously, λ_T is preserved by all translations in $T(\widehat{A})$, so by Lemma 2.6 T consists of congruences of $\widehat{A}$. It follows now that q is a power of p. If $q > p$, then (2.1.d) trivially holds, while if $q = p$, then by Lemma 2.7 and by the simplicity of $\mathbf{A}$ we have $\Theta_T = \Delta$ and condition (2.1.c) holds for $\mathbf{A}$. $\diamond$

Concluding remarks

1. For an elementary Abelian p-group $\widehat{A} = (A; +)$ (p prime) let $\mathcal{Q}(\widehat{A})$ denote the clone consisting of all operations on A preserving the relation $Q_{\widehat{A}}$; in other words, $\mathcal{Q}(\widehat{A})$ is the largest one among the clones of those algebras on A that are semi-affine with respect to $\widehat{A}$. These clones constitute one of the six classes of maximal clones in Rosenberg's theorem [3]. Making use of Theorem 2.1 one can easily determine the maximal subclones of $\mathcal{Q}(\widehat{A})$. There are three types:

- — those containing the operation $x - y + z$; to find them explicitly one can apply the description of the clones of affine algebras (cf. [9], [10; 2.6]);
- — the inverse images of $\operatorname{Clo}(p; L_p)^{[m]}$ under all isomorphisms $\widehat{A} \to (p; +)^m$; and

— for each q-regular family T of congruences of $\widehat{A}$ with $q > p$, the clone of all operations in $\mathcal{Q}(\widehat{A})$ preserving λ_T.

2. Let $\mathbf{A}$ be a surjective, finite, simple algebra that is semi-affine with respect to an elementary Abelian p-group $\widehat{A} = (A; +)$ (p prime). Combining Corollary 2.2 and the claims in Lemma 1.1 (1.1.iii) we get that either $\mathbf{A}$ is affine with respect to $\widehat{A}$, or it is isomorphic to a reduct of $(q; S_q)^{[m]}$ for some power q of p. An application of this observation yields an alternative proof for the result shown in [12] stating that all surjective, finite, simple algebras of type $\mathbf{2}$ are affine.

Acknowledgement. I am indebted to P. P. Pálfy for pointing out reference [7] for the proof of Lemma 2.5.

References

[1] A. F. Pixley, Functionally complete algebras generating distributive and permutable classes, *Math. Z.* **114** (1970), 361–372.

[2] A. F. Pixley, The ternary discriminator function in universal algebra, *Math. Ann.* **191** (1971), 167–180.

[3] I. G. Rosenberg, Über die funktionale Vollständigkeit in den mehrwertigen Logiken (Struktur der Funktionen von mehreren Veränderlichen auf endlichen Men-

gen), *Rozpravy Československe Akad. Věd Řada Mat. Přírod. Věd* **80** (1970), 9–93.

[4] I. G. Rosenberg, Functional completeness of single generated or surjective algebras, in: *Finite Algebra and Multiple-Valued Logic* (Proc. Conf. Szeged, 1979), Colloq. Math. Soc. J. Bolyai, vol. 28, North-Holland, Amsterdam, 1981; pp. 635–652.

[5] I. G. Rosenberg, Á. Szendrei, Degrees of clones and relations, *Houston J. Math.* **9** (1983), 545–580.

[6] G. Rousseau, Completeness in finite algebras with a single operation, *Proc. Amer. Math. Soc.* **18** (1967), 1009–1013.

[7] M. Suzuki, *Group Theory I*, Grundlehren der mathematischen Wissenschaften 247, Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[8] L. Szabó, Á. Szendrei, Słupecki-type criteria for quasilinear functions over a finite dimensional vector space, *Elektron. Informationsverarbeit. Kybernetik* **17** (1981), 601–611.

[9] Á. Szendrei, On closed classes of quasilinear functions, *Czechoslovak Math. J.* **30** (105) (1980), 498–509.

[10] Á. Szendrei, *Clones in Universal Algebra*, Séminaire de Mathématiques Supérieures, vol. 99, Les Presses de l'Université de Montréal, Montréal, 1986.

[11] Á. Szendrei, The primal algebra characterization theorem revisited, *Algebra Universalis*, to appear.

[12] Á. Szendrei, Simple Abelian algebras, *J. Algebra*, to appear.