BIFURCATION OF PERIODIC DELAY DIFFERENTIAL EQUATIONS AT POINTS OF 1:4 RESONANCE *

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Abstract. The time-periodic scalar delay differential equation $\dot{x}(t) = \gamma f(t, x(t-1))$ is considered, which leads to a resonant bifurcation of the equilibrium at critical values of the parameter. Using Floquet theory, spectral projection and center manifold reduction, we give conditions for the stability properties of the bifurcating invariant curves and four-periodic orbits. The coefficients of the third order normal form are derived explicitly. We show that the 1:4 resonance has no effect on equations of the form $\dot{z}(t) = -\gamma r(t)g(x(t-1))$.

Key Words. bifurcation of maps, periodic delay equation, Floquet multipliers, spectral projection, center manifold, projection method, 1:4 resonance

AMS(MOS) subject classification. 34K18, 37M20

1. Introduction. The generic bifurcation of planar discrete dynamical systems is the Neimark-Sacker bifurcation, where a complex conjugate pair of multipliers crosses the unit circle at critical values of the parameter and an invariant curve bifurcates from the equilibrium. These results can be extended to higher dimensional systems by center manifold theorems and projection methods. The case, when the critical multipliers are fourth roots of unity, is called 1:4 strong resonance. At strong resonances the Neimark-Sacker bifurcation theorem is not valid any more and we need further studies. The modern theory of strong resonances is due to Arnold ([1],[2]). We can not expect the appearance of an invariant curve in general, and intricate behavior of the system is possible (see e.g. [4],[11]). The most complicated case of resonances is the 1:4. In this paper we use some related results of Iooss ([14]),

Wan ([21]) and Lemaire ([17]). In a previous paper ([18]) the bifurcation of the time-one map of a scalar periodic delay differential equation was studied. For the general case, the bifurcation analysis is performed by using Floquet theory, center manifold reduction and a spectral projection method, but a wide class of delay differential equations leads to a strong 1:4 resonance. The purpose of this paper is to broaden the results to this case.

Consider the non-autonomous scalar delay differential equation

$$(1) \qquad\qquad \dot{x}(t) = \gamma f(t, x(t-1)),$$

where γ is a real parameter, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^4-smooth function satisfying

$$f(t+1, \xi) = f(t, \xi)$$

and

$$f(t, 0) = 0$$

for all $t, \xi \in \mathbb{R}$. Such equations arise very naturally in several applications, i.e. in population dynamics. A nice overview of related models can be found in [19]. The periodicity is due to the periodic fluctuation of the environment. Denote the Banach space of continuous real and complex valued functions on the interval [-1,0] by C and $C_{\mathbb{C}}$, respectively, with the norm

$$||\phi|| = \sup_{-1 \le t \le 0} |\phi(t)|.$$

Every $\phi \in C$ determines a unique continuous function $x^\phi : [-1, \infty) \to \mathbb{R}$, which is differentiable on $(0, \infty)$, satisfies (1) for all $t > 0$ and $x^\phi(t) = \phi(t)$ for all $t \in [-1, 0]$. Such a function x^ϕ is called the solution of (1) with the initial value ϕ. The time-one map $F : C \to C$ is defined by the relations

$$F(\phi) = x_1^\phi, x_t(s) = x(t+s), s \in [-1, 0].$$

The notation F_γ emphasizes the dependence of the time-one map on the parameter. The spectrum $\sigma(U)$ of the monodromy operator U (the derivative of the time-one map F at 0) determines the behavior of solutions close to the equilibrium 0. The monodromy operator is a linear continuous map and with the relation $U(\psi) = U(\text{Re } \psi) + iU(\text{Im } \psi)$ considered as an operator $C_{\mathbb{C}} \to C_{\mathbb{C}}$ and given by $U(\psi) = y_1^\psi$, where y^ψ is the solution of the linear variational equation

$$(2) \qquad \dot{y}(t) = \gamma f_\xi(t, 0)y(t - 1),$$

where $y^\psi|_{[-1,0]} \equiv \psi$. The operator U is compact, therefore all the non-zero points of the spectrum are isolated points and eigenvalues of finite multiplicity with finite dimensional range of the associated eigenprojection $P_\mu : C_\mathbb{C} \to C_\mathbb{C}$, where $\mu \in \sigma(U), \mu \neq 0$. These eigenvalues are called Floquet multipliers. The spectral theory and other properties of different types of delay differential equations were extensively studied in [5] and [12].

In [18], the equation

$$(3) \qquad \dot{x}(t) = \gamma(a(t)x(t) + f(t, x(t - 1)))$$

was studied. Varying γ, Floquet multipliers cross the unit circle and bifurcation of an invariant curve occurs, supposing that the critical Floquet multipliers are not third or fourth roots of unity. For equation (1), the critical eigenvalues are i and $-i$, that is a strong resonance, and the results of [18] are not valid anymore. In the case of strong resonance, in general one can not expect the appearance of the invariant curve (see [1] or [2]). In [21], a condition was given which guarantees the appearance of the invariant curve for 2-dimensional maps even at points of resonance. Independently, a similar result was presented in [17]. Roughly speaking, if there are no bifurcating four-periodic points, the invariant curve occurs. The stability of the four-periodic points was treated in [14]. To apply this to our infinite-dimensional system, we use center manifold reduction.

The classical process of computing the dynamical system restricted to the center manifold using bilinear forms for delay differential equations (see [12] for the theory and [13] for applications) can not be applied directly to periodic equations. Faria ([6] and [7]) presented the method of normal forms for periodic functional differential equations with autonomous linear part. We established a spectral projection method in [18] for periodic scalar equations. The spectral projection is represented by a Riesz-Dunford integral. The resolvent of the monodromy operator of a periodic delay differential equation and corresponding spectral projections were calculated in the paper of Frasson and Verduyn Lunel ([9, Section 6.2.]) in a more general setting. Certain computations done in [18] for equation (3), can be used for equation (1), simply taking $a(t) \equiv 0$. We remark that these arguments work only if the period and the delay are the same. If the delay is not a multiple of the period, then we can not compute the Floquet multipliers by the characteristic equation. Some information can be obtained on the Floquet multipliers in a similar

problem in [20], there the period is three and the delay is one. The most difficult case, when the delay is incommensurable with the period, there are no results in this direction.

The paper is organized as follows. In *Section 2* we summarize some previous results, follow by the general theory ([5],[12]) or obtained in [18]. *Section 3* is devoted to the bifurcation analysis of strong resonance. We give an explicit condition in terms of f and its partial derivatives to ensure the bifurcation of an invariant curve or four-periodic points, and determine the direction of the appearance and the stability properties. We apply our results to equations with periodic coefficient in *Section 4*, showing that the resonance does not cause any "anomalies" for this class of equations. In *Section 5* we illustrate the results on the example of the celebrated Wright equation with periodic coefficient.

2. Preliminary results. A non-zero point μ of the spectrum of the monodromy operator U is called a Floquet multiplier of equation (2) and any λ for which $\mu = e^\lambda$ is called a Floquet exponent of equation (2). By the Floquet theory ([12, p. 237]), $\mu = e^\lambda$ is a Floquet multiplier of equation (2) if and only if there is a nonzero solution of equation (2) of the form $y(t) = p(t)e^{\lambda t}$, where $p(t+1) = p(t)$. Substituting this solution into equation (2), one can easily deduce that the Floquet exponents are the zeros of the characteristic function

(4) $$h(\lambda) = \lambda - \gamma\beta e^{-\lambda},$$

where

$$\beta = \int_{-1}^{0} f_\xi(t,0)dt.$$

We assume that $\beta \neq 0$. The eigenfunctions have the form

$$\chi_\mu(t) : [-1,0] \ni t \mapsto e^{\gamma e^{-\lambda} \int_{-1}^{t} f_\xi(s,0)ds} \in \mathbb{C}.$$

For any root of the characteristic equation $h(\lambda) = 0$, the corresponding $\chi_\mu(t)$ defines a Floquet solution of equation (2), hence the Floquet exponents coincide with the roots of the characteristic function .

Let

$$\Delta(z) = z - e^{\frac{\gamma\beta}{z}}.$$

The equation $\Delta(z) = 0$ is equivalent to the characteristic equation. Any complex number $\mu = e^\lambda$ is a root of $\Delta(z)$ if and only if λ is a Floquet exponent. Applying Theorem 3.1. of [12, p. 247] to equation (1), one finds that the Floquet multipliers consist of the roots of $\Delta(z)$ and the algebraic multiplicity of an eigenvalue μ equals to the order of μ as a zero of $\Delta(z)$. When this number is 1, we call μ a simple eigenvalue. According to the Riesz-Schauder Theorem, if $U : C_\mathbb{C} \to C_\mathbb{C}$ is a compact operator with a simple eigenvalue μ, then there are two closed subspaces E_μ and Q_μ such that E_μ is one-dimensional, $E_\mu \oplus Q_\mu = C_\mathbb{C}$, furthermore the relations $U(E_\mu) \subset E_\mu$ and $U(Q_\mu) \subset Q_\mu, \sigma(U|E_\mu) = \{\mu\}$ and $\sigma(U|Q_\mu) = \sigma(U)\backslash\{\mu\}$ hold. The spectral projection P_μ onto E_μ along Q_μ can be represented by the Riesz-Dunford integral

$$P_\mu = \frac{1}{2\pi i} \int_{\Gamma_\mu} (zI - U)^{-1}dz = \operatorname*{Res}_{z=\mu} (zI - U)^{-1},$$

where Γ_μ is a small circle around μ such that μ is the only singularity of $(zI - U)^{-1}$ inside Γ_μ.

For simplicity, let $b(t) = \gamma f_\xi(t, 0)$ and $B(t) = \int_{-1}^{t} b(s)ds$. With this notation the linearized equation takes the form

$$\dot{y}(t) = b(t)y(t - 1),$$

$\beta = \frac{1}{\gamma} \int_{-1}^{0} b(t)dt$. By the variation-of-constants formula for ordinary differential equations we find the following representation of the time-one map

$$(5) \qquad F(\phi)(t) = \phi(0) + \int_{-1}^{t} \gamma f(s, \phi(s))ds, \quad t \in [-1, 0],$$

which implies for the monodromy operator

$$(6) \qquad U(\phi)(t) = \phi(0) + \int_{-1}^{t} b(s)\phi(s)ds, \quad t \in [-1, 0].$$

We need the derivatives of the operator F up to order three, evaluated at 0. Let $V = D^2F(0)$ and $W = D^3F(0)$. V and W are n-linear operators with $n = 2$ and $n = 3$, respectively. By the representation (5), one has

$$V(\phi_1, \phi_2)(t) = \int_{-1}^{t} \gamma f_{\xi\xi}(s, 0)\phi_1(s)\phi_2(s)ds, \quad t \in [-1, 0],$$

and

$$W(\phi_1, \phi_2, \phi_3)(t) = \int_{-1}^{t} \gamma f_{\xi\xi\xi}(s, 0)\phi_1(s)\phi_2(s)\phi_3(s)ds, \quad t \in [-1, 0].$$

The following statements are special cases of Lemma 4 and Theorem 2 of [18], setting $a(t) \equiv 0$.

PROPOSITION 1. *([18]) The resolvent of the monodromy operator can be expressed as*

$$(7) \qquad (zI - U)^{-1}(\psi)(t) = e^{\int_{-1}^{t} \frac{b(u)}{z} du} H(t), \quad t \in [-1, 0],$$

where

$$H(t) = \left(\frac{1}{z}\psi(0) + e^{\int_{-1}^{0} \frac{b(u)}{z} du} \int_{-1}^{0} \frac{1}{z^2} e^{-\int_{-1}^{s} \frac{b(u)}{z} du} b(s)\psi(s)ds\right) \cdot$$
$$\cdot \left(\left(z - e^{\int_{-1}^{0} \frac{b(u)}{z} du}\right)^{-1} + \frac{1}{z}e^{-\int_{-1}^{t} \frac{b(u)}{z} du}\psi(t) + \int_{-1}^{t} \frac{1}{z^2} e^{-\int_{-1}^{s} \frac{b(u)}{z} du} b(s)\psi(s)ds\right).$$

The spectral projection operator, corresponding to a simple eigenvalue μ, has the representation

$$P_\mu(\psi) = \chi_\mu R_\mu(\psi),$$

where

$$(8) \qquad R_\mu(\psi) = \left(\frac{1}{\mu + \gamma\beta}\right)\left(\psi(0) + \int_{-1}^{0} \frac{b(s)\psi(s)}{\chi_\mu(s)} ds\right).$$

Notice that $R_\mu(\chi_\mu) = 1$. Consider the decomposition

$$C = T^c \oplus T^{su},$$

where $T^c = \operatorname{Re} E_\mu \oplus \operatorname{Im} E_\mu$ is the critical 2-dimensional realified center eigenspace corresponding to μ and spanned by $\{\operatorname{Re} \chi_\mu, \operatorname{Im} \chi_\mu\}$, moreover $T^{su} = \operatorname{Re} Q_\mu \oplus \operatorname{Im} Q_\mu$ is the 2-codimensional realified stable-unstable subspace corresponding to the other part of $\sigma(U)$. The idea of the projection method is that we introduce new variables x, y and use them as coordinates on these subspaces. Suppose we have a map

$$\tilde{x} = A_1(x) + g(x, y),$$

$$\tilde{y} = A_2(y) + h(x, y),$$

where A_1 and A_2 are linear maps on the corresponding subspaces and

$$g(0,0) = 0, \quad Dg(0,0) = 0,$$

$$h(0,0) = 0, \quad Dh(0,0) = 0.$$

For $y = M(x)$ we have

$$\tilde{x} = A_1(x) + g(x, M(x))$$

$$\tilde{y} = A_2(M(x)) + h(x, M(x)).$$

If $M(x)$ denotes the center manifold then by the invariance $\tilde{y} = M(\tilde{x})$, and thus

(9) $$M(A_1(x) + g(x, M(x))) = A_2(M(x)) + h(x, M(x)).$$

The coefficients of the Taylor-expansion of $M(x)$ can be calculated by this formula. For details and examples we refer to [16] and [23]. The computations in the infinite dimensional case can be found in [18]. Represent the Taylor-expansion of F in the form

$$F(\phi) = U(\phi) + \frac{1}{2}V(\phi, \phi) + \frac{1}{6}W(\phi, \phi, \phi) + O(||\phi||^4).$$

Let $Z(\phi) = F(\phi) - U(\phi)$ be the nonlinear part of F. Now decompose $\phi \in C$ as

$$\phi = z\chi_\mu + \bar{z}\bar{\chi}_\mu + \psi,$$

where $z = R_\mu(\phi) \in \mathbb{C}$, $z\chi_\mu + \bar{z}\bar{\chi}_\mu \in T^c$ and $\psi \in T^{su}$. The complex variable z is a coordinate on the 2-dimensional real eigenspace T^c and the function ψ is a variable in T^{su}. The subspaces T^c and T^{su} are invariant under U. For any real ϕ, $\phi \in T^{su}$ if and only if $P_\mu(\phi) = 0$. $U(\chi_\mu) = \mu\chi_\mu$ implies $U(\bar{\chi}_\mu) = \bar{\mu}\bar{\chi}_\mu$, $\overline{R_\mu} = R_{\bar{\mu}}$.

PROPOSITION 2. ([18]) *The restricted map can be written as*

(10) $$\tilde{z} = \mu z + \frac{1}{2}\rho_{20}z^2 + \rho_{11}z\bar{z} + \frac{1}{2}\rho_{02}\bar{z}^2 + \frac{1}{2}\rho_{21}z^2\bar{z} + \frac{1}{6}\rho_{03}\bar{z}^3 + ...,$$

where

$$(11) \quad \rho_{20} = R_\mu(V(\chi_\mu, \chi_\mu))$$
$$\rho_{11} = R_\mu(V(\chi_\mu, \bar\chi_\mu))$$
$$\rho_{02} = R_\mu(V(\bar\chi_\mu, \bar\chi_\mu))$$
$$\rho_{21} = R_\mu(W(\chi_\mu, \chi_\mu, \bar\chi_\mu)) + 2R_\mu(V(\chi_\mu, (1-U)^{-1}V(\chi_\mu, \bar\chi_\mu))) +$$
$$+ R_\mu(V(\bar\chi_\mu, (\mu^2 - U)^{-1}V(\chi_\mu, \chi_\mu))) +$$
$$+ \frac{\frac{1}{\mu}(1-2\mu)}{1-\mu} R_\mu(V(\chi_\mu, \chi_\mu))R_\mu(V(\chi_\mu, \bar\chi_\mu)) -$$
$$- \frac{2}{1-\frac{1}{\mu}}|R_\mu(V(\chi_\mu, \bar\chi_\mu))|^2 - \frac{\mu}{\mu^3 - 1}|R_\mu(V(\bar\chi_\mu, \bar\chi_\mu))|^2,$$
$$\rho_{03} = R_\mu(W(\bar\chi_\mu, \bar\chi_\mu, \bar\chi_\mu)) + 3R_\mu\Big(V(\bar\chi_\mu, (\mu^{-2}I - U)^{-1} \cdot$$
$$\cdot (V(\bar\chi_\mu, \bar\chi_\mu - R_\mu(V(\bar\chi_\mu, \bar\chi_\mu))\chi_\mu - R_{\bar\mu}(V(\bar\chi_\mu, \bar\chi_\mu))\bar\chi_\mu)\Big).$$

The coefficients $\rho_{20}, \rho_{11}, \rho_{02}, \rho_{21}$ are computed in [18, *Section 5*]. In the non-resonant case ρ_{03} is not needed, but can be obtained completely analogously, hence the computation is omitted here.

3. Bifurcation of the Time-One Map at Points of Resonance. Two conditions are formulated in the Neimark-Sacker bifurcation theorem: the transversality condition, and the non-resonance condition, viz. $\frac{\partial \mu(\gamma)}{\partial \gamma}|_{\gamma_j} \neq 0$ and $\mu_j^3 \neq 1, \mu_j^4 \neq 1$, where γ_j is a critical parameter value and μ_j is a corresponding critical multiplier. The following two lemmas show that the transversality condition is always fulfilled for equation (1), while $\mu_j^4 = 1$. This situation is a 1:4 strong resonance.

LEMMA 1. *The critical values of (2) are*

$$\gamma_j = \frac{-\frac{\pi}{2} + 2j\pi}{\beta}, \quad j \in \mathbb{Z},$$

and the corresponding critical Floquet multipliers are $\mu_j = e^{\lambda_j} = i$ and $\bar\mu_j = e^{\bar\lambda_j} = -i$. These Floquet-multipliers are simple eigenvalues and the critical eigenfunctions are

$$\chi_{\pm i}(t) : [-1, 0] \ni t \mapsto e^{\mp iB(t)} \in \mathbb{C}.$$

Proof One can check easily that i and $-i$ can not be a double root of $\Delta(z)$, thus if i or $-i$ is a Floquet-multiplier, then it is always a simple eigenvalue. Suppose that $\lambda = i\theta$ is a critical Floquet-exponent, then by the real part of (4) we have $\cos(\theta) = 0$, hence $\theta = \frac{\pi}{2} + 2k\pi$ or $\theta = -\frac{\pi}{2} + 2k\pi$, where $k \in \mathbb{Z}$. Taking into account the imaginary part of (4), both options lead to the statements of the lemma by simple calculations.

□

Introduce the notation $B = B(0) = \gamma\beta$.

LEMMA 2.

$$\frac{\partial \mu(\gamma)}{\partial \gamma}\Big|_{\gamma_j} = \frac{\beta}{1 + \lambda(\gamma_j)} = \frac{\beta}{1 + B^2}(1 + iB)$$

Proof By the characteristic equation and the Implicit Function Theorem $\mu(\gamma) = e^{\frac{\gamma\beta}{\mu(\gamma)}}$ is defined in a neighborhood of γ_j. Differentiating with respect to γ gives

$$\mu'(\gamma) = e^{\frac{\gamma\beta}{\mu(\gamma)}}\left(\frac{\beta\mu(\gamma) - \beta\gamma\mu'(\gamma)}{\mu^2(\gamma)}\right) = \beta - \lambda(\gamma)\mu'(\gamma).$$

This yields $\mu'(\gamma) = \frac{\beta}{1 + \lambda(\gamma)}$. Setting $\gamma = \gamma_j$ one has $\lambda = -i\gamma_j\beta = -iB$ and the lemma is proved.

□

The next proposition is the Poincaré normal form map for 1:4 resonance.

PROPOSITION 3. ([16, p. 436]) *Suppose that we have a map* $g = g(\gamma) : \mathbb{C} \mapsto \mathbb{C}$, *depending on the parameter* $\gamma \in \mathbb{R}$, *and* g *has the form*

$$(12) \qquad g(z) = \mu z + \frac{\rho_{20}}{2}z^2 + \rho_{11}z\bar{z} + \frac{\rho_{02}}{2}\bar{z}^2 + \frac{\rho_{30}}{6}z^3$$
$$+ \frac{\rho_{21}}{2}z^2\bar{z} + \frac{\rho_{12}}{2}z\bar{z}^2 + \frac{\rho_{03}}{6}\bar{z}^3 + \mathcal{O}(|z|^4),$$

where $\mu = \mu(\gamma)$ *and* $\rho_{kl} = \rho_{kl}(\gamma)$ *depends on the parameter smoothly and* $\mu(\gamma_j) = i$ *for some critical value* $\gamma = \gamma_j$. *Then by a coordinate transformation depending smoothly on the parameter, in the critical case the transformed map takes the form*

$$\tilde{g}(w) = iw + c_1 w^2\bar{w} + c_2\bar{w}^3 + \mathcal{O}(|w|^4),$$

where

$$c_1 = \frac{1 + 3i}{4}\rho_{20}\rho_{11} + \frac{1 - i}{2}\rho_{11}\bar{\rho}_{11} + \frac{-1 - i}{4}\rho_{02}\bar{\rho}_{02} + \frac{\rho_{21}}{2}$$

and

$$c_2 = \frac{i-1}{4}\rho_{11}\rho_{02} + \frac{-i-1}{4}\rho_{02}\bar{\rho}_{20} + \frac{\rho_{03}}{6}.$$

Note that similar, but different formulas are presented for c_2 in [14, Chapter IV] and [21]. These formulas are miscalculated and false. One can check directly by a straightforward, but rather elaborative computation that the formula of [16, p. 436], presented in Proposition 3 is the correct one. However, in the literature the wrong formula of [14] is spreading, see for example the recent papers [10] and [22], where applications of the resonant normal form to mechanical systems are presented. Since the applied formula is not correct, the obtained results may not be correct as well.

Define $a_1 = \frac{c_1}{i}$, $a_2 = \frac{c_2}{i}$ and $d = \frac{\partial|\mu(\gamma)|}{\partial\gamma}|_{\gamma=\gamma_j}$.

PROPOSITION 4 (RESONANT BIFURCATION THEOREM,[14],[21]). *Suppose that we have a map $g(z) : \mathbb{C} \mapsto \mathbb{C}$ of the form (12), depending smoothly on the parameter γ, satisfying $d \neq 0$ and $\mu(\gamma_j) = i$.*

If $|\mathrm{Im}\left(\frac{a_1}{d}\right)| > |\frac{a_2}{d}|$, then a unique invariant curve bifurcates (and no periodic points of order 4) from the equilibrium 0 as the parameter γ passes through γ_j. The cases $\mathrm{Re}\, a_1 < 0$ and $\mathrm{Re}\, a_1 > 0$ are called supercritical and subcritical bifurcations. In the supercritical case a stable invariant curve appears for $\gamma > \gamma_j$, while in the subcritical case an unstable invariant curve disappears when γ increases through γ_j.

If $|\mathrm{Im}\left(\frac{a_1}{d}\right)| < |\frac{a_2}{d}|$, then two families of periodic points of order 4 bifurcate (and no invariant curve). Moreover, if $|a_1| > |a_2|$, the two families bifurcate on the same side and at least one of them is unstable. If $|a_1| < |a_2|$, then the two families bifurcate on opposite sides and both of them are unstable.

LEMMA 3. *For the restricted map of the time-one map corresponding to equation (1) we have*

$$\begin{aligned} a_1 &= \frac{3-i}{4}\rho_{20}\rho_{11} - \frac{1+i}{2}|\rho_{11}|^2 - \frac{1-i}{4}|\rho_{02}|^2 - \frac{i}{2}\rho_{21} = \\ &= -\frac{i}{2}\Big[R_i(W(\chi_i,\chi_i,\bar{\chi}_i)) + 2R_i(V(\chi_i,(1-U)^{-1}V(\chi_i,\bar{\chi}_i))) + \\ &\quad + R_i(V(\bar{\chi}_i,(i^2-U)^{-1}V(\chi_i,\chi_i))) \Big], \end{aligned}$$

and

$$a_2 = \frac{-1+i}{4}\bar{\rho}_{20}\rho_{02} + \frac{1+i}{4}\rho_{11}\rho_{02} - \frac{i}{6}\rho_{03} =$$

$$= -\frac{i}{6}\left[R_i(W(\bar{\chi}_i, \bar{\chi}_i, \bar{\chi}_i)) + 3R_i\Big(V(\bar{\chi}_i, (i^{-2}I - U)^{-1}(V(\bar{\chi}_i, \bar{\chi}_i)))\Big)\right].$$

Proof Apply Proposition 2 and Proposition 3 with $\mu = i$. We obtain the lemma by a simple calculation.

$\Box$

Let us define

(13) $$\delta = |\text{Im }(a_1) - B\text{Re }(a_1)| - |a_2|\sqrt{1 + B^2}.$$

Remark that δ depends on the parameter. Some additional computation yields

$$|\text{Im }(\frac{a_1}{d})| > |\frac{a_2}{d}| \Leftrightarrow |\text{Im }(a_1(1 - iB)| > |a_2(1 - iB)|,$$

that is $\delta > 0$.

We apply the center manifold theorem for maps in Banach-spaces to the time-one map F. See [15] for the existence and [8] for the smoothness result. Summarizing all the previous lemmas and propositions of *Section 2* and *Section 3*, combining with the center manifold theorem and the reduction principle (for details see [3],[16] and [23]), we obtain our main theorem.

THEOREM 1. *The family of time-one maps F_γ, corresponding to equation (1), has at the critical value $\gamma = \gamma_j$ the fixed point $\phi = 0$ with exactly two simple Floquet-multipliers $\mu_j = i$ and $\bar{\mu}_j = -i$ on the unit circle. This is a 1:4 strong resonance. The transversality condition is fulfilled. There is a neighborhood of 0 in which a unique invariant curve (and no 4-periodic points) bifurcates from 0, providing that $\delta > 0$. The direction of the bifurcation is determined by the sign of $\text{Re }(a_1)$. If $\delta < 0$, then two families of 4-periodic points (and no invariant curve) bifurcate from the equilibrium in a neighborhood of 0. Furthermore, if $|a_1| > |a_2|$, the two families bifurcate on the same side and at least one of them is unstable. If $|a_1| < |a_2|$, then the two families bifurcate on the opposite side and both of them are unstable.*

The conditions given in the theorem can be checked for any given equation, we can compute γ_j, a_1, a_2 and B explicitly by terms of $f(t, \xi)$ and its partial derivatives.

4. Equations with Periodic Coefficient. In this section we consider the equation

(14) $$\dot{z}(t) = -\gamma r(t)g(z(t-1)),$$

where γ is a real parameter, $r : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $r(t+1) = r(t)$ for all $t \in \mathbb{R}$, $g(\xi)$ is a C^4-smooth function satisfying $g(0) = 0$. Without loss of generality we may suppose that

$$g(\xi) = \xi + \frac{S}{2}\xi^2 + \frac{T}{6}\xi^3 + \mathcal{O}(\xi^4),$$

where $S, T \in \mathbb{R}$. With our previous notations we have

$$f(t, \xi) = -r(t)g(\xi),$$

$$f_\xi(t, 0) = -r(t),$$

$$f_{\xi\xi}(t, 0) = -Sr(t),$$

$$f_{\xi\xi\xi}(t, 0) = -Tr(t),$$

and

$$b(t) = -\gamma r(t).$$

We show that this equation behaves at the bifurcation points as a nonresonant equation: an invariant curve bifurcates and no 4-periodic points from the equilibrium 0. The following lemma is used many times during the detailed computations.

LEMMA 4. *Let* $B(t) = \int_{-1}^{t} b(s)ds$. *Then*

$$\int_{-1}^{t} e^{B(s)}b(s)ds = e^{B(t)} - 1,$$

$$\int_{-1}^{t} e^{B(s)}b(s)B(s)ds = e^{B(t)}B(t) - e^{B(t)} + 1.$$

Proof The first identity is obvious, the second can be deduced from the first by a partial integration.

$\square$

THEOREM 2. *For any family of time-one maps corresponding to equation (14), if* $T \neq \frac{11S^2}{5}$ *then a unique invariant curve bifurcates from the equilibrium*

0 *as the parameter γ passes through γ_j. The bifurcation is supercritical if $T < S^2(\frac{11B+2}{5B})$ and subcritical if $T > S^2(\frac{11B+2}{5B})$.*

Proof Let us fix $\gamma = \gamma_j$ to be a critical parameter value. Using Lemma 4, we have

$$V(\chi_i, \chi_i)(t) = \int_{-1}^{t} Sb(s)e^{-2iB(s)}ds = \frac{S}{-2i}(e^{-2iB(t)} - 1) = \frac{iS}{2}(e^{-2iB(t)} - 1),$$

$$V(\chi_i, \bar{\chi}_i)(t) = SB(t),$$

$$V(\bar{\chi}_i, \bar{\chi}_i)(t) = \int_{-1}^{t} Sb(s)e^{2iB(s)}ds = \frac{S}{2i}(e^{2iB(t)} - 1) = \frac{-iS}{2}(e^{2iB(t)} - 1),$$

$$W(\chi_i, \chi_i, \bar{\chi}_i) = \int_{-1}^{t} Tb(s)e^{-iB(s)}ds = iT(e^{-iB(t)} - 1),$$

and

$$W(\bar{\chi}_i, \bar{\chi}_i, \bar{\chi}_i) = \int_{-1}^{t} Tb(s)e^{3iB(s)}ds = \frac{-iT}{3}(e^{3iB(t)} - 1).$$

Notice that $B = -\frac{\pi}{2} + 2j\pi$, hence $e^{iB} = \cos B + i\sin B = -i$, and $e^{imB} = (-i)^m$ for all $m \in Z$. Taking into account this fact, one obtains

$$(15) \quad R_i(e^{miB(t)}) = \left(\frac{1}{i+B}\right)\left(e^{miB} + \int_{-1}^{0} b(s)e^{(m+1)iB(s)}ds\right) =$$

$$= \left(\frac{1}{i+B}\right)\left(e^{miB} + \frac{1}{(m+1)i}(e^{(m+1)iB} - 1)\right) =$$

$$= \left(\frac{1}{i+B}\right)\left((-i)^m - i\frac{(-i)^{m+1} - 1}{m+1}\right) =$$

$$= \frac{m(-i)^m + i}{(i+B)(m+1)}$$

for any $m \neq -1$. Observe that $R_i(e^{3iB(t)}) = \frac{i}{i+B} = R_i(1)$. If $m = -1$, we get the eigenfunction $e^{-iB(t)}$, and as in general,

$$R_i(e^{-iB(t)}) = R_i(\chi_i(t)) =$$

$$= \left(\frac{1}{i+B}\right)\left(e^{-iB} + \int_{-1}^{0} b(s)ds\right) = \frac{i+B}{i+B} = 1.$$

Now let us evaluate the resolvent by Proposition 1 and Lemma 4:

$$
\begin{aligned}
(1-U)^{-1}V(\chi_i,\bar{\chi}_i) &= e^{B(t)}\Big((SB+e^B\int_{-1}^0 e^{-B(s)}b(s)SB(s)ds)(1-e^B)^{-1}\\
&\quad +e^{-B(t)}SB(t)+\int_{-1}^t e^{-B(s)}b(s)SB(s)ds\Big)\\
&= Se^{B(t)}\Big((B+e^B(-e^{-B}B-e^{-B}+1))(1-e^B)^{-1}\\
&\quad +e^{-B(t)}B(t)-e^{-B(t)}B(t)-e^{-B(t)}+1\Big)\\
&= Se^{B(t)}(-1-e^{-B(t)}+1)=-S.
\end{aligned}
$$

Referring to Lemma 3 and $(i^2-U)^{-1}=(i^{-2}-U)^{-1}=(-1-U)^{-1}$, we still need

$$(16) \qquad (-1-U)^{-1}(e^{miB(t)})=e^{-B(t)}H_1(t),$$

where

$$
\begin{aligned}
H_1(t) &= (-1-U)^{-1}(e^{miB(t)})=e^{-B(t)}\Big((-e^{miB}+e^{-B}\int_{-1}^0 b(s)e^{(mi+1)B(s)}ds)\cdot\\
&\quad \cdot(-1-e^{-B})^{-1}-e^{(1+mi)B(t)}+\int_{-1}^t b(s)e^{(1+mi)B(s)}ds\Big)=\\
&= e^{-B(t)}\Big((-e^{miB}+e^{-B}\tfrac{e^{(mi+1)B}-1}{mi+1})(-1-e^{-B})^{-1}-e^{(1+mi)B(t)}+\tfrac{e^{(mi+1)B(t)}-1}{mi+1}\Big)=\\
&= e^{-B(t)}\tfrac{(-i)^m mi-1}{(1+e^{-B})(mi+1)}-e^{miB(t)}\tfrac{mi}{mi+1}.
\end{aligned}
$$

Particularly,

$$
\begin{aligned}
(-1-U)^{-1}(e^{2iB(t)}-1) &=\\
= e^{-B(t)}\tfrac{(-i)^2 2i-1}{(1+e^{-B})(2i+1)}-e^{2iB(t)}\tfrac{2i}{2i+1}-e^{-B(t)}\tfrac{-1}{1+e^{-B}} &=\\
= -e^{2iB(t)}\tfrac{2i}{2i+1}. &
\end{aligned}
$$

Similarly, one finds

$$(17) \qquad (-1-U)^{-1}(e^{-2iB(t)}-1)=e^{-2iB(t)}\frac{2i}{-2i+1}.$$

Now we are ready to compute the coefficients of the normal form given in Lemma 3, namely

$$(18) \quad a_2 = -\frac{i}{6}R_i\Big[W(\bar{\chi}_i, \bar{\chi}_i, \bar{\chi}_i) + 3V(\bar{\chi}_i, (-1-U)^{-1}(V(\bar{\chi}_i, \bar{\chi}_i)))\Big] =$$

$$= -\frac{i}{6}R_i\Big[\frac{-iT}{3}(e^{3iB(t)} - 1) - \frac{3iS}{2}V(\bar{\chi}_i, (-1-U)^{-1}(e^{2iB(t)} - 1)\Big]$$

$$= -\frac{T}{18}R_i\Big[(e^{3iB(t)} - 1)\Big] - \frac{S}{4}R_i\Big[V(\bar{\chi}_i, -e^{2iB(t)}\frac{2i}{2i+1}\Big] = 0,$$

where we used the linearity of R_i and the identity

$$\frac{1}{S}R_i(V(\bar{\chi}_i, e^{2iB(t)})) = \frac{1}{T}R_i(W(\bar{\chi}_i, \bar{\chi}_i\bar{\chi}_i) = \frac{1}{3i}R_i(e^{3iB(t)} - 1) = 0.$$

We use (15) and (16) to conclude

$$(19) \quad a_1 = -\frac{i}{2}R_i\Big[W(\chi_i, \chi_i, \bar{\chi}_i) + 2V(\chi_i, -S) +$$

$$+ V(\bar{\chi}_i, (-1-U)^{-1}V(\chi_i, \chi_i))\Big]$$

$$= -\frac{i}{2}R_i\Big[Ti(e^{-iB(t)} - 1) + 2(-i)S^2(e^{-iB(t)} - 1)\Big]$$

$$-\frac{i}{2}R_i\Big[V(\bar{\chi}_i, (-1-U)^{-1}\Big(\frac{iS}{2}(e^{-2iB(t)} - 1))\Big)\Big]$$

$$= \frac{T - 2S^2}{2}R_i[e^{-iB(t)} - 1] + \frac{S}{4}R_i\Big[V(\bar{\chi}_i, e^{-2iB(t)}\frac{2i}{-2i+1}\Big]$$

$$= \frac{(T - 2S^2)B}{2(i+B)} + \frac{Si}{2 - 4i}R_i[V(\bar{\chi}_i, e^{-2iB(t)})]$$

$$= \frac{(T - 2S^2)B}{2(i+B)} + \frac{Si}{2 - 4i}R_i[\frac{S}{-i}(e^{-iB(t)} - 1)]$$

$$= \frac{(T - 2S^2)B}{2(i+B)} - \frac{S^2B}{(2 - 4i)(i+B)} = \frac{B}{2(i+B)}(T - S^2\frac{11 + 2i}{5}).$$

Applying (18),(19) and $\frac{1}{i+B} = \frac{B-i}{1+B^2}$ to (13), we find

$$(20) \qquad\qquad 2(1 + B^2)\text{Re}\,(a_1) = TB^2 - BS^2\frac{11B + 2}{5},$$

$$2(1 + B^2)\text{Im}\,(a_1) = -TB + BS^2\frac{11 - 2B}{5}.$$

The sign of Re (a_1) determines the direction of the bifurcation, as formulated in the theorem, which is the same as the sign of $T - S^2(\frac{11B+2}{5B})$. Substituting the previous two formulas into (13), we deduce

$$\begin{aligned} \delta &= \frac{1}{2(1+B^2)}|B|\cdot|-T-TB^2+\frac{S^2}{5}(11-2B+2B+11B^2)| \\ &= \frac{|B|}{2}\cdot|T-S^2\frac{11}{5}|. \end{aligned}$$

Since $B \neq 0$, the condition $T \neq \frac{11S^2}{5}$ guarantees that $\delta > 0$ and Theorem 2 is proved.

□

5. An example. The classical form of the celebrated Wright-Hutchinson equation (or delayed logistic equation) is

$$\dot{y}(t) = -\alpha y(t-1)(1+y(t)).$$

The change of variable $z(t) = \ln(1+y(t))$ transforms Wright's equation into

$$\dot{z}(t) = -\alpha(e^{z(t-1)}-1).$$

Since the pioneer works of Wright ([24]), a huge amount of papers concerned with the dynamical properties of this equation and its generalizations. Here we consider this equation with a periodic coefficient:

$$(21) \qquad\qquad \dot{z}(t) = -\alpha r(t)(e^{z(t-1)}-1),$$

where $\alpha > 0$ and $r(t)$ is a continuous function satisfying $r(t+1) = r(t)$ for all $t \in \mathbb{R}$. Without loss of generality we may suppose $\int_{-1}^{0} r(s)ds = 1$. We have $g(\xi) = \xi + \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \mathcal{O}(\xi^4)$, that is $S = 1$ and $T = 1$. The next theorem is a direct application of Theorem 2 and Lemma 1.

THEOREM 3. *The family of time-one maps corresponding to equation (21), undergoes a supercritical bifurcation and a unique invariant curve bifurcates from the equilibrium 0 as the parameter α passes through $\frac{\pi}{2}$.*

Remark that taking $r(t) \equiv 1$ we get back the autonomous case. For the autonomous Wright equation it is well known that at the value $\alpha = \frac{\pi}{2}$ a periodic solution emerges from the equilibrium by a supercritical Hopf bifurcation. This is consistent with Theorem 3.

REFERENCES

[1] V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 250, Springer-Verlag, New York, 1988, Translated from the Russian by Joseph Szücs [József M. Szűcs].

[2] V. I. Arnold, V. S. Afrajmovich, Yu. S. Ilyashenko, and L. P. Shilnikov, *Bifurcation theory and catastrophe theory*, Springer-Verlag, Berlin, 1999, Translated from the 1986 Russian original by N. D. Kazarinoff, Reprint of the 1994 English edition from the series Encyclopaedia of Mathematical Sciences [*Dynamical systems. V*, Encyclopaedia Math. Sci., 5, Springer, Berlin, 1994].

[3] J. Carr, *Applications of the centre manifold theory*, Applied Mathematical Sciences, vol. 35, Springer-Verlag, New York, 1981.

[4] Ch.-Q. Cheng, *Hopf bifurcations in nonautonomous systems at points of resonance*, Sci. China Ser. A **33** (1990), no. 2, 206–219.

[5] O. Diekmann, S. A. Van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay equations. Functional-, complex-, and nonlinear analysis*, Applied Mathematical Sciences, vol. 110, Springer-Verlag, New York, 1995.

[6] T. Faria, *Normal forms for periodic retarded functional-differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 1, 21–46.

[7] *Normal forms on centre manifolds for periodic functional-differential equations*, International Conference on Differential Equations (Lisboa, 1995), World Sci. Publishing, River Edge, NJ, 1998, pp. 322–326.

[8] T. Faria, W. Huang, and J. Wu, *Smoothness of center manifolds for maps and formal adjoints for semilinear FDEs in general Banach spaces*, SIAM J. Math. Anal. **34** (2002), no. 1, 173–203.

[9] M. V. S. Frasson and S. M. Verduyn Lunel, *Large time behaviour of linear functional differential equations*, Integral Equations Operator Theory **47** (2003), no. 1, 91–121.

[10] Luo G. and Xie J., *Bifurcation and chaos in a system with impacts*, Phys. D **148** (2001), 183–200.

[11] J.-M. Gambaudo, *Perturbation of a Hopf bifurcation by an external time-periodic forcing*, J. Differential Equations **57** (1985), no. 2, 172–199.

[12] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993.

[13] B. D. Hassard, N. D. Kazarinoff, and Y.-H. Wan, *Theory and applications of Hopf bifurcation*, London Mathematical Society Lecture Note Series, vol. 41, Cambridge University Press, 1981.

[14] G. Iooss, *Bifurcation of maps and applications*, Mathematics Studies, vol. 36, North-Holland, Amsterdam, 1979.

[15] T. Krisztin, H.-O. Walther, and J. Wu, *Shape, smoothness and invariant stratification of an attracting set for delayed monotone positive feedback*, Fields Institute Monographs, vol. 11, Amer. Math. Soc., Providence, RI, 1999.

[16] Yu. A. Kuznetsov, *Elements of applied bifurcation theory*, second ed., Applied Mathematical Sciences, vol. 112, Springer-Verlag, New York, 1998.

[17] F. Lemaire-Body, *Bifurcation de Hopf pour les applications dans un cas résonnant*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), no. 9, A727–A730.

[18] G. Röst, *Neimark-Sacker bifurcation for periodic delay differential equations*, Nonlinear Anal. **60** (2005), no. 6, 1025–1044.

[19] S. Ruan, *Delay differential equations in single species dynamics*, Proceedings of the NATO Advanced Study Institute held in Marrakech, Morocco, 9-21 September 2002 (O. Arino, M.L. Hbid, and E. A. Dads, eds.), NATO Science Series II: Mathematics, Physics and Chemistry, vol. 205, Springer-Verlag, New York, 2005.

[20] H.-O. Walther and A. L. Skubachevskiĭ, *On Floquet multipliers for slowly oscillating periodic solutions of nonlinear functional-differential equations*, Tr. Mosk. Mat. Obs. **64** (2003), 3–53.

[21] Y.-H. Wan, *Bifurcation into invariant tori at points of resonance*, Arch. Rational Mech. Anal. **68** (1978), no. 4, 343–357.

[22] G. Wen, D. Xu, and J.H. Xie, *Controlling Hopf bifurcations of discrete-time systems in resonance*, Chaos Solitons Fractals **23** (2005), 1865–1877.

[23] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Texts in Applied Mathematics, vol. 2, Springer-Verlag, New York, 1990.

[24] E. M. Wright, *A non-linear difference-differential equation*, J. Reine Angew. Math. **194** (1955), 66–87.