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Stability Threshold for Scalar Linear Periodic Delay Differential Equations

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Abstract. We prove that for the linear scalar delay differential equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t-1)$$

with non-negative periodic coefficients of period $P > 0$, the stability threshold for the trivial solution is $r := \int_0^P (b(t) - a(t))\mathrm{d}t = 0$, assuming that $b(t+1) - a(t)$ does not change its sign. By constructing a class of explicit examples, we show the counter-intuitive result that, in general, $r = 0$ is not a stability threshold.

1 Introduction

We investigate the scalar periodic delay-differential equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t-1), \tag{1.1}$$

where a, b are assumed to be P-periodic continuous real functions with $a(t) \geq 0$ and $b(t) \geq 0$. Equation (1.1) has been studied as the linear variational equation of

$$x'(t) = g(t, x(t), x(t-1)),$$

where $g(t, 0, 0) = 0$ and $g(t, \xi, \eta) = g(t + P, \xi, \eta)$ for all $t, \xi, \eta \in \mathbb{R}$. Similarly, for a smooth nonlinearity $f(x, y)$, the linearization of $u'(t) = f(u(t), u(t-1))$ around a periodic orbit $p(t)$ is

$$u'(t) = f_x(p(t), p(t-1))u(t) + f_y(p(t), p(t-1))u(t-1),$$

having the same form as (1.1). This type of equation arises in several mathematical models, such as neural networks [3], or transmission dynamics of vector-borne diseases [2], and population growth models [6, 10] with seasonality. One can interpret (1.1) as a population model of a single species with periodically varying recruitment and mortality rates and fixed length juvenile period. Then the non-negativity assumptions on the coefficients $a(t)$ and $b(t)$ are biologically natural.

Let $\Omega := C([-1, 0], \mathbb{R})$ be the Banach space of real valued continuous functions on $[-1, 0]$ with the usual supremum norm. For any $\phi \in \Omega$, a unique solution $x(t; \phi)$ exists for all $t \geq 0$ with $x(\theta) = \phi(\theta)$, $-1 \leq \theta \leq 0$. From the non-negativity of the coefficients, it follows that the non-negative cone $\Omega_+ := C([-1, 0], \mathbb{R}_+)$ is positively invariant as

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non-negative solutions remain non-negative. We use the notation $x_t = x_t^{\phi} \in \Omega$ for the function $x_t(\theta) = x(t+\theta)$, $\theta \in [-1, 0]$. Let $\mathcal{U}: \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \Omega$ be the solution operator of (1.1). That is, $\mathcal{U}(t, \sigma, \phi) = x_{t+\sigma}$, where $x_{t+\sigma}$ is the segment of the solution of the initial value problem

$$\begin{aligned} \dot{x}(t) &= -a(t)x_t(0) + b(t)x_t(-1), \quad t \geq \sigma, \\ x_\sigma &= \phi, \end{aligned}$$

at time $t+\sigma$. We now define the monodromy operator (also referred to in the literature as the Poincaré-map, time-one map, period map) $\mathcal{M}: \Omega \to \Omega$ by $\mathcal{M}(\psi) = \mathcal{U}(P, 0, \psi)$. The stability of zero is determined by the spectral radius of $\mathcal{M}$ [4].

In the special case when $a(t) = a_*$ and $b(t) = b_*$ are constants, the sharp stability condition $a_* \geq b_*$ is very well known [8]. Equation (1.1) with general time dependent bounded continuous coefficients was addressed in [4], where it was shown that the solution $x = 0$ of (1.1) is uniformly asymptotically stable if $\sup_t b(t) < k \inf_t a(t)$ for some $0 \leq k < 1$. This has been applied to the periodic case in [2], and further related investigations can be found in [5]. In the periodic case, for $P = 1$, the characteristic equation was derived in [7] using Floquet theory as

$$\lambda + \int_0^1 a(s)\mathrm{d}s = \int_0^1 b(s)\mathrm{d}s e^{-\lambda},$$

and it immediately follows (see [8]) that the stability threshold in this case is $r = 0$ where $r := \int_0^1 (b(s) - a(s))\mathrm{d}s$. This result also extends naturally to the case $\tau = 1 = kP$, $k \in \mathbb{N}$. The same conclusion was derived using a different approach in [10] as well, where the authors studied a competitive population model with stage structure in a seasonal environment (see also [6]).

The special case of $a(t)$ being a constant function, but P is arbitrary, was considered recently by Chen and Wu [3]. Using a discrete Lyapunov functional and the variation of constants formula, they found that for any $b(t) > 0$ there is a critical $a^+ > 0$ that is the stability threshold. Some estimates were provided for a^+, but the exact value was not determined. In Section 2, we derive the explicit threshold formula, determining the stability of zero for (1.1), which is valid even when the period P is not related to the delay (generalizing the implications of [7, 10]), assuming P-periodic $a(t) \geq 0$, $b(t) \geq 0$ such that $b(t+1) - a(t)$ does not change its sign. Our theorem provides some new results compared to the one in [4], since, for example, the following simple case does not fit there but will be covered here.

$$\begin{aligned} a(t) &= t(P-t) + 1, \quad 0 \leq t \leq P, \\ b(t+1) &= t(P-t) + 1 - \epsilon, \quad 0 \leq t+1 \leq P, \end{aligned}$$

where a and b are extended to the real line periodically, so that they are P-periodic functions with $P > 1$ and $\epsilon < P - 1$. Moreover, unlike in [3], our stability threshold is given explicitly.

Knowing that r is the stability threshold whenever $a(t)$ and $b(t)$ are non-negative continuous periodic functions with the same period $1/k$, $k \in \mathbb{N}$, and when they are non-negative, the period is arbitrary and $b(t+1) - a(t)$ does not change its sign, one may conjecture that r being the stability threshold is the general property of (1.1).

This conjecture is also supported by the fact that r can be interpreted as a Malthusian parameter of a population model, being the time averaged difference of birth and death rates (the relation of r to the basic reproduction number defined for periodic systems is addressed in Section 3). However, in Section 4, we construct a family of non-negative periodic coefficients for which the sign of r does not determine the stability of zero in (1.1). We compute the exact stability threshold for this family as well.

2 Stability Theorem

Without loss of generality, we can assume $P > 1$. Define

$$r := \int_0^P (b(s) - a(s))\mathrm{d}s. \tag{2.1}$$

Theorem 2.1 *For* (1.1), *the following hold if the sign of* $b(t+1) - a(t)$ *does not change.*

(i) *If* $r > 0$, *zero is unstable.*
(ii) *If* $r = 0$, *zero is stable, but not asymptotically stable.*
(iii) *If* $r < 0$, *zero is asymptotically stable.*

Remark 2.2 Note that the conditions for (i)–(iii) can be written in a more explicit way. For example, the condition in (i) is the same as assuming $b(t+1) \geq a(t) \geq 0$ for all $t \in \mathbb{R}$ and $b(s+1) \not\equiv a(s)$ for some $s \in \mathbb{R}$. We stated the theorem in a way that stresses the threshold property of r.

Proof (i) It is sufficient to show that $\lim_{t\to\infty} x(t;\phi) = \infty$ for $\phi \in \Omega_+$ with $\phi(\theta) > 0$ for all $\theta \in [-1, 0]$. We first prove that $x^\infty := \limsup_{t\to\infty} x(t;\phi) > 0$. For simplicity, we write $x(t)$ for $x(t;\phi)$. Suppose $\limsup_{t\to\infty} x(t) = 0$. It implies

$$\lim_{t\to\infty} x(t) = 0 \tag{2.2}$$

by the non-negativity of $x(t)$. We define the function $V: \mathbb{R} \to \mathbb{R}$ by

$$V(t) := \int_{t-1}^t b(u+1)x(u)\mathrm{d}u + x(t). \tag{2.3}$$

The boundedness of $b(t)$ and (2.2) imply

$$\lim_{t\to\infty} V(t) = 0. \tag{2.4}$$

One can see from (2.3) that $\dot{V}(t) = (b(t+1) - a(t))x(t)$ and

$$V(t) = V(0) + \int_0^t (b(u+1) - a(u))x(u)\mathrm{d}u. \tag{2.5}$$

For any integer $n \geq 1$, using the integral mean-value theorem, one has

$$\begin{aligned} V(nP) &= V(0) + \sum_{k=1}^{n} \int_{(k-1)P}^{kP} (b(u+1) - a(u))x(u)\mathrm{d}u \\ &= V(0) + \sum_{k=1}^{n} x(u_k^*) \int_{(k-1)P}^{kP} (b(u+1) - a(u))\mathrm{d}u \\ &= V(0) + \sum_{k=1}^{n} x(u_k^*) \int_{0}^{P} (b(u+1) - a(u))\mathrm{d}u \\ &= V(0) + r\sum_{k=1}^{n} x(u_k^*) \qquad (2.6) \end{aligned}$$

for some $u_k^* \in ((k-1)P, kP)$. Positivity of $x(t)$ and $r > 0$ imply $\{V(nP)\}_{n\in\mathbb{N}}$ is strictly increasing with $V(0) \geq 0$, which contradicts (2.4). Hence, $x^\infty > 0$.

Now we will show that $\lim_{t\to\infty} x(t) = \infty$. Non-negativity of $x(t)$ on (1.1) implies

$$\dot{x}(t) \geq -a(t)x(t)$$

for all $t \geq 0$. By the comparison method described in [8, Theorem 3.6], for $t_2 \geq t_1$,

$$x(t_2) \geq x(t_1)e^{-\int_{t_1}^{t_2} a(u)\mathrm{d}u}.$$

Since $x(t)$ is continuous, it has a minimum m_k and a maximum M_k on each interval $[(k-1)P, kP]$, attained as points $t_{m_k}, t_{M_k} \in [(k-1)P, kP]$, $k = 1, 2, \ldots$. Comparing m_{k+1} and M_k to $x(kP)$, from the previous inequality one can deduce

$$m_{k+1} \geq x(kP)e^{-\int_{kP}^{t_{m_{k+1}}} a(u)\mathrm{d}u} \geq x(kP)e^{-\int_{kP}^{(k+1)P} a(u)\mathrm{d}u}$$

and

$$x(kP) \geq M_k e^{-\int_{t_{M_k}}^{kP} a(u)\mathrm{d}u} \geq M_k e^{-\int_{(k-1)P}^{kP} a(u)\mathrm{d}u}.$$

Hence

$$m_{k+1} \geq M_k e^{-\int_{(k-1)P}^{(k+1)P} a(u)\mathrm{d}u} = M_k e^{-2\int_0^P a(u)\mathrm{d}u}, \qquad (2.7)$$

and finally

$$\limsup_{k\to\infty} m_k \geq \limsup_{k\to\infty} M_k e^{-2\int_0^P a(u)\mathrm{d}u} = x^\infty e^{-2\int_0^P a(u)\mathrm{d}u} > 0. \qquad (2.8)$$

Since $\{V(nP)\}_{n\in\mathbb{N}}$ is strictly increasing, either it converges or $\lim_{n\to\infty} V(nP) = \infty$. If it converges, by (2.6), $x(u_k^*) \to 0$ as $k \to \infty$, which contradicts (2.8). Therefore, $\lim_{n\to\infty} V(nP) = \infty$. Applying (2.3) to $t = nP$, we have

$$\begin{aligned} V(nP) &= \int_{nP-1}^{nP} b(u+1)x(u)\mathrm{d}u + x(nP) \\ &= x(t_n^*) \int_{nP-1}^{nP} b(u+1)\mathrm{d}u + x(nP) \\ &\leq M_n\Big(1 + \int_{P-1}^{P} b(u+1)\mathrm{d}u\Big) \end{aligned}$$

for some $t_n^* \in [nP-1, nP] \subset [(n-1)P, nP]$. The boundedness of $b(t)$ and

$$\lim_{n\to\infty} V(nP) = \infty$$

imply $\lim_{n\to\infty} M_n = \infty$. Now it follows from (2.7) that $\lim_{n\to\infty} m_n = \infty$. Thus

$$\lim_{t\to\infty} x(t) = \infty.$$

(ii) Assume that $r = 0$. By the following equality

$$0 = \int_0^P (b(u) - a(u))\mathrm{d}u = \int_0^P (b(u+1) - a(u))\mathrm{d}u,$$

together with the assumption that $b(u+1)-a(u)$ does not change its sign, we conclude

$$b(u+1) - a(u) = 0 \quad \text{for all } u \in R.$$

By (2.5), we obtain

$$V(t) = V(0) \quad \text{for all } t. \tag{2.9}$$

If $\phi \ge 0$, by (2.3), $0 \le x(t) \le V(t) = V(0) \le (b_{\max}+1)\|\phi\|$, where $b_{\max}$ denotes the maximum of $b(t)$ and by $\phi \ge 0$ we mean that the inequality $\phi(\theta) \ge 0$ holds for any $\theta \in [-1, 0]$. If $\phi \le 0$, by (2.3),

$$0 \ge x(t) \ge V(t) = V(0) \ge -(b_{\max}+1)\|\phi\|.$$

Now for any $\phi \in \Omega$, let initial functions $\xi \ge 0$ and $\psi \le 0$ such that $\psi < \phi < \xi$. By the comparison principle [8],

$$-(b_{\max}+1)\|\phi\| \le x(t;\psi) \le x(t;\phi) \le x(t;\xi) \le (b_{\max}+1)\|\phi\|.$$

Therefore, the zero is stable. One can easily see that zero is not asymptotically stable by (2.9) and (2.3).

(iii) It is sufficient to prove that $\lim_{t\to\infty} x(t;\phi) = 0$ for any $\phi \in \Omega$. We first prove it for $\phi \ge 0$, and we show that it also holds for $\phi \le 0$. Finally we prove it for general ϕ.

If $\phi \ge 0$, since $r < 0$, one can see from (2.6) that $\{V(nP)\}_{n\in\mathbb{N}}$ is decreasing, with lower bound 0. Therefore, $\{V(np)\}$ converges, implying $x(u_k^*) \to 0$ as $k \to \infty$. Meanwhile,

$$x(u_{k+1}^*) \ge m_{k+1} \ge M_k e^{-2\int_0^P a(u)\mathrm{d}u},$$

which implies $M_k \to 0$ as $k \to \infty$. Hence, $x(t) \to 0$ as $t \to \infty$.

Consider the case with non-positive ϕ. One can see from (1.1) that $x(t;-\phi) = -x(t;\phi)$ and $\lim_{t\to\infty} x(t;\phi) = -\lim_{t\to\infty}(-x(t;\phi)) = -\lim_{t\to\infty} x(t;-\phi) = 0$. Now for any $\phi \in \Omega$, we can choose initial functions $\xi \ge 0$ and $\psi \le 0$ such that $\psi < \phi < \xi$. By the comparison principle, $x(t;\psi) \le x(t;\phi) \le x(t;\xi)$. We know that $\lim_{t\to\infty} x(t;\xi) = 0 = \lim_{t\to\infty} x(t;\psi)$. Therefore, $\lim_{t\to\infty} x(t;\phi) = 0$. ■

3 Relation of r to the Basic Reproduction Number

In a biological context, r can be interpreted as an averaged Malthusian parameter, and

$$R = \frac{\int_0^P b(s)\mathrm{d}s}{\int_0^P a(s)\mathrm{d}s}$$

can be interpreted as an averaged reproduction number, and then $R > 1$ is equivalent to $r > 0$. However, this naive approach does not give us the adequate basic reproduction number for periodic systems or periodic equations with delays, and as we show in Section 4, there are examples when $R = 1$ is not a stability threshold as we might expect. The definition of the basic reproduction number R_0 for periodic systems is more involved (see [1,9]), and in the sequel of this Section we follow the definition and notation of Zhao [11]. In particular, [11, §3] deals with a periodic delay SEIR model, and linearizing around the disease-free periodic solution, one obtains a scalar periodic linear delay differential equation for the infectives, namely [11, (3.5)], which has exactly the same form as our equation (1.1). Let C_P be the Banach space of continuous P-periodic functions from $\mathbb{R} \to \mathbb{R}$, equipped with the supremum norm. Define the linear operator $L: C_P \to C_P$ by

$$[Lv](t) = \int_\tau^\infty e^{-\int_{t-s+\tau}^t a(u)\mathrm{d}u} b(t-s+\tau)v(t-s)\mathrm{d}s,$$

where $v \in C_P$. Then the basic reproduction number is defined as the spectral radius of the operator L, *i.e.*, $R_0 := \rho(L)$ (see [11]). Since (1.1) is in the class of [11, (2.1)], we can apply [11, Theorem 2.1] combined with our Theorem 2.1 to obtain the following.

Corollary 3.1 *Assume that the sign of $b(t+1) - a(t)$ does not change. Then $r < 0$ if and only if $R_0 < 1$, $r = 0$ if and only if $R_0 = 1$, and $r > 0$ if and only if $R_0 > 1$.*

As stated in the final comments of [11], in general it is not easy to numerically compute R_0 for time delayed periodic population models, therefore our results here can be particularly useful in many situations.

4 The Case of r Not Being a Stability Threshold

In this section, we present a particular example showing that the assumption in Theorem 2.1 is critical.

Consider a special case $a(t) = \alpha \in \mathbb{R}_+$ and $b(t)$ a continuous function such that

$$\begin{cases} b(t) = 0 & \text{if } kP \le t \le kP + L, \quad k = 0, 1, 2, \dots, \\ b(t) > 0 & \text{elsewhere,} \end{cases} \tag{4.1}$$

where $1 \le L < P < L + 1$.

Lemma 4.1 *Let*

$$\mathcal{A} := \left\{ \psi \in \Omega \;\middle|\; \psi(\theta) = \begin{cases} \psi(-1)e^{-\alpha(1+\theta)} & \text{if } \theta \in [-1, L-P] \\ \psi(-1)e^{-\alpha(1+\theta)}\left(e^\alpha \int_{L-P}^\theta b(s)\mathrm{d}s + 1\right) & \text{if } \theta \in (L-P, 0] \end{cases} \right\}.$$

Then $\mathcal{M}(\Omega) \subset \mathcal{A}$. Consequently, $\mathcal{A}$ is forward invariant under $\mathcal{M}$.

Proof Let $\psi \in \Omega$. Then $\mathcal{M}(\psi) = \mathcal{U}(P, 0, \psi) = x_P$ where x_P is the solution of

$$\dot{x}(t) = -\alpha x_t(0) + b(t)x_t(-1), \quad t \ge 0,$$
$$x_0 = \psi.$$

For $P - 1 \le t < L$, $x'(t) = -\alpha x(t)$ and $x(t) = x(P-1)e^{-\alpha(t-(P-1))}$. Hence, for $-1 \le \theta < L - P$, $x_P(\theta) = x_P(-1)e^{-\alpha(\theta+1)}$. For $L \le t < P$, $0 \le t - 1 < L$, and we have

$$x(t-1) = x(P-1)e^{-\alpha(t-P)}.$$

Therefore, $x'(t) = -\alpha x(t) + b(t)x(P-1)e^{-\alpha(t-P)}$, and the solution is

$$x(t) = x(P-1)e^{-\alpha(t+1-P)}\Big(e^{\alpha}\int_L^t b(s)\mathrm{d}s + 1\Big).$$

Thus, for $L - P \le \theta < 0$,

$$x_P(\theta) = x_P(-1)e^{-\alpha(1+\theta)}\Big(e^{\alpha}\int_{L-P}^{\theta} b(s)\mathrm{d}s + 1\Big). \tag{4.2}$$

■

Theorem 4.2 *Let* $\gamma := -\alpha + \frac{1}{P}\ln\big(e^{\alpha}\int_{L-P}^{0} b(s)\mathrm{d}s + 1\big)$. *The solution* $x = 0$ *of equation* (1.1) *with* (4.1) *is stable if and only if* $\gamma \le 0$.

Proof From the calculations of the proof of Lemma (4.1), we find that for any $\phi \in \mathcal{M}$,

$$x(P;\phi) = x(0;\phi)e^{\gamma} = \phi(0)e^{\gamma}.$$

Inductively, for any n, we have $x(nP;\phi) = \phi(0)e^{\gamma n}$. If there exists a $K > 0$ such that for any solution, $x_{nP}(\theta) \le Kx_{(n-1)P}(0)$ for all $\theta \in [-1, 0]$, the stability result follows and $\gamma < 0$ gives asymptotic stability. For $(n-1)P \le t < nP - 1 < (n-1)P + L$, $x'(t) = -\alpha x(t)$ and $x(nP-1) = x((n-1)P)e^{-\alpha(P-1)}$. By (4.2), for $L - P \le \theta < 0$,

$$\begin{aligned}
x_{nP}(\theta) &= x_{nP}(-1)e^{-\alpha(1+\theta)}\Big(e^{\alpha}\int_{L-P}^{\theta} b(s)\mathrm{d}s + 1\Big)\\
&= x_{(n-1)P}(0)e^{-\alpha(P-1)}e^{-\alpha(1+\theta)}\Big(e^{\alpha}\int_{L-P}^{\theta} b(s)\mathrm{d}s + 1\Big)\\
&\le x_{(n-1)P}(0)e^{-\alpha(P-1)}e^{-\alpha(1+L-P)}\Big(e^{\alpha}\int_{L-P}^{0} b(s)\mathrm{d}s + 1\Big)\\
&= x_{(n-1)P}(0)e^{-\alpha L}\Big(e^{\alpha}\int_{L-P}^{0} b(s)\mathrm{d}s + 1\Big),
\end{aligned}$$

so we can choose $K = e^{-\alpha L}(e^{\alpha}\int_{L-P}^{0} b(s)\mathrm{d}s + 1)$. The instability is obvious for $\gamma > 0$. ■

Finally, we address an example where the sign of r does not always coincide with the sign of γ. Consider the special case of (4.1),

$$b(t) = \begin{cases} 0 & \text{if } kP \le t \le kP + L\\ \dfrac{4\beta}{P-L}\Big(-\Big|t - \dfrac{P+L}{2}\Big| + \dfrac{P-L}{2}\Big) & \text{if } kP + L \le t \le (k+1)P, \end{cases} \tag{4.3}$$

where $k = 0, 1, 2, \ldots$. In this case,

$$\gamma = -\alpha + \frac{1}{P}\ln(e^{\alpha}\beta(P-L) + 1) \quad \text{and} \quad r = \beta(P-L) - \alpha P.$$

The following four scenarios: (i) $r > 0, \gamma > 0$ (unstable), (ii) $r < 0, \gamma > 0$ (unstable), (iii) $r > 0, \gamma < 0$ (stable), and (iv) $r < 0, \gamma < 0$ (stable) are all possible. Figure 1 shows the parameter sets of each case. The area with $\gamma < 0$ but $r > 0$, and the area with $\gamma > 0$ but $r < 0$ are the regions where r in (2.1) does not work as a stability threshold. Figures 2 and 3 show situations when the stability is just the opposite that one would expect from the sign of r. Overall, our results show that for a large class of scalar periodic delay differential equations, time averaging of the coefficients preserves the stability property of zero, however it is not always the case. This suggests that in practical problems, one needs to think about periodic variations in the model parameters very carefully.

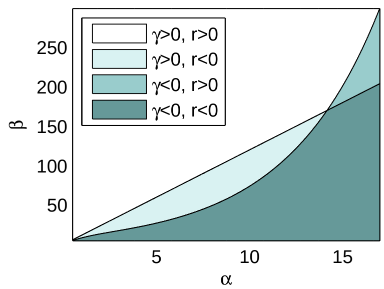

Figure 1: Special case of (4.1) with function $b(t)$ as in (4.3) with $P = 1.2$ and $L = 1.1$. Distinctive $\alpha - \beta$ parameter regions are determined by the signs of γ and r.

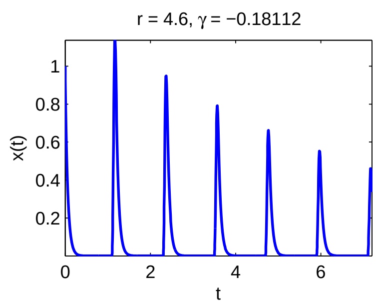

Figure 2: Solution with parameters $\alpha = 17$ and $\beta = 250$, which implies $r > 0$ but $\gamma < 0$. Zero solution is stable. Initial function is given by $\phi(\theta) = 1$ for all $\theta \in [-1, 0]$.

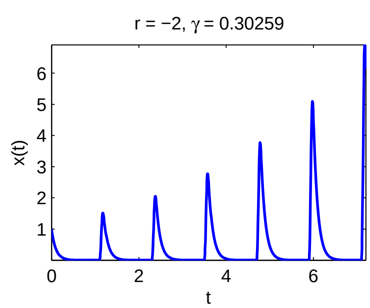

Figure 3: Solution with parameters $\alpha = 10$ and $\beta = 100$, which implies $r < 0$ but $\gamma > 0$. Zero solution is unstable. Initial function is given by $\phi(\theta) = 1$ for all $\theta \in [-1, 0]$.

References

[1] N. Bacaër and S. Guernaoui, *The epidemic threshold of vector-borne diseases with seasonality: the case of cutaneous leishmaniasis in Chichaoua, Morocco*. J. Math. Biol. **53**(2006), 421–436. http://dx.doi.org/10.1007/s00285-006-0015-0

[2] S. Busenberg and K. L. Cooke, *Periodic solutions of a periodic nonlinear delay differential equation*. SIAM J. Appl. Math. **35**(1978), no. 4, 704–721. http://dx.doi.org/10.1137/0135059

[3] Y. Chen and J. Wu, *Threshold dynamics of scalar linear periodic delay-differential equations*. In: Infinite dimensional dynamical systems. Fields Inst. Commun. 64. Springer, New York, 2013, pp. 269–278.

[4] J. K. Hale and S. Verduyn-Lunel, *Introduction to functional differential equations*. Applied Mathematical Sciences 99. Springer-Verlag, New York, 1993.

[5] L. Hatvani and T. Krisztin, *Asymptotic stability for a differential-difference equation containing terms with and without a delay*. Acta Sci. Math. (Szeged) **60**(2009), 371–384.

[6] Y. Lou and X.-Q. Zhao, *Threshold dynamics in a time-delayed periodic SIS epidemic model*. Discrete Contin. Dyn. Syst. Ser. B **12**(2009), 169–186. http://dx.doi.org/10.3934/dcdsb.2009.12.169

[7] G. Röst, *Neimark–Sacker bifurcation for periodic delay differential equations*. Nonlinear Anal. **60**(2005), no. 6, 1025–1044. http://dx.doi.org/10.1016/j.na.2004.08.043

[8] H. L. Smith, *An introduction to delay differential equations with applications to the life sciences*. Texts in Applied Mathematics 57. Springer, New York, 2011.

[9] W. Wang and X.-Q. Zhao, *Threshold dynamics for compartmental epidemic models in periodic environments*. J. Dynam. Differential Equations **20**(2008), no. 3, 699–717. http://dx.doi.org/10.1007/s10884-008-9111-8

[10] D. Xu and X.-Q. Zhao, Dynamics in a periodic competitive model with stage structure. J. Math. Anal. Appl. **311**(2005), no. 2, 417–438. http://dx.doi.org/10.1016/j.jmaa.2005.02.062

[11] X.-Q. Zhao. *Basic reproduction ratios for periodic compartmental models with time delays*. J. Dynam. Differential Equations, to appear. doi:10.1007/s10884-015-9425-2, 2016

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