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ON A SEIR EPIDEMIC MODEL WITH DELAY

GERGELY RÖST, SHAO YUAN HUANG, AND LÁSZLÓ SZÉKELY

Bolyai Institute, University of Szeged, Szeged, Hungary
Department of Mathematics, Tsing Hua University, Hsinchu,
Taiwan 30043, R. O. China
Institute of Mathematics and Informatics, Szent István University
Gödöllő, Hungary
(szekely.laszlo@gek.szie.hu)

ABSTRACT. We improve previous results for an SEIR-type disease transmission model with fixed latency period, standard incidence and variable population size. Infected individuals are assumed to be unable to give birth, and individuals recover from the disease and acquire permanent immunity with probability f, and dies from the disease with probability $1 - f$. In particular, by a novel application of the envelope method we give a complete classification of the equilibria, we show the stability of the disease free equilibrium for $R_0 < 1$, and the proportional persistence of the disease for $R_0 > 1$.

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1. INTRODUCTION

Compartmental models are commonly used in analysing and predicting the dynamics of a spreading infectious disease. One of the classical types of such model family is the SEIR model. If on adequate contact with an infectious individual, a susceptible becomes exposed for a while; that is, infected but not yet infectious, then it is realistic to introduce a latent compartment, leading to an SEIR-model. This model has several variants, depending on the modelling assumptions [1]. In many cases, the dynamics have been completely described a long time ago [6], [7], but certain SEIR models have still been actively studied recently, for example considering varying infectivity depending on age since infection [10], time delay and nonlinear incidence [5], varying population size and vaccination [12], various latent and infectious periods [15], etc. It is important to know whether the standard results (such as threshold dynamics, global asymptotic stability, or persistence) are still valid for different modelling assumptions or not.

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The following particular model has been considered by Yan and Liu in [16]:

$$S'(t) = bS(t) + bE(t) + bR(t) - \mu S(t) - \gamma \frac{S(t)I(t)}{N(t)}, \tag{1.1}$$

$$E'(t) = \gamma \frac{S(t)I(t)}{N(t)} - \gamma \frac{S(t-\tau)I(t-\tau)}{N(t-\tau)} e^{-\mu\tau} - \mu E(t), \tag{1.2}$$

$$I'(t) = -\mu I(t) + \gamma \frac{S(t-\tau)I(t-\tau)}{N(t-\tau)} e^{-\mu\tau} - \alpha I(t), \tag{1.3}$$

$$R'(t) = -\mu R(t) + f\alpha I(t), \tag{1.4}$$

where $N(t) = S(t)+E(t)+I(t)+R(t)$ is the total population size. Here $S(t), E(t), I(t)$ and $R(t)$ denote the number of susceptible, latent, infectious and recovered individuals at time t, respectively. The parameter b is the per capita birth rate with $b > \mu$, which is the natural mortality rate, thus we have an exponentially growing population structure. Many animal diseases are known to reduce fertility or parental care ability that is necessary for the newborn to survive. Here we assume that infectious individuals don't reproduce. The transmission rate is denoted by γ, the fixed length of the latency period is τ, the removal rate from the infectious class is α. An infected individual recovers and acquires permament immunity with probability $f \in (0, 1)$, and the disease is lethal with probability $1 - f$. All parameters are assumed to be non-negative.

Now we transform the model by taking the proportions of the number of individuals in the given compartment with regard to the total population size, i.e. let $s(t) = S(t)/N(t)$, $e(t) = E(t)/N(t)$, $i(t) = I(t)/N(t)$, $r(t) = R(t)/N(t)$. Then the model in the new variables reads as

$$s'(t) = b - bi(t) - m(t)s(t) - \gamma s(t)i(t), \tag{1.5}$$

$$e'(t) = \gamma s(t)i(t) - \gamma s(t-\tau)i(t-\tau)e^{-\int_{t-\tau}^{t} m(s)ds} - m(t)e(t), \tag{1.6}$$

$$i'(t) = \gamma s(t-\tau)i(t-\tau)e^{-\int_{t-\tau}^{t} m(s)ds} - (m(t)+\alpha)i(t), \tag{1.7}$$

$$r'(t) = f\alpha i(t) - m(t)r(t), \tag{1.8}$$

with

$$s(t), e(t), i(t), r(t) \geq 0, \quad t \in [-\tau, 0],$$

$$s(t) + e(t) + i(t) + r(t) = 1, \quad t \in [-\tau, 0],$$

$$m(t) = b - (b + (1-f)\alpha)i(t).$$

The non-negativity of solutions for all $t > 0$ can be guaranteed by imposing the additional condition

$$e(0) = \int_{-\tau}^{0} \gamma s(u)i(u)e^{-\int_{u}^{0} m(z)dz} du.$$

The modified contact rate

$$R_0 = \gamma \frac{e^{-b\tau}}{b+\alpha} \tag{1.9}$$

is an important threshold parameter for the above system. Yan and Liu [16] proved several results for this model, in particular they showed that there is at least one endemic equilibrium when $R_0 > 1$, all solutions converge to the disease free equilibrium $(1, 0, 0, 0)$ if $\gamma < b + \alpha$, and the disease persists in the population if $R_0 > 1$. We improve the results of [16] in three aspects:

- we give a complete classification of the equilibria, and we show that there is no endemic equilibrium if $R_0 < 1$ (i.e. backward bifurcation is not possible), and there is a unique endemic equilibrium if $R_0 > 1$;
- we provide a stability analysis for the disease free equilibrium;
- we think that the proof of the persistence result in [16] is not complete, hence we give an alternative proof for the persistence of the disease by applying a result of Hale and Waltman [4].

2. EQUILIBRIA

Denote by (s^*, e^*, i^*, r^*) the endemic equilibrium of (1.5)–(1.8), i.e. an equlibrium with $i^* > 0$. Recall that $s^* + e^* + i^* + r^* = 1$. The endemic equilibrium has to satisfy the algebraic equations

$$0 = b - bi^* - m^* s^* - \gamma s^* i^*, \tag{2.1}$$

$$0 = \gamma s^* i^* (1 - e^{-\tau m^*}) - m^* e^*, \tag{2.2}$$

$$0 = \gamma s^* i^* e^{-\tau m^*} - (m^* + \alpha) i^*, \tag{2.3}$$

$$0 = f\alpha i^* - m^* r^*, \tag{2.4}$$

where $m^* = b - (b + (1-f)\alpha)i^*$. From (2.4) we have

$$r^* = \frac{f\alpha i^*}{b - (b + (1-f)\alpha)i^*}. \tag{2.5}$$

Since $0 \leq r^*$ and $0 < f\alpha i^*$, the denominator of (2.5) has to be positive, that is $i^* < b/(b+(1-f)\alpha)$. Furthermore, $r^* < 1$ must hold for all $i^* \in (0, b/(b+(1-f)\alpha))$, which gives $i^* < b/(b+\alpha)$. Therefore, for an endemic equilibrium of system (1.5)–(1.8) we have

$$i^* \in \left(0, \frac{b}{b+\alpha}\right).$$

By equation (2.1) it follows

$$s^* = \frac{b(1-i^*)}{b + [\gamma - (b + (1-f)\alpha)]i^*}. \tag{2.6}$$

Since $s^* > 0$, the relation $b + [\gamma - (b + (1-f)\alpha)]i^* > 0$ must hold as well. If $R_0 \geq 1$ then $\gamma > b + \alpha$, thus the inequality is satisfied for all points of the interval

$(0, b/(b+\alpha))$. In the case $R_0 < 1$ it is easy to check that the inequality holds for $i^* = 0$ and $i^* = b/(b+\alpha)$. Since the denominator is a linear function of i^*, it is also positive for all $i^* \in (0, b/(b+\alpha))$. Also, $s^* < 1$, which implies that the relation $\gamma > (1-f)\alpha$ must hold in order to have an endemic equilibrium.
Furthermore, by equation (2.3) we have

$$s^* = \frac{(b+\alpha) - (b+(1-f)\alpha)i^*}{\gamma \exp(-b\tau)\exp\{(b+(1-f)\alpha)\tau i^*\}}, \tag{2.7}$$

thus

$$\frac{(b+\alpha) - (b+(1-f)\alpha)i^*}{\gamma \exp(-b\tau)\exp\{(b+(1-f)\alpha)\tau i^*\}} = \frac{b(1-i^*)}{b + [\gamma - (b+(1-f)\alpha)]i^*}, \tag{2.8}$$

which is

$$\frac{1}{R_0}\frac{1-(b+(1-f)\alpha)i^*/(b+\alpha)}{\exp\{(b+(1-f)\alpha)\tau i^*\}} = \frac{1-i^*}{1+[\gamma-(b+(1-f)\alpha)]i^*/b}. \tag{2.9}$$

Denote by $H(i^*)$ the nominator of the difference of the right- and left-hand sides of (2.9), that is

$$\begin{aligned} H(i^*) :=& R_0 e^{(b+(1-f)\alpha)\tau i^*}(1-i^*) \\ & - \left(1 - \frac{b+(1-f)\alpha}{b+\alpha} i^*\right)\left(1 + \frac{\gamma-(b+(1-f)\alpha)}{b} i^*\right). \end{aligned}$$

The number of zeros of the function H on the interval $(0, b/(b+\alpha))$ equals the number of endemic equilibria. First, we show that $H(b/(b+\alpha)) < 0$.

$$\begin{aligned} H\left(\frac{b}{b+\alpha}\right) &= \gamma \frac{e^{-b\tau}}{b+\alpha} e^{\frac{b+(1-f)\alpha}{b+\alpha} b\tau} \frac{\alpha}{b+\alpha} \\ & \quad - \left(1 - \frac{b(b+(1-f)\alpha)}{(b+\alpha)^2}\right)\left(1 + \frac{\gamma-(b+(1-f)\alpha)}{b+\alpha}\right) \\ &\le \frac{1}{(b+\alpha)^2}\left[\gamma\alpha - (b(1+f)\alpha + \alpha^2)\frac{\gamma+f\alpha}{b+\alpha}\right] \\ &= \frac{\alpha}{(b+\alpha)^2}\left[\gamma - (\gamma+f\alpha)\frac{b(1+f)+\alpha}{b+\alpha}\right] < 0. \end{aligned}$$

We also have that $H(0) = R_0 - 1$.

Theorem 2.1. *If $R_0 \le 1$, there exists no endemic equilibrium of system* (1.5)–(1.8)*.*

Proof. We shall prove that $H(i^*)$ is negative on the interval $(0, b/(b+\alpha))$, or equivalently

$$\begin{aligned} R_0 e^{(b+(1-f)\alpha)\tau i^*}(1-i^*) < & \left(1 - \frac{b+(1-f)\alpha}{b+\alpha} i^*\right) \\ & \times \left(1 + \frac{[\gamma-(b+(1-f)\alpha)]}{b} i^*\right). \end{aligned}$$

Observe that on this interval the inequality

$$1 - i^* \leq 1 - \frac{b + (1-f)\alpha}{b+\alpha} i^* \tag{2.10}$$

holds, therefore it is sufficient to show that

$$R_0 e^{(b+(1-f)\alpha)\tau i^*} < 1 + \frac{\gamma - (b + (1-f)\alpha)}{b} i^*. \tag{2.11}$$

If $i^* = 0$ then the left side of (2.11) equals R_0 and the right side equals 1, while at the endpoint $i^* = b/(b+\alpha)$ we have

$$\frac{\gamma}{b+\alpha} e^{\left(\frac{b+(1-f)\alpha}{b+\alpha} - 1\right) b\tau} < \frac{\gamma + f\alpha}{b+\alpha}, \tag{2.12}$$

which holds because the left side is less than $\gamma/(b+\alpha)$. Since the left side of (2.11) is strictly convex and the right side is linear, furthermore at the endpoints the linear function has at least the same values as the exponential function, we obtain that the inequality holds for all points of $(0, b/(b+\alpha))$. $\square$

Next we turn our attention to the case $R_0 > 1$. Since $H(0)H(b/(b+\alpha)) < 0$ holds, H has at least one zero value on the interval $(0, b/(b+\alpha))$, therefore, the existence of at least one endemic equilibrium is guaranteed. We have the following exact result.

Theorem 2.2. *If $R_0 > 1$ then system* (1.5)–(1.8) *has a unique endemic equilibrium.*

The proof is based on the Cheng-Lin envelope method which is developed systematically by Cheng and Lin in [2]. The idea of this method was applied earlier by Wright [14] to study transcendental characteristic equations arisen from delay differential equations. Then similar ideas were used several times by Cheng and Lin to study the roots of different characteristic equations. Later, a similar method, called parametric representation method was applied to find bifurcation curves for various problems motivated by chemical and biological processes ([9], [11]). Here we use the Cheng-Lin theory as presented in [2].

Given a family of straight lines L_λ which can be expressed in the the form

$$L_\lambda : f(\lambda)x + g(\lambda)y = h(\lambda) \tag{2.13}$$

where λ belongs to an interval I, and f, g, h are differentiable on I, we may sometimes be able to associate exactly one point P_λ in each L_λ such that the totality of these points form a curve S. Such an associated curve S is called an envelope of the family $\{L_\lambda : \lambda \in I\}$ if the straight lines L_λ and S share a common tangent line at the common point P_λ. By Theorem 2.3 in [2], if $f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda) \neq 0$ for $\lambda \in I$, then the envelope S can be described by parameter functions $X(\lambda)$ and $Y(\lambda)$ where

$$X(\lambda) = \frac{g'(\lambda)h(\lambda) - g(\lambda)h'(\lambda)}{f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)} \text{ and } Y(\lambda) = \frac{f(\lambda)h'(\lambda) - f'(\lambda)h(\lambda)}{f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)} \text{ for } \lambda \in I \tag{2.14}$$

is obtained by solving the linear system

$$\begin{cases} f(\lambda)x + g(\lambda)y = h(\lambda), \\ f'(\lambda)x + g'(\lambda)y = h'(\lambda). \end{cases}$$

The following Theorem is a corollary of the Fundamental Theorem of the Cheng-Lin method (Theorem 2.6 in [2]).

Theorem 2.3. *Let f, g, h are differentiable functions on $(0,1)$ such that $f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda) \neq 0$ for $\lambda \in (0,1)$. Let the curve S be the envelope of the the family $\{L_\lambda : \lambda \in (0,1)\}$ which is of the form (2.13). Then for any $\overline{X}$ and $\overline{Y}$, the function*

$$f(\lambda)\overline{X} + g(\lambda)\overline{Y} - h(\lambda)$$

has exactly m mutually distinct roots in $(0,1)$ if and only if S has exactly m mutually distinct tangents that also pass through the point $(\overline{X}, \overline{Y})$.

For the sake of convenience, let us introduce the notations

$$D = b + (1-f)\alpha, \qquad z = \frac{\gamma - D}{b+\alpha}, \qquad d = \frac{Db}{(b+\alpha)^2},$$

$$A = \frac{(\gamma - D)Db}{(b+\alpha)^3}, \qquad B = \frac{Db - (b+\alpha)(\gamma - D)}{(b+\alpha)^2},$$

$$\sigma = \frac{D}{b+\alpha} b\tau, \qquad \overline{X} = R_0, \qquad \overline{Y} = R_0 \frac{b}{b+\alpha}.$$

Observe that $A = zd$ and $B = d - z$. It is easy to see that $0 < d < 1$, furthermore, if $R_0 > 1$, then $\gamma > b + \alpha \geq D$ thus $z > 0$ and $A > 0$. Let

$$\lambda = \frac{b+\alpha}{b} i^*.$$

We note that $\lambda \in (0,1)$ if, and only if, $i^* \in (0, b/(b+\alpha))$. Substituting $i^* = b\lambda/(b+\alpha)$ into $H(i^*)$, we obtain

$$\begin{aligned} \overline{H}(\lambda) &= \left[\frac{(\gamma - D)Db}{(b+\alpha)^3} \lambda^2 + \left(\frac{Db - (b+\alpha)(\gamma - D)}{(b+\alpha)^2} \right) \lambda - 1 \right] \\ &\quad + \left(R_0 - \frac{bR_0}{b+\alpha} \lambda \right) e^{\frac{D}{b+\alpha} b\tau\lambda} \\ &= \left[A\lambda^2 + B\lambda - 1 \right] + (\overline{X} - \overline{Y}\lambda) e^{\sigma\lambda}, \end{aligned}$$

therefore, $\overline{H}(\lambda)$ has a unique real root in $(0,1)$ if, and only if, $H(i^*)$ has a unique real root in $(0, b/(b+\alpha))$.

To apply the Cheng-Lin envelope method, using the notation of [2], we consider the function

$$\overline{H}(\lambda | x, y) = \left[A\lambda^2 + B\lambda - 1 \right] + (x - y\lambda) e^{\sigma\lambda}. \tag{2.15}$$

For $\lambda \in (0,1)$, let L_λ be a straight line in the plane defined by

$$L_\lambda : x - \lambda y = -e^{-\sigma\lambda}\left(A\lambda^2 + B\lambda - 1\right), \tag{2.16}$$

which is of the form (2.13). According to (2.14), we find the envelope S of the family of straight lines $\{L_\lambda : 0 < \lambda < 1\}$:

$$X(\lambda) = e^{-\sigma\lambda}\left\{-A\sigma\lambda^3 + (A - B\sigma)\lambda^2 + \sigma\lambda + 1\right\}, \tag{2.17}$$

and

$$Y(\lambda) = e^{-\sigma\lambda}\left\{-A\sigma\lambda^2 + (2A - B\sigma)\lambda + (\sigma + B)\right\}. \tag{2.18}$$

By Theorem 2.3, it is sufficient to verify that the curve S has unique tangent line which passes through the point $(\overline{X}, \overline{Y})$. In order to see this, observe that $(X(0^+), Y(0^+)) = (1, \sigma + B)$,

$$X'(\lambda) = \lambda Y'(\lambda) = \lambda e^{-\sigma\lambda} G(\lambda), \tag{2.19}$$

furthermore

$$\frac{dY}{dX} = \frac{1}{\lambda}, \text{ and } \frac{d^2Y}{dX^2} = -\frac{1}{\lambda^2 X'(\lambda)}, \tag{2.20}$$

for $0 < \lambda < 1$, where

$$G(\lambda) = A\sigma^2\lambda^2 + (B\sigma^2 - 4A\sigma)\lambda + (2A - 2B\sigma - \sigma^2).$$

From (2.20) it follows that the tangent line of S at $(X(0^+), Y(0^+))$ is vertical. We have the following lemma.

Lemma 2.4. *Let $\alpha, \beta \in \mathbb{R}$ and the quadratic polynomial $P(\lambda) = \lambda^2 + \alpha\lambda + \beta$.*
(i) $P(\lambda)$ has no roots in $(0,1)$ if and only if one of the following conditions holds:

(1-1) $\alpha \geq 0$ *and* $\beta \geq 0$;
(1-2) $-2 < \alpha < 0$ *and* $\beta > \alpha^2/4$;
(1-3) $\alpha \leq -2$ *and* $\beta \geq -\alpha - 1$;
(1-4) $\beta \leq 0$ *and* $\beta \leq -\alpha - 1$.

(ii) $P(\lambda)$ has unique root in $(0,1)$ if and only if one of the following conditions holds:

(2-1) $-\alpha - 1 < \beta < 0$;
(2-2) $0 < \beta < -\alpha - 1$;
(2-3) $-2 < \alpha < 0$ *and* $\beta = \alpha^2/4$.

(iii) $P(\lambda)$ has two distinct real roots in $(0,1)$ if and only if $-2 < \alpha < 0$, $0 < \beta < \alpha^2/4$, and $\beta > -\alpha - 1$.

We may refer to section 5.1 in [2] to prove this result by means of Cheng-Lin envelope method or can be proved by elementary calculations.

Lemma 2.5.

(1) *$G(\lambda)$ either has no roots in $(0,1)$, or has a unique root in $(0,1)$.*
(2) *$G(\lambda)$ has a unique root in $(0,1)$ if and only if $0 < w < -s-1$, where*

$$s = \frac{B\sigma - 4A}{A\sigma} \text{ and } w = \frac{2A - 2B\sigma - \sigma^2}{A\sigma^2}. \tag{2.21}$$

Proof. Note that $G(\lambda) = A\sigma^2(\lambda^2 + s\lambda + w)$. We first claim that if $s > -2$, then $w < -s-1$. Indeed,

$$A + B - 1 = -(1+z)(1-d) < 0. \tag{2.22}$$

Since $s > -2$, we have

$$-8A + 4A\sigma > -2B\sigma. \tag{2.23}$$

Then

$$\begin{aligned} w + s + 1 &= \frac{1}{A\sigma^2}(2A - 2B\sigma - 4A\sigma + (A+B-1)\,\sigma^2) \\ &< \frac{1}{A\sigma^2}(-6A + (A+B-1)\,\sigma^2) < 0. \end{aligned}$$

By our claim and Lemma 2.4 (iii), $G(\lambda)$ can not have two roots in $(0,1)$, and the proof of statement (1) is complete. Furthermore, by our claim, we see that s and w can not satisfy (2-1) and (2-3) in Lemma 2.4 (ii), therefore the proof of statement (2) is also complete. □

Let

$$L_{1^-}(x) := x - X(1^-) + Y(1^-) = x + e^{-\sigma}\,(A + B - 1)\,,$$

which is obtained by taking the left limit of L_λ as $\lambda \to 1^-$. Note that $L_{1^-}(x)$ is the tangent line of S at $\lambda = 1^-$. The following two lemmas are dealing with the mutual positions of $L_{1^-}(x)$ and $(\overline{X}, \overline{Y}) = (R_0, R_0 b/(b+\alpha))$.

Lemma 2.6. *The point $(\overline{X}, \overline{Y})$ lies above the line $L_{1^-}(x)$, that is*

$$R_0 \frac{b}{b+\alpha} > L_{1^-}(R_0).$$

Proof.

$$\begin{aligned} R_0 \frac{b}{b+\alpha} - L_{1^-}(R_0) &= R_0 \frac{b}{b+\alpha} - e^{-\sigma}\,(A+B-1) - R_0 \\ &= \frac{1}{b+\alpha}\left\{ bR_0 + e^{-\frac{D}{b+\alpha}b\tau}\,(1-d)\,(\gamma + f\alpha) \right\} - R_0 \\ &\geq \frac{1}{b+\alpha}\left\{ bR_0 + \gamma e^{-b\tau}\,(1-d) \right\} - R_0 \\ &= \frac{\gamma e^{-b\tau}}{b+\alpha}\left\{ 1 + \frac{fb\alpha}{(b+\alpha)^2} \right\} - R_0 \\ &> R_0 - R_0 = 0. \end{aligned}$$

□

Lemma 2.7. *Let s and w be defined by (2.21) and assume that s and w satisfy $w < -s-1$. If $R_0 > 1$ then $X(1^-) - R_0 > 0$.*

Proof. In view of $w < -s-1$, we have

$$(1-B-A)\sigma^2 + (4A+2B)\sigma - 2A > 0. \tag{2.24}$$

Let

$$F(t) = (1-B-A)\,t^2 + (4A+2B)\,t - 2A. \tag{2.25}$$

By (2.22) and $F(0) = -2A < 0$, we see that $F(t)$ has unique positive root $\overline{\sigma}$. Since $\sigma \geq 0$ and $F(\sigma) > 0$, we have $\overline{\sigma} < \sigma$. Observe that

$$\begin{aligned} F\left(\frac{z}{1+z}\right) &= (1-B-A)\left(\frac{z}{1+z}\right)^2 + (4A+2B)\,\frac{z}{1+z} - 2A \\ &= (1-d+z-zd)\,\frac{z^2}{(1+z)^2} + (4zd+2d-2z)\,\frac{z}{1+z} - 2zd \\ &= -\frac{1-d}{1+z}z^2 < 0, \end{aligned}$$

therefore

$$\frac{z}{1+z} < \overline{\sigma} < \sigma. \tag{2.26}$$

Now we can estimate $X(1^-) - R_0$.

$$\begin{aligned} X(1^-) - R_0 &= e^{-\frac{D}{b+\alpha}b\tau}\left\{(1-A-B)\sigma + A + 1\right\} - \frac{\gamma}{b+\alpha}e^{-b\tau} \\ &> e^{-b\tau}\left\{(1-A-B)\sigma + A + 1 - \frac{\gamma}{b+\alpha}\right\} \\ &= e^{-b\tau}\left\{(1-A-B)\,\sigma + A - \frac{\gamma - D}{b+\alpha} + 1 - \frac{D}{b+\alpha}\right\} \\ &= e^{-b\tau}\left\{(1-d)\,(1+z)\left(\sigma - \frac{z}{1+z}\right) + 1 - \frac{D}{b+\alpha}\right\} > 0, \end{aligned}$$

the proof is complete. □

Finally, we are in the position to prove Theorem 2.2.

Proof. Note that if $G(\lambda)$ has unique root in $(0,1)$, then $w > 0$ and $G(0) = A\sigma^2 w > 0$ by Lemma 2.5 (2). So, by Lemma 2.5, we see that $G(\lambda)$ has three possible situations.

Case 1. $G(\lambda) > 0$ on $(0,1)$;

Case 2. $G(\lambda) < 0$ on $(0,1)$;

Case 3. $G(\lambda) > 0$ on $(0,\lambda^*)$ and $G(\lambda) < 0$ on $(\lambda^*,1)$ for some $\lambda^* \in (0,1)$.

Case 1. Since $G(\lambda) > 0$ on $(0, 1)$, by (2.19) and (2.20) both $X(\lambda)$ and $Y(\lambda)$ are strictly increasing on $(0, 1)$, and the graph of S is concave. By Lemma 2.6, we see that

$$\overline{Y} = R_0 b/(b + \alpha) > L_{1^-}(R_0) = L_{1^-}(\overline{X}).$$

Also, $\overline{X} = R_0 > 1$, and the situation is depicted in Figure 1(a). Then, by Theorem 3.5 in [2] S has unique tangent line which passes through the point $(\overline{X}, \overline{Y})$.

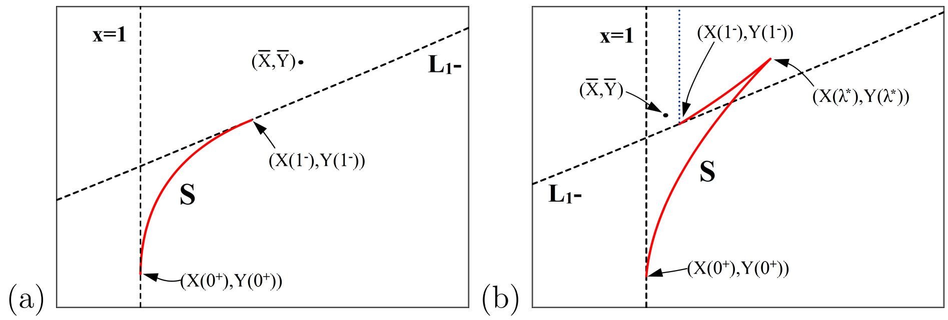

FIGURE 1. The relative geometric positions of the vertical line $x = 1$, the line L_{1^-}, the point $(\overline{X}, \overline{Y})$ and the envelope S are depicted. Case 1 corresponds to (a), case 3 corresponds to (b). Since in both cases there is a unique tangent line from $(\overline{X}, \overline{Y})$ to S, the Cheng-Lin envelope method guarantees that there is a unique endemic equilibrium.

Case 2. Since $G(\lambda) < 0$ on $(0, 1)$, by (2.19) and (2.20) we obtain that $X(\lambda)$ and $Y(\lambda)$ are strictly decreasing on $(0, 1)$. In this case $G(1) = A\sigma^2(1 + s + w) < 0$ implies $w < -s - 1$. Then by Lemma 2.7,

$$R_0 < X(1^-) < X(0^+) = 1,$$

which is a contradiction. Hence, when $R_0 > 1$ it is impossible that $G(\lambda) < 0$ on $(0, 1)$.

Case 3. $G(\lambda) > 0$ on $(0, \lambda^*)$, thus $X(\lambda)$ and $Y(\lambda)$ is strictly increasing on $(0, \lambda^*)$, in addition the graph of S over $(0, \lambda^*)$ is concave. Furthermore, since $G(\lambda) < 0$ on $(\lambda^*, 1)$, by (2.19) and (2.20), then $X(\lambda)$ and $Y(\lambda)$ is strictly decreasing on $(\lambda^*, 1)$, and the graph of S over $(\lambda^*, 1)$ is convex. We recall s and w defined by (2.21). Since $G(\lambda)$ has unique root in $(0, 1)$, by Lemma 2.5 (2), we have $0 < w < -s - 1$. Furthermore, Lemma 2.6 and 2.7 imply $L_{1^-}(\overline{X}) < \overline{Y}$ and $1 < \overline{X} < X(1^-)$. The geometric situation is depicted in Figure 1 (b). By Theorem 3.7 in [2], the curve S over $[\lambda^*, 1)$ has no tangent line which passes through the point $(\overline{X}, \overline{Y})$, in addition, according to Theorem 3.5 in [2], the curve S over $(0, \lambda^*)$ has a unique tangent line which passes through the point $(\overline{X}, \overline{Y})$. Thus, the curve S has a unique tangent line which passes through the point $(\overline{X}, \overline{Y})$, therefore, by Theorem 2.3 the proof is concluded. □

Remark We assumed $f > 0$ throughout the paper. However, allowing $f = 0$ admits an additional equilibrium $(0, 0, 1, 0)$, which corresponds to the situation when the disease is lethal for everyone and recovery is not possible. Then this equilibrium describes a population where everyone is infectious and remains so, and since they don't reproduce, they will die out exponentially.

3. STABILITY OF THE DISEASE FREE EQUILIBRIUM

Theorem 3.1. *The disease free equilibrium is locally asymptotically stable if* $R_0 < 1$, *and unstable if* $R_0 > 1$.

Proof. Linearizing the system (1.5)–(1.8) around $(1, 0, 0, 0)$ gives

$$s'(t) = -bs(t) - (\gamma - (1-f)\alpha)i(t), \tag{3.1}$$

$$e'(t) = -be(t) - \gamma e^{-b\tau} i(t-\tau), \tag{3.2}$$

$$i'(t) = -(b+\alpha)i(t) + \gamma e^{-b\tau} i(t-\tau), \tag{3.3}$$

$$r'(t) = f\alpha i(t) - br(t). \tag{3.4}$$

The associated characteristic function is

$$\begin{aligned} C(\lambda) &= (b+\lambda)^3(\lambda + b + \alpha - \gamma e^{-b\tau} e^{-\lambda\tau}) \\ &= (b+\lambda)^3(\lambda + (b+\alpha)(1 - R_0 e^{-\lambda\tau})). \end{aligned}$$

The first factor of $C(\lambda)$ shows that $\lambda = -b$ is a triple characteristic root with negative real part, so it is sufficient to investigate the second factor

$$C_2(\lambda) := \lambda + (b+\alpha)(1 - R_0 e^{-\lambda\tau}). \tag{3.5}$$

First, consider the case $R_0 < 1$. For the contrary, suppose that there exists a characteristic root λ_* with positive real part. Then, from the real part of (3.5), we obtain

$$0 = \mathrm{Re}\lambda_* + (b+\alpha)(1 - R_0 e^{-\tau \mathrm{Re}\lambda_*} \cos \tau \mathrm{Im}\lambda_*) > \mathrm{Re}\lambda_* > 0, \tag{3.6}$$

which is a contradiction.
Now suppose that $R_0 > 1$. We shall prove that there exists at least one real positive characteristic root. Since $C_2(0)$ is negative and on the real line the limit $\lim_{x\to\infty} C_2(x)$ equals plus infinity, the continuity of C_2 implies the existence of such a root. □

4. PERSISTENCE

In [16] the authors proved that the disease is uniformly persistent if $R_0 > 1$ in the sense that there exists an $\eta > 0$ such that $\liminf i(t) > \eta$. While this statement is true, we think that their proof is not completely correct: on page 127 in the proof of their Theorem 4.2 in [16] the authors claim that

$$X \cap B_\delta X_2 = \{x \in X; d(x, X_2) < \delta\}$$

has a compact closure $K := \{x \in X; d(x, X_2) \leq \delta\}$, and they use this property to invoke a general abstract persistence result. Here $X = C^+([-\tau, 0], R_+^2)$ and

$$X_2 = \{(\phi_0, \phi_1) \in X : \phi_i(s) > 0, \phi_{1-i}(s) \equiv 0 \quad \text{for all } s \in [-\tau, 0] \text{ and some } i \in \{0, 1\}\}.$$

However, the set K is not compact in X, which is the positive cone of the Banach space of continuous functions mapping from $[-\tau, 0]$ to R^2. Hence, in what follows we give an alternative proof.

It is sufficient to restrict our attention to the $s - i$ subsystem (1.5),(1.7), where we shall use the natural phase space

$$Y := C_+([-\tau, 0], R_+) \times C_+([-\tau, 0], R_+)$$

for our variables $s(t), i(t)$.

Theorem 4.1. *If $R_0 > 1$, then the disease persists, i.e.* $\liminf i(t) > \eta$ *for some* $\eta > 0$.

Proof. We shall apply the following persistence result of Hale and Waltman [4]:

Suppose that we have the following:

(i) X^0 is open and dense in Y with $X^0 \cup X_0 = Y$ and $X^0 \cap X_0 = \emptyset$;
(ii) the solution operators $T(t)$ satisfy

$$T(t) : X^0 \to X^0, \quad T(t) : X_0 \to X_0;$$

(iii) $T(t)$ is point dissipative in Y;
(iv) there is a $t_0 \geq 0$ such that $T(t)$ is compact for all $t \geq t_0$;
(v) $\mathcal{A} = \bigcup_{x \in A_b} \omega(x)$ is isolated and has an acyclic covering N, where A_b is the global attractor of $T(t)$ restricted to X_0 and $N = \cup_{i=1}^k N_i$;
(vi) for each $N_i \in N$, $W^s(N_i) \cap X^0 = \emptyset$, where W^s refers to the stable set.

Then, $T(t)$ is a uniform repeller with respect to X^0, i.e. there is an $\eta > 0$ such that for any $x \in X^0$, $\liminf_{t \to \infty} d(T(t)x, X_0) \geq \eta$.

We are checking all these six conditions. Let $X_0 = \{x \in Y : i(t) \equiv 0\}$, then

(i) follows from the definition of X_0 and the continuous dependence of solutions on initial conditions;
(ii) is obvious;
(iii) is obvious;
(iv) follows from the boundedness and the Arzela-Ascoli theorem with $t_0 = \tau$.

To prove (v), first we analyze the behaviour of the system restricted to X_0, where $i(t) = 0$ and $m(t) = b$:

$$s'(t) = b - bs(t), \tag{4.1}$$

$$i'(t) = 0, \tag{4.2}$$

This linear system has the global attractor $P = (1, 0)$ on X_0. Therefore $\mathcal{A} = \{P\}$ and we have the trivial covering $N = \{P\}$.

Before proving that P is isolated, we show (vi). Suppose the contrary, that there is a solution such that $i(t) > 0$, $i(t) \to 0$, $s(t) \to 1$. Then we also have $m(t) \to b$. Since $R_0 > 1$ is equivalent with $\gamma e^{-b\tau} > b + \alpha$, there exists an $\varepsilon > 0$ such that

$$\gamma(1 - \varepsilon)e^{-b\tau} > b + \alpha.$$

Let us fix such an ε, then there is a t_0 such that $s(t - \tau) > 1 - \varepsilon$ for all $t > t_0$. Note that $m(t) < b$ and $e^{-\int_{t-\tau}^{t} m(s)ds} > e^{-b\tau}$ when $i(t) > 0$. Define

$$J(t) = i(t) + \big(\gamma(1 - \varepsilon)e^{-b\tau}\big) \int_{t-\tau}^{t} i(s)ds.$$

Then $J(t) \to 0$ follows from $i(t) \to 0$ as $t \to \infty$. Differentiating with respect to time, we obtain

$$\frac{dJ}{dt} = i'(t) + \big(\gamma(1 - \varepsilon)e^{-b\tau}\big)(i(t) - i(t - \tau)),$$

and from (1.7) we have

$$i'(t) > \gamma(1 - \varepsilon)i(t - \tau)e^{-b\tau} - (b + \alpha)i(t),$$

which yields

$$\frac{dJ}{dt} > \Big(\gamma(1 - \varepsilon)e^{-b\tau} - (b + \alpha)\Big)i(t) > 0$$

contradicting to $J(t) \to 0$, thus we obtain $W^s(P) \cap X^0 = \emptyset$.

We finish with showing (v), by taking advantage of the functional $J(t)$. There is a neighborhood U_ε of P where the functional $J(t)$ is increasing along solutions. Suppose that there is a compact invariant set M in this neighborhood. Then the continuous functional takes its maximum at some point $p \in M$. Clearly $p \in X_0$, otherwise J is strictly increasing, contradicting to the invariance of M and the maximum property of p. Hence $M \subset X_0$ where P is asymptotically stable, therefore the trivial covering $N = \{P\}$ is isolated in Y (see Chapter 8.3 in [13] for more detailed explanation), and also acyclic, since there is no orbit that connects it to itself. Therefore, we have that if $R_0 > 1$, then the disease persists in the population, more precisely there is an $\eta > 0$ such that $\liminf i(t) > \eta$.

Note that we obtained this persistence result for the transformed system (1.5)–(1.8), which means for the original model that in the population the fraction of infectious individuals will remain above some positive constant on the long run, what we called proportional persistence in the abstract of the paper. □

5. DISCUSSION

In this paper we improve the results of [16] for an SEIR model with fixed latency period and exponentially growing population. In particular, in [16] it was shown that there is at least one endemic equilibrium if the threshold parameter $R_0 = \gamma\frac{e^{-b\tau}}{b+\alpha} > 1$, but further analysis has not been given regarding the equilibria. Here we demonstrate that there is no endemic equilibrium whenever $R_0 \leq 1$, thus backward bifurcation (which has been observed in some epidemic models, see for example [3]) is not possible for this model. The characterization of equilibria is usually a straightforward task in epidemic models, however, in this case, it turned out to be surprisingly difficult. To overcome this unexpected difficulty and to prove that there is exactly one endemic equilibrium in the case $R_0 > 1$, we need to employ a rather technically subtle application of the envelope method investigated in details in [2].

Furthermore, by standard linearization technique we provide a local stability analysis for the disease free equilibrium, showing that it is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.

In [16], the authors proved a persistence result for $R_0 > 1$. While their statement is correct, we believe that their proof is not complete, hence we give an alternative proof for the persistence of the disease by applying a result of Hale and Waltman [4].

There are still open problems remained for this model. In [16], the authors showed by applying Lyapunov's direct method that the disease free equilibrium is globally asymptotically stable if $\gamma < b + \alpha$. We know that it is unstable if $R_0 > 1$, or equivalently if $\frac{\gamma}{b+\alpha} < e^{-b\tau}$. Thus, the global asymptotic stability is still unknown for the parameter regions

$$1 < \frac{\gamma}{b+\alpha} < e^{-b\tau}.$$

Nevertheless, the nonexistence of endemic equilibrium and the local stability of the disease free equilibrium for this case suggests the conjecture that the disease free equilibrium is globally asymptotically stable.

For $R_0 > 1$ it is now known that the disease persists in the population. In [16] the authors concluded the global asymptotic stability of the endemic equilibrium in the special case when the length of the latency period is zero. However, when the delay is positive, it is unsolved whether the unique endemic equilibrium is globally asymptotically stable, or the model can exhibit sustained oscillations, which is a common consequence of introducing time delays.

It seems that for this model the recently - in the context of epidemic models with mass action - very successfully applied Lyapunov-functions (see, for instance [8] or

[5] and references thereof), which are various combinations of functions of the form $f(x) = 1 + x - \ln x$, are not applicable here, because of the term

$$e^{-\int_{t-\tau}^{t} m(s)ds}$$

in (1.6), thus the global asymptotic stability problem might be challenging.

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