The Classification of Commutative Minimal Clones*

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Abstract

We describe all commutative minimal clones.

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1 Introduction

A clone is **trivial** if every operation is a projection. A clone is **minimal** if it is not trivial, but its only proper subclone is trivial. If C is a clone generated by the operation f, then an f**-representation** of C is a pair $\mathbf{A} = \langle A; f^{\mathbf{A}} \rangle$ where A is a set, $f^{\mathbf{A}}$ is an operation on A, and the assignment $f \mapsto f^{\mathbf{A}}$ extends to a clone homomorphism from C to the concrete clone of operations on A. A representation is **faithful** if the assignment $f \mapsto f^{\mathbf{A}}$ extends to a clone isomorphism from C to the clone of operations on A generated by $f^{\mathbf{A}}$. A representation $\langle A; f^{\mathbf{A}} \rangle$ is **finite** if A is.

If f is an m-ary operation of a clone C and g is an n-ary operation of C, then f and g **commute** if the following relation holds:

$$g\left(f\begin{pmatrix} x_{11} \\ \vdots \\ x_{m1} \end{pmatrix}, \ldots, f\begin{pmatrix} x_{1n} \\ \vdots \\ x_{mn} \end{pmatrix}\right) = f\begin{pmatrix} g(x_{11}, \ldots, x_{1n}) \\ \vdots \\ g(x_{m1}, \ldots, x_{mn}) \end{pmatrix}.$$

If any two operations in C commute, then C is said to be a **commutative clone**. (Commutative clones are sometimes called **entropic** clones.) By a **commutative operation** we mean an operation which commutes with itself. It is easy to see that a commutative operation generates a commutative clone.

In this paper we classify all commutative minimal clones up to isomorphism. The statement of the classification theorem is contained in Lemma 2.1 and Theorems 2.2, 2.3, 3.20 and 4.6. Most of the minimal clones we list were known before (cf. [5]), although our classification does include new minimal clones constructed from irreducible representations of

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symmetric groups. Aside from these new clones, the main contribution of this paper is the proof of the completeness of the list.

It is possible to give a shorter proof of the classification theorem if one is willing to consider only minimal clones that have finite faithful representations. We do not assume finiteness. However, although there exist minimal clones that fail to have a finite faithful representation, it is a corollary of the classification that there are no such *commutative* minimal clones.

2 Elementary Observations

The purpose of this section is to reduce our problem to two special cases, which we treat in following sections.

LEMMA 2.1 ([9], Proposition 1.12) *Let C be a nontrivial clone. If $f \in C$ is an operation whose arity is minimal among operations in C which are not projections, then up to a permutation of variables f is one of the following operations:*

(I) A unary operation which is not a projection.

(II) An idempotent essentially binary operation.

(III) An operation $M(x, y, z)$ satisfying the majority laws:

$$M(x, x, y) = M(x, y, x) = M(y, x, x) = x.$$

(IV) An operation $m(x, y, z)$ satisfying the minority laws:

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = y.$$

(V) An operation $s(x, \mathbf{y}) = s(x, y_1, \ldots, y_n)$ of arity 3 or more, which depends on all variables, and which satisfies the semiprojection laws: for any $0 \leq i < j \leq n$

$$s(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n) = x_0. \quad \square$$

The kind of operation which characterizes each of the types $(III) - (V)$ will be called a **majority operation**, a **minority operation**, or a **semiprojection**, respectively. In this paper, "semiprojection" always means "first-variable semiprojection".

It is not hard to see that a minimal clone which contains an operation of one of these five types cannot contain one of the other types. Let us therefore divide minimal clones into types $(I) - (V)$ according to which case we are in in the previous lemma.

The only types of minimal clones which have been thoroughly classified are types (I) and (IV). The nontrivial part of this classification (type (IV)) was done by I. Rosenberg [8]. Proofs of the following can be found in [3] and [9].

THEOREM 2.2 *Up to isomorphism, the only minimal clones of type (I) are the following:*

(1) The clone generated by a unary operation f such that $f^p(x) = x$ for some prime p, but $f(x) \neq x$.

(2) The clone generated by a unary operation f such that $f(x) = f(y)$, but $f(x) \neq x$.

(3) The clone generated by a unary operation f such that $f^2(x) = f(x)$, but $f(x) \neq f(y)$ and $f(x) \neq x$.

Up to isomorphism, the only minimal clone of type (IV) is the clone of affine Boolean groups. (I.e., the clone of $\langle\{0,1\};\ x+y+z \pmod 2\rangle$.) □

Notice that all the clones in Theorem 2.2 are commutative, and all have finite faithful representations.

There are no commutative minimal clones of type (III), as we now show.

THEOREM 2.3 *A majority operation $M(x,y,z)$ on a set of size > 1 is not commutative.*

Proof: A majority operation does not commute with itself on any array of the following form:

$$\begin{bmatrix} x & x & x \\ x & y & y \\ y & y & x \end{bmatrix}, \qquad x \neq y. \quad \square$$

We have reduced our problem to the consideration of minimal clones of types (II) and (V) only. We shall follow roughly the same plan to classify the commutative minimal clones of these two types: we shall climb the lattice of varieties of the appropriate signature which have a commutative clone, identifying minimal clones as we go along, until we run out of minimal clones. Fortunately, we will not have far to climb.

3 Binary Clones

In this section we consider commutative minimal clones of type (II). These are the minimal clones of algebras defined with one basic operation which is binary, idempotent, and which commutes with itself. An algebra defined with one basic operation which is binary is called a **groupoid**. The clone of a groupoid is commutative precisely when the equation $(xu)(vy) = (xv)(uy)$ holds. To avoid any confusion with groupoids satisfying the equation $xy = yx$, which is another kind of commutative law, we shall call a groupoid whose clone is commutative an **entropic groupoid** in this section.

Our strategy in this section is to look at the lattice of subvarieties of the variety of idempotent entropic groupoids until we have located all varieties whose clone is minimal.

If $\mathcal{V}$ is a variety and $\mathcal{E}$ is an equationally complete subvariety, then we call $\mathcal{E}$ a **completion** of $\mathcal{V}$. If $\mathcal{V}$ has a minimal clone and $\mathcal{E}$ is a completion of $\mathcal{V}$, then the clone of $\mathcal{E}$ must be minimal or trivial. To restate our strategy more precisely, we will classify varieties of idempotent entropic groupoids with minimal clone by considering, in a case-by-case way, which of these varieties have a given set of completions. For this task, we need to begin by considering the equationally complete varieties of idempotent entropic groupoids. A description of these varieties follows from the next theorem, which restates a result from [2].

THEOREM 3.1 *An equationally complete idempotent variety with commutative clone is term equivalent to the variety of sets, the variety of semilattices, or a variety of affine vector spaces over a field.* □

In this theorem, an **affine vector space** is the idempotent reduct of a vector space.

Theorem 3.1 is only a description of the possible completions up to term equivalence, and we need a more precise description for our arguments. Therefore, note that a variety $\mathcal{V}$ of groupoids is term equivalent to the variety of sets if and only if $\mathcal{V}$ is the variety of left zero groupoids or the variety of right zero groupoids, denoted $\mathcal{LZ}$ and $\mathcal{RZ}$ respectively. Both varieties have trivial clones. Note that the only variety of groupoids term equivalent to the variety of semilattices is the variety of semilattices, which we denote $\mathcal{S}$. This variety has a minimal clone. Next, if a variety $\mathcal{V}$ of groupoids is term equivalent to the variety of affine $\mathbf{F}$-vector spaces, then the groupoid operation must be equal to an affine operation $rx+(1-r)y$ for some $r \in F-\{0,1\}$ which generates $\mathbf{F}$ as a ring. Moreover, if $r \in F-\{0,1\}$ is such that $rx+(1-r)y$ generates the clone of all affine $\mathbf{F}$-vector space operations, and $r' \in F'-\{0,1\}$ is such that $r'x+(1-r')y$ generates the clone of all affine $\mathbf{F}'$-vector space operations, then $rx+(1-r)y$ and $r'x+(1-r')y$ determine the same variety of groupoids if and only if there is an isomorphism from $\mathbf{F}$ to $\mathbf{F}'$ which takes r to r'. Finally, if the subclone generated by the affine vector space operation $x-y+z$ is not a proper (nontrivial) subclone, then $\mathbf{F}$ must be a finite field with a prime number of elements. The prime involved must be odd for clones of type (II). Therefore finite prime fields of odd cardinality will be the only ones that interest us. Such fields are rigid, so each $r \in F-\{0,1\}$ will yield a different variety of groupoids. We will denote the different varieties by $\mathcal{A}(\mathbf{F},r)$ where $\mathbf{F}$ is a finite prime field of odd cardinality and $r \in F-\{0,1\}$. Each of these varieties has a minimal clone. $\mathcal{A}(\mathbf{F},r)$ and $\mathcal{A}(\mathbf{F}',r')$ have isomorphic clones if and only if $\mathbf{F} \cong \mathbf{F}'$.

THEOREM 3.2 *The equationally complete varieties of idempotent entropic groupoids which have a minimal or trivial clone are* $\mathcal{LZ}, \mathcal{RZ}, \mathcal{S}$ *and* $\mathcal{A}(\mathbf{F},r)$ *for* $\mathbf{F}$ *a finite prime field of odd cardinality and* $r \in F-\{0,1\}$. □

This theorem can also be deduced from Theorem 3.3 of [1].

Now let $\mathcal{V}$ be an entropic variety of groupoids which has a minimal clone. According to the theorem, the completions of $\mathcal{V}$ are among $\mathcal{LZ}, \mathcal{RZ}, \mathcal{S}$ and $\mathcal{A}(\mathbf{F},r)$ for $\mathbf{F}$ a finite prime field of odd cardinality and $r \in F-\{0,1\}$. For each possible subset

$$K \subseteq \{\mathcal{LZ}, \mathcal{RZ}, \mathcal{S}\} \cup \{\mathcal{A}(\mathbf{F},r) \mid \mathbf{F} \text{ a finite prime field of odd cardinality and } r \in F-\{0,1\}\}$$

we will find the possibilities for such $\mathcal{V}$ given that the set of completions of $\mathcal{V}$ is K. Our work is reduced by the main result of [3], which implies the following result.

THEOREM 3.3 *Let* $\mathcal{V}$ *be a variety which has a minimal clone. If* $\mathbf{A} \in \mathcal{V}$ *is an abelian algebra whose clone is not trivial, then* $\mathbf{A}$ *generates* $\mathcal{V}$. □

The variety $\mathcal{LZ} \vee \mathcal{RZ} = \mathcal{LZ} \times \mathcal{RZ}$ is the variety of rectangular bands. Every rectangular band is abelian. A rectangular band which is neither a left zero groupoid nor a right zero groupoid has a nontrivial clone. Thus, each algebra in $(\mathcal{LZ} \vee \mathcal{RZ}) - (\mathcal{LZ} \cup \mathcal{RZ})$ is an abelian algebra whose clone is not trivial. Moreover, each nontrivial algebra in $\mathcal{A}(\mathbf{F},r)$ is an abelian algebra whose clone is not trivial. Therefore we deduce the following.

COROLLARY 3.4 *let $\mathcal{V}$ be a variety of groupoids which has a minimal clone.*

(1) If $\mathcal{V}$ contains $\mathcal{LZ}$ and $\mathcal{RZ}$, then $\mathcal{V} = \mathcal{LZ} \vee \mathcal{RZ}$.

(2) If $\mathcal{V}$ contains some $\mathcal{A}(\mathbf{F}, r)$, then $\mathcal{V} = \mathcal{A}(\mathbf{F}, r)$. □

Thus, we no longer need to consider varieties containing any $\mathcal{A}(\mathbf{F}, r)$, or varieties containing both $\mathcal{LZ}$ and $\mathcal{RZ}$. In fact, because of left-right symmetry, we only have to understand the structure of varieties $\mathcal{V}$ whose set of completions is a subset of $\{\mathcal{LZ}, \mathcal{S}\}$.

3.1 $\mathcal{S}$ is the Only Completion

This is the easiest of the cases we must consider.

THEOREM 3.5 *If $\mathcal{V}$ is a variety of idempotent entropic groupoids whose clone is minimal, and the only completion of $\mathcal{V}$ is $\mathcal{S}$, then $\mathcal{V} = \mathcal{S}$.*

Proof: Assume that $\mathcal{V}$ is a variety of idempotent entropic groupoids whose clone is minimal, and which has $\mathcal{S}$ as its only completion. If $\mathbf{A} \in \mathcal{V}$ is abelian, and has a nontrivial clone, then $\mathcal{V}$ is generated by $\mathbf{A}$ according to Theorem 3.3. But the varieties which are generated by abelian algebras with minimal clones are listed at the end of [3], and none of these varieties has a completion term equivalent to $\mathcal{S}$. Therefore $\mathcal{V}$ can have no abelian algebra with a nontrivial clone. $\mathcal{V}$ also cannot have a nontrivial abelian algebra $\mathbf{B}$ with a trivial clone, or else a completion of $\mathsf{HSP}(\mathbf{B}) \subseteq \mathcal{V}$ would have a trivial clone. But by hypothesis the only possible completion is $\mathcal{S}$, whose clone is nontrivial, hence $\mathcal{V}$ contains no nontrivial abelian algebra whatsoever.

Corollary 3.4 of [4] proves that a variety of idempotent entropic groupoids which contains no nontrivial abelian algebras has a semilattice term. Thus, $\mathcal{V}$ has a semilattice term which (by minimality) generates its clone. This forces $\mathcal{V}$ to be term equivalent to the variety of semilattices. As mentioned before, the only variety of groupoids term equivalent to the variety of semilattices is the variety of semilattices, so $\mathcal{V} = \mathcal{S}$. □

3.2 $\mathcal{LZ}$ is the Only Completion

Our goal in this subsection is to prove the following theorem.

THEOREM 3.6 *If $\mathcal{V}$ is a variety of idempotent entropic groupoids whose clone is minimal, and the only completion of $\mathcal{V}$ is $\mathcal{LZ}$, then $\mathcal{V}$ is one of the following varieties:*

(1) The variety of **right semilattices**, *which is axiomatized by the idempotent and entropic laws together with $x(xy) = x$ and $(xy)y = xy$.*

*(2) A variety of p-***cyclic groupoids**, *which is axiomatized by the idempotent and entropic laws together with $x(xy) = x$ and $\underbrace{(\cdots((xy)y)\cdots)y}_{p\ y\text{'s}} = x$, where p is a prime number.*

We prove this theorem in two steps. That which is logically the first step, but which we shall postpone until the end of this subsection, is a proof that if $\mathcal{V}$ is a variety of idempotent entropic groupoids whose clone is minimal, and the only completion of $\mathcal{V}$ is $\mathcal{LZ}$, then $\mathcal{V} \models x(xy) = x$. The second step, which we shall deal with immediately, is a proof that describes the structure of $\mathcal{V}$ *assuming* that $\mathcal{V} \models x(xy) = x$. Our arguments will require a good understanding of the terms of $\mathcal{V}$. We have been informed that everything we need in this section can be found in [7]. However, since the more complicated arguments used in Section 4 are modeled on the arguments we use here, we include proofs in this section rather than references to proofs.

LEMMA 3.7 *Let $\mathcal{V}$ be a variety of idempotent entropic groupoids satisfying $x(xy) = x$. Then $\mathcal{V}$ satisfies the equations*

(1) $x(yz) = xy$ and

(2) $(xy)z = (xz)y$.

Proof: For the first item, we use the entropic law as follows: $x(yz) = (x(xy))(yz) = (xy)((xy)z) = xy$. For the second item, we use idempotence, entropicity and the first item: $(xy)z = (xy)(zz) = (xz)(yz) = (xz)y$. □

In fact, it is easy to see that a variety of groupoids satisfying $x(xy) = x$ is idempotent and entropic *if and only if* it satisfies equations (1) and (2) of Lemma 3.7.

LEMMA 3.8 *If $\mathcal{V}$ is a variety of idempotent entropic groupoids satisfying $x(xy) = x$ which has a minimal clone, then $\mathcal{V}$ is one of the following varieties:*

(1) The variety of right semilattices.

(2) A variety of p-cyclic groupoids.

Proof: Let's try to understand the structure of the terms of $\mathcal{V}$. Since $\mathcal{V}$ is a variety of groupoids, every term has an associated binary composition tree, where each leaf is labelled with a variable and each internal node represents an application of the groupoid operation. (In particular, each internal node has exactly two children.) The equations of $\mathcal{V}$ allow us to reduce each term to a $\mathcal{V}$-equivalent term with a particularly simple composition tree.

The equation $x(yz) = xy$, which we established in Lemma 3.7, implies that the following reduction can be performed locally in any tree.

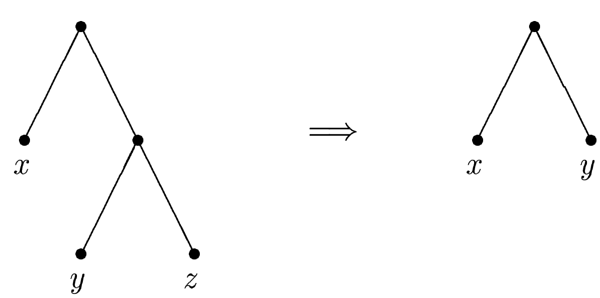

FIGURE 1: $x(yz) = xy$.

Since this reduction can be performed locally, it is always possible to reduce any term to a $\mathcal{V}$-equivalent term where no node has a right-right grandchild. Equivalently, we can reduce until we have a tree where the right child of any internal node is a leaf. Therefore any term may be assumed to be reduced to one whose tree has the following structure.

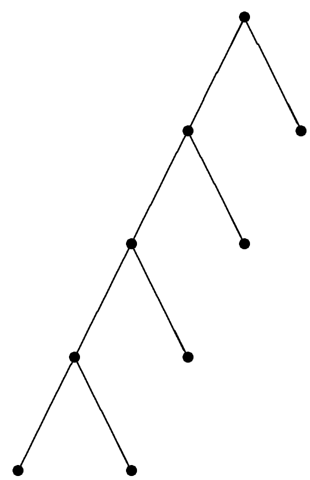

FIGURE 2: Reduced form.

This implies that any term is $\mathcal{V}$-equivalent to a left-associated product of variables, and therefore is a specialization of a product of the form

$$xy_1y_2\cdots y_n := (\cdots((xy_1)y_2)\cdots)y_n$$

where the variables are distinct. The equation $(xy)z = (xz)y$, from Lemma 3.7, implies that any permutation of the order of the variables in a left-associated product which fixes the leftmost variable produces a $\mathcal{V}$-equivalent term. We may therefore assume that the leftmost variable occurs at most once, for in the left-associated product $xy_1y_2\cdots x\cdots y_n$ we can permute the variables in a way that fixes the leftmost x and moves the other x adjacent to it: $xxy_1y_2\cdots y_n$. Such a term reduces to $xy_1y_2\cdots y_n$ via the idempotent law.

Using the notation xy^e to denote the left-associated product $xyy\cdots y$, with e occurrences of y, we summarize our observations as follows.

Claim 3.9 If $\mathcal{V}$ is a variety of idempotent entropic groupoids satisfying the equation $x(xy) = x$, then every $(k+1)$-ary term with leftmost variable x is $\mathcal{V}$-equivalent to a term of the form $xy_1^{e_1}\cdots y_k^{e_k}$ for some $e_1,\ldots,e_k \geq 0$, where the order of the variables $y_1,\ldots,y_k$ is irrelevant.

Note that $\mathcal{V}$-equivalent terms have the same leftmost variable, namely the unique variable x for which the terms interpret in the subvariety $\mathcal{LZ}$ of $\mathcal{V}$ as x. Therefore it makes sense to talk about the leftmost variable of a term of $\mathcal{V}$. If a term of $\mathcal{V}$ is given in the form described in Claim 3.9, then its leftmost variable is clearly x.

Let us consider now the set M of all terms of $\mathcal{V}$ whose set of variables is $\{x,y\}$ and whose leftmost variable x. By Claim 3.9 M consists of all binary terms $\mathsf{m}(x,y) := xy^m$ $(m \geq 0)$, where by definition $\mathsf{0}(x, y) = x$. For any $\mathsf{m}, \mathsf{n} \in M$ we have $\mathsf{n}(\mathsf{m}(x, y), y) = xy^{m+n}$. Therefore we can introduce an addition on M by defining

$$(\mathsf{m} + \mathsf{n})(x, y) := \mathsf{n}(\mathsf{m}(x, y), y) = xy^{m+n}.$$

The algebra $\mathbf{M} = \langle M; +, \mathsf{0}\rangle$ is a monoid which is generated by the single element $1 \in \mathbf{M}$.

Making use of the assumption that the clone of $\mathcal{V}$ is minimal, we will now show that $\mathbf{M}$ has no nontrivial proper submonoids. Monoids with this property will be called **minimal monoids**.

Claim 3.10 If $\mathcal{V}$ satisfies all assumptions of the lemma, then $\mathbf{M}$ is a minimal monoid.

Proof of Claim: We know from Claim 3.9 that every binary term of $\mathcal{V}$ is of the form $\mathsf{m}(x, y)$ or $\mathsf{m}(y, x)$ for some $\mathsf{m} \in \mathbf{M}$. Furthermore, the term reductions discussed prior to that claim yield that for arbitrary elements $\mathsf{m}, \mathsf{n}, \mathsf{k} \in \mathbf{M}$ and for both $(u, v) = (x, y)$ and $(u, v) = (y, x)$ we have

$$\mathsf{m}(\mathsf{n}(u, v), \mathsf{k}(u, v)) = \mathsf{n}(u, v) \qquad \text{and} \qquad \mathsf{m}(\mathsf{n}(u, v), \mathsf{k}(v, u)) = (\mathsf{m} + \mathsf{n})(u, v).$$

This implies that if $\mathbf{N}$ is a submonoid of $\mathbf{M}$, then the subclone of the clone of $\mathcal{V}$ generated by all terms in $\mathbf{N}$ has the property that its only binary terms are those $\mathcal{V}$-equivalent to $\mathsf{n}(x, y)$ or $\mathsf{n}(y, x)$ for some $\mathsf{n} \in \mathbf{N}$. Since the clone of $\mathcal{V}$ is minimal, $\mathbf{N}$ must be equal to $\mathbf{M}$ or $\{\mathsf{0}\}$. Hence the monoid $\mathbf{M}$ is minimal.

Claim 3.11 Every minimal monoid $\mathbf{M} = \langle M; +, 0\rangle$ is isomorphic to the 2-element semilattice or a cyclic group of prime order.

Proof of Claim: Assume that $\mathbf{M}$ has an idempotent element e different from 0. Then since $\{0, e\}$ is a nontrivial submonoid we must have $M = \{0, e\}$, and $+$ must be a semilattice operation on M.

Now assume that 0 is the only idempotent of $\mathbf{M}$. Since $\mathbf{M}$ has no proper nontrivial submonoids, any $g \in M - \{0\}$ generates $\mathbf{M}$. Fix such a generator g. If g has infinite order, then $M = \{0, g, 2g, 3g, \ldots\}$. But now $N = \{0, 2g, 4g, \ldots\}$ is a proper nontrivial submonoid. This is a impossible, so g must have finite order. Because of this, the sequence $g, 2g, 3g, \ldots$ contains an idempotent, which can only be 0. This implies that $\mathbf{M}$ is a cyclic group. Since $\mathbf{M}$ has no proper nontrivial subgroups, it must be a cyclic group of prime order.

We apply the previous claim to the minimal monoid $\mathbf{M}$ corresponding to the variety $\mathcal{V}$. If $\mathbf{M}$ is the 2-element semilattice, then $M = \{\mathsf{0}, \mathsf{1}\}$. By the equations listed earlier,

$$(xy)y = (\mathsf{1} + \mathsf{1})(x, y) = \mathsf{1}(x, y) = xy.$$

Since the hypotheses of this lemma include that $\mathcal{V}$ is an idempotent entropic variety of groupoids satisfying $x(xy) = x$, this additional equation $(xy)y = xy$ implies that $\mathcal{V}$ is a subvariety of the variety of right semilattices.

If $\mathbf{M}$ is a cyclic group of prime order p, then $M = \{0, 1, \ldots, p-1\}$. By the equations listed earlier,

$$xy^p = (\underbrace{1 + \cdots + 1}_{p\ \text{1's}})(x, y) = 0(x, y) = x.$$

It follows that $\mathcal{V}$ satisfies the equations that axiomatize the variety of p-cyclic groupoids, so $\mathcal{V}$ is a subvariety of the variety of p-cyclic groupoids.

We now know that $\mathcal{V}$ has a minimal (hence nontrivial) clone and that $\mathcal{V}$ is a subvariety of the variety of right semilattices or a variety of p-cyclic groupoids. The following claim completes the proof of this lemma.

Claim 3.12 The variety of right semilattices has a minimal clone and the property that its only nontrivial proper subvariety is $\mathcal{LZ}$. The same statement is true for any variety of p-cyclic groupoids.

Proof of Claim: The proof of this claim for any variety of p-cyclic groupoids appears at the end of the proof of Lemma 3.5 of [3]. Therefore we prove the claim for the variety of right semilattices only.

Assume that $\mathcal{W}$ is a subvariety of the variety of right semilattices. Since $\mathcal{W}$ satisfies the assumptions of Claim 3.9, the terms of $\mathcal{W}$ can be written in the form described there. The equation $(xy)y = xy$ allows us to further reduce terms where some variable occurs more than once. Thus, a typical term is $\mathcal{W}$-equivalent to a left-associated product $xy_1 \cdots y_{n-1}y_n$ where all variables are distinct, and the order of $y_1, \ldots, y_n$ is irrelevant. Observe that the clone generated by each such term with $n \geq 1$ contains $xx \cdots xy = xy$. This proves that the clone of $\mathcal{W}$ is minimal.

Now assume that $\mathcal{W}$ is a proper subvariety of the variety of right semilattices. Then there is an equation $r = s$ which holds in $\mathcal{V}$ but fails in the variety of right semilattices. We may assume that both r and s are left-associated products of distinct variables. If r and s have the same leftmost variable, then it must be that the set of variables occurring in r is not the same as the set which occurs in s. Assume that y occurs in s but not in r. If we set all variables other than y equal to x, then $r = s$ specializes to $x = xy$. In the case when r and s have different leftmost variables, say x and y respectively, we can apply equation (1) from Lemma 3.7 to conclude that $x = xr = xs = xy$. The equation $x = xy$ defines $\mathcal{LZ}$, so in both cases we get that $\mathcal{W} \subseteq \mathcal{LZ}$. □

Now we proceed to show that $x(xy) = x$ must hold in $\mathcal{V}$. For this purpose, the following lemma will be useful.

LEMMA 3.13 (Lemma 3.5 of [4]) *Let $\mathbf{A}$ be an idempotent entropic groupoid and let $U \subseteq A$ be a subset. If an equation $r(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$ holds whenever all x_i lie in U, then the same equation holds when all x_i lie in the subgroupoid of $\mathbf{A}$ generated by U.* □

LEMMA 3.14 *If $\mathcal{V}$ is an idempotent entropic variety of groupoids whose clone is minimal, and the only completion of $\mathcal{V}$ is $\mathcal{LZ}$, then $\mathcal{V} \models x(xy) = x$.*

Proof: Assume instead that there is $\mathcal{V}$ satisfying all hypotheses, except $\mathcal{V} \not\models x(xy) = x$. Then $\mathcal{V}$ contains a groupoid $\mathbf{A}$ generated by a pair of elements $\{a, b\}$ such that either $a(ab) \neq a$ or $b(ba) \neq b$, or both. By factoring by a maximal congruence which maintains at least one of these inequalities, we may assume that $\mathbf{A}$ is subdirectly irreducible with monolith μ, and that $a(ab) \equiv_\mu a$ and $b(ba) \equiv_\mu b$.

Let B be any nontrivial μ-class, and let $\mathbf{B}$ be the subgroupoid of $\mathbf{A}$ supported by B. According to the results in [4], $\mathbf{B}$ must be term equivalent to a two-element semilattice, a quasi-affine algebra with a cancellative binary term, an affine Boolean group, or to a set. Moreover, it is shown in [4] that if $\mathbf{B}$ is a two-element semilattice, then $\mathbf{A}$ has a semilattice term. Necessarily this term generates the clone of $\mathbf{A}$, so $\mathbf{A}$ is in fact a subdirectly irreducible semilattice; i.e., $\mathbf{A} = \mathbf{B}$. This is impossible since $\mathcal{S}$ is not a completion of $\mathcal{V}$. $\mathbf{B}$ cannot be affine either, for the same kind of reason, so $\mathbf{B}$ is not an affine Boolean group *nor is* $\mathbf{B}$ *quasi-affine with a cancellative binary term.* The last part of this claim follows from the main result of [3], which implies that a quasi-affine algebra with a cancellative binary term which has a minimal clone is in fact affine. We are left only with the possibility that $\mathbf{B}$ is term equivalent to a set. Thus $\mathbf{B}$ lies in a completion of $\mathcal{V}$, and therefore $\mathbf{B} \in \mathcal{LZ}$. This statement holds for any nontrivial μ-class of $\mathbf{A}$.

Since a and $a(ab)$ are μ-related, the operation xy is left zero on the set $\{a, a(ab)\}$. This forces $a(a(ab)) = a$, and a similar argument shows that $b(b(ba)) = b$. Since $\mathbf{A}$ is idempotent, this implies that the equation $x(x(xy)) = x$ is satisfied for all $x, y \in \{a, b\} =: U$. Now we use Lemma 3.13 to conclude that the subgroupoid generated by $U = \{a, b\}$ (which is $\mathbf{A}$) also satisfies the equation $x(x(xy)) = x$.

We claim that $x * y := x(xy)$ generates the clone of $\mathbf{A}$. Since the clone of $\mathbf{A}$ is minimal, it suffices to verify that $x * y$ is not a projection. Since $\mathcal{LZ} \models xy = x$, we have $\mathcal{LZ} \models x * y = x$. But $\mathcal{LZ}$ is the only completion of $\mathsf{HSP}(\mathbf{A})$, therefore it is impossible for $\mathbf{A}$ to satisfy the equation $x * y = y$. It is also impossible for $\mathbf{A}$ to satisfy the equation $x * y = x$, since we chose $\mathbf{A}$ so that $\mathbf{A} \not\models x(xy) = x$. Since $\mathbf{A} \not\models x * y = x$ or y, therefore $x * y$ generates the clone of $\mathbf{A}$.

Let $\mathcal{U} = \mathsf{HSP}(\mathbf{A})$. So far we know that xy is a term of $\mathcal{U}$ for which the following conditions hold:

(a) $\mathcal{LZ} \models xy = x$,

(b) $\mathcal{U} \not\models x(xy) = x$ and

(c) $\mathcal{U} \models x(x(xy)) = x$.

We shall obtain a contradiction by showing that $\mathcal{U}$ can have no term with these properties. First observe that condition (c) implies that

$$\mathcal{U} \models x * (x * y) = x(x(x(xy))) = xx = x.$$

Thus $\mathcal{U}$ is term equivalent to a variety (defined in terms of the operation $*$) to which we can apply Lemma 3.8: $\mathcal{U}$ is term equivalent to either the variety of right semilattices or a variety of p-cyclic groupoids for some prime p. If the operation $*$ is a right semilattice term, then the $\mathcal{U}$-inequivalent binary terms are represented by: $x, y, x * y$ and $y * x$. The terms $t(x, y)$ from this list for which $\mathcal{LZ} \models t(x, y) = x$ are just $t(x, y) = x$ and $t(x, y) = x * y$, and for both of these choices it is the case that $\mathcal{U} \models t(x,t(x,y)) = x$. Thus the variety of right semilattices has no binary term for which conditions (a) and (b) both hold. Now assume that $\mathcal{U}$ is term equivalent to a variety of p-cyclic groupoids. In this case, a representative set of binary terms of $\mathcal{U}$ are the left-associated products of the form $x * y^i$ or $y * x^i$ where $0 \leq i < p$. Those for which condition (a) from above hold are just the terms of the form $t(x,y) = x * y^i$. But again we have $\mathcal{U} \models t(x,t(x,y)) = x * x^i = x$, so no variety of p-cyclic groupoids has a binary term xy for which conditions (a) and (b) both hold. We have established the contradiction that concludes the proof. □

Theorem 3.6 is a consequence of Lemmas 3.8 and 3.14.

3.3 $\mathcal{LZ}$ and $\mathcal{S}$ are Completions

In this subsection our goal is to prove the following.

THEOREM 3.15 *If $\mathcal{V}$ is a variety of idempotent entropic groupoids whose clone is minimal, and $\mathcal{V}$ has both $\mathcal{LZ}$ and $\mathcal{S}$ as completions, then $\mathcal{V} = \mathcal{LZ} \vee \mathcal{S}$.*

Our method for proving this will be similar to what we did in the previous subsection. Let $\mathcal{V}$ be an arbitrarily chosen idempotent entropic variety of groupoids with minimal clone whose only completions are $\mathcal{LZ}$ and $\mathcal{S}$. We shall prove later that $\mathcal{V} \models x(xy) = xy = (xy)y$, but we precede that argument with a proof that if $\mathcal{V}$ satisfies these equations, then $\mathcal{V} = \mathcal{LZ} \vee \mathcal{S}$.

LEMMA 3.16 *Let $\mathcal{V}$ be a nontrivial variety of idempotent entropic groupoids such that $\mathcal{V} \models x(xy) = xy = (xy)y$. If the completions of $\mathcal{V}$ are among $\{\mathcal{LZ}, \mathcal{S}\}$, then $\mathcal{V} = \mathcal{LZ}$, $\mathcal{S}$ or $\mathcal{LZ} \vee \mathcal{S}$.*

Proof: First we show that the equations stated in the theorem imply the associative law.

$$\begin{aligned} x(yz) &= (x(yz))(yz) \\ &= (xy)((yz)z) \\ &= (xy)(yz) \\ &= (x(xy))(yz) \\ &= (xy)((xy)z) \\ &= (xy)z. \end{aligned}$$

This shows that $\mathcal{V}$ is a variety of idempotent entropic semigroups, i.e., **normal bands**. It is known that there are only eight varieties of normal bands, and they are different possible joins of subsets of $\{\mathcal{LZ}, \mathcal{RZ}, \mathcal{S}\}$ (cf. [6]). Those joins which are nontrivial and whose completions are among $\{\mathcal{LZ}, \mathcal{S}\}$ are just $\mathcal{LZ}$, $\mathcal{S}$ and $\mathcal{LZ} \vee \mathcal{S}$. The lemma follows from this. □

Now we prove that the equations $x(xy) = xy = (xy)y$ must hold if $\mathcal{LZ}$ and $\mathcal{S}$ are completions of $\mathcal{V}$.

LEMMA 3.17 *If $\mathcal{V}$ is an idempotent entropic variety of groupoids whose clone is minimal, and the completions of $\mathcal{V}$ are $\mathcal{LZ}$ and $\mathcal{S}$, then $\mathcal{V} \models x(xy) = xy = (xy)y$.*

Proof: To fix some terminology for this proof, we will say that a binary term $t(x,y)$ is **idempotent in its first variable** if $\mathcal{V} \models t(t(x,y),y) = t(x,y)$ and **idempotent in its second variable** if $\mathcal{V} \models t(x,t(x,y)) = t(x,y)$. If both hold, then $t(x,y)$ is **idempotent in both variables**. The conclusion of the theorem is that xy is idempotent in both variables.

We shall assume that xy is not idempotent in both variables and argue to a contradiction. Since xy is not idempotent in both variables, $\mathcal{V}$ contains a groupoid $\mathbf{A}$ generated by a pair of elements $\{a,b\}$ where at least one of the following inequalities holds:

$$a(ab) \neq ab, \quad b(ba) \neq ba, \quad (ab)b \neq ab, \quad (ba)a \neq ba.$$

As explained in the proof of Lemma 3.14, we may assume that $\mathbf{A}$ is subdirectly irreducible with monolith μ, and that $a(ab) \equiv_\mu ab \equiv_\mu (ab)b$ and $b(ba) \equiv_\mu ba \equiv_\mu (ba)a$. Let B be any nontrivial μ-class, and let $\mathbf{B}$ be the subgroupoid of $\mathbf{A}$ supported by B. As argued in the proof of Lemma 3.14, if $\mathbf{B}$ is a two-element semilattice, then $\mathbf{A} = \mathbf{B}$. But this is impossible if $\mathbf{A} \not\models x(xy) = xy = (xy)y$, since these equations hold in a semilattice. Also, as argued in Lemma 3.14, $\mathbf{B}$ cannot be a quasi-affine algebra with a cancellative binary term nor an affine Boolean group. Thus we can conclude that $\mathbf{B} \in \mathcal{LZ}$ here, just as we did in the proof of Lemma 3.14. This holds for any μ-class B, so $u \equiv_\mu v \Longrightarrow uv = u$.

Claim 3.18 For any $x, y \in \{a,b\}$, either $x(xy) = xy$ or $x(xy) = x$.

Proof of Claim: Assume that $a(ab) \neq ab$. Then, since $a(ab) \equiv_\mu ab$ and xy is left zero on μ-classes, we have

$$[a(ab)](ab) = a(ab) \neq ab = (ab)(ab).$$

If $R_{ab}(x)$ is the mapping from A to A defined by $x \mapsto x(ab)$, then this shows that $R_{ab}(a(ab)) = R_{ab}(a) \neq R_{ab}(ab)$. But $R_{ab}(x)$ is an endomorphism of $\mathbf{A}$, since $\mathbf{A}$ is idempotent and entropic. Thus $\ker(R_{ab})$ is a congruence which does not contain $\mathrm{Cg}(a(ab), ab) = \mu$. Since μ is the least nonzero congruence of $\mathbf{A}$, R_{ab} is one-to-one. From $R_{ab}(a(ab)) = R_{ab}(a)$ we deduce that $a(ab) = a$. This proves the claim in the case where $x = a$ and $y = b$. The same argument works if $x = b$ and $y = a$. The claim is true if $x = y = a$ or $x = y = b$, since the groupoid operation is idempotent.

It follows from the previous claim that for any $x, y \in \{a,b\}$ we have $x(x(xy)) = x(xy)$. Thus, since $\{a,b\}$ generates $\mathbf{A}$, Lemma 3.13 shows that $\mathbf{A} \models x(x(xy)) = x(xy)$.

Claim 3.19 $\mathbf{A} \not\models x(xy) = x$ and $\mathbf{A} \not\models x(xy) = y$.

Proof of Claim: Assume otherwise that $\mathbf{A} \models x(xy) = x$ or $\mathbf{A} \models x(xy) = y$. Then the same equation would hold in some completion of $\mathsf{HSP}(\mathbf{A}) \subseteq \mathcal{V}$, therefore in $\mathcal{LZ}$ or $\mathcal{S}$. Neither equation holds in $\mathcal{S}$, and only the first equation holds in $\mathcal{LZ}$. Therefore let's show that the assumption that $\mathbf{A} \models x(xy) = x$ leads to a contradiction.

If $\mathbf{2} = \langle \{0,1\}; xy \rangle \in \mathcal{V}$ is the two-element semilattice, then $\mathbf{A} \times \mathbf{2} \in \mathcal{V}$. The operation $x * y := x(xy)$ does not generate a trivial subclone of the clone of $\mathbf{A} \times \mathbf{2}$, since $x * y$ does not generate a trivial subclone on the second factor algebra. However, if $\mathbf{A} \models x * y = x(xy) = x$, then $x * y$ cannot generate the total clone of $\mathbf{A} \times \mathbf{2}$, since it does not generate the total clone of the first factor. Thus if $\mathbf{A} \models x(xy) = x$, then $x * y$ generates a proper nontrivial subclone of $\mathbf{A} \times \mathbf{2}$ (and hence $\mathcal{V}$). This is impossible if the clone of $\mathcal{V}$ is minimal.

The previous claim implies that $x*y := x(xy)$ generates the clone of $\mathbf{A}$. What we proved before this claim shows that $\mathbf{A} \models x * (x * y) = x(x(x(xy))) = x(xy) = x * y$, so $x * y$ is idempotent in its second variable. We now argue that it is also idempotent in its first variable. For this, we must show that $\mathbf{A} \models (x * y) * y = x * y$. Recall that $\mathbf{A}$ is generated by $\{a, b\}$ and that $a(ab) \equiv_\mu ab \equiv_\mu (ab)b$. Therefore

$$(a * b) * b = [a(ab)][[a(ab)]b] \equiv_\mu [a(ab)][(ab)b] \equiv_\mu [a(ab)][a(ab)] = a(ab) = a * b.$$

Since xy is left zero on μ-classes, the same is true of $x * y$, and consequently we deduce from the last calculation that $(a * b) * [(a * b) * b] = a * b$. Therefore, using that $x * y$ is idempotent in its second variable, we have

$$(a * b) * b = (a * b) * [(a * b) * b] = a * b.$$

Using the same argument with a and b interchanged shows that $x * y = (x * y) * y$ whenever $x, y \in \{a, b\}$. Therefore, by Lemma 3.13, the equation $(x * y) * y = x * y$ holds in $\mathbf{A}$.

Let $\mathcal{U} = \mathsf{HSP}(\mathbf{A}) \subseteq \mathcal{V}$. The clone of $\mathcal{U}$ is generated by $x * y$, which is idempotent in both variables. Since xy and $x * y$ agree in $\mathcal{LZ}$ and $\mathcal{S}$, $\mathcal{U}$ is term equivalent to a variety of idempotent entropic groupoids (defined in terms of the operation $*$) which has a minimal clone, has $\mathcal{LZ}$ and $\mathcal{S}$ as completions, and which satisfies $x * (x * y) = x * y = (x * y) * y$. It follows from Lemma 3.16 that $\mathcal{U} \subseteq \mathcal{LZ} \vee \mathcal{S}$. This implies that $\{x, y, x * y, y * x\}$ is a set of terms representing all binary terms up to $\mathcal{U}$-equivalence. But this is impossible, since all four of these are idempotent in both variables, while xy is a binary term of $\mathcal{U}$ which is not idempotent in at least one of its variables. This contradiction concludes the proof. □

Theorem 3.15 is a consequence of Lemmas 3.16 and 3.17.

3.4 The List of Binary Clones

We summarize the results of the preceding subsections now concerning commutative minimal clones of type (II). We shall describe only one presentation of each clone, and we shall list clones up to isomorphism or anti-isomorphism only.

THEOREM 3.20 *If C is a commutative minimal clone of type (II), then C is isomorphic or anti-isomorphic to the clone of one of the following varieties:*

(1) Affine spaces: $\mathcal{A}(\mathbf{F}, r)$ for some prime field of odd cardinality and $r \in F - \{0, 1\}$,

(2) Rectangular bands: $\mathcal{LZ} \vee \mathcal{RZ}$,

(3) Semilattices: $\mathcal{S}$,

(4) The variety of right semilattices,

(5) A variety of p-cyclic groupoids for some prime p, or

(6) Left normal bands: $\mathcal{LZ} \vee \mathcal{S}$. □

Note that each variety is generated by a finite algebra, so each clone on the list has a finite faithful representation.

4 Clones Generated by a Semiprojection

Although our strategy for type (V) minimal clones is the same as for type (II), there are fewer cases to consider in type (V). The reason for this is that the only possible completion of a variety whose clone is generated by a semiprojection $s(x, \mathbf{y})$ of arity ≥ 3 is the subvariety defined by the equation $s(x, \mathbf{y}) = x$. (Reason: If $\mathcal{V}$ is the variety of s-representations and $\mathbf{A}$ is a nontrivial member of some completion of $\mathcal{V}$, then any two-element subset of $\mathbf{A}$ is a subuniverse on which $s(x, \mathbf{y}) = x$ holds. The corresponding two-element subalgebra generates the same completion as $\mathbf{A}$. Thus $\mathbf{A}$ must lie in the equationally complete variety where the $s(x, \mathbf{y}) = x$ holds.) Therefore our arguments in this section compare only to the arguments in Subsection 3.2.

Let $\mathcal{V}$ be a variety whose clone is generated by a semiprojection s. The fact established in the previous paragraph implies that in $\mathcal{V}$ every term $f(x_1, \ldots, x_k)$ has a distinguished variable, namely the variable x_i for which the identity $f(x_1, \ldots, x_k) = x_i$ holds in the completion of $\mathcal{V}$. This variable x_i is uniquely determined since the variety $\mathcal{V}$ is not trivial. If we express f in any way as a composition of s, then x_i is the variable that occurs at the leftmost leaf of the composition tree. Therefore x_i will be referred to as the **leftmost variable** of f.

Our first lemma is the analogue of Lemma 3.7.

LEMMA 4.1 *Let $\mathcal{V}$ be the variety of s-representations, where $s(x, \mathbf{y})$ is an $(n+1)$-ary commutative semiprojection. Then $\mathcal{V}$ satisfies the equations*

(1) $s(x, y_1, \ldots, y_{i-1}, s(y_i, \mathbf{z}), y_{i+1}, \ldots, y_n) = s(x, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)$, *and*

(2) $s(s(x, \mathbf{y}), \mathbf{z}) = s(s(x, \mathbf{z}), \mathbf{y})$.

Proof: The first equation follows from applying s to itself on the following matrix, and then applying the semiprojection laws and the commutative law.

$$\begin{bmatrix} x & y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & \cdots & y_n \\ x & x & \cdots & x & z_1 & x & \cdots & x \\ x & x & \cdots & x & z_2 & x & \cdots & x \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x & x & \cdots & x & z_n & x & \cdots & x \end{bmatrix}.$$

The second equation follows in the same way from

$$\begin{bmatrix} x & y_1 & \cdots & y_n \\ z_1 & x & \cdots & x \\ z_2 & x & \cdots & x \\ \vdots & \vdots & & \vdots \\ z_n & x & \cdots & x \end{bmatrix}. \quad \square$$

LEMMA 4.2 *Let C be a clone generated by a commutative $(n+1)$-ary semiprojection.*

(1) *For* $m \geq n$ *every* $(m+1)$*-ary member* $f(x, \mathbf{y})$ *of* C *with leftmost variable* x *is of the form*

$$t_r(\cdots t_2(t_1(x, \mathbf{z}_1), \mathbf{z}_2) \cdots, \mathbf{z}_r)$$

where $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_r$ *is an enumeration of the subsequences of* $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ *of length* n*, and* $t_1, t_2, \ldots, t_r$ *are uniquely determined* $(n+1)$*-ary members of* C *with leftmost variable* x*. Each* t_i *arises from* f *by setting some variables from* $\mathbf{y}$ *equal to* x*.*

(2) *Every* $(n+1)$*-ary member* $f(x, \mathbf{y})$ *of* C *with leftmost variable* x *is of the form*

$$s(\cdots s(s(x, \mathbf{y}_1), \mathbf{y}_2) \cdots, \mathbf{y}_k)$$

where each $\mathbf{y}_i$ *is a permutation of* $\mathbf{y} = (y_1, \ldots, y_n)$*. Hence every* $(n+1)$*-ary member of* C *with leftmost variable* x *is either a semiprojection or the projection* x*.*

Proof: The equation of Lemma 4.1 (1) implies that the following type of reduction can be performed locally in any composition tree.

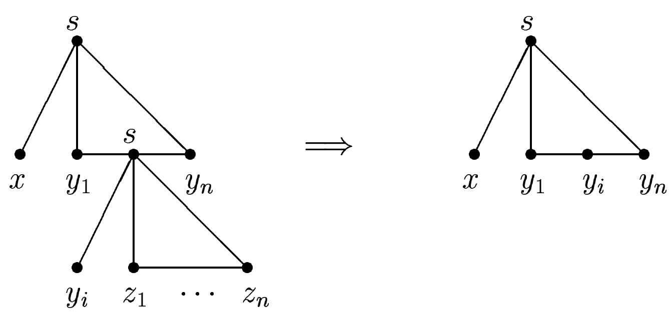

FIGURE 3: $s(x, \mathbf{y}s(y_i, \mathbf{z})\mathbf{y}') = s(x, \mathbf{y}y_i\mathbf{y}')$.

Using reductions of this type, any operation $f(x, \mathbf{y}) \in C$ with leftmost variable x can be reduced to one of the form displayed in Figure 4.

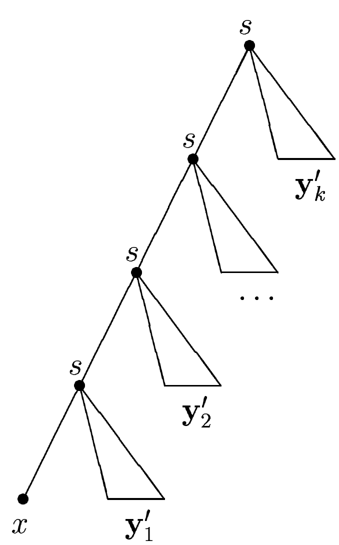

FIGURE 4: A typical operation.

Here each n-tuple $\mathbf{y}'_i$ has coordinates from the set $\{x, y_1, \ldots, y_m\}$ where $\mathbf{y} = (y_1, \ldots, y_m)$. If some $\mathbf{y}'_i$ has a repetition among its coordinates, then by the semiprojection laws the i-th application of s can be deleted from the tree. Therefore we may assume that there are no repetitions among coordinates in any $\mathbf{y}'_i$. Observe also that there can be no x in the string $\mathbf{y}'_1$, or by the semiprojection laws we could delete the first (= leftmost = innermost) application of s from the tree. Lemma 4.1 (2) states that $s(s(x,\mathbf{y}),\mathbf{z}) = s(s(x,\mathbf{z}),\mathbf{y})$, which implies that in the representation of f depicted in Figure 4 the order of the strings $\mathbf{y}'_1, \ldots, \mathbf{y}'_k$ relative to one another is irrelevant. Therefore, our argument that the string $\mathbf{y}'_1$ contains no occurrence of x applies to all other strings, as well. In particular, we can assume that each $\mathbf{y}'_i$ is a permutation of one of the length-n subsequences $\mathbf{z}_j$ of $(y_1, \ldots, y_m)$.

In the special case when $m = n$ this establishes that each $\mathbf{y}'_i$ is a permutation of $\mathbf{y}$, so f is of the form described in (2). It is easy to see that in this case f satisfies the semiprojection laws, so f is either a semiprojection or the projection x, as claimed.

Returning to the general case we will again make use of the fact that in the representation of f depicted in Figure 4 the order of the strings $\mathbf{y}'_1, \ldots, \mathbf{y}'_k$ relative to one another is irrelevant. Hence we can rearrange them so that the list $\mathbf{y}'_1, \ldots, \mathbf{y}'_k$ starts with the permutations of $\mathbf{z}_1$ (if any), followed by the permutations of $\mathbf{z}_2$ (if any), and so on. Now we get for f the representation required in (1) as follows. Suppose that among $\mathbf{y}'_1, \ldots, \mathbf{y}'_k$ exactly the first l strings are permutations of $\mathbf{z}_1$. If $l = 0$ then let t_1 be the projection x. Otherwise let t_1 be the $(n+1)$-ary operation determined by the following property: $t_1(x, \mathbf{z}_1)$ is the operation corresponding to the subtree at the l-th internal node in the tree in Figure 4 representing f. Pruning this subtree off we can repeat the same construction with $\mathbf{z}_2$ in place of $\mathbf{z}_1$ to get t_2, etc.

Our next task is to show the uniqueness of $t_1, t_2, \ldots, t_r$. Assume that in C we have

$$t_r(\cdots t_2(t_1(x,\mathbf{z}_1),\mathbf{z}_2)\cdots,\mathbf{z}_r) = t'_r(\cdots t'_2(t'_1(x,\mathbf{z}_1),\mathbf{z}_2)\cdots,\mathbf{z}_r).$$

We know that the $(n+1)$-ary operations $t_1, t_2, \ldots t_r$ and $t'_1, t'_2, \ldots t'_r$ are semiprojections. Furthermore, for each $j \neq r$ the string $\mathbf{z}_j$ contains a variable distinct from all variables in $\mathbf{z}_r$. Therefore setting each variable not occurring in $\mathbf{z}_r$ to x and applying the semiprojection laws we get from the displayed equality that $t_r(x,\mathbf{z}_r) = t'_r(x,\mathbf{z}_r)$. As we saw earlier, the order of the operations $t_i(x,\mathbf{z}_i)$, resp. $t'_i(x,\mathbf{z}_i)$ in the composition is irrelevant. Therefore we can move $t_i(x,\mathbf{z}_i)$ and $t'_i(x,\mathbf{z}_i)$ for any fixed i to the outermost position to conclude, as before, that $t_i(x,\mathbf{z}_i) = t'_i(x,\mathbf{z}_i)$.

The argument in the previous paragraph shows also that if $f(x,\mathbf{y})$ has representation $t_r(\cdots t_2(t_1(x,\mathbf{z}_1),\mathbf{z}_2)\cdots,\mathbf{z}_r)$, then each $t_i(x,\mathbf{z}_i)$ arises from f by setting some variables from $\mathbf{y}$ equal to x. This completes the proof of the lemma. □

This lemma implies that if a clone is generated by an $(n+1)$-ary commutative semiprojection s, then every nontrivial subclone has an $(n+1)$-ary member which is not a projection, and that member has a special form. This special form shows that, up to a permutation of variables, every $(n+1)$-ary member of the clone is a semiprojection or a projection. The lemma also shows that to establish minimality of a clone generated by a commutative semiprojection, it suffices to show that every $(n+1)$-ary semiprojection in the clone generates every other $(n+1)$-ary semiprojection.

These observations encourage us to focus on those $(n+1)$-ary members of the clone C generated by s which satisfy the semiprojection laws. These are the $(n+1)$-ary semiprojections together with the projection x. The subset of C consisting of all these operations will be denoted by M. In other words, M is the set of $(n+1)$-ary members of C with leftmost variable x.

Now we define some operations on M. Addition is defined as follows:

$$(t+u)(x,\mathbf{y}) := u(t(x,\mathbf{y}),\mathbf{y}).$$

Lemma 4.1 (2) and Lemma 4.2 (2) imply that this operation is commutative. It is also associative, since for any $(n+1)$-ary operations $t(x,\mathbf{y}), u(x,\mathbf{y})$ and $v(x,\mathbf{y})$ in M the sums $t+(u+v)$ and $(t+u)+v$ both represent $v(u(t(x,\mathbf{y}),\mathbf{y}),\mathbf{y})$. It is clear that the projection x is a neutral element with respect to $+$.

Let S_n denote the group of permutations of $\{1,\ldots,n\}$. For any n-tuple $\mathbf{y}=(y_1,\ldots,y_n)$ and for any permutation $\sigma\in S_n$ the n-tuple $(y_{\sigma(1)},\ldots,y_{\sigma(n)})$ will be denoted by $\mathbf{y}_\sigma$. With this notation we have $(\mathbf{y}_\sigma)_\tau=\mathbf{y}_{\sigma\tau}$ for all $\sigma,\tau\in S_n$. If $\sigma\in S_n$ and $t=t(x,\mathbf{y})$ is an arbitrary member of M, then we define the member σt of M as follows:

$$(\sigma t)(x,\mathbf{y}) := t(x,\mathbf{y}_\sigma).$$

It is an immediate consequence of the definitions that

$$\sigma(t+u)=\sigma t+\sigma u \qquad \text{and} \qquad (\sigma\tau)t=\sigma(\tau t)$$

for all $t,u\in M$ and $\sigma,\tau\in S_n$. The straightforward computations verifying these equalities are as follows:

$$(\sigma(t+u))(x,\mathbf{y})=(t+u)(x,\mathbf{y}_\sigma)=u(t(x,\mathbf{y}_\sigma),\mathbf{y}_\sigma)=(\sigma u)((\sigma t)(x,\mathbf{y}),\mathbf{y})=(\sigma t+\sigma u)(x,\mathbf{y})$$

and

$$((\sigma\tau)t)(x,\mathbf{y})=t(x,\mathbf{y}_{\sigma\tau})=t(x,(\mathbf{y}_\sigma)_\tau)=(\tau t)(x,\mathbf{y}_\sigma)=(\sigma(\tau t))(x,\mathbf{y}).$$

Thus M has a natural algebraic structure with an underlying commutative monoid, and an action of S_n as a group of monoid endomorphisms. We introduce some definitions regarding such structures.

A **semiring** is an algebra $\mathbf{R}=\langle R;\cdot,+,1,0\rangle$ where $\langle R;\cdot,1\rangle$ and $\langle R;+,0\rangle$ are monoids, $x+y=y+x$, $0\cdot x=0=x\cdot 0$, $x\cdot(y+z)=x\cdot y+x\cdot z$, and $(x+y)\cdot z=x\cdot z+y\cdot z$. If $\mathbf{M}$ is a commutative monoid, then the set $End(\mathbf{M})$ of endomorphisms of $\mathbf{M}$ has a natural semiring structure: multiplication of endomorphisms is given by composition, addition is pointwise addition, 1 is the identity endomorphism, 0 is the zero endomorphism. If $\mathbf{S}=\langle S;\cdot,1\rangle$ is a monoid, then an **S-semimodule** is a commutative monoid $\mathbf{M}=\langle M;+,0\rangle$ for which there is a prescribed homomorphism of $\mathbf{S}$ into the multiplicative monoid of the semiring $End(\mathbf{M})$. If $\mathbf{R}$ is the subsemiring of $End(\mathbf{M})$ generated by the image of $\mathbf{S}$, then we may identify the $\mathbf{S}$-semimodule structure on $\mathbf{M}$ with the $\mathbf{R}$-semimodule structure. In this way, it is meaningful for us to write sums of elements of $\mathbf{S}$: such sums are to be interpreted as elements of $End(\mathbf{M})$.

As we have seen above, for any clone C generated by an $(n+1)$-ary commutative semiprojection the set M of those $(n+1)$-ary operations which satisfy the semiprojection laws comprise an S_n-semimodule $\mathbf{M} = \langle M; +, S_n \rangle$ which we will refer to as the **associated semimodule**. We will show that such a clone C is minimal if and only if the associated semimodule has no proper nontrivial subsemimodule. We will call a semimodule with this property a **minimal semimodule**. After showing that a C is minimal if and only if its associated semimodule is minimal, we classify minimal semimodules and use them to give an explicit description of minimal commutative clones generated by a semiprojection.

It is easy to see from the defintion that a clone is trivial exactly when all operations are projections. There are two types of trivial clones: one in which the projections of the same arity are distinct, which we call the **initial clone**, and one in which the projections of the same arity coincide, which we call the **terminal clone**. The reason for this terminology is that the initial and terminal clones are initial and terminal objects in the category **Clone** of all clones and clone homomorphisms. It is easy to see that a clone is not terminal if and only if it has a subclone which is isomorphic to the initial clone. Let $\mathcal{C}$ denote the full subcategory of **Clone** consisting of all nonterminal clones which are generated by a commutative $(n+1)$-ary operation satisfying the semiprojection laws. Furthermore, let $\mathcal{M}$ denote the category of all one-generated S_n-semimodules and all semimodule homomorphisms. The relationship between $\mathcal{C}$ and $\mathcal{M}$ is described by the next lemma, which is the crucial result of this section.

LEMMA 4.3 *There exists a categorical equivalence between $\mathcal{C}$ and $\mathcal{M}$, which assigns to each clone in $\mathcal{C}$ its associated semimodule.*

Proof: Let C be a clone from $\mathcal{C}$, which is generated by the semiprojection s, and let $\mathbf{M}$ be the associated semimodule of C. By Lemma 4.2 (2) every member of $\mathbf{M}$ is of the form $\sigma_1 s + \sigma_2 s + \cdots + \sigma_k s$ for some k and some permutations $\sigma_i \in S_n$. Therefore $\mathbf{M}$ is generated by s, and hence belongs to $\mathcal{M}$.

Let C' be another clone from $\mathcal{C}$, $\mathbf{M}'$ its associated semimodule, and consider a clone homomorphism $\varphi: C \to C'$. If $t \in \mathbf{M}$ then t is a semiprojection or the projection x. This property is preserved by any clone homomorphism, therefore $t\varphi$ belongs to $\mathbf{M}'$. Thus φ restricts to a mapping $\varphi^-: \mathbf{M} \to \mathbf{M}'$. The operations of the associated semimodules are special clone operations, hence it follows that φ^- is a homomorphism of S_n-semimodules. Clearly, the restriction of a composition of homomorphisms equals the composition of the restrictions, and the restriction of each identity homomorphisms is an identity homomorphism.

This proves that there is a functor $F: \mathcal{C} \to \mathcal{M}$ which assigns to each clone in $\mathcal{C}$ its associated semimodule and to each homomorphism in $\mathcal{C}$ its restriction to the respective associated semimodules. To prove that F is a categorical equivalence it will be sufficient to verify that F is full, faithful, and representative. The faithfulness of F is easy to see. Let C, C' be clones from $\mathcal{C}$, and let s be a semiprojection generating C. If $\varphi, \psi: C \to C'$ are distinct homomorphisms, then $\varphi(s)$ and $\psi(s)$ must be different. But s lies in the semimodule associated to C, therefore φ^-, ψ^- are also distinct.

Next we prove that F is representative, that is, for each semimodule $\mathbf{N}$ from $\mathcal{M}$ there exists a clone C in $\mathcal{C}$ with associated semimodule isomorphic to $\mathbf{N}$. We will distinguish the semimodule operations of $\mathbf{N}$ notationally from the operations of a semimodule associated to a clone by writing $x \oplus y$ and $x \mapsto \sigma \circ x$ for the operations of $\mathbf{N}$ instead of $x + y$ and $x \mapsto \sigma x$.

Let $\mathbf{N}$ be any S_n-semimodule from $\mathcal{M}$ with generating element s. The clone C we are going to construct will be defined over the universe $U = N \cup \{a_1, \ldots, a_n\}$, where we assume that the elements $a_1, \ldots, a_n$ are pairwise distinct and the set $\{a_1, \ldots, a_n\}$ is disjoint from N. The n-tuple $(a_1, \ldots, a_n)$ will be denoted by $\mathbf{a}$. The generating element of C, which will also be denoted by s, is defined as follows:

$$s(x, \mathbf{y}) = \begin{cases} x \oplus (\pi \circ s) & \text{if } x \in N \text{ and } \mathbf{y} = \mathbf{a}_\pi, \\ x & \text{otherwise.} \end{cases}$$

We define C to be the clone on U generated by s.

It is easy to see that s satisfies the semiprojection laws. To verify that s commutes with itself, consider an $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} x & \mathbf{u} \\ \mathbf{v} & * \end{bmatrix}$$

of elements from U where the entries of the $n \times n$ submatrix denoted $*$ are irrelevant. Applying s first to the columns and then to the resulting row we get $s(s(x, \mathbf{v}), \mathbf{u}')$, where $\mathbf{u}'$ is of the form $\mathbf{a}_\pi$ for some $\pi \in S_n$ if and only if $\mathbf{u} = \mathbf{a}_\pi$. Therefore this element is equal to

$$\begin{cases} x \oplus (\rho \circ s) \oplus (\pi \circ s) & \text{if } x \in N,\ \mathbf{u} = \mathbf{a}_\pi, \text{ and } \mathbf{v} = \mathbf{a}_\rho, \\ x \oplus (\pi \circ s) & \text{if } x \in N,\ \mathbf{u} = \mathbf{a}_\pi, \text{ and } \mathbf{v} \text{ is not of the form } \mathbf{a}_\rho \text{ for any } \rho \in S_n, \\ x \oplus (\rho \circ s) & \text{if } x \in N,\ \mathbf{u} \text{ is not of the form } \mathbf{a}_\pi \text{ for any } \pi \in S_n, \text{ and } \mathbf{v} = \mathbf{a}_\rho, \\ x & \text{otherwise.} \end{cases}$$

We get the same result if we apply s first to the rows of the matrix and then to the resulting column. Thus s is a commutative $(n+1)$-ary semiprojection, or the projection x. Since C is not the terminal clone (as $|U| > 1$), this proves that C belongs to $\mathcal{C}$.

In the same way as we defined the operation s corresponding to the element $s \in N$, we can define an $(n+1)$-ary semiprojection $t = t(x, \mathbf{y})$ for every element t of N:

$$t(x, \mathbf{y}) = \begin{cases} x \oplus (\pi \circ t) & \text{if } x \in N \text{ and } \mathbf{y} = \mathbf{a}_\pi, \\ x & \text{otherwise.} \end{cases}$$

Let M denote the set of all these semiprojections. (Later we shall show that M is the universe of the associated semimodule of C.) It is straightforward to compute that for $t, u \in N$ and $\sigma \in S_n$ the operations corresponding to the elements $t \oplus u$ and $\sigma \circ t$ of N are

$$\begin{aligned} (t \oplus u)(x, \mathbf{y}) &= \begin{cases} x \oplus \pi \circ (t \oplus u) = x \oplus (\pi \circ t) \oplus (\pi \circ u) & \text{if } x \in N \text{ and } \mathbf{y} = \mathbf{a}_\pi, \\ x & \text{otherwise} \end{cases} \\ &= u(t(x, \mathbf{y}), \mathbf{y}) \end{aligned}$$

and

$$\begin{aligned} (\sigma \circ t)(x, \mathbf{y}) &= \begin{cases} x \oplus (\pi \circ (\sigma \circ t)) = x \oplus ((\pi\sigma) \circ t) & \text{if } x \in N \text{ and } \mathbf{y} = \mathbf{a}_\pi, \text{ i.e., if } x \in N \\ & \qquad \text{and } \mathbf{y}_\sigma = (\mathbf{a}_\pi)_\sigma = \mathbf{a}_{\pi\sigma}, \\ x & \text{otherwise.} \end{cases} \\ &= t(x, \mathbf{y}_\sigma). \end{aligned}$$

Now refer back to Lemma 4.2 (2) to see how the $(n+1)$-ary members of the clone C which satisfy the semiprojection laws can be expressed in terms of the generating operation s. Using this representation and the last two displayed equalities we get that all these members of C belong to M. Conversely, since s generates the S_n-semimodule $\mathbf{N}$, therefore every element $t \in \mathbf{N}$ has the form $t = (\sigma_1 \circ s) \oplus \cdots \oplus (\sigma_k \circ s)$ for some k and some permutations $\sigma_1, \ldots, \sigma_k \in S_n$. Hence the same equalities imply also that every operation $t(x, \mathbf{y})$ from M can be expressed by s in the form that occurs in Lemma 4.2 (2). Thus every operation from M belongs to C and satisfies the semiprojection laws. This implies that M is exactly the set of those $(n+1)$-ary members of C which satisfy the semiprojection laws. By definition this means that M is the underlying set of the associated semimodule $\mathbf{M}$ of C.

It is clear that the mapping $\mathbf{N} \to \mathbf{M}$, $t \mapsto t = t(x, \mathbf{y})$ is a bijection. The last two displayed equalities show that this mapping is an isomorphism. In fact, in the first equality the left hand side is the operation corresponding to the sum $t \oplus u$ in $\mathbf{N}$, while the right hand side is the operation which, by definition, is $t + u$ in the associated semimodule $\mathbf{M}$. Similarly, in the second equality the left hand side is the operation corresponding to the element $\sigma \circ t$ in $\mathbf{N}$, while the right hand side is the operation which is σt in the associated semimodule $\mathbf{M}$. This completes the proof that the functor F is representative.

Before proving that F is a full functor, we introduce some notation. Let C be a clone from $\mathcal{C}$, let s be a generating semiprojection, and let $\mathbf{M}$ denote the semimodule associated to C. We will write the members $t(x, \mathbf{y})$ of $\mathbf{M}$ in the form $x + t[\mathbf{y}]$. This will allow us to write any composite operation $u_m(\cdots u_2(u_1(x, \mathbf{y}_1), \mathbf{y}_2) \cdots, \mathbf{y}_m)$ in the form $x + u_1[\mathbf{y}_1] + u_2[\mathbf{y}_2] + \cdots + u_m[\mathbf{y}_m]$, or briefly $x + \sum_{i=1}^{m} u_i[\mathbf{y}_i]$. As we have seen, we don't have to be particular about associativity or the order of the summands following x. With this notation the unique representation for the elements of C described in Lemma 4.2 (1) takes the form $x + \sum_{i=1}^{r} t_i[\mathbf{z}_i]$ $(t_i \in \mathbf{M})$.

To prove that the functor F is full let us consider arbitrary clones C, C' from $\mathcal{C}$, their associated semimodules $\mathbf{M}, \mathbf{M}'$, and an arbitrary homomorphism $\chi\colon \mathbf{M} \to \mathbf{M}'$. We have to verify that χ extends to a homomorphism $\chi^+\colon C \to C'$. We define χ^+ as follows: if a member of C has arity at most n, then it is a projection, so its image under χ^+ will be the corresponding projection in C'; if $g(x, \mathbf{y})$ is an $(m+1)$-ary operation $(m \geq n)$ from C and its representation given in Lemma 4.2 (1) is

$$g(x, \mathbf{y}) = x + \sum_{i=1}^{r} t_i[\mathbf{z}_i] \qquad (t_i \in \mathbf{M}), \tag{1}$$

then the image of g under χ^+ is defined to be the operation

$$\chi^+(g(x, \mathbf{y})) = x + \sum_{i=1}^{r} \chi(t_i)[\mathbf{z}_i].$$

Clearly, χ^+ is indeed an extension of χ. Notice also that χ^+ has the property that for every member g of C the operations g and $\chi^+(g)$ have the same leftmost variable.

Let s be an $(n+1)$-ary operation generating C such that s is a semiprojection or the projection x. Then $s \in \mathbf{M}$ and hence $\chi(s)$ is an element of $\mathbf{M}'$. In fact, $\chi(s)$ is the generating element of the subsemimodule $\chi(\mathbf{M})$ of $\mathbf{M}'$. It follows from this that every operation in $\chi(\mathbf{M})$

can be expressed in terms of $\chi(s)$. Hence the definition of χ^+ yields that for each operation $g \in C$ the operation $\chi^+(g)$ belongs to the subclone of C' generated by $\chi(s)$.

Now we prove that χ^+ is a clone homomorphism. It suffices to verify that for all $m \geq 0$, for every $(m+1)$-ary operation $g(x, \mathbf{y}) \in C$ with leftmost variable x, and for arbitrary $(m+1)$-ary operations $g_1(x, \mathbf{y})$, ..., $g_n(x, \mathbf{y})$ from C the following equality holds:

$$\chi^+(s(g(x,\mathbf{y}), g_1(x,\mathbf{y}), \ldots, g_n(x,\mathbf{y}))) = (\chi(s))(\chi^+(g(x,\mathbf{y})), \chi^+(g_1(x,\mathbf{y})), \ldots, \chi^+(g_n(x,\mathbf{y}))). \tag{2}$$

If $m < n$, then the operations $g, g_1, \ldots, g_n$ are projections, so (2) trivially holds. From now on let $m \geq n$, and for each i let x_i denote the common leftmost variable of $g_i(x, \mathbf{y})$ and $\chi^+(g_i(x, \mathbf{y}))$. As we have seen at the beginning of the proof of Lemma 4.2, a repeated application of the equation in Lemma 4.1 (1) reduces the composite operation on the left hand side of (2) to $s(g(x,\mathbf{y}), x_1, \ldots, x_n)$. On the right hand side of (2) we can use the same argument with $\chi(s)$ in place of s, since the operations $\chi^+(g), \chi^+(g_1), \ldots, \chi^+(g_n)$ belong to the subclone of C' generated by $\chi(s)$. This reduces the right hand side of (2) to $(\chi(s))(\chi^+(g(x,\mathbf{y})), x_1, \ldots, x_n)$. Therefore the equality to be proved is

$$\chi^+(s(g(x,\mathbf{y}), x_1, \ldots, x_n)) = (\chi(s))(\chi^+(g(x,\mathbf{y})), x_1, \ldots, x_n). \tag{3}$$

If one of the variables $x_1, \ldots, x_n$ is x, which is the leftmost variable of $g(x, \mathbf{y})$, or if these variables are not pairwise distinct, then both sides of (3) reduce to $\chi^+(g(x, \mathbf{y}))$, so (3) holds. Otherwise, there is a $\mathbf{z}_j$ among the enumeration of the subsequences of $\mathbf{y}$ of length n such that $(x_1, \ldots, x_n)$ coincides with $(\mathbf{z}_j)_\sigma$ for some permutation $\sigma \in S_n$. Suppose that the representation of $g(x, \mathbf{y})$ given in Lemma 4.2 (1) has the form displayed in (1). Then the operation on the left hand side of (3) is

$$\begin{aligned} s(g(x,\mathbf{y}), x_1, \ldots, x_n) &= (x + \textstyle\sum_{i=1}^r t_i[\mathbf{z}_i]) + s[(\mathbf{z}_j)_\sigma] \\ &= (x + \textstyle\sum_{i=1}^r t_i[\mathbf{z}_i]) + \sigma s[\mathbf{z}_j] \\ &= (x + \textstyle\sum_{i\neq j} t_i[\mathbf{z}_i]) + (t_j + \sigma s)[\mathbf{z}_j], \end{aligned}$$

while the operation on the right hand side of (3) is

$$\begin{aligned} \chi(s)(\chi^+(g(x,\mathbf{y})), x_1, \ldots, x_n) &= (x + \textstyle\sum_{i=1}^r \chi(t_i)[\mathbf{z}_i]) + \chi(s)[(\mathbf{z}_j)_\sigma] \\ &= (x + \textstyle\sum_{i=1}^r \chi(t_i)[\mathbf{z}_i]) + \sigma\chi(s)[\mathbf{z}_j] \\ &= (x + \textstyle\sum_{i\neq j} \chi(t_i)[\mathbf{z}_i]) + (\chi(t_j) + \sigma\chi(s))[\mathbf{z}_j]. \end{aligned}$$

The latter operation is the image of the preceding one under χ^+, because in the semimodule $\mathbf{M}'$ we have $\chi(t_j + \sigma s) = \chi(t_j) + \sigma\chi(s)$. This completes the proof of (3). □

COROLLARY 4.4 *Assume that $s(x, \mathbf{y})$ is a commutative semiprojection of arity $(n+1)$ and $\mathcal{V}$ is the variety of s-representations. Let $\mathbf{M}$ be the associated S_n-semimodule. The clone of $\mathcal{V}$ is minimal if and only if $\mathbf{M}$ is a minimal semimodule.* □

Now we determine the minimal S_n-semimodules.

LEMMA 4.5 *The minimal S_n-semimodules are*

(1) the two-element, lower bounded, join semilattice $\langle\{0,1\};+,0\rangle$ with trivial S_n-action, and

(2) the irreducible S_n-modules over fields of prime order.

Proof: It is easy to see that the S_n-semimodules of types (1) and (2) are minimal: those of type (1) because the semimodule has only two elements; those of type (2) because a subsemimodule of an S_n-module of characteristic greater than zero is in fact a submodule, and by definition the only proper submodule of an irreducible S_n-module is the trivial one. Therefore, the nontrivial claim of the lemma is that any minimal S_n-semimodule is of one of the two types.

Let $\mathbf{M}$ be a minimal S_n-semimodule. Let $\sigma_1,\ldots,\sigma_{n!}$ be an enumeration of S_n where σ_1 is the identity. Let Ω denote the sum $\sigma_1+\cdots+\sigma_{n!}$ in $End(\mathbf{M})$. In $End(\mathbf{M})$ we have $\sigma\Omega=\Omega=\Omega\sigma$ for all $\sigma\in S_n$; in particular, the additive endomorhism Ω commutes with every element of S_n. Hence Ω is a semimodule endomorphism of $\mathbf{M}$. By the minimality of $\mathbf{M}$, it is either the zero map or it is onto. We first consider the case that it is onto.

Choose any $\sigma\in S_n$ and any $m\in M$. Then, since Ω is onto, there is an element $n\in M$ such that $m=\Omega n$. Hence

$$\sigma m=\sigma(\Omega n)=(\sigma\Omega)n=\Omega n=m.$$

This proves that S_n acts trivially on $\mathbf{M}$, so $\mathbf{M}$ is simply a minimal monoid with trivial S_n-action. But we have already determined the minimal monoids in Claim 3.11 of the proof of Lemma 3.8: they are the two-element semilattice and the cyclic groups of prime order. The cyclic groups of prime order considered as S_n-semimodules are precisely the (irreducible) unit representations of S_n over the fields of prime order. Therefore, when Ω is onto, the structure of $\mathbf{M}$ is as described in the lemma.[1]

Now we assume that Ω is the zero map. For any $m\in M$ we have

$$0=\Omega m=\sigma_1 m+\sigma_2 m+\cdots+\sigma_{n!}m=m+\sigma_2 m+\cdots+\sigma_{n!}m.$$

This shows that every $m\in M$ has an additive inverse $-m=\sigma_2 m+\cdots+\sigma_{n!}m$. Thus $\mathbf{M}$ is an S_n-module which is minimal as a semimodule. Assume that some nonzero element of $\mathbf{M}$ has finite additive order. Then there is an $m_0\in M$ whose additive order is p for some prime p. The map

$$x\mapsto px:=\underbrace{x+x+\cdots+x}_{p\ x\text{'s}}$$

is a semimodule endomorphism which annihilates m_0, so it annihilates the subsemimodule generated by m_0, which is all of $\mathbf{M}$. Thus, in the case where $\mathbf{M}$ has an element of finite additive order, $\mathbf{M}$ is additively a vector space over the p-element field. The minimality of $\mathbf{M}$ as an S_n-(semi)module implies that it is an irreducible representation of S_n over the p-element field.

[1]For Ω to be onto in the case when $\mathbf{M}$ is a cyclic group of prime order p with trivial S_n-action, then we must have $p>n$. But the case where $p\leq n$ also yields minimal semimodules; they are included in the argument for the case where Ω is zero.

The final case to consider is the case where $\mathbf{M}$ is an S_n-module where every nonzero element has infinite additive order; i.e., where the underlying additive group $\langle M; +, -, 0\rangle$ is torsion-free. We will show that this case cannot occur by showing that it leads to the following contradictory conclusions: (i) $\langle M; +, -, 0\rangle$ is a finitely generated abelian group and (ii) $\langle M; +, -, 0\rangle$ is a divisible group. That these are contradictory conclusions follows immediately from the structure theorem for finitely generated abelian groups.

To reach conclusion (i), choose some $m_1 \in M - \{0\}$. The set $\{\sigma_1 m_1, \sigma_2 m_1, \ldots, \sigma_{n!} m_1\}$ is closed under the action of S_n, so the additive subgroup of $\mathbf{M}$ that it generates is a subsemimodule of $\mathbf{M}$. Since the subsemimodule contains $m_1 \neq 0$, it is all of $\mathbf{M}$. Thus, $\mathbf{M}$ is finitely generated as a group.

To reach conclusion (ii), choose any $m_2 \in M - \{0\}$ and any positive integer n. We must show that the equation $nx = m_2$ has a solution in $\mathbf{M}$. The map $x \mapsto nx$ is an endomorphism of the module $\mathbf{M}$ which is one-to-one, since $\langle M; +, -, 0\rangle$ is torsion-free. Therefore it is onto, by the minimality of $\mathbf{M}$. This implies the existence of a solution to $nx = m_2$, and so the proof is finished. □

Let $s(x, \mathbf{y})$ be a commutative semiprojection which generates a minimal clone. If the associated minimal semimodule is a two-element semilattice with trivial S_n-action, then we call any s-representation a **right semilattice**. This does not conflict with our previous use of the term. If the associated minimal semimodule is an irreducible S_n-module, then we call any s-representation a **right irreducible S_n-module**.

Combining Corollary 4.4 and Lemma 4.5, we get the following list of all minimal clones which are generated by a commutative semiprojection.

THEOREM 4.6 *Let $\mathcal{V}$ be the variety of s-representations, where s is a commutative semiprojection. If $\mathcal{V}$ has a minimal clone, then $\mathcal{V}$ is one of the following varieties:*

(1) The variety of right semilattices.

(2) A variety of right irreducible S_n-modules over a field of prime order. □

We know from Lemma 4.3 that a minimal clone generated by an $(n+1)$-ary commutative semiprojection is determined up to isomorphism by the isomorphism type of its associated semimodule. The result of Lemma 4.5 shows that every minimal S_n-semimodule is simple. Thus, any commutative clone generated by a semiprojection which satisfies relations implying that the associated semimodule is minimal must be a minimal or trivial clone. This means that

- the idempotent and commutative laws,
- the semiprojection laws, and
- any set of relations which constitute a presentation of the associated (minimal) semimodule

constitute a full set of defining relations for any minimal commutative clone generated by a semiprojection.

Each of the varieties described in Theorem 4.6 is generated by its free algebra on $n+1$ generators, where $n+1$ is the arity of s. This free algebra is finite, in fact it has cardinality $(n+1)\cdot|M|$ where $\mathbf{M}$ is the associated semimodule, so each clone listed has a finite faithful representation.

In light of Lemma 2.1, Theorems 2.2, 2.3, 3.20 and 4.6 constitute a description of all commutative minimal clones.

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