Mathematical Modelling of Zombies

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BANELING DYNAMICS IN *LEGEND OF THE SEEKER*

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Abstract

We propose a model of baneling dynamics, following the description in the television series *Legend of the Seeker*. A deceased human being may return to life as a baneling, allowed to remain in the world so long as it has killed at least one human in the previous day. We consider two intervention strategies, one in which humans have an effective way of killing banelings (wizard's fire), and one in which a cure is available but in very limited quantities (shadow water). With our model, we predict the future baneling dynamics and evaluate the two intervention strategies.

14.1 Introduction

PREVIOUS chapters have dealt directly with the zombie threat, with aspects including, but not limited to, spatial diffusion, government decision-making, the estimation of parameters, etc. However, in order to understand the zombie epidemic from all perspectives, it is instructive to examine the effects of similar undead creatures on humanity. Although banelings are mere fiction—unlike the zombies we all live in fear of—they are still helpful in showcasing the rise of an undead species.

In the *Legend of the Seeker* television series [1, 2], banelings are those who have died and been offered a second chance at life in service to the Keeper. The Keeper is the lord of the Underworld and his hatred for life is endless. The boundary between the world of the dead and the world of the living is called the veil, which prevents the Keeper from being loosed on the world of life. Due to some unfortunate events, the veil has been breached, allowing the Keeper to send the dead back to the world of the living. When a person dies, the Keeper may grant him a second chance in life in exchange for the person becoming a baneling. This will not stop until the veil is repaired.

Banelings are required to have killed at least one person sometime in the preceding twenty-four hours in order to stay in the world of the living. The clock is reset after each kill. Therefore killing multiple people at once will not result in multiple days of life for the baneling. Banelings are removed from the world of the living if they are unable to kill within any time interval of length one day. Humans have no effective way of killing the banelings. The only way to prevent banelings from remaining in the world of the living is burning their physical bodies.

The purpose of this paper is to create a dynamical model that takes into account the essential features of the baneling phenomenon to predict the future of the human population. Though banelings are fictional characters, our model resembles some real-life situations. These are discussed in the concluding section.

Banelings are undead corpses, just like zombies. However, previous models of zombie outbreaks [3] are not applicable to banelings. First, a person bitten by a zombie will become a zombie inevitably, while any dead person may become a baneling by accepting the evil offer of the Keeper regardless of the cause of death; i.e., anyone who has died, by natural or other causes, may become a baneling as a result of the choice made by that person. The other major difference that characterizes baneling dynamics compared to zombie outbreak models is the removal term, which has to reflect the requirement that a baneling must kill someone every day.

There are major differences in the appearance and behavior of zombies and banelings. Though these are not relevant to the mathematical modelling, for the sake of clear distinction and to provide adequate context to the nonexpert reader, we briefly outline the major similarities and differences between the characteristics of banelings and standard contemporary zombies.

Zombies display visible signs of desiccation, decay and emaciation on their face and body, and have blank, expressionless faces; on the other hand, banelings look like normal people. Thus, while zombies are easily recognizable by visual inspection after their incubation period, one can never be sure whether someone is a baneling or not. Banelings show symptoms such as rotting and deterioration of flesh only when they are near the end of their one-day limit without having killed anyone. When a zombie attacks, it will tear out organs from the living belly and feast upon them, while making moaning and guttural sounds. Banelings, on the contrary, prefer to slash the throat quickly, quietly and effectively, because their purpose is to stay in the world of the living as long as they can. Zombies have limited intelligence and they are mostly dangerous when gathered in relentless hordes, but a single baneling can outsmart and trick his opponent. Zombies stay in their pitiable state indefinitely (however, some speculate that eventually a decaying zombie will break down to the point where it cannot function anymore), and it is very difficult to kill them, since they are already dead; they are vulnerable only to attacks that chop the head off or crush the brain. By contrast, banelings will perish if they do not manage to kill, though, like zombies, they can only be killed with great difficulty and through specific methods (in this case, burning). For further details on zombies, banelings and the world they belong to, we refer to [4], [5], [6] and [1].

14.2 Model Derivation

In our model, we use the following state variables:

- $H(t)$ denotes the number of humans at time t,
- $B(t)$ denotes the number of banelings at time t,
- $P(t)$ denotes the number of permanently deceased humans at time t.

Since the events are happening in a relatively short time period, natural human demographics are neglected: there are no births and all deaths are attributed to the killing spree of the banelings. Let D be the rate humans are being killed by banelings, while $q \in [0, 1]$ is the fraction of killed persons who accept the Keeper's evil offer and become banelings. Those who refuse the Keeper's evil offer move into the class of permanently deceased humans and remain there for all time. Let R be the baneling removal rate: those banelings who were not able to kill in the last day must return to the Underworld, so they move into the class of permanently deceased humans and remain there. In this initial model, the humans have no proactive way of removing banelings, so the only negative effect on the baneling population arises from their inability to kill a human in the last twenty-four hours.

Then the basic model has the structure

$$\begin{aligned} H'(t) &= -D, \\ B'(t) &= qD - R, \\ P'(t) &= (1 - q)D + R. \end{aligned}$$

Next we determine how D and R depend on the population sizes. The number of encounters between banelings and people is assumed to be proportional to both the human and baneling densities with a searching-efficiency parameter α. Let κ express the chance that such an encounter results in the killing of the human by the baneling. Then we have $D = \kappa\alpha H(t)B(t)$.

Given a baneling, it has a time window of one day to kill, or else it is removed from the baneling population. For the removal term R, we need to determine the fraction of banelings who were unsuccessful and unable to kill in the time interval $[t-1, t]$. The number of encounters of the baneling with humans is assumed to follow a Poisson distribution, and the expected number of encounters is proportional to the average human population size in this time interval and is equals to $\alpha\bar{H}_t$, where

$$\bar{H}_t \equiv \int_{t-1}^{t} H(u)du.$$

Hence the parameter of the Poisson distribution is $\lambda = \alpha\bar{H}_t$. The probability that a baneling had k encounters with humans in the time interval $[t - 1, t]$ is then

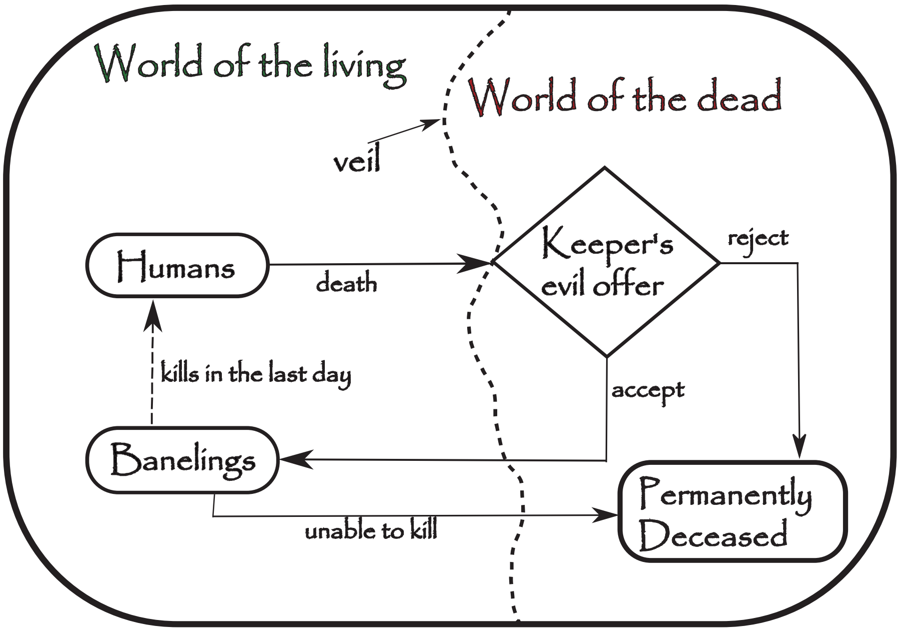

Figure 14.1: Baneling dynamics diagram. There are three population groups: humans, banelings and the permanently deceased. A dead human becomes a baneling (by accepting the Keeper's evil offer) or remains deceased permanently (by refusing the offer). Banelings stay in the world of the living as long as they have killed a human in the last day. If they fail to do so, then they move to the permanently deceased class.

$p_k = \frac{e^{-\lambda}\lambda^k}{k!}$ and the probability that all of these encounters were unsuccessful (from the baneling's perspective) is $(1-\kappa)^k$. The chance that a baneling was unable to kill is

$$\sum_{k=0}^{\infty}(1-\kappa)^k p_k = e^{-\lambda}e^{\lambda(1-\kappa)} = e^{-\lambda\kappa} = e^{-\alpha\bar{H}_t\kappa}.$$

Thus our baneling model becomes

$$H'(t) = -\kappa\alpha H(t)B(t) \tag{14.1}$$

$$B'(t) = q\kappa\alpha H(t)B(t) - B(t)\exp\left[-\alpha\kappa\int_{t-1}^{t} H(u)du\right] \tag{14.2}$$

$$P'(t) = (1-q)\kappa\alpha H(t)B(t) + B(t)\exp\left[-\alpha\kappa\int_{t-1}^{t} H(u)du\right]. \tag{14.3}$$

Since $H(t)+$ $B(t)+$ $P(t)$ is constant, and the equation for $P'(t)$ decouples, we omit the variable $P(t)$ from the further analysis.

14.3 Analysis of a Simplified Baneling Model

We first consider a simplification of our model by approximating $\int_{t-1}^{t} H(u)du$ with $H(t)$. Since $H(t)$ is decreasing, this estimation results in a smaller value of $B'(t)$, so that the model will underestimate the number of banelings compared to the model (14.1)-(14.2). The simplified model takes the form

$$H'(t) = -\kappa\alpha H(t)B(t) \tag{14.4}$$
$$B'(t) = q\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t)]. \tag{14.5}$$

Note that this model is similar to some of the classical predator–prey, host–parasite and disease-spread models [7], but not identical to any of them. Given the biological meaning of the model, we consider solutions only with nonnegative initial values (H_0, B_0). Notice that the system has a continuum of baneling-free equilibria (BFE) given by $(H_{0,}0) \in \mathcal{R}^2$ (since we are assuming that there are no natural deaths). We summarize the basic properties of system (14.4)-(14.5) in the following theorem.

Theorem 1 *The solutions of (14.4)-(14.5) remain positive and $H(t)$ is monotone decreasing. The limits $H_\infty \equiv \lim_{t\to\infty} H(t)$ and $B_\infty \equiv \lim_{t\to\infty} B(t)$ exist with $B_\infty = 0$.*

Proof. From (14.4)-(14.5), we can express $H(t)$ and $B(t)$ as

$$H(t) = H_0 \exp\left[-\int_0^t \kappa\alpha B(s)ds\right]$$

and

$$B(t) = B_0 \exp\left[\int_0^t q\kappa\alpha H(s) - \exp[-\alpha\kappa H(s)]ds\right].$$

Hence they remain positive for all $t > 0$ if the initial values are positive. It follows that $H(t)$ is monotone decreasing and thus converges. Note that

$$\frac{d}{dt}[H(t) + B(t)] = -(1-q)\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t)] \leq 0.$$

It follows that the combined human-baneling population decreases during a baneling outbreak in our model. Considering

$$\frac{d}{dt}[qH(t) + B(t)] = -B(t)\exp[-\alpha\kappa H(t)] \leq -B(t)\exp[-\alpha\kappa H_0],$$

we conclude that $B(t)$ converges to zero since otherwise $qH(t)+B(t)$ would become negative, which is not possible. $\square$

Now we define the baneling reproduction number $\mathcal{B}_0$: in the initial phase when banelings first appear, the number of humans is approximately H_0. The sojourn time of one single baneling in the world of the living is $1/\exp[-\alpha\kappa H_0]$ on average and it kills $\kappa\alpha H_0$ humans per day, resulting a total number of

$$\mathcal{B}_0 \equiv q\kappa\alpha H_0 \exp[\alpha\kappa H_0]$$

new banelings. We say that a baneling outbreak is occurring when the number of banelings starts to increase rapidly. By preventing the outbreak, we mean that (possibly because of human intervention) the number of banelings cannot increase from the very beginning and the baneling population drops to zero with minimal human deaths.

Theorem 2 *System (14.4)-(14.5) has threshold parameter $\mathcal{B}_0$: there is an outbreak of banelings if and only if the baneling reproduction number $\mathcal{B}_0$ is greater than* 1*.*

Proof. One can see that $B(t)$ is increasing if and only if

$$q\kappa\alpha H(t) - \exp[-\alpha\kappa H(t)] > 0.$$

It is easy to see that $f(z) \equiv q\kappa\alpha z - \exp[-\alpha\kappa z]$ is a monotone increasing function of z with a unique positive zero z^*. Since $H(t)$ is decreasing, $f(H(t))$ is also decreasing as time elapses, so we can conclude that there are threshold dynamics. There is no baneling outbreak if

$$f(H_0) = q\kappa\alpha H_0 - \exp[-\alpha\kappa H_0] < 0,$$

or equivalently $\mathcal{B}_0 < 1$. In the opposite case, the number of banelings will initially increase, until $H(t)$ drops below z^*. In the special situation $\mathcal{B}_0 = 1$, $B'(0) = 0$, but, for any $h > 0$, the monotonicity of $H(t)$ and $f(z)$ imply that

$$q\kappa\alpha H(h) - \exp[-\alpha\kappa H(h)] < 0$$

so $B'(h) < 0$. Thus we observe an outbreak of banelings if and only if the baneling reproduction number satisfies $\mathcal{B}_0 > 1$. □

Since $\mathcal{B}_0 \equiv q\kappa\alpha H_0 \exp[\alpha\kappa H_0]$, one can see that if $\kappa\alpha H_0 \exp[\alpha\kappa H_0] < 1$, then $\mathcal{B}_0 < 1$ regardless of the value of q; i.e., it does not matter how many people accept the Keeper's evil offer, there will be no outbreak. For example, if $\alpha\kappa = 0.01$ and the population size is $H_0 = 400$, then the critical value of q is approximately 0.005 (that is, only one out of every 200 deceased humans becomes a baneling). If q is larger than that, then there will be a baneling outbreak. Besides q, humans may decrease the searching parameter α (by hiding from the banelings) or the chance κ of being killed upon an encounter with a baneling (by improving their defence skills). For example, if $q = 0.2$, then the humans can prevent the baneling outbreak by reducing $\alpha\kappa$ below 0.003.

Theorem 3 *The fraction of the human population that survives the baneling outbreak can be determined from the final-size relation*

$$a\kappa q H_0 - Ei(-a\kappa H_0) = a\kappa q H_\infty - Ei(-a\kappa H_\infty),$$

where $Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$ *is the first exponential integral.*

Proof. For the properties of the first exponential integral $Ei(x)$, see [8]. We have $Ei'(x) = \exp[x]/x$. Consider the function

$$W(H, B) = B + qH - Ei(-\alpha\kappa H)/\alpha\kappa.$$

Then

$$\begin{aligned}\frac{dW}{dt} &= B'(t) + qH'(t) - \big(\exp[-\alpha\kappa H(t)]/\alpha\kappa H(t)\big)H'(t) \\ &= q\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t)] - q\kappa\alpha H(t)B(t) \\ &\quad - \big(\exp[-\alpha\kappa H(t)]/\alpha\kappa H(t)\big)(-\kappa\alpha H(t)B(t)) \\ &= 0.\end{aligned}$$

We can conclude that W is a first integral of our system.

Using $W(H_0, B_0) = W(H_\infty, 0)$, we obtain the final-size relation

$$\alpha\kappa q H_0 - Ei(-\alpha\kappa H_0) = \alpha\kappa q H_\infty - Ei(-\alpha\kappa H_\infty).$$

The final-size relation is graphically represented in Figure 14.2. □

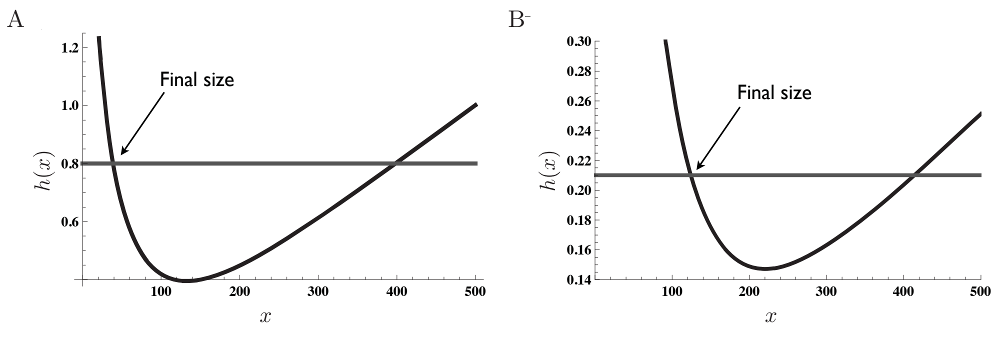

Figure 14.2: Plot of the function $h(x) = q\alpha\kappa x - Ei(-\alpha\kappa x)$. The final size is determined by the relation $h(H_0) = h(H_\infty)$. A: Parameter values are $q = 0.2$ and $\alpha\kappa = 0.01$. For an initial population of $H_0 = 400$, we have $h(H_0) \approx 0.8$, so the final size is given by the other intersection of $h(x)$ with the horizontal line; i.e., by a value x_* for which $h(x_*) = 0.8$, so $H_\infty \approx 50$. B: Parameter values are $q = 0.05$ and $\alpha\kappa = 0.01$. For an initial population of 400, the final size is 130.

14.4 Intervention 1: Wizard's Fire

To be removed from the world of the living, a baneling's body must be burned. In the television series, a particularly potent method of burning large quantities of bodies is the so-called wizard's fire [1], which is a possible intervention strategy. To incorporate the effect of the wizard's fire into our model in a simplified way, we assume that banelings can be killed and burned at rate w. Thus our model takes the form

$$H'(t) = -\kappa\alpha H(t)B(t) \tag{14.6}$$
$$B'(t) = q\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t)] - wB(t). \tag{14.7}$$

This modified model can be treated analogously to system (14.4)-(14.5). Define the wizard-baneling reproduction number

$$\mathcal{B}_w \equiv \frac{q\kappa\alpha H_0}{\exp[-\alpha\kappa H_0] + w},$$

which expresses the expected number of banelings generated by a single baneling in the early phase of the outbreak, when wizard's fire intervention is present.

Theorem 4 *The solutions of (14.6)-(14.7) remain positive and $H(t)$ is monotone decreasing. The limits $H_\infty \equiv \lim_{t\to\infty} H(t)$ and $B_\infty \equiv \lim_{t\to\infty} B(t)$ exist with $B_\infty = 0$. System (14.6)-(14.7) has threshold parameter $\mathcal{B}_w$: there is an outbreak of banelings if and only if the wizard-baneling reproduction number satisfies $\mathcal{B}_w > 1$.*

Proof. This can be shown analogously to Theorems 1 and 2; the only difference is that, instead of $f(z)$, we use $g(z) \equiv q\kappa\alpha z - \exp[-\alpha\kappa z] - w = f(z) - w$. □

Theorem 4 implies that banelings can be controlled by sufficiently intense wizard's fire. If w is sufficiently large; or, more precisely, if

$$w > f(H_0) = q\kappa\alpha H_0 - \exp[-\alpha\kappa H_0],$$

then $g(H_0) < 0$ and $\mathcal{B}_w < 1$, so the baneling outbreak is contained.

14.5 Intervention 2: Treatment by Shadow Water

In the television series, banelings can be cured (turned back into humans) by shadow water, but this resource is extremely scarce. In episode 34 ("Hunger," Episode 12 of Season 2; see [9] and [2]) a source of shadow water is discovered and used to cure banelings, but the Keeper finds out and demolishes the source. Taking that episode as a guide, we will suppose that, once the treatment by shadow water starts, the Keeper will learn the location of the source very quickly and destroy it. Therefore the cure will be available only for a limited amount of time. For the modelling of this very brief window of opportunity for initiating treatment, we assume that all available cures are administered at once at some time T, healing a fraction σ of the banelings, who then become normal humans.

Thus our model can be formulated as an impulsive system with one impulse at $t = T$. More precisely, for $t \neq T$, we have the basic model

$$H'(t) = -\kappa\alpha H(t)B(t) \tag{14.8}$$
$$B'(t) = q\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t)], \tag{14.9}$$

while, for $t = T$, we have the discontinuity

$$H(T+) = H(T) + \sigma B(t) \tag{14.10}$$
$$B(T+) = (1-\sigma)B(T), \tag{14.11}$$

where $0 < \sigma < 1$.

For further details on the theory and simulations of impulsive systems in life sciences, we refer to [10]. Note, however, that cured banelings are susceptible to becoming banelings in the future, at the same rate as the rest of the human population. The shadow treatment impulse decreases the number of banelings and increases the number of humans, but the long-term effect of such a healing impulse is not immediately obvious. The following theorem provides an encouraging result.

A B

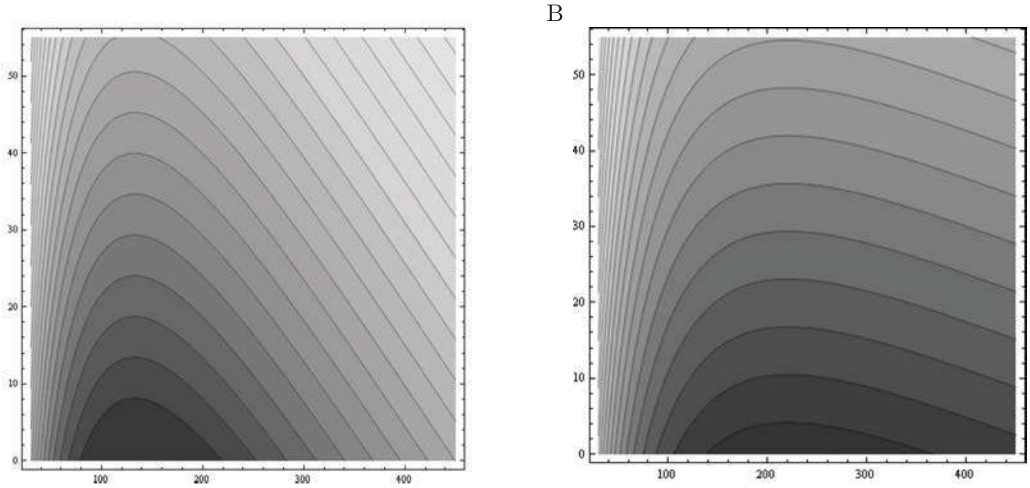

Figure 14.3: Level curves of the first integral $W(H,B) = B + qH - Ei(-a\kappa H)/a\kappa$ on the (H,B)-plane. The phase curves of the solutions move left along the level curves, since $H(t)$ is decreasing. A: Parameter values are $q = 0.2$ and $\alpha\kappa = 0.01$. B: $q = 0.05$ and $\alpha\kappa = 0.01$.

Theorem 5 *Even a single impulse of shadow water healing increases H_∞, the final size of the human population.*

Proof. The downward-pointing normal vector of the level curve of the first integral W (see Figure 14.3) is:

$$\mathbf{n} = (-W_{H,} - W_B) = \left(-q + \frac{\exp[-\alpha\kappa H]}{\alpha\kappa H}, -1\right).$$

An impulse at time T pushes the state of the planar system from (H, B) to $(H + \sigma B, (1 - \sigma)B)$ by the shadow-water treatment vector $\mathbf{s} = (\sigma B, -\sigma B)$. We can see that the scalar product $\mathbf{n} \cdot \mathbf{s} = 1 - q + \frac{\exp[-\alpha\kappa H]}{\alpha\kappa H} > 0$, which means that, after the impulse, the state of the system jumps to a level set that corresponds to a lower value of W and thus a larger final population size. □

Besides increasing the final human population size, shadow-water impulses have an additional potential benefit: the baneling peak can be delayed. See, for example, the numerical simulations in Section 7.

14.6 The Time-Delay Baneling Model

Our original model (14.1)-(14.3) is a delay differential system with distributed delay, because the second equation involves the history of $H(t)$. The analysis of such systems is fairly technical [11]. To define a dynamical system, we use the phase space $C \times R$, where C denotes the Banach space of continuous functions on the interval $[-1, 0]$ together with the norm

$$||\phi|| = \sup_{t\in[-1,0]} |\phi(t)|, \quad \phi \in C.$$

It is possible to define a semiflow on our phase space, and the standard existence and uniqueness results hold, for specified initial values $H_0 \in C$ and $B_0 \in R$. Although the detailed analysis of this system is beyond the scope of this study, we can nevertheless draw some conclusions without using the advanced tools of the theory of delay differential equations.

The positivity of solutions can be shown for this system as well, completely analogously to the case of the simplified system of ODEs. Then it follows that $H(t)$ is monotone decreasing for $t \geq 0$ and therefore converges; we also obtain

$$H(t-1) \geq \int_{t-1}^{t} H(u)du \geq H(t).$$

This implies

$$\exp[-\alpha\kappa H(t)] \geq \exp\left[-\alpha\kappa \int_{t-1}^{t} H(u)du\right] \geq \exp\left[-\alpha\kappa H(t-1)\right].$$

To prove that $B(t)$ tends to zero, consider

$$\begin{aligned}\frac{d}{dt}[qH(t) + B(t)] &= -B(t)\exp\left[-\alpha\kappa \int_{t-1}^{t} H(u)du\right] \\ &\leq -B(t)\exp[-\alpha\kappa H(t-1)] \\ &\leq -B(t)\exp[-\alpha\kappa H_0],\end{aligned}$$

for $t \geq 1$; hence $B(t)$ must converge to zero, or else the left-hand side becomes negative. Since

$$\begin{aligned}B'(t) &= q\kappa\alpha H(t)B(t) - B(t)\exp\left[-\alpha\kappa \int_{t-1}^{t} H(u)du\right] \\ &\geq q\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t)]\end{aligned}$$

for $t \geq 1$, we can use standard comparison arguments to deduce that the simplified baneling model predicts a lower number of banelings than the model with distributed delay.

If we approximate $\int_{t-1}^{t} H(u)du$ by $H(t-1)$, then we obtain the system

$$H'(t) = -\kappa\alpha H(t)B(t), \tag{14.12}$$

$$B'(t) = q\kappa\alpha H(t)B(t) - B(t)\exp[-\alpha\kappa H(t-1)] \tag{14.13}$$

with a single constant delay. The basic properties of this system can be deduced similarly to Theorem 2, but the further analysis—for example, finding an appropriate invariant that replaces the first integral—is again beyond the scope of this study. Nevertheless, from the monotonicity of $H(t)$ and standard comparison arguments, we can conclude that the single-delay model overestimates the baneling population.

14.7 Numerical Simulations

We have run computer simulations for various parameter values. For the sake of comparison, we consider two outbreaks of different severity. For both scenarios, we also examine the effect of interventions. We choose our population size to be 400. In the first case, let $q = 0.2$ and $\alpha\kappa = 0.01$. These parameter values appear in Figures 14.2 and 14.3 as well. In the more severe case, we assume that half of the deceased accept the Keeper's evil offer and that the banelings are three times more effective; i.e., $q = 0.5$ and $\alpha\kappa = 0.03$. The parameters were chosen because they make it easy to highlight the interesting features of the model.

The time course of these two baneling outbreaks can be seen in Figure 14.4. We can see that, in the first case, a small number of people survive, while in the more serious case the banelings reduce the population to a negligible size.

Figure 14.5 shows the application of wizard's fire with $w = 0.6$. We can see that the wizard's fire was very effective in the first case and prevented almost half

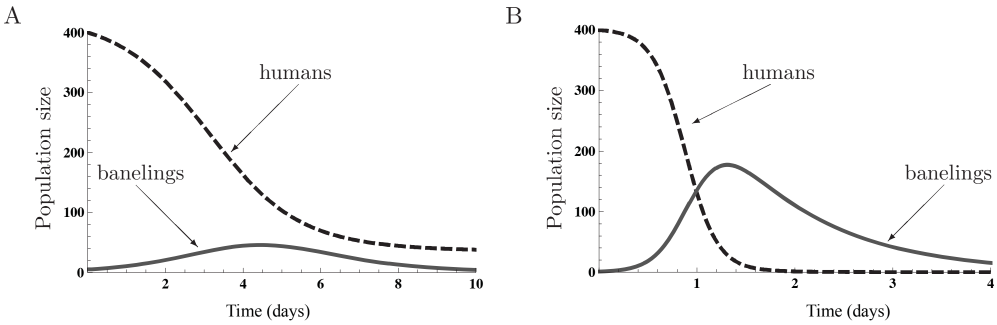

Figure 14.4: Time course of two baneling outbreaks without intervention in a population of 400. A: $q = 0.2$ and $\alpha\kappa = 0.01$. B: $q = 0.5$ and $\alpha\kappa = 0.03$.

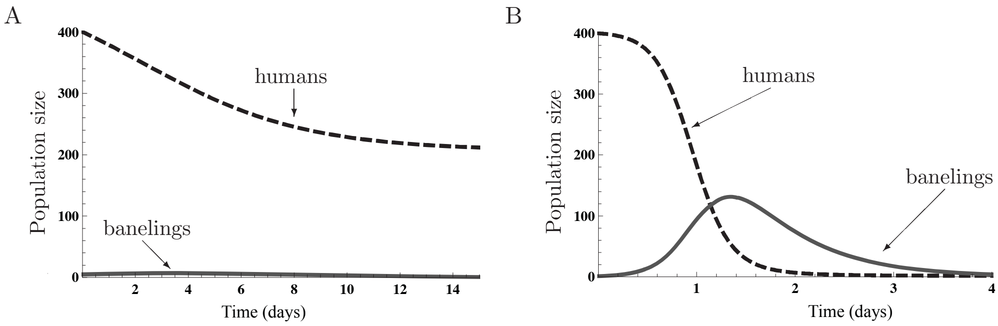

Figure 14.5: Time course of two baneling outbreaks in a population of 400, with wizard's fire $w = 0.6$. A: $q = 0.2$ and $\alpha\kappa = 0.01$. B: $q = 0.5$ and $\alpha\kappa = 0.03$.

of the baneling murders, but it only slightly mitigated the damage in the second case. To prevent the outbreak (i.e., to ensure that $\mathcal{B}_w < 1$), the necessary intensity of wizard's fire must be at least $w = q\kappa\alpha H(0) - \exp[-\alpha\kappa H(0)] = 0.78$ in the first case, and $w = 6$ in the second case.

Figures 14.6 and 14.7 demonstrate the effect of shadow-water treatment in various situations. We can see that the the impact of the impulsive healing depends crucially on the timing of the cure. It seems that, for maximal benefit, shadow-water treatments should be administered at the peak time of the baneling outbreak. However, this conjecture is still to be proven. Conversely, if the outbreak is too severe, the impulsive intervention has no long-term benefit.

14.8 Summary and Conclusions

We have developed a dynamic population model to analyze the time course of a baneling outbreak, following the description in the television series *Legend of the Seeker*. Our model allowed us to make predictions for various values of the key parameters, such as a) the efficiency of banelings, b) the relative number of people

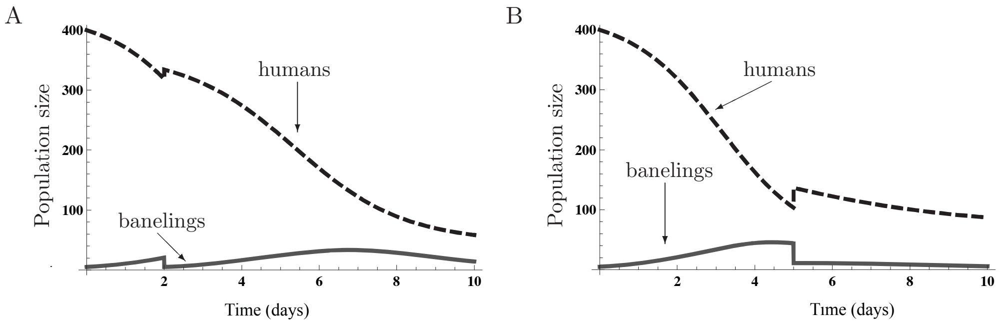

Figure 14.6: The effect of shadow water treatment impulse with $\sigma = 0.75$ in a population of 400, when $q = 0.2$ and $\alpha\kappa = 0.01$. A: Healings occurred at $T = 2$. B: Healings occurred at $T = 4.7$.

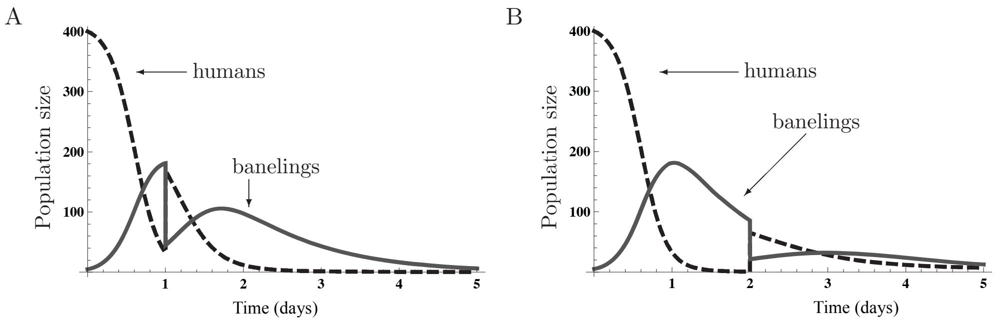

Figure 14.7: The effect of shadow water treatment impulse with $\sigma = 0.75$ in a population of 400, when $q = 0.5$ and $\alpha\kappa = 0.03$. A: Healings occurred at $T = 1$. B: Healings occurred at $T = 2$.

who accept the Keepers evil offer and return as banelings upon their deaths, and c) the human population density.

The major conclusion of our research is that an outbreak of banelings occurs if and only if the baneling reproduction number is greater than one. The baneling reproduction number expresses the average number of new banelings produced by the deeds of one single baneling in the initial phase of the outbreak. This number depends on how effectively the banelings are able to find and kill humans, and how many humans accept the Keeper's evil offer and become banelings themselves. Taking all these paremeters into account, our model is able to predict whether there will be an outbreak of banelings or their numbers will drop quickly without causing significant damage to the human population.

In any case, banelings will eventually disappear. This makes sense, since they perish if humans can defend themselves effectively; but they also perish if they are so successful that they manage to kill most humans. In this latter case, there remain no available victims to fulfill the requirement of the Keeper to kill someone every day.

The most important question is how many people survive the baneling attacks. We derived a formula that gives a precise answer to this question. Our results show that banelings can never completely eradicate the human population, since the final population size is always positive. However, this positive final size can be too small to be viable.

Humans can increase the odds of surviving a baneling encounter by training and improving their defence skills. The number of such encounters can be reduced by hiding from the banelings. These are reflected in the model parameters; by changing those, we can calculate how the final population size changes. We have also examined two possible intervention strategies: the application of wizard's fire and shadow water. Wizard's fire is an effective way of removing banelings. The outbreak can be controlled by sufficiently intensive baneling killing combined with wizard's fire. Shadow-water treatment effectively cures banelings, but cannot be applied continuously because this resource is very scarce and the Keeper destroys the resources shortly after treatments start. However, even one impulse of shadow water healing delays the baneling peak, and also increases the final size of the human population.

On a demographic time scale, if the veil cannot be repaired and the humans are capable of sustaining a viable population, we can expect a constant baneling supply by natural deaths, so the banelings are expected to become endemic.

Though banelings are fictional characters, our model resembles some real-life situations. For example, one can think of a severe disease, where a patient who would otherwise perish needs care every day, but there is a chance that the caregivers die or get infected. Then wizard's fire plays a similar role to permanent isolation of the sick, and shadow-water treatment corresponds to a cure that is available in very limited quantities.

Another analogue from real life is the rapid spread of a fashion, political agenda or evangelization. Then banelings are activists who spread some ideas. If they cannot recruit new followers in a time interval of some length, then they become disappointed, lose enthusiasm and give up the idea themselves. Unsuccessfully approached individuals will be dismissive of the idea forever. In this context, wizard's fire corresponds to some counter-propaganda which constantly reduces the number of followers, while shadow-water treatment can be some particular event that reflects bad light on the given idea and makes some followers turn away from it (but later they can be convinced to rejoin).

The television series *Legend of the Seeker* is based on *The Sword of Truth* fantasy novel series by Terry Goodkind [5]. In the books, banelings are the agents of the Keeper, doing his bidding. In this chapter, we have chosen to follow the representation of the TV series, where banelings are undead people who are required to kill a human every day. And finally, an obvious real-life application of the work herein is the very zombie epidemic we are all facing. Good luck.

Appendices

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Competing interests. The manuscript recommends the application of wizard's fire and shadow water. The author has never received any funds from the Magic Guild or the shadow water industry and has no potential conflict of interest (except his personal admiration for one specific Mord-Sith).

B Glossary

Baneling. A fictional creatures in the TV series *Legend of The Seeker*. They are deceased persons who accepted the offer of the evil ruler of the underworld so that they can return to the world of the living and remain there as long as they kill at least one person each day.

Baneling reproduction number. In epidemiology, the basic reproduction number is a key concept, expressing the expected number of secondary infections generated by a single infected individual introduced into a susceptible population. Analogously, the baneling reproduction number represents how many banelings will appear due to the activities of a single initial baneling.

Final-size relation. The final size relation is an equation that relates the initial population sizes, the model parameters and the final population sizes. Having known the former ones, one can use this equation to predict the final population size.

First integral. A function of the model variables, which is a conserved quantity (which in many physical systems can be interpreted as energy); i.e., it does not change as time elapses, although the variables themselves change in time. The first integral tells you very important information about the behaviour of solutions.

Impulse. A discontinuous jump.

Impulsive system. A set of differential equations with discontinuities representing interventions that happen in a short time with a large effect. Impulses generate discontinuous solutions in models where the state changes smoothly between impulses.

Legend of the Seeker. An epic fantasy television series, based on the series of novels *The Sword of Truth* by US fantasy writer Terry Goodkind. The show premiered in November 2008 and ran for two seasons.

Shadow water. A scarce substance that can cure banelings; i.e., turn them into normal human beings again.

Wizard's fire. Wizard's fire is a particularly strong fire, shrieking through the air as if shot from the fingers of high-level wizards, which can effectively burn to ashes a large number of bodies, thus preventing them from returning as banelings.

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