ON THE GLOBAL ATTRACTOR OF DELAY DIFFERENTIAL EQUATIONS WITH UNIMODAL FEEDBACK

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Abstract. We give bounds for the global attractor of the delay differential
equation $\dot{x}(t) = -\mu x(t) + f(x(t-\tau))$, where f is unimodal and has negative
Schwarzian derivative. If f and μ satisfy certain condition, then, regardless
of the delay, all solutions enter the domain where f is monotone decreasing
and the powerful results for delayed monotone feedback can be applied to
describe the asymptotic behaviour of solutions. In this situation we determine
the sharpest interval that contains the global attractor for any delay. In the
absence of that condition, improving earlier results, we show that if the delay
is sufficiently small, then all solutions enter the domain where f' is negative.
Our theorems then are illustrated by numerical examples using Nicholson's
blowflies equation and the Mackey-Glass equation.

1. **Introduction.** This note is motivated by a recent paper by G. Röst and J. Wu
[17] about the so-called delayed recruitment model defined by the delay differential
equation

$$x'(t) = -\mu x(t) + f(x(t-\tau)), \tag{1.1}$$

where $\mu > 0$, $\tau > 0$, and $f : [0, \infty) \to [0, \infty)$ is a continuous function.

In particular, they consider the case when f is a unimodal function, which is
the situation for the famous Nicholson's blowflies equation and the Mackey-Glass
model.

In that reference, the authors have proved several results on the global dynamics
of Eq. (1.1), and they also formulated some open problems. It is our purpose
to prove new results in the direction initiated in [17], and also to answer some of
the open questions. We show the applicability of our results for different cases of
the Nicholson's blowflies equation proposed by Gurney *et al.* [3] to explain the
experimental data obtained by Nicholson [16] for the population of the Australian
sheep-blowfly

$$x'(t) = -\mu x(t) + p x(t-\tau) e^{-\gamma x(t-\tau)}, \tag{1.2}$$

where μ, p, γ are positive parameters; and the Mackey-Glass equation in *hematopoiesis* [13]

$$x'(t) = -\mu x(t) + \frac{px(t-\tau)}{1 + x(t-\tau)^n}.$$ (1.3)

Following [17], we assume that f is unimodal. More precisely, the following hypothesis will be required:

(U) $f(x) \geq 0$ for all $x \geq 0$, $f(0) = 0$, and there is a unique $x_0 > 0$ such that $f'(x) > 0$ if $0 \leq x < x_0$, $f'(x_0) = 0$, and $f'(x) < 0$ if $x > x_0$. Moreover, $f''(x) < 0$ if $0 < x \leq x_0$, and $\lim_{x\to\infty} f(x) = 0$.

Much is known about the global picture of the dynamics of Eq. (1.1), when f is a monotone function. However, unimodal feedback may lead to very complicated and still not completely understood dynamics. See [7, 8, 9, 11] and references thereof.

We shall use the function $g(x) = \mu^{-1} f(x)$. Notice that the equilibria of (1.1) are the fixed points of g, and that, under condition (U), function g has at most two fixed points $x = 0$ and $x = K > 0$.

As in [17], we consider nonnegative solutions of (1.1). We recall that for each nonnegative and nonzero function $\phi \in C = C([-\tau, 0], \mathbb{R})$, there exists a unique solution $x^\phi(t)$ of (1.1) such that $x^\phi = \phi$ on $[-\tau, 0]$. Moreover, $x^\phi(t) > 0$, $\forall t > 0$. (See, e.g., [4, Corollary 12].)

If $g'(0) \leq 1$, then zero is the only nonnegative equilibrium of Eq. (1.1) under condition (U). Moreover, it is well known (see, e. g., [4, 17]) that in this case all solutions of (1.1) converge to 0 as t tends to infinity. On the other hand, the positive equilibrium K is globally attracting for Eq. (1.1) if $g'(K) \geq 0$ (equivalently, if $K \leq x_0$).

Thus, we shall assume that $g'(0) > 1$ and $K > x_0$. Ivanov and Sharkovsky [6, Theorem 2.3] proved that an invariant and attracting interval $[\alpha, \beta]$ for g is also invariant and attracting for (1.1) for all values of the delay τ, that is,

$$\alpha \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \beta,$$

for any nonzero solution x of (1.1). Similar results were proven using a slightly different approach in [4, 17].

It is clear that we can choose $\beta = g(x_0)$, $\alpha = g(\beta) = g^2(x_0)$ to get an attracting invariant interval $[\alpha, \beta]$ for the map g (here, and in the following, g^2 denotes the composition $g \circ g$). Thus, this interval contains the global attractor associated to Eq (1.1) for all values of τ. Using this fact, Röst and Wu obtain sufficient conditions to ensure that every solution of (1.1) enters the domain where f' is negative. In this case the asymptotic behaviour of the solutions is governed by monotone delayed feedback and the comprehensive theory of monotone dynamics is applicable, as it was demonstrated in [17]. In particular, since a Poincaré-Bendixson type theorem is available for (1.1) when f' is negative, conditions of this kind exclude the possibility of solutions with complicated asymptotic behaviour (the ω-limit set can only be the positive equilibrium K or a periodic orbit). We include here the main results of [17] in this direction.

Theorem 1. *Every solution of (1.1) enters the domain where f' is negative if any of the following conditions holds:*

(L) $\alpha = g(\beta) = g^2(x_0) > x_0$.

(L$_\tau$) $\tau < \tau^* := \dfrac{\Pi(x_0) - x_0}{\mu(g(x_0) - g^2(x_0))}$, *where* $\Pi : (0, K) \to [K, \infty)$ *is the inverse of the restriction of g to the interval* $[K, \infty)$.
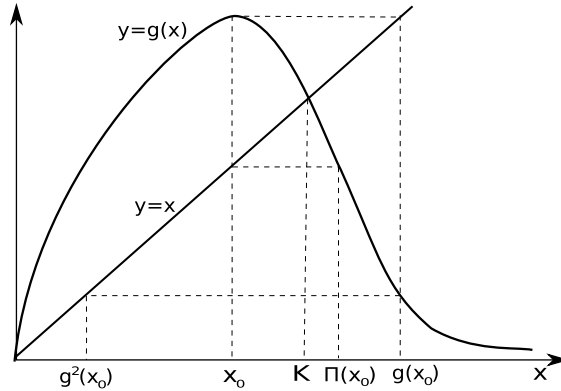

FIGURE 1. The profile of the function g in a situation where condition (L) does not hold.

Notice that the first condition in Theorem 1 is independent of the delay, while condition (L$_\tau$) shows that even if f is unimodal, the solutions of (1.1) have the same asymptotic behaviour as in the case of monotone decreasing feedback for all sufficiently small delays τ.

An open problem suggested in [17] is the following: under condition (L), find the sharpest invariant and attracting interval containing the global attractor of (1.1) for all τ. (Numerical experiments performed in [17] show that $J = [\alpha, \beta]$ seems to be a very sharp bound). To avoid confusion we remark that the global attractor $\mathcal{A}$ is a subset of the function space $C([-\tau, 0], \mathbb{R})$, and saying that an interval $[a, b]$ contains the global attractor we mean that for each $\phi \in \mathcal{A}$, we have $a \leq \phi(s) \leq b$ for any $s \in [-\tau, 0]$.

Our main results in this note are the following:

1. We completely solve this problem in the case of Nicholson's blowflies and Mackey-Glass equations, obtaining an interesting dichotomy result for (1.2) and (1.3) when condition (L) holds (Theorem 6).
2. We give a weaker delay-dependent condition different from (L$_\tau$) under which the statement of Theorem 1 remains valid. In other words, we can determine a τ_* that is larger than τ^* in Theorem 1, such that all solutions enter the domain where f' is negative, if $\tau < \tau_*$. Moreover, we provide some examples showing that the new condition significantly improves (L$_\tau$) in certain situations.

2. **Main results.** In this section, we assume that condition (U) holds, $g'(0) > 1$ and $K > x_0$, where $g = \mu^{-1}f$. Denote

$$\bar{\alpha} = \inf\{A > 0 : g^2(A) = A\} \quad ; \quad \bar{\beta} = \sup\{B > 0 : g^2(B) = B\}. \qquad (2.1)$$

Since $g'(0) > 1$, $\bar{\alpha}$ and $\bar{\beta}$ are well defined real numbers. Assuming that (L) is satisfied, it is clear that $\bar{\alpha} > x_0$ and hence g is decreasing on $\bar{J} := [\bar{\alpha}, \bar{\beta}]$. Moreover, $g(\bar{\alpha}) = \bar{\beta}$, and $g(\bar{\beta}) = \bar{\alpha}$. As a consequence, $\bar{J}$ is invariant for the map g.

Lemma 2. *Assume that (L) holds. Then, $\bar{J} = [\bar{\alpha}, \bar{\beta}]$ is an attracting invariant interval for the map g.*

Proof. We have already proved that $\bar{J}$ is invariant. Next, we prove that it is attracting.

Since $h = g^2$ is monotone increasing in $\bar{J}$, and $\bar{\alpha}$ and $\bar{\beta}$ are respectively the minimal and the maximal fixed points of h in $[\alpha, \beta]$, it follows that

$$\lim_{n \to \infty} h^n(x) = \bar{\alpha}, \, \forall \, x \in [\alpha, \bar{\alpha}] \quad ; \quad \lim_{n \to \infty} h^n(x) = \bar{\beta}, \, \forall \, x \in [\bar{\beta}, \beta].$$

Since $g([\alpha, \bar{\alpha}]) = [\bar{\beta}, \beta]$ and $g([\bar{\beta}, \beta]) = [\alpha, \bar{\alpha}]$, the result follows from the fact that $[\alpha, \beta]$ is attracting for g. □

An application of the above mentioned Theorem 2.3 in [6] gives the following result:

Corollary 3. *Under condition (L), the interval $\bar{J} = [\bar{\alpha}, \bar{\beta}]$ is an attracting and invariant interval for (1.1) for all values of the delay τ.*

Remark 1. If the equilibrium K is globally attracting for g, then $\bar{\alpha} = \bar{\beta} = K$, and hence K is also a global attractor for (1.1) for all values of the delay. This result is [6, Theorem 2.2].

Remark 2. If (L) does not hold, the interval $\bar{J}$ does not need to be globally attracting for (1.1). For example, for the Nicholson's blowflies equation considered in [17]

$$x'(t) = -0.05x(t) + x(t - \tau)e^{-x(t-\tau)}, \tag{2.2}$$

the interval $\bar{J}$ is given by

$$\bar{J} = [\bar{\alpha}, \bar{\beta}] \approx [0.4261, 5.5653].$$

It is easy to check that this interval does not attract every orbit associated to the map g, since there is a period four orbit given by

$$0.24286 \to 3.80991 \to 1.6878 \to 6.24235 \to 0.24286.$$

Numerical experiments from [17] suggest that $\bar{J}$ also does not attract every orbit of (2.2) for large values of τ.

We recall here that the nonlinearity f in some important examples of Eq. (1.1) (including the Mackey-Glass and Nicholson's blowflies models) fulfills the following additional assumption:

(S) f is three times differentiable, and $(Sf)(x) < 0$ whenever $f'(x) \neq 0$, where Sf denotes the Schwarzian derivative of f, defined by

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2.$$

The following proposition is a consequence of Singer's results [18].

Proposition 4. *Assume that $g : [a, b] \to [a, b]$ satisfies (S) and $g'(x) < 0$ for all $x \in [a, b]$. Let K be the unique fixed point of g in $[a, b]$. Then,*

- *If $|g'(K)| \leq 1$, then $\lim_{n \to \infty} g^n(x) = K$, $\forall \, x \in [a, b]$.*
- *If $|g'(K)| > 1$, then there exists a globally attracting $2-$cycle $\{p, q\}$. More precisely, $g(p) = q$, $g(q) = p$, $p \neq q$, and*

$$\lim_{n \to \infty} g^{2n}(x) = \begin{cases} p \, \text{ if } \, x < K \\ q \, \text{ if } \, x > K. \end{cases}$$

For a detailed proof of the first statement in a more general situation, see, e.g., [12, Proposition 3.3]. The second statement can be easily proved using the same arguments.

Remark 3. As noticed above, condition (S) holds for the Mackey-Glass and Nicholson's blowflies models, among others. See, for example, [2, 4, 15]. Thus, under condition (L), in these models the interval $\bar{J}$ given in Lemma 2 reduces to $\{K\}$ if $|g'(K)| \leq 1$, whereas $\bar{J} = [p, q]$ if $|g'(K)| > 1$.

The next result shows that $\bar{J}$ is actually the sharpest invariant and attracting interval containing the global attractor of (1.1) for all τ if (L) and (S) hold. We emphasize that in the limit case $g^2(x_0) = x_0$ in (L), we have $\bar{\alpha} = \alpha = x_0$, $\bar{\beta} = \beta = g(x_0)$. Thus, as suggested in [17], the intervals $\bar{J}$ and J coincide in this special situation.

Proposition 5. *Assume that (L) is fulfilled, and (S) holds in the interval $[\bar{\alpha}, \bar{\beta}]$. If $|g'(K)| > 1$ then, for any $\xi > 0$, there exists a sufficiently large τ_ξ such that the interval $[\bar{\alpha} + \xi, \bar{\beta} - \xi]$ is not attracting for (1.1) if $\tau > \tau_\xi$.*

Proof. We make use of Theorems 2.1 and 2.2 in [15] (see also [14]). Define $\varepsilon = (\mu\tau)^{-1}$. A direct application of [15, Theorem 2.1] provides an $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$, Eq. (1.1) possesses a slowly oscillating periodic solution $p(t)$ and there exist constants $\zeta, \omega > 0$ satisfying $p(t) \in \bar{J}$, for all t, $p(0) = p(\zeta) = p(\omega) = K$, $p(t) > K$ in $(0, \zeta)$, $p(t) < K$ in (ζ, ω), $p(t) = p(t + \omega)$ for all t.

Next, [15, Theorem 2.2] states that, given any $\delta > 0$, there exist $\varepsilon_1 > 0$, $k > 0$ such that $|p(t) - \bar{\alpha}| \leq \delta$ in $[\varepsilon k, \zeta - \varepsilon k]$ for all $\varepsilon < \varepsilon_1$. Choosing $\delta = \xi/2$, it is clear that the periodic solution $p(t)$ is not attracted by the interval $[\bar{\alpha} + \xi, \bar{\beta} - \xi]$ if $\tau > \tau_\xi = (\mu\varepsilon_1)^{-1}$. □

Remark 4. What is important in the proof of Proposition 5 is the fact that g has a unique globally attracting 2-periodic solution. Condition (S) implies this fact, although it is not a necessary condition. For example, function

$$g(x) = 30\frac{x + x^{5/2}}{2 + 35x^3}$$

satisfies (L) and (U), $|g'(K)| > 1$, and it has a unique globally attracting 2-cycle defined by $\bar{\alpha} = 0.728449$, $\bar{\beta} = 2.2822$. The conclusion of Proposition 5 holds although $(Sg)(x) > 0$ for $x > 2.02$. Notice that $[\alpha, \beta] = [0.515162, 3.62133]$.

On the other hand, if g does not have a unique globally attracting 2-periodic solution, under condition (L), interval $\bar{J}$ is still the smallest globally attracting interval for the difference equation $x_{n+1} = g(x_n)$, $n = 0, 1, \ldots$ We conjecture that the conclusion of Proposition 5 remains valid without assumption (S).

As a consequence of Corollary 3, Remark 1 and Propositions 4 and 5, we get the following dichotomy for Equation (1.1) under conditions (L) and (S):

Theorem 6. *Assume that (L) is fulfilled and (S) holds in the interval $[\alpha, \beta]$. Then exactly one of the following holds:*

(1) *$|g'(K)| \leq 1$ and the global attractor of (1.1) for all values of the delay τ is $\{K\}$.*

(2) *$|g'(K)| > 1$ and the sharpest invariant and attracting interval containing the global attractor of (1.1) for all values of the delay τ is $[\bar{\alpha}, \bar{\beta}]$, where $\{\bar{\alpha}, \bar{\beta}\}$ is the unique 2-cycle of g in $[\alpha, \beta]$.*

According to Theorem 1, when condition (L) does not hold, it is still possible to find a delay-dependent condition (L_τ) under which every solution of (1.1) enters the domain where f' is negative and the theory of monotone delayed feedback can be applied to describe the asymptotic behaviour of our equation with unimodal feedback [17]. Next we give a different condition in the same direction. The proof is based on the following lemma proved in [4] (see also [12, Lemma 5.1]).

Lemma 7. *Assume that (U) holds, $g'(0) > 1$, and $K > x_0$. Then, for every nonnegative and nonzero solution $x(t)$ of (1.1) there exist finite positive limits*

$$M = \limsup_{t\to\infty} x(t)\,,\ m = \liminf_{t\to\infty} x(t).$$

Moreover, $[m, M] \subset g_1([m, M])$, where

$$g_1(x) := \left(1 - e^{-\mu\tau}\right) g(x) + e^{-\mu\tau} K. \qquad (2.3)$$

Theorem 8. *Assume that the following condition holds:*
(L'_τ) $g_1^2(x_0) > x_0$, where $g_1(x) = (1 - e^{-\mu\tau}) g(x) + e^{-\mu\tau} K$.

Then, every solution of (1.1) enters the domain where f' is negative.

Proof. Let $x(t)$ be a nonnegative nonzero solution of (1.1), $m = \liminf_{t\to\infty} x(t)$ and $M = \limsup_{t\to\infty} x(t)$. Notice that $g_1(x)$ lies between $g(x)$ and K. In particular, $g_1(x_0) > x_0$. On the other hand, since f satisfies (U), it is clear that g_1 also meets the same condition, except that $g_1(0) > 0$. Moreover, g_1 has the only fixed point K. Since, by Lemma 7, $[m, M] \subset g_1([m, M])$, we obtain $m \leq K \leq M$, thus $m \leq g_1(m)$, $M \geq g_1(M)$ and the following inequalities must hold:

$$\min\{g_1(M), g_1^2(x_0)\} \leq m, \quad M \leq g_1(x_0).$$

Since g_1 is decreasing in $[M, g_1(x_0)]$, $M \leq g_1(x_0)$ implies $g_1^2(x_0) \leq g_1(M)$ and then

$$g_1^2(x_0) = \min\{g_1(M), g_1^2(x_0)\} \leq m.$$

Finally, (L'_τ) implies that $m \geq g_1^2(x_0) > x_0$, and therefore $x(t) > x_0$ for all sufficiently large t. The proof is complete. $\qquad\square$

As a byproduct of the proof of Theorem 8, we get the following result, which is of independent interest to obtain sharper bounds for the global attractor of Eq. (1.1).

Corollary 9. *Let g_1 be the function defined in (2.3). If (U) holds, $g'(0) > 1$, and $K > x_0$, then the interval $[g_1^2(x_0), g_1(x_0)]$ contains the global attractor of (1.1).*

Notice that (L'_τ) always holds if (L) is satisfied, since $g_1^2(x_0) > \alpha = g^2(x_0)$. Next we show that when (L) does not hold, (L'_τ) is sharper than (L_τ), and therefore Theorem 8 improves Theorem 3.8 of [17].

Proposition 10. *Assume that (L) does not hold. Then (L_τ) implies (L'_τ). In other words, (L'_τ) gives a better estimate than (L_τ) for the possible delays that still guarantee that every solution enters the domain where f' is negative.*

Proof. If (L) fails, then $g^2(x_0) < x_0$. Thus we have $g^2(x_0) < x_0 < K < \Pi(x_0) < g(x_0)$ (as depicted in Figure 1). Using the notation $\theta = \mu\tau \geq 0$, (L_τ) is equivalent with

$$x_0 + \theta(g(x_0) - g^2(x_0)) < \Pi(x_0)$$

and (L'_τ) can be written as

$$(1 - e^{-\theta})g[(1 - e^{-\theta})g(x_0) + e^{-\theta}K] + e^{-\theta}K > x_0.$$

First we suppose that $(1 - e^{-\theta})g(x_0) + e^{-\theta}K \leq \Pi(x_0)$. Notice that $g(x_0) > (1 - e^{-\theta})g(x_0) + e^{-\theta}K > K$, and g is decreasing in $[K, g(x_0)]$, so

$$g[(1 - e^{-\theta})g(x_0) + e^{-\theta}K] \geq g(\Pi(x_0)) = x_0,$$

and by

$$(1 - e^{-\theta})g[(1 - e^{-\theta})g(x_0) + e^{-\theta}K] + e^{-\theta}K \geq (1 - e^{-\theta})x_0 + e^{-\theta}K > x_0,$$

(L'_τ) holds. So far we have not used $(L\tau)$, $(1 - e^{-\theta})g(x_0) + e^{-\theta}K \leq \Pi(x_0)$ always implies (L'_τ).

Now we consider the remaining case $(1 - e^{-\theta})g(x_0) + e^{-\theta}K > \Pi(x_0)$. Since (L_τ) holds, we have $(1 - e^{-\theta})g(x_0) + e^{-\theta}K > x_0 + \theta(g(x_0) - g^2(x_0))$. On the other hand,

$$g[(1 - e^{-\theta})g(x_0) + e^{-\theta}K] > g^2(x_0).$$

Therefore,

$$\begin{aligned}
(1 - e^{-\theta})g[(1 - e^{-\theta})g(x_0) + e^{-\theta}K] + e^{-\theta}K &> (1 - e^{-\theta})g^2(x_0) + e^{-\theta}K = \\
&= (1 - e^{-\theta})g(x_0) + e^{-\theta}K + (1 - e^{-\theta})(g^2(x_0) - g(x_0)) > \\
&> x_0 + \theta(g(x_0) - g^2(x_0)) + (1 - e^{-\theta})(g^2(x_0) - g(x_0)) = \\
&= x_0 + (g(x_0) - g^2(x_0))(\theta + e^{-\theta} - 1) > x_0,
\end{aligned}$$

where in the last step we used that $g(x_0) - g^2(x_0) > 0$ (this follows because (L) does not hold), and that the function $h(\theta) = \theta + e^{-\theta} - 1$ is nonnegative for $\theta \geq 0$. □

Remark 5. Notice that (L_τ) always fails if $\tau \geq 1/\mu$ (or equivalently $\theta \geq 1$), but we have not used this fact in the proof.

3. **Examples.** In this section, we use the Nicholson's blowflies equation and the Mackey-Glass equation with different parameters in order to illustrate our results in Section 2.

After a change of variables, one can always write (1.2) in the form

$$x'(t) = -\mu x(t) + x(t - \tau)e^{-x(t-\tau)}. \tag{3.1}$$

It is well known that there is a unique equilibrium $x = 0$ if $\mu \geq 1$, and it attracts all nonnegative solutions. If $\mu < 1$, there is a positive equilibrium $K = -\ln(\mu)$, and $x = 0$ becomes unstable. Moreover, K is globally attracting for all values of the delay if $\mu \in (e^{-2}, 1) \approx (0.13533, 1)$. Next, using Theorem 9.3 in [15], it follows that condition (L) holds for $\mu \in (\nu, e^{-2})$, where $\nu \approx 0.10472$ is defined by the relation $\nu = e^{-1}v^{-1}$, v being the unique solution greater than 2 of equation $v^2 = e^{v-1}$.

For $\mu = 0.13$, condition (L) holds, and the invariant and attracting interval for (3.1) given by Theorem 3.5 in [17] is

$$[\alpha, \beta] = [g^2(x_0), g(x_0)] = [1.2848, 2.8298].$$

This interval may be strengthened, since the smallest invariant and attracting interval for (3.1) independent of the delay τ is given by the unique 2-cycle of g, that is,

$$[\bar{\alpha}, \bar{\beta}] \approx [1.54796, 2.53248].$$

According to Proposition 5, there exists a slowly oscillating periodic solution of (3.1) whose minimum and maximum values get closer and closer to $\bar{\alpha}$ and $\bar{\beta}$ as τ tends to infinity. See Figure 2.A, where two distinct solutions are presented and the horizontal lines indicate $\bar{\alpha}$ and $\bar{\beta}$.

For $\mu = 1/16 = 0.0625$, condition (L) does not hold (indeed, $g^2(x_0) = 0.2616 < x_0 = 1$). Since $\Pi(x_0) = 4.21007$, we get

$$\tau^* = \frac{\Pi(x_0) - x_0}{\mu(g(x_0) - g^2(x_0))} = 0.570734.$$

On the other hand, one can check that condition (L'_τ) in Theorem 8 holds for $\tau < \tau_* = 1.46534$, showing how (L'_τ) gives an estimate significantly sharper than (L_τ).

It is interesting to notice that from Theorem 2.1 in [12] it follows that the positive equilibrium K is globally attracting for (3.1) if $\tau < 1.1935$. Hence, the information provided by Theorem 1 is not very useful here, whereas Theorem 8 can be applied for τ between 1.1935 and 1.46534.

Our next example is the Mackey-Glass equation

$$x'(t) = -\mu x(t) + \frac{2x(t - \tau)}{1 + x(t - \tau)^{20}}. \tag{3.2}$$

The function $f(x) = 2x(1 + x^{20})^{-1}$ satisfies the unimodal condition (U) with $x_0 = 0.863$. For $\mu \geq 2$, there is a unique equilibrium $x = 0$ which attracts all nonnegative solutions. If $\mu < 2$, there is also a positive equilibrium $K = (-1 + 2\mu^{-1})^{1/20}$. Next, $K \leq x_0$ for $\mu \in [1.9, 2)$ (and hence K is globally attracting), while $K > x_0$ for $\mu < 1.9$.

Denote, as usual, $g = \mu^{-1} f$. One can check that (L) holds if $\mu > 1.774$ so in the interval $(1.774, 1.9)$ the dichotomy stated in Theorem 6 applies. Since $|g'(K)| \leq 1$ if and only if $\mu \geq 1.8$, the equilibrium K attracts all positive solutions for $\mu \in [1.8, 1.9)$, whereas for each $\mu \in (1.774, 1.8)$ we can determine the sharpest delay-independent interval $[\bar{\alpha}, \bar{\beta}]$ containing the global attractor of (3.2) by finding the unique 2-cycle of g in the interval $[\alpha, \beta] = [g^2(x_0), g(x_0)]$. For example, setting $\mu = 1.79$, we have $K = 0.898$, $[\alpha, \beta] = [0.872, 0.916]$ and the interval $[\bar{\alpha}, \bar{\beta}] = [0.876, 0.914]$ contains the global attractor of (3.2) for all values of the delay. See Figure 2 B, where the horizontal lines represent $\bar{\alpha}$ and $\bar{\beta}$.

When (L) does not hold, we still can use Theorem 8. For example, for $\mu = 1$ the positive equilibrium $K = 1$ loses its asymptotic stability for $\tau > 0.188$. One can check that condition (L'_τ) in Theorem 8 holds for $\tau < \tau_* = 0.195$, so for $\tau \in (0, \tau_*)$ every solution of (3.2) enters the domain where f' is negative, while
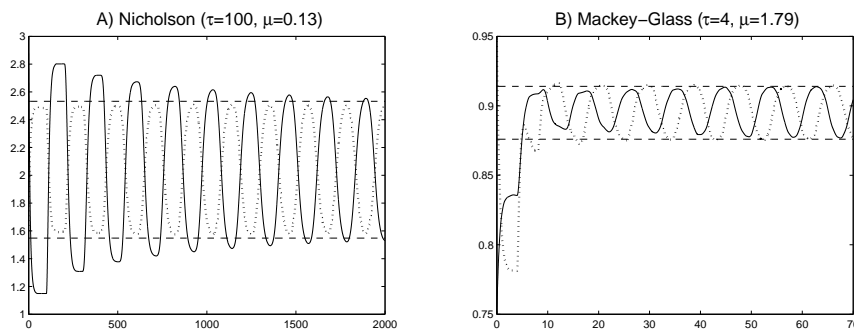

FIGURE 2. Nicholson's blowflies equation and Mackey-Glass equation when the condition (L) holds
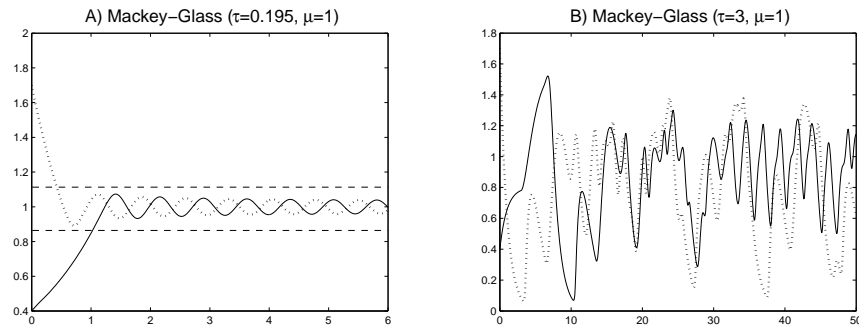
FIGURE 3. Illustration of Theorem 8: Mackey-Glass equation with different delays

(L_τ) is satisfied only for $\tau < \tau^* = 0.092$. In this situation apparently chaotic behavior can be observed for large enough delays (see Figure 3 B with $\tau = 3$), but Theorem 8 and known results for Eq. (1.1) with decreasing f (see, e.g., [7, 17]) guarantee that complicated behavior is not possible if $\tau < \tau_* = 0.195$, as illustrated in Figure 3 A. Moreover, we get a good bound for the global attractor of (3.2) from Corollary 9. Indeed, in this case the interval $[\alpha, \beta] = [0.00016, 1.639]$ is improved up to $[g_1^2(x_0), g_1(x_0)] = [0.864, 1.113]$. See Figure 3 A, where the horizontal lines indicate $g_1^2(x_0)$ and $g_1(x_0)$.

Remark 6. We think that the analysis made in this paper and in [17] may be generalized to equation

$$x'(t) = -h(x(t)) + f(x(t - \tau)), \tag{3.3}$$

where the linear instantanous decay term in (1.1) is replaced by a nonlinear increasing function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$, and $\lim_{x \to \infty} h(x) > f(x_0)$. For motivations and some results on equation (3.3), see, e.g., [10, Section 4.8], [1, Section 3.3], and [5].

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