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Neimark–Sacker bifurcation for periodic delay differential equations

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Abstract

In this paper we study the delay differential equation

$$\dot{x}(t) = \gamma(a(t)x(t) + f(t, x(t-1))),$$

where γ is a real parameter, the functions $a(t)$, $f(t, \xi)$ are C^4-smooth and periodic in the variable t with period 1. Varying the parameter, eigenvalues of the monodromy operator (the derivative of the time-one map at the equilibrium 0) cross the unit circle and bifurcation of an invariant curve occurs. To detect the critical parameter-values, we use Floquet theory. We give an explicit formula to compute the coefficient that determines the direction of the bifurcation. We extend the center manifold projection method to our infinite-dimensional Banach space using spectral projection represented by a Riesz–Dunford integral. The Neimark–Sacker Bifurcation Theorem implies the appearance of an invariant torus in the space $C \times S^1$. We apply our results to an equation used in neural network theory.

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1. Introduction

Consider the non-autonomous scalar delay differential equation

$$\dot{x}(t) = \gamma(a(t)x(t) + f(t, x(t-1))), \tag{1}$$

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where γ is a real parameter, $a : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are C^4-smooth functions satisfying

$$a(t+1) = a(t),$$

$$f(t+1, \xi) = f(t, \xi)$$

and

$$f(t, 0) = 0,$$

for all $t, \xi \in \mathbb{R}$. Every element ϕ of the Banach space C of continuous real functions on the initial interval $[-1, 0]$ determines a unique continuous function $x^\phi : [-1, \infty) \to \mathbb{R}$, which is differentiable on $(0, \infty)$, satisfies (1) for all $t > 0$ and $x^\phi(t) = \phi(t)$ for all $t \in [-1, 0]$. We call such a function x^ϕ the solution of (1) with the initial value ϕ. The time-one map $F : C \to C$ is defined by the relations

$$F(\phi) = x_1^\phi, \quad x_t(s) = x(t+s), \quad s \in [-1, 0].$$

When we want to emphasize the dependence of the time-one map on the parameter, we write F_γ. Denote the Banach space of continuous complex valued functions on the interval $[-1, 0]$ by $C_{\mathbb{C}}$. The spaces C and $C_{\mathbb{C}}$ are equipped with the norm

$$\|\phi\| = \sup_{-1 \leqslant t \leqslant 0} |\phi(t)|.$$

The behavior of solutions close to the equilibrium 0 is determined by the spectrum $\sigma(U)$ of the monodromy operator U. This is the derivative of the time-one map F at 0. The monodromy operator is a linear continuous map and with the relation $U(\psi) = U(\operatorname{Re}\psi) + iU(\operatorname{Im}\psi)$ considered as an operator $C_{\mathbb{C}} \to C_{\mathbb{C}}$ and given by $U(\psi) = y_1^\psi$, where $y^\psi : [-1, \infty) \to \mathbb{C}$ is the solution of the linear variational equation

$$\dot{y}(t) = \gamma(a(t)y(t) + f_\xi(t, 0)y(t-1)) \tag{2}$$

with $y_0^\psi = \psi \in C_{\mathbb{C}}$. The operator U is compact, therefore all the non-zero points of the spectrum are isolated points and eigenvalues of finite multiplicity with finite dimensional range of the associated eigenprojection $P_\mu : C_{\mathbb{C}} \to C_{\mathbb{C}}$, where $\mu \in \sigma(U)$, $\mu \neq 0$. These eigenvalues are called Floquet multipliers. The spectral theory and other properties of different types of delay differential equations are extensively studied in [3,5] and [14].

Varying γ, the Floquet multipliers cross the unit circle and bifurcation of an invariant curve occurs. In *Section* 2 we analyze how the Floquet multipliers depend on the parameter γ. The conditions for the Neimark–Sacker Bifurcation Theorem (it is also known as Hopf-bifurcation for maps) are checked in *Section* 3. A detailed proof of this theorem can be found in [7], under generalized conditions in [1]. For applications of the Hopf-bifurcation to autonomous functional differential equations, see [6]. In the critical case, all the essential qualitative features of our dynamical system are captured by the center manifold. Unfortunately, the classical process of computing the dynamical system restricted to the center manifold using bilinear forms for delay differential equations [5,6] cannot be applied directly to periodic equations. In [4], the method of normal forms is presented for periodic functional differential equations, but that works only for equations with autonomous linear part. In this paper we use a functional analytic approach to extend the projection method to our Banach space. The computations are analogue to the one in [12], but performed in the infinite dimensional space C. The arguments of this paper work only for the case when the delay is the same as the period. When the delay is a multiple of the period, we are in the same situation again. A different case, when the delay is not a multiple of the period, but commensurable with that. Then the Floquet multipliers can be deduced by the explicit solution of a system of ordinary differential equations, which is not possible in general. In [17], some information was obtained on the Floquet multipliers in a similar problem: a linearization along a periodic solution of an autonomous equation. There the delay is 1, the period of the solution is 3, and under additional conditions a criterion was formulated for the hyperbolicity of the periodic solution. The most difficult case when the delay is incommensurable with the period.

We give an explicit formula for the coefficient that determines the direction of the bifurcation in *Section* 5, which is important for specific applications. To do this, we have to replace the scalar product in $\mathbb{R}^n$ by the spectral projection represented by a Riesz–Dunford integral. The spectral projection is the residuum of the resolvent of U at the corresponding eigenvalue. An explicit expression for the resolvent and the spectral projection operator is computed in *Section* 4, solving an ordinary differential equation with boundary condition. *Section* 6 summarizes our achievements. We show an example in *Section* 7, applying our results to an equation arisen in neural network theory.

2. The characteristic equation and the Floquet multipliers

A non-zero point μ of the spectrum of the monodromy operator U is called a Floquet multiplier of Eq. (2) and any λ for which $\mu = e^{\lambda}$ is called a Floquet exponent of Eq. (2). By the Floquet theory ([5, p. 237]), $\mu = e^{\lambda}$ is a Floquet multiplier of Eq. (2) if and only if there is a nonzero solution of equation (2) of the form

$$y(t) = p(t)e^{\lambda t},$$

where $p(t+1) = p(t)$. Substituting this solution into Eq. (2) and using $p(t-1) = p(t)$ we obtain for $p(t)$ the linear ordinary differential equation

$$\dot{p}(t) = p(t)(\gamma a(t) + \gamma f_{\xi}(t, 0)e^{-\lambda} - \lambda),$$

which has the solution

$$p(t) = p(t_0)e^{\int_{t_0}^{t}[\gamma a(s) + \gamma f_{\xi}(s,0)e^{-\lambda} - \lambda]\,ds}.$$

We can choose t_0 such that $p(t_0) \neq 0$. Setting $t = t_0 + 1$, by the periodicity of $p(t)$ we deduce that the Floquet exponents are the zeros of the characteristic function

$$h(\lambda) = \gamma\alpha + \gamma\beta e^{-\lambda} - \lambda, \tag{3}$$

where

$$\alpha = \int_{t_0}^{t_0+1} a(t)\,\mathrm{d}t = \int_{-1}^{0} a(t)\,\mathrm{d}t,$$

$$\beta = \int_{t_0}^{t_0+1} f_\xi(t,0)\,\mathrm{d}t = \int_{-1}^{0} f_\xi(t,0)\,\mathrm{d}t.$$

The eigenfunctions have the form

$$\chi_\mu(t) : [-1,0] \ni t \mapsto \mathrm{e}^{\int_{-1}^{t}[\gamma a(s)+\gamma f_\xi(s,0)\mathrm{e}^{-\lambda}]\,\mathrm{d}s} \in \mathbb{C}.$$

For any root λ of the characteristic function, the corresponding $\chi_\mu(t)$ defines a Floquet solution of Eq. (2), hence the Floquet exponents coincide with the roots of the characteristic function. Characteristic functions of the same type occur linearizing around a periodic orbit of an autonomous equation. With this object these characteristic functions were widely discussed. A detailed analysis can be found e.g. in [3, Chapter XI]. We recall some basic facts.

Define the intervals I_k for $k = 0, 1, 2, \ldots$ by

$$I_k = ((2k-1)\pi, (2k+1)\pi).$$

Each I_k is divided by the point $2k\pi$ into intervals I_k^- and I_k^+:

$$I_k^- = ((2k-1)\pi, 2k\pi), \quad I_k^+ = (2k\pi, (2k+1)\pi).$$

Let the curves $C_k^\pm$ parametrized by v in the (u, v)-plane be defined as

$$C_k^\pm = \left\{ (u,v) = \left(\frac{v\cos(v)}{\sin(v)}, -\frac{v}{\sin(v)} \right) \middle| \, v \in I_k^\pm \right\}.$$

The curves $C_k^\pm$ are drawn in Fig. 1, lie in the sections $v > |u|$ and $v < -|u|$, do not intersect, are ordered as is shown in Fig. 1 and asymptotic to the lines $v = \pm u$. These curves and the line $L = \{(u, v)|v = -u\}$ divide the plane into regions. Taking $u = \gamma\alpha$ and $v = \gamma\beta$, the number of roots of the characteristic function outside the unit circle is constant in these regions. These numbers are also indicated in Fig. 1.

The conjugate of a root of the characteristic function is also a root. The derivative of $h(\lambda)$ with respect to λ is $h'(\lambda) = -v\mathrm{e}^{-\lambda} - 1$. From this we obtain that λ is a double root of $h(\lambda)$ only when $v = -\mathrm{e}^{u-1}$ and $\lambda = u - 1$, which is real. The curve D in Fig. 1 is given by $v = -\mathrm{e}^{u-1}$ and intersects the curves C_k^+. Double zeros of $h(\lambda)$ occur only on D.

We have the double root zero at the point $(1, -1)$, at the other intersection points $u > 1$ and the double root is positive. We restrict our attention to the case $v > |u|$ or $v < -|u|$, that is $\beta^2 > \alpha^2$, because all the Neimark–Sacker bifurcation phenomena appear in this region, where the Floquet multipliers on the unit circle are always simple. The lines $u = 0$ and $v = 2u$ represent the case of resonance (Lemma 2). For a fixed α and β the critical parameter values can be obtained easily from the equation

$$(\gamma\alpha, \gamma\beta) = \left(\frac{v\cos(v)}{\sin(v)}, -\frac{v}{\sin(v)} \right)$$

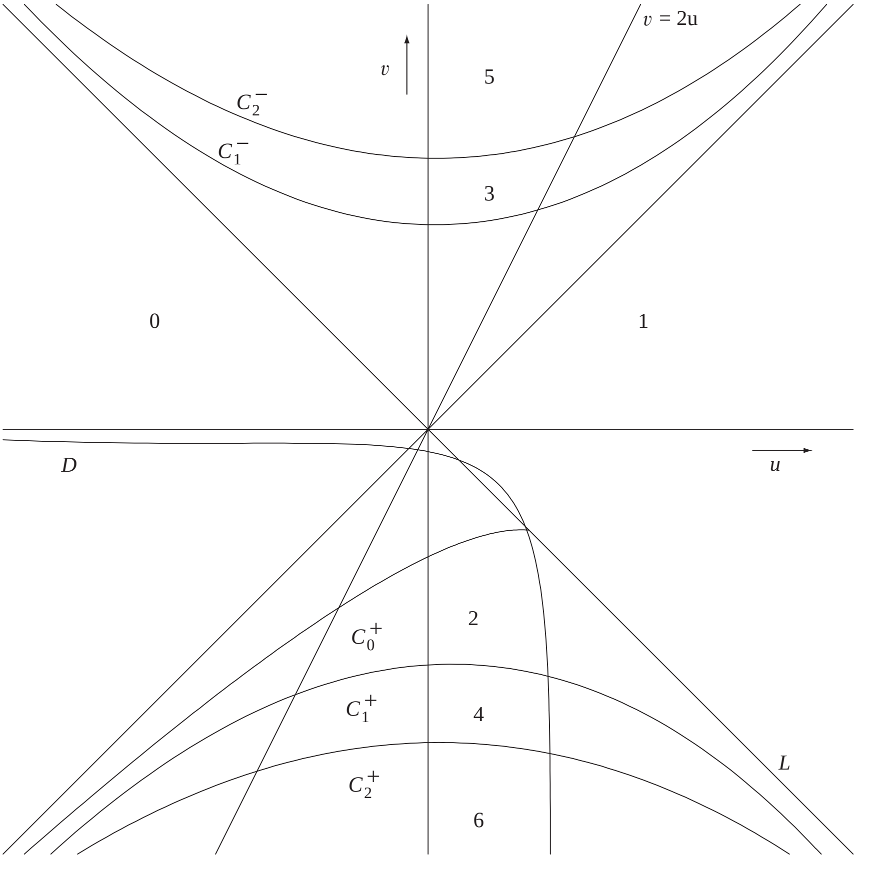

Fig. 1.

as

$$\gamma_{\pm n} = -\frac{\pm \arccos(-\frac{\alpha}{\beta}) + 2n\pi}{\pm\beta \sin(\arccos(-\frac{\alpha}{\beta}))}, \quad n \in \mathbb{N}.$$

Lemma 1. *The critical Floquet multipliers are* $\mu_j = \mathrm{e}^{\lambda_j} = \mathrm{e}^{\mathrm{i}\gamma_j\sqrt{\beta^2-\alpha^2}} = -\frac{\alpha}{\beta} - \mathrm{i}\sqrt{1-\frac{\alpha}{\beta}^2}$ *and* $\bar{\mu}_j = \mathrm{e}^{\bar{\lambda}_j} = \mathrm{e}^{-\mathrm{i}\gamma_j\sqrt{\beta^2-\alpha^2}} = -\frac{\alpha}{\beta} + \mathrm{i}\sqrt{1-\frac{\alpha^2}{\beta^2}}$.

Proof. Since $|\mu_j| = 1$, $\lambda_j = \mathrm{i}\theta$ for some $\theta \in \mathbb{R}$. In that case the real and imaginary parts of the characteristic equation (3) are

$$\begin{cases} 0 = \gamma_j\alpha + \gamma_j\beta\cos(\theta), \\ \theta = -\gamma_j\beta\sin(\theta), \end{cases} \tag{4}$$

thus

$$\begin{cases} \frac{\alpha^2}{\beta^2} = \cos^2(\theta), \\ \frac{\theta^2}{\gamma_j^2\beta^2} = \sin^2(\theta), \end{cases} \tag{5}$$

follows. The sum of the last two equations gives

$$\frac{\alpha^2}{\beta^2} + \frac{\theta^2}{\gamma_j^2\beta^2} = 1$$

and

$$\theta^2 = \gamma_j^2(\beta^2 - \alpha^2).$$

From (4) we obtain

$$\begin{cases} \cos(\theta) = -\frac{\alpha}{\beta}, \\ \sin(\theta) = -\frac{\theta}{\gamma_j\beta} \end{cases} \tag{6}$$

and finally

$$e^{\lambda}_j = e^{i\theta} = \cos(\theta) + i\,\sin(\theta) = -\frac{\alpha}{\beta} - i\sqrt{1 - \frac{\alpha^2}{\beta^2}}. \qquad \square$$

3. Neimark–Sacker bifurcation

Combining the Center Manifold Theorem, the Reduction Principle and the Neimark–Sacker Bifurcation Theorem (for details see [2,12] and [18]) we can state the following:

Theorem A. *Suppose that the one-parameter family of time-one maps $F_\gamma : C \to C$ corresponding to Eq.* (1) *has at the critical value $\gamma = \gamma_j$ the fixed point $\phi = 0$ with exactly two simple Floquet multipliers* $e^{i\theta}$, $e^{-i\theta}$ *on the unit circle. Then there is a neighborhood of* 0 *in which a unique closed invariant curve bifurcates from* 0 *as γ passes through γ_j, providing that the transversality condition*

$$\left.\frac{\partial|\mu(\gamma)|}{\partial\gamma}\right|_{\gamma_j} \neq 0$$

and the nonresonance conditions

$$\mu_j^4 \neq 1, \quad \mu_j^3 \neq 1$$

hold.

Now we propose the main result. The definition of the terms in Theorem 1 (R_μ, V, W) and the detailed calculation can be found in *Section* 5.

Theorem 1. *The direction of the appearance of the invariant curve is determined by the sign of the coefficient*

$$\delta(\gamma_j) = \tfrac{1}{2}\mathrm{Re}\left(\frac{1}{\mu}R_\mu(W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu) + 2V(\chi_\mu, (1-U)^{-1}V(\chi_\mu, \bar{\chi}_\mu)) \right.$$
$$\left. + V(\bar{\chi}_\mu, (\mu^2 - U)^{-1}V(\chi_\mu, \chi_\mu)))\right),$$

where all the terms can be computed explicitly by (1).

Let us mention that the cases $\delta(\gamma_j) < 0$ and $\delta(\gamma_j) > 0$ are called supercritical and subcritical Neimark–Sacker bifurcations. In the supercritical case a stable (only in a restricted sense, inside the invariant center manifold) invariant curve appears for $\gamma > \gamma_j$, while in the subcritical case an unstable invariant curve disappears when γ increasingly crosses γ_j.

When $\delta(\gamma_j) = 0$, we need further investigations (see [1]). Here we suppose that the nondegeneracy condition $\delta(\gamma_j) \neq 0$ is fulfilled. The smoothness of F_γ is guaranteed by the smoothness of $a(t)$ and $f(t, \xi)$.

Lemma 2. *The transversality condition holds.*

Proof. Let μ_j be a Floquet multiplier with modulus 1 for the critical parameter value γ_j. By the Implicit Function Theorem there is a smooth function $\mu(\gamma) = \mathrm{e}^{\lambda(\gamma)}$ in a neighborhood of γ_j with $\mu(\gamma_j) = \mu_j$, where $\lambda(\gamma_j)$ satisfies (3). Using the notation $\lambda(\gamma) = k(\gamma) + \mathrm{i}l(\gamma)$ the transversality condition is equivalent to

$$k'(\gamma_j) \neq 0.$$

It is clear that $k(\gamma_j) = 0$, $l(\gamma_j) \neq 0$ and $\gamma_j \neq 0$. The real and the imaginary parts of the characteristic equation are

$$\begin{cases} k(\gamma) = \gamma\alpha + \gamma\beta\mathrm{e}^{-k(\gamma)}\cos(l(\gamma)), \\ l(\gamma) = -\gamma\beta\mathrm{e}^{-k(\gamma)}\sin(l(\gamma)). \end{cases} \tag{7}$$

Differentiating (7) with respect to γ gives

$$\begin{cases} k'(\gamma) = \alpha + \beta\mathrm{e}^{-k(\gamma)}\cos(l(\gamma)) - \gamma\beta\mathrm{e}^{-k(\gamma)}k'(\gamma)\cos(l(\gamma)) - \gamma\beta\mathrm{e}^{-k(\gamma)}\sin(l(\gamma))l'(\gamma), \\ l'(\gamma) = -\beta\mathrm{e}^{-k(\gamma)}\sin(l(\gamma)) + \gamma\beta\mathrm{e}^{-k(\gamma)}k'(\gamma)\sin(l(\gamma)) - \gamma\beta\mathrm{e}^{-k(\gamma)}\cos(l(\gamma))l'(\gamma). \end{cases} \tag{8}$$

At critical values (7) becomes

$$\begin{cases} 0 = \alpha + \beta\cos(l(\gamma_j)), \\ l(\gamma_j) = -\gamma_j\beta\sin(l(\gamma_j)). \end{cases} \tag{9}$$

Substituting (9) into (8) yields

$$\begin{cases} k'(\gamma_j) = \gamma_j\alpha k'(\gamma_j) + l(\gamma_j)l'(\gamma_j), \\ l'(\gamma_j) = \frac{l(\gamma_j)}{\gamma_j} - k'(\gamma_j)l(\gamma_j) + \alpha\gamma_j l'(\gamma_j). \end{cases}$$

Supposing $k'(\gamma_j) = 0$ we get $l'(\gamma_j) = 0$ and finally $l(\gamma_j) = 0$, a contradiction, hence the transversality conditions hold. Notice that this fact is intuitively clear from Fig. 1. $\square$

Lemma 3. *$\mu_j^4 = 1$ if and only if $\alpha = 0$, and $\mu_j^3 = 1$ if and only if $\beta = 2\alpha$.*

Proof. Elementary calculations show substituting the appropriate values of the roots of unity into the real and the imaginary parts of the characteristic equation. See Lemma 1. □

4. Resolvent and spectral projection

Let

$$\varDelta(z) = z - \mathrm{e}^{[\gamma\alpha + \frac{\gamma\beta}{z}]}.$$

The equation $\varDelta(z) = 0$ is equivalent to the characteristic equation. Any complex number $\mu = \mathrm{e}^{\lambda}$ is a root of $\varDelta(z)$ if and only if λ is a Floquet exponent. Applying Theorem 3.1 of [5, p. 247] to Eq. (1) one finds that the Floquet multipliers consist of the roots of $\varDelta(z)$, and the algebraic multiplicity of an eigenvalue μ equals the order of μ as a zero of $\varDelta(z)$. When this number is 1, we call μ a simple eigenvalue. We state the Riesz–Schauder Theorem for the case when the eigenvalue is simple.

Theorem B. *Let $U : C_{\mathbb{C}} \to C_{\mathbb{C}}$ be a compact operator. If $z = \mu$ is a simple eigenvalue of U, then there are two closed subspaces E_μ and Q_μ such that*

(1) *E_μ is one-dimensional*;
(2) *$E_\mu \oplus Q_\mu = C_{\mathbb{C}}$*;
(3) *$U(E_\mu) \subset E_\mu$ and $U(Q_\mu) \subset Q_\mu$*;
(4) *$\sigma(U|E_\mu) = \{\mu\}$, $\sigma(U|Q_\mu) = \sigma(U)\backslash\{\mu\}$*;
(5) *the spectral projection P_μ onto E_μ along Q_μ can be represented by a Riesz–Dunford integral*;

$$P_\mu = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\mu} (zI - U)^{-1}\, \mathrm{d}z = \operatorname*{Res}_{z=\mu} (zI - U)^{-1},$$

where Γ_μ is a small circle around μ such that μ is the only singularity of $(zI - U)^{-1}$ inside Γ_μ.

For simplicity, let $b(t) = \gamma f_\xi(t, 0)$ and $c(t) = \gamma a(t)$. With this notation the linearized equation takes the form

$$\dot{y}(t) = c(t)y(t) + b(t)y(t-1),$$

$\alpha = \frac{1}{\gamma}\int_{-1}^{0} c(t)\,\mathrm{d}t$, $\beta = \frac{1}{\gamma}\int_{-1}^{0} b(t)\,\mathrm{d}t$ and the eigenfunctions are

$$\chi_\mu(t) : [-1, 0] \ni t \mapsto \mathrm{e}^{\int_{-1}^{t}[c(s) + \frac{b(s)}{\mu}]\,\mathrm{d}s} \in \mathbb{C}.$$

By the variation-of-constants formula for ordinary differential equations we find the following representation of the time-one map

$$F(\phi)(t) = \mathrm{e}^{\int_{-1}^{t} c(u)\,\mathrm{d}u}\left(\phi(0) + \int_{-1}^{t} \mathrm{e}^{-\int_{-1}^{s} c(u)\,\mathrm{d}u}\gamma f(s,\phi(s))\,\mathrm{d}s\right),$$
$$t \in [-1,0], \tag{10}$$

which implies for the monodromy operator

$$U(\phi)(t) = \mathrm{e}^{\int_{-1}^{t} c(u)\,\mathrm{d}u}\left(\phi(0) + \int_{-1}^{t} \mathrm{e}^{-\int_{-1}^{s} c(u)\,\mathrm{d}u} b(s)\phi(s)\,\mathrm{d}s\right), \quad t \in [-1,0]. \tag{11}$$

Lemma 4. *The resolvent of the monodromy operator can be expressed as*

$$\begin{aligned}
&(zI-U)^{-1}(\psi)(t)\\
&\quad = \mathrm{e}^{\int_{-1}^{t}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u}\\
&\qquad \times \left(\left(\frac{1}{z}\psi(0) + \mathrm{e}^{\int_{-1}^{0}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u}\int_{-1}^{0}\frac{1}{z^2}\mathrm{e}^{-\int_{-1}^{s}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u} b(s)\psi(s)\,\mathrm{d}s\right)\right.\\
&\qquad \times \left(z-\mathrm{e}^{\int_{-1}^{0}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u}\right)^{-1} + \frac{1}{z}\mathrm{e}^{-\int_{-1}^{t}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u}\psi(t)\\
&\qquad \left. + \int_{-1}^{t}\frac{1}{z^2}\mathrm{e}^{-\int_{-1}^{s}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u} b(s)\psi(s)\,\mathrm{d}s\right).
\end{aligned} \tag{12}$$

Proof. Set $\phi = (zI-U)^{-1}\psi$, or equivalently

$$\psi(t) = z\phi(t) - U(\phi)(t), \quad t \in [-1,0]. \tag{13}$$

To compute the resolvent we have to solve this equation. Let

$$\hat{\phi}(t) = \mathrm{e}^{-\int_{-1}^{t} c(u)\,\mathrm{d}u}\phi(t), \tag{14}$$

then by the representation (11) and the notation in (14), multiplying (13) with $\mathrm{e}^{-\int_{-1}^{t} c(u)\,\mathrm{d}u}$ gives

$$\hat{\psi}(t) = z\hat{\phi}(t) - \phi(0) - \int_{-1}^{t} b(s)\hat{\phi}(s)\,\mathrm{d}s. \tag{15}$$

First we suppose that ψ is differentiable. Differentiating (15) one has

$$\hat{\phi}'(t) = \frac{b(t)}{z}\hat{\phi}(t) + \frac{1}{z}\hat{\psi}'(t). \tag{16}$$

Taking $t=-1$ in (15) we obtain

$$\hat{\psi}(-1) = z\hat{\phi}(-1) - \mathrm{e}^{\int_{-1}^{0} c(u)\,\mathrm{d}u}\hat{\phi}(0). \tag{17}$$

The Eqs. (16) and (17) define a boundary value problem. By the variation-of-constants formula from (16) we find the solution

$$\hat{\phi}(t) = \mathrm{e}^{\int_{-1}^{t} \frac{b(u)}{z}\,\mathrm{d}u} \left(\hat{\phi}(-1) + \int_{-1}^{t} \frac{1}{z} \mathrm{e}^{-\int_{-1}^{s} \frac{b(u)}{z}\,\mathrm{d}u} \hat{\psi}'(s)\,\mathrm{d}s \right). \tag{18}$$

Letting $t = 0$ we have

$$\hat{\phi}(0) = \mathrm{e}^{\int_{-1}^{0} \frac{b(u)}{z}\,\mathrm{d}u} \left(\hat{\phi}(-1) + \int_{-1}^{0} \frac{1}{z} \mathrm{e}^{-\int_{-1}^{s} \frac{b(u)}{z}\,\mathrm{d}u} \hat{\psi}'(s)\,\mathrm{d}s \right).$$

Substituting this into the boundary condition, (17) becomes

$$\hat{\psi}(-1) = z\hat{\phi}(-1) - \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \left(\hat{\phi}(-1) + \int_{-1}^{0} \frac{1}{z} \mathrm{e}^{-\int_{-1}^{s} \frac{b(u)}{z}\,\mathrm{d}u} \hat{\psi}'(s)\,\mathrm{d}s \right),$$

which yields

$$\begin{aligned}\hat{\phi}(-1) = &\left(\hat{\psi}(-1) + \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \int_{-1}^{0} \frac{1}{z} \mathrm{e}^{-\int_{-1}^{s} \frac{b(u)}{z}\,\mathrm{d}u} \hat{\psi}'(s)\,\mathrm{d}s \right) \\ &\times (z - \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u})^{-1},\end{aligned} \tag{19}$$

whenever

$$(z - \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u}) \neq 0,$$

that is $z \notin \sigma(U)$. Substituting (19) into (18) and returning to $\phi(t)$ we get

$$\begin{aligned}\phi(t) = \mathrm{e}^{\int_{-1}^{t} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} &\left(\left(\hat{\psi}(-1) + \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \int_{-1}^{0} \frac{1}{z} \mathrm{e}^{-\int_{-1}^{s} \frac{b(u)}{z}\,\mathrm{d}u} \hat{\psi}'(s)\,\mathrm{d}s \right) \right. \\ &\left. \times (z - \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u})^{-1} + \int_{-1}^{t} \frac{1}{z} \mathrm{e}^{-\int_{-1}^{s} \frac{b(u)}{z}\,\mathrm{d}u} \hat{\psi}'(s)\,\mathrm{d}s \right).\end{aligned}$$

Now we integrate by parts and return to $\psi(t)$ to obtain the expression

$$\begin{aligned}&(zI - U)^{-1}(\psi)(t) \\ &\quad = \mathrm{e}^{\int_{-1}^{t} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \left(\left(\psi(-1) + \frac{1}{z}\psi(0) - \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \frac{1}{z}\psi(-1) \right.\right. \\ &\qquad \left. + \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \int_{-1}^{0} \frac{1}{z^2} \mathrm{e}^{-\int_{-1}^{s} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} b(s)\psi(s)\,\mathrm{d}s \right) \\ &\qquad \times (z - \mathrm{e}^{\int_{-1}^{0} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u})^{-1} + \frac{1}{z} \mathrm{e}^{-\int_{-1}^{t} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} \psi(t) \\ &\qquad \left. - \frac{1}{z}\psi(-1) + \int_{-1}^{t} \frac{1}{z^2} \mathrm{e}^{-\int_{-1}^{s} [c(u) + \frac{b(u)}{z}]\,\mathrm{d}u} b(s)\psi(s)\,\mathrm{d}s \right),\end{aligned} \tag{20}$$

which can be simplified to (12). This formula is valid for any differentiable ψ. The differentiable functions form a dense set in $C_{\mathbb{C}}$, therefore the continuity of the formula implies that (12) is valid for any $\psi \in C_{\mathbb{C}}$, we obtained an explicit expression for the resolvent. $\square$

Theorem 2. *The spectral projection operator has the representation*

$$P_\mu(\psi) = \chi_\mu R_\mu(\psi),$$

where

$$R_\mu(\psi) = \left(\frac{1}{\mu + \gamma\beta}\right)\left(\psi(0) + \int_{-1}^{0} \frac{b(s)\psi(s)}{\chi_\mu(s)}\,\mathrm{d}s\right).$$

Proof. A straightforward computation proves the theorem as follows:

$$\begin{aligned}
P_\mu(\psi)(t) &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\mu}(zI - U)^{-1}\,\mathrm{d}z(\psi)(t) = \operatorname*{Res}_{z=\mu}(zI - U)^{-1}(\psi)(t)\\
&= \lim_{z\to\mu}((z-\mu)(zI-U)^{-1}(\psi)(t))\\
&= \mathrm{e}^{\int_{-1}^{t}[c(u)+\frac{b(u)}{\mu}]\,\mathrm{d}s}\left(\psi(-1) + \frac{1}{\mu}\psi(0) - \mathrm{e}^{\int_{-1}^{0}[c(u)+\frac{b(u)}{\mu}]\,\mathrm{d}u}\frac{1}{\mu}\psi(-1)\right.\\
&\quad\left. +\mathrm{e}^{\int_{-1}^{0}[c(u)+\frac{b(u)}{\mu}]\,\mathrm{d}u}\int_{-1}^{0}\frac{1}{\mu^2}\mathrm{e}^{-\int_{-1}^{s}[c(u)+\frac{b(u)}{\mu}]\,\mathrm{d}u}b(s)\psi(s)\,\mathrm{d}s\right)\\
&\quad\times \lim_{z\to\mu}((z-\mu)(z-\mathrm{e}^{\int_{-1}^{0}[c(u)+\frac{b(u)}{z}]\,\mathrm{d}u})^{-1})\\
&= \chi_\mu(t)(\lim_{z\to\mu}((z-\mu)\Delta(z)^{-1}))\left(\psi(-1) + \frac{1}{\mu}\psi(0)\right.\\
&\quad\left. -\mathrm{e}^{[\gamma\alpha+\frac{\gamma\beta}{\mu}]}\frac{1}{\mu}\psi(-1) + \mathrm{e}^{[\gamma\alpha+\frac{\gamma\beta}{\mu}]}\frac{1}{\mu^2}\int_{-1}^{0}\frac{b(s)\psi(s)}{\chi_\mu(s)}\,\mathrm{d}s\right)\\
&= \chi_\mu(t)\frac{1}{\Delta'(\mu)}\left(\frac{1}{\mu}\psi(0) + \frac{1}{\mu}\int_{-1}^{0}\frac{b(s)\psi(s)}{\chi_\mu(s)}\,\mathrm{d}s\right)\\
&= \chi_\mu(t)\left(\frac{1}{\mu+\gamma\beta}\right)\left(\psi(0) + \int_{-1}^{0}\frac{b(s)\psi(s)}{\chi_\mu(s)}\,\mathrm{d}s\right). \qquad \square
\end{aligned}$$

5. The direction of the bifurcation

Consider the decomposition

$$C = T^c \oplus T^{su},$$

where $T^c = \mathrm{Re}\,E_\mu \oplus \mathrm{Im}\,E_\mu$ is the critical 2-dimensional realified center eigenspace corresponding to μ and spanned by $\{\mathrm{Re}\,\chi_\mu, \mathrm{Im}\,\chi_\mu\}$, $T^{su} = \mathrm{Re}\,Q_\mu \oplus \mathrm{Im}\,Q_\mu$ is the 2-codimensional realified stable-unstable subspace corresponding to the other part of $\sigma(U)$. The idea of the projection method is that we introduce new variables $x \in T^c$, $y \in T^{su}$ and use them as coordinates on these subspaces, which are invariant for $DF(0)$. Suppose we have the map F in the form

$$\begin{cases} \tilde{x} = A(x) + g(x, y), \\ \tilde{y} = B(y) + h(x, y), \end{cases}$$

where $(\tilde{x}, \tilde{y}) = F(x, y)$; $x, \tilde{x} \in T^c$; $y, \tilde{y} \in T^{su}$; A and B are linear maps on the corresponding subspaces and

$$g(0, 0) = 0, \quad Dg(0, 0) = 0,$$

$$h(0, 0) = 0, \quad Dh(0, 0) = 0.$$

For a manifold $y = M(x)$ we have

$$\begin{cases} \tilde{x} = A(x) + g(x, M(x)), \\ \tilde{y} = B(M(x)) + h(x, M(x)). \end{cases}$$

If $M(x)$ denotes the center manifold then by the invariance $\tilde{y} = M(\tilde{x})$, and thus

$$M(A(x) + g(x, M(x))) = B(M(x)) + h(x, M(x)). \tag{21}$$

The coefficients of the Taylor-expansion of $M(x)$ can be calculated by this formula. For details and examples we refer to [12] and [18]. The computation below follows [12], but the space T^{su} is not finite dimensional and the scalar product is replaced by the representation of the spectral projection given in Theorem 2. We need the derivatives of the operator F up to order three, which can be found from the representation (10), respectively,

$$DF(\eta)(\phi)(t) = \mathrm{e}^{\int_{-1}^{t} c(u)\,\mathrm{d}u} \left(\phi(0) + \int_{-1}^{t} \mathrm{e}^{-\int_{-1}^{s} c(u)\,\mathrm{d}u} \gamma f_{\xi}(s, \eta(s))\phi(s)\,\mathrm{d}s \right),$$

$$D^2F(\eta)(\phi_1, \phi_2)(t) = \mathrm{e}^{\int_{-1}^{t} c(u)\,\mathrm{d}u} \int_{-1}^{t} \mathrm{e}^{-\int_{-1}^{s} c(u)\,\mathrm{d}u} \gamma f_{\xi\xi}(s, \eta(s))\phi_1(s)\phi_2(s)\,\mathrm{d}s$$

and

$$D^3F(\eta)(\phi_1, \phi_2, \phi_3)(t) = \mathrm{e}^{\int_{-1}^{t} c(u)\,\mathrm{d}u} \times \int_{-1}^{t} \mathrm{e}^{-\int_{-1}^{s} c(u)\,\mathrm{d}u} \gamma f_{\xi\xi\xi}(s, \eta(s))\phi_1(s)\phi_2(s)\phi_3(s)\,\mathrm{d}s.$$

Evaluating at $\eta = 0$, $F(0) = 0$ and $DF(0) = U$. Let $V = D^2F(0)$ and $W = D^3F(0)$. V and W are multilinear operators. Represent the Taylor-expansion of F in the form

$$F(\phi) = U(\phi) + \tfrac{1}{2}V(\phi, \phi) + \tfrac{1}{6}W(\phi, \phi, \phi) + O(\|\phi\|^4).$$

Let $Z(\phi) = F(\phi) - U(\phi)$ the nonlinear part of F. Now decompose $\phi \in C$ as

$$\phi = z\chi_\mu + \bar{z}\bar{\chi}_\mu + \psi,$$

where $z = R_\mu(\phi) \in \mathbb{C}$, $z\chi_\mu + \bar{z}\bar{\chi}_\mu \in T^c$ and $\psi \in T^{su}$. The complex variable z is a coordinate on the 2-dimensional real eigenspace T^c and the function ψ is a variable in T^{su}.

The subspaces T^c and T^{su} are invariant under U. For any real ϕ, $\phi \in T^{su}$ if and only if $P_\mu(\phi)=0$. $U(\chi_\mu)=\mu\chi_\mu$ implies $U(\bar\chi_\mu)=\bar\mu\bar\chi_\mu$, $\overline{R_\mu}=R_{\bar\mu}$. After the above remarks we can write

$$\begin{cases} \tilde z = \mu z + R_\mu(Z(\phi)),\\ \tilde\psi = U(\psi)+Z(\phi)-R_\mu(\phi)\chi_\mu - R_{\bar\mu}(\phi)\bar\chi_\mu, \end{cases}$$

moreover

$$\begin{cases} \tilde z = \mu z + R_\mu(Z(z\chi_\mu+\bar z\bar\chi_\mu+\psi)),\\ \tilde\psi = U(\psi)+Z(z\chi_\mu+\bar z\bar\chi_\mu+\psi)-R_\mu(Z(z\chi_\mu+\bar z\bar\chi_\mu+\psi)\chi_\mu\\ \qquad -R_{\bar\mu}(Z(z\chi_\mu+\bar z\bar\chi_\mu+\psi)\bar\chi_\mu. \end{cases} \tag{22}$$

The Taylor-expansion of $Z(z\chi_\mu+\bar z\bar\chi_\mu+\psi)$ around 0 with respect to z, $\bar z$ and ψ in terms of V and W is given by

$$\begin{cases} Z(z\chi_\mu+\bar z\bar\chi_\mu+\psi) = \frac12 V(\chi_\mu,\chi_\mu)z^2 + V(\chi_\mu,\bar\chi_\mu)z\bar z + \frac12 V(\bar\chi_\mu,\bar\chi_\mu)\bar z^2\\ \qquad +V(\chi_\mu,\psi)z + V(\bar\chi_\mu,\psi)\bar z + \frac12 V(\psi,\psi)\\ \qquad +\frac16 W(\chi_\mu,\chi_\mu,\chi_\mu)z^3 + \frac12 W(\chi_\mu,\chi_\mu,\bar\chi_\mu)z^2\bar z+\cdots\ . \end{cases}$$

Thus we can rewrite (22) using the linearity of R_μ:

$$\begin{cases} \tilde z = \mu z + \frac12 R_\mu(V(\chi_\mu,\chi_\mu))z^2 + R_\mu(V(\chi_\mu,\bar\chi_\mu))z\bar z + \frac12 R_\mu(V(\bar\chi_\mu,\bar\chi_\mu))\bar z^2\\ \qquad +R_\mu(V(\chi_\mu,\psi))z + R_\mu(V(\bar\chi_\mu,\psi))\bar z + \frac12 R_\mu(V(\psi,\psi))\\ \qquad +\frac16 R_\mu(W(\chi_\mu,\chi_\mu,\chi_\mu))z^3 + \frac12 R_\mu(W(\chi_\mu,\chi_\mu,\bar\chi_\mu))z^2\bar z+\cdots\\ \tilde\psi = U(\psi) + \frac12\zeta_{20}z^2 + \zeta_{11}z\bar z + \frac12\zeta_{02}\bar z^2+\cdots, \end{cases}$$

where

$$\zeta_{20} = V(\chi_\mu,\chi_\mu) - R_\mu(V(\chi_\mu,\chi_\mu))\chi_\mu - R_{\bar\mu}(V(\chi_\mu,\chi_\mu))\bar\chi_\mu,$$

$$\zeta_{11} = V(\chi_\mu,\bar\chi_\mu) - R_\mu(V(\chi_\mu,\bar\chi_\mu))\chi_\mu - R_{\bar\mu}(V(\chi_\mu,\bar\chi_\mu))\bar\chi_\mu,$$

$$\zeta_{02} = V(\bar\chi_\mu,\bar\chi_\mu) - R_\mu(V(\bar\chi_\mu,\bar\chi_\mu))\chi_\mu - R_{\bar\mu}(V(\bar\chi_\mu,\bar\chi_\mu))\bar\chi_\mu.$$

The center manifold is tangent to T^c at 0 and has the representation

$$\psi = M(z,\bar z) = \frac12 v_{20}z^2 + v_{11}z\bar z + \frac12 v_{02}\bar z^2 + O(|z|^3),$$

where $P_\mu(v_{ij})=0$. In view of (21), the coefficients v_{ij} can be computed from

$$(\mu^2 I - U)v_{20} = \zeta_{20},$$
$$(I-U)v_{11} = \zeta_{11},$$
$$(\mu^{-2}I-U)v_{02} = \zeta_{02}.$$

Remark that μ and $\bar\mu$ are the only eigenvalues on the unit circle and not third roots of unity, so the operators $(I-U)$, $(\mu^2 I-U)$, $(\mu^{-2}I-U)$ are invertible and given by Lemma 5. The restricted map

$$\tilde z = \mu z + R_\mu(Z(z\chi_\mu + \bar z\bar\chi_\mu + M(z,\bar z)))$$

can be written as

$$\tilde{z} = \mu z + \tfrac{1}{2}\rho_{20}z^2 + \rho_{11}z\bar{z} + \tfrac{1}{2}\rho_{02}\bar{z}^2 + \tfrac{1}{2}\rho_{21}z^2\bar{z} + \cdots, \tag{23}$$

where only that cubic term retained which is necessary to analyze the Neimark–Sacker bifurcation (see [12]), and

$$\begin{aligned}
\rho_{20} &= R_\mu(V(\chi_\mu, \chi_\mu)),\\
\rho_{11} &= R_\mu(V(\chi_\mu, \bar{\chi}_\mu)),\\
\rho_{02} &= R_\mu(V(\bar{\chi}_\mu, \bar{\chi}_\mu)),
\end{aligned}$$

$$\begin{aligned}
\rho_{21} &= R_\mu(W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu)) + 2R_\mu(V(\chi_\mu, v_{11})) + R_\mu(V(\bar{\chi}_\mu, v_{20}))\\
&= R_\mu(W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu)) + R_\mu(V(\chi_\mu, (1-U)^{-1}\zeta_{11}))\\
&\quad + R_\mu(V(\bar{\chi}_\mu, (\mu^2-U)^{-1}\zeta_{20}))\\
&= R_\mu(W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu)) + R_\mu(V(\chi_\mu, (1-U)^{-1}V(\chi_\mu, \bar{\chi}_\mu))\\
&\quad - R_\mu(V(\chi_\mu, \bar{\chi}_\mu))\chi_\mu - R_{\bar{\mu}}(V(\chi_\mu, \bar{\chi}_\mu))\bar{\chi}_\mu\\
&\quad + R_\mu(V(\bar{\chi}_\mu, (\mu^2-U)^{-1}V(\chi_\mu, \chi_\mu)) - R_\mu(V(\chi_\mu, \chi_\mu))\chi_\mu\\
&\quad - R_{\bar{\mu}}(V(\chi_\mu, \chi_\mu))\bar{\chi}_\mu)).
\end{aligned} \tag{24}$$

Taking into account the identities

$$z\bar{z} = |z|^2, \quad \bar{\mu} = \frac{1}{\mu}, \quad \overline{R_{\bar{\mu}}(V(\phi, \psi))} = R_\mu(V(\bar{\phi}, \bar{\psi})),$$

$$(1-U)^{-1}\chi_\mu = \frac{1}{1-\mu}\chi_\mu, \quad (\mu^2-U)^{-1}\chi_\mu = \frac{1}{\mu(\mu-1)}\chi_\mu,$$

$$(1-U)^{-1}\bar{\chi}_\mu = \frac{1}{1-\frac{1}{\mu}}\bar{\chi}_\mu, \quad (\mu^2-U)^{-1}\bar{\chi}_\mu = \frac{\mu}{\mu^3-1}\bar{\chi}_\mu,$$

we obtain

$$\begin{aligned}
\rho_{21} &= R_\mu(W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu)) + 2R_\mu(V(\chi_\mu, (1-U)^{-1}V(\chi_\mu, \bar{\chi}_\mu)))\\
&\quad + R_\mu(V(\bar{\chi}_\mu, (\mu^2-U)^{-1}V(\chi_\mu, \chi_\mu)))\\
&\quad + \frac{\frac{1}{\mu}(1-2\mu)}{1-\mu}R_\mu(V(\chi_\mu, \chi_\mu))R_\mu(V(\chi_\mu, \bar{\chi}_\mu))\\
&\quad - \frac{2}{1-\frac{1}{\mu}}|R_\mu(V(\chi_\mu, \bar{\chi}_\mu))|^2 - \frac{\mu}{\mu^3-1}|R_\mu(V(\bar{\chi}_\mu, \bar{\chi}_\mu))|^2.
\end{aligned} \tag{25}$$

By [12], restricted maps of the form (23) can be transformed into the form

$$\tilde{z} = \mu z(1 + d(\gamma_j)|z|^2) + O(|z|^4),$$

where the real number $\delta(\gamma_j) = \operatorname{Re} d(\gamma_j)$ determines the direction of the bifurcation and can be computed by

$$\delta(\gamma_j) = \operatorname{Re}\left(\frac{\frac{1}{\mu}\rho_{21}}{2}\right) - \operatorname{Re}\left(\frac{(1-2\mu)\frac{1}{\mu^2}}{2(1-\mu)}\rho_{20}\rho_{11}\right) - \tfrac{1}{2}|\rho_{11}|^2 - \tfrac{1}{4}|\rho_{02}|^2.$$

Using this formula with the coefficients given in (24) and (25), we arrive at the invariant expression

$$\begin{aligned}\delta(\gamma_j) = \tfrac{1}{2}\operatorname{Re}\Bigg(&\frac{1}{\mu}R_\mu(W(\chi_\mu, \chi_\mu, \bar{\chi}_\mu) + 2V(\chi_\mu, (1-U)^{-1}V(\chi_\mu, \bar{\chi}_\mu)) \\ &+ V(\bar{\chi}_\mu, (\mu^2 - U)^{-1}V(\chi_\mu, \chi_\mu)))\Bigg),\end{aligned} \tag{26}$$

and the proof of Theorem 1 is complete. In (26) all the terms can be expressed explicitly by $a(t)$, the partial derivatives of $f(t, \xi)$ and $\mu = \mathrm{e}^\lambda$, for example

$$\begin{aligned}V(\chi_\mu, \chi_\mu)(t) &= \mathrm{e}^{\int_{-1}^t c(u)\,\mathrm{d}u}\int_{-1}^t \mathrm{e}^{-\int_{-1}^s c(u)\,\mathrm{d}u}\gamma f_{\xi\xi}(s, 0)\chi_\mu(s)^2\,\mathrm{d}s \\ &= \mathrm{e}^{\int_{-1}^t \gamma a(u)\,\mathrm{d}u}\int_{-1}^t \gamma f_{\xi\xi}(s, 0)(\mathrm{e}^{\int_{-1}^s \gamma a(u) + 2\gamma f_\xi(u,0)\mathrm{e}^{-\lambda}\,\mathrm{d}u})\mathrm{d}s \quad \text{etc.}\end{aligned}$$

6. The invariant torus

The evolutionary system associated with the translation along the solutions of the Eq. (1) is given by the relation

$$T(t, s)\varphi = x_t^{\varphi, s},$$

where $t > s$, $T(t, s) : C \to C$ and $x^{\varphi, s}$ is the solution of (1) satisfying $x_s^{\varphi, s} = \varphi$. Let $F^\tau = T(\tau + 1, \tau)$, then $F^0 = F$. The periodicity of (1) yields that $F^\tau = F^{\tau+1}$ and the system can be considered in the space $C \times S^1$ as an autonomous system with the solution maps

$$G(t) : C \times S^1 \ni (\varphi, s) \mapsto (x_t^{\varphi, s}, t + s \bmod 1) \in C \times S^1.$$

The characteristic equation and the Floquet-multipliers of the monodromy operators $U^\tau = DF^\tau(0)$ are independent of τ hence an invariant curve occurs at the same parameter-values for all F^τ. Denote these curves by Γ^τ. While $T(t, s)(\Gamma^s)$ is an invariant curve with respect to F^t, the uniqueness property shows that $T(t,s)(\Gamma^s)=\Gamma^t$ and the set

$$\mathbb{T}=\bigcup_{\tau\in[0,1)}(\Gamma^\tau,\tau)$$

forms an invariant torus in the space $C\times S^1$ under the dynamics generated by the associated maps $G(t)$.

Theorem 3. *If the conditions of the Neimark–Sacker bifurcation are satisfied, then for any family of systems generated by the solution maps $G_\gamma(t): C\times S^1\to C\times S^1$ corresponding to Eq.* (1) *a unique invariant torus bifurcates from the periodic solution* $(0,t)$ *as the parameter γ passes through the critical value γ_j. The direction of the appearance of the invariant torus is determined by the sign of the coefficient $\delta(\gamma_j)$, which can be computed explicitly.*

7. An application

In this section we consider the equation

$$\dot z(t)=\gamma r(t)(-mz(t)+g(z(t-1))), \tag{27}$$

where γ is a real parameter, $m>0$, $r:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are C^4-smooth functions, r satisfies $r(t+1)=r(t)$ for all $t\in\mathbb{R}$ and $g(0)=0$. When the function $r(t)$ is a constant function, (27) is an autonomous equation. A comprehensive study of that equation can be found in [8–11] and [15,16] under both positive feedback ($\xi g(\xi)>0$ for all $\xi\neq 0$) and negative feedback ($\xi g(\xi)<0$ for all $\xi\neq 0$) conditions. Monotonicity properties of the feedback are also required for certain results. The structure of the global attractor was described for a wide range of parameters, but even complicated behavior of the solutions is possible [13]. The periodic orbits, appearing when the system goes through Hopf bifurcation, play an important role in the dynamics. In some specific applications in neural network theory the same equation occurs with $g(\xi)=k_1\tanh(k_2\xi)$, $k_1>0$, $k_2>0$ (see [19] and references therein).

A more realistic model can be (27), where $r(t)$ is a periodic function.

Theorem 4. *Assume that* $0<\int_{-1}^0 r(s)\,\mathrm{d}s$, $g\prime(0)<0$, $g''(0)=0$ *and* $g'''0)\neq 0$. *Then in the case of* $g'''(0)<0$ *the dynamical system generated by the time-one map related to Eq.* (27) *undergoes a subcritical, in the case of* $g'''(0)>0$ *a supercritical Neimark–Sacker bifurcation as the parameter γ passes through γ_0 increasingly.*

Remark that the condition $g''(0)=0$ is satisfied by the functions $g(\xi)=\arctan(\xi)$ or $g(\xi)=\tanh(\xi)$. The theorem describes a situation when only the sign of $g'''(0)$ determines the direction of the appearance of the invariant circle. Let us define

$$r=\int_{-1}^0 r(s)\,\mathrm{d}s,\qquad q(t)=\int_{-1}^t r(s)\,\mathrm{d}s.$$

Using the notations of the previous sections we have

$$a(t) = -mr(t), \quad f(t, x(t-1)) = r(t)g(x(t-1)),$$

$$f_{\xi}(t,0) = r(t)g'(0), \quad f_{\xi\xi}(t,0) = r(t)g''(0), \quad f_{\xi\xi\xi}(t,0) = r(t)g'''(0),$$

$$\alpha = -mr, \quad \beta = g'(0)r, \quad c(u) = -\gamma_0 mr(u), \quad b(u) = \gamma_0 g'(0)r(u).$$

Lemma 5. *Suppose that* $|\mu| = 1$. *Then the eigenfunction corresponding to the center subspace satisfies* $|\chi_\mu(t)| = 1$ *and* $\chi_\mu(t)\bar{\chi}_\mu(t) = 1$ *for all* $t \in [-1, 0]$.

Proof. The characteristic equation of (27) is

$$\lambda = -mr\gamma + g'(0)r\gamma \mathrm{e}^{-\lambda} \tag{28}$$

and the eigenfunctions have the form

$$\chi_\mu(s) = \mathrm{e}^{\gamma[-m+g'(0)\mathrm{e}^{-\lambda}q(s)]}, \tag{29}$$

thus it is enough to show that

$$\mathrm{Re}\,(-m + g'(0)\mathrm{e}^{-\lambda}) = 0.$$

By the characteristic equation (28)

$$-m + g'(0)\mathrm{e}^{-\lambda} = \frac{\lambda}{r\gamma} = \mathrm{i}\frac{\theta}{r\gamma},$$

a purely imaginary number. □

If $0 < \gamma < \gamma_0$ then all the Floquet multipliers are inside the unit circle and the stationary solution 0 is stable. We study the first bifurcation, when γ passes through γ_0. Using (29) and Lemma 5, we conclude

$$\begin{aligned} W(\chi_{\mu_0}, \chi_{\mu_0}, \bar{\chi}_{\mu_0})(t) &= \mathrm{e}^{\int_{-1}^{t} -\gamma_0 mr(u)\,\mathrm{d}u} \int_{-1}^{t} \mathrm{e}^{-\int_{-1}^{s} -\gamma_0 mr(u)\,\mathrm{d}u} \gamma_0 r(s) \\ &\quad \times g'''(0)\chi_{\mu_0}(s)\chi_{\mu_0}(s)\bar{\chi}_{\mu_0}(s)\,\mathrm{d}s \\ &= \gamma_0 g'''(0)\mathrm{e}^{-\gamma_0 mq(t)} \int_{-1}^{t} \mathrm{e}^{\gamma_0 g'(0)\bar{\mu}_0 q(s)} r(s)\,\mathrm{d}s \\ &= \gamma_0 g'''(0)\mathrm{e}^{-\gamma_0 mq(t)} (\mathrm{e}^{\gamma_0 g'(0)\bar{\mu}_0 q(t)} - 1)\frac{1}{\gamma_0 g'(0)\bar{\mu}_0} \\ &= \frac{g'''(0)\mu_0}{g'(0)}(\chi_{\mu_0}(t) - \mathrm{e}^{-\gamma_0 mq(t)}). \end{aligned} \tag{30}$$

Denote the above function by $W_0(t)$ for short. Let $x \cong y$ if and only if $xy > 0$. The assumption $g''(0) = 0$ implies that $V \equiv 0$. Now by Theorem 1, Theorem 2 and (30) we get

$$\begin{aligned}
\delta(\gamma_0) &= \tfrac{1}{2}\mathrm{Re}\left(\frac{1}{\mu_0}R_{\mu_0}(W_0(t))\right)\\
&\cong \mathrm{Re}\left(\frac{1}{\mu_0}\left(\frac{1}{\mu_0+\gamma_0 r g'(0)}\right)\left(W_0(0)+\int_{-1}^{0}\frac{\gamma_0 g'(0) r(s) W_0(s)}{\chi_{\mu_0}(s)}\,\mathrm{d}s\right)\right)\\
&= \mathrm{Re}\left(\frac{1}{\mu_0}\left(\frac{\bar\mu_0+\gamma_0 r g'(0)}{|\mu_0+\gamma_0 r g'(0)|^2}\right)\left(\frac{g'''(0)\mu_0}{g'(0)}(\chi_{\mu_0}(0)-\mathrm{e}^{-\gamma_0 m q(0)})\right.\right.\\
&\quad\left.\left.+\int_{-1}^{0}\frac{\gamma_0 g'(0) r(s) g'''(0)\mu_0(\chi_{\mu_0}(s)-\mathrm{e}^{-\gamma_0 m q(s)})\bar\chi_{\mu_0}(s)}{g'(0)|\chi_{\mu_0}(s)|^2}\,\mathrm{d}s\right)\right)\\
&\cong -g'''(0)\,\mathrm{Re}\left((\bar\mu_0+\gamma_0 r g'(0))\left(\mu_0-\mathrm{e}^{-\gamma_0 m r}\right.\right.\\
&\quad\left.\left.+\int_{-1}^{0}\gamma_0 g'(0) r(s)(1-\mathrm{e}^{-\gamma_0 m q(s)}\bar\chi_{\mu_0})\,\mathrm{d}s\right)\right)\\
&= -g'''(0)\,\mathrm{Re}\left((\bar\mu_0+\gamma_0 r g'(0))\left(\mu_0-\mathrm{e}^{-\gamma_0 m r}+\gamma_0 g'(0) r\right.\right.\\
&\quad\left.\left.+\frac{1-\bar\mu_0\mathrm{e}^{-\gamma_0 m r}}{\mu_0-2\frac{m}{g'(0)}}\right)\right).
\end{aligned}$$

Introduce the notations

$$w = \frac{m}{g'(0)} \in (-1,0), \qquad \widetilde{w} = \frac{\arccos(w)}{\sin(\arccos(w))} > 0,$$

then

$$\gamma_0 = \frac{-\widetilde{w}}{r g'(0)} > 0.$$

By Lemma 1, one has

$$\mu_0 = w - \mathrm{i}\sqrt{1-w^2}, \qquad \bar\mu_0 = w + \mathrm{i}\sqrt{1-w^2}.$$

Adverting the relations

$$\gamma_0 g'(0) r = -\widetilde{w}, \qquad \gamma_0 m r = -\widetilde{w} w,$$

the expression in the argument of Re can be written as

$$\begin{aligned}
&(w+\mathrm{i}\sqrt{1-w^2}-\widetilde{w})\left(w-\mathrm{i}\sqrt{1-w^2}-\mathrm{e}^{\widetilde{w}w}-\widetilde{w}+\frac{1-(w+\mathrm{i}\sqrt{1-w^2})\mathrm{e}^{\widetilde{w}w}}{w-\mathrm{i}\sqrt{1-w^2}-2w}\right)\\
&\quad= (w+\mathrm{i}\sqrt{1-w^2}-\widetilde{w})(w-\mathrm{i}\sqrt{1-w^2}-\mathrm{e}^{\widetilde{w}w}-\widetilde{w}-(w-\mathrm{i}\sqrt{1-w^2})+\mathrm{e}^{\widetilde{w}w})\\
&\quad= (w+\mathrm{i}\sqrt{1-w^2}-\widetilde{w})(-\widetilde{w}),
\end{aligned}$$

which has the real part

$$\widetilde{w}(\widetilde{w}-w).$$

To finish the proof we only need some elementary calculus. The function

$$f(\xi) = \frac{\arccos(\xi)}{\sin(\arccos(\xi))} - \xi$$

has the derivative

$$f'(\xi) = -1 - \frac{1}{1-\xi^2} + \frac{\xi \arccos(\xi)}{(1-\xi^2)^{3/2}},$$

which is negative on $(-1, 0]$. Since $f(0) = \frac{\pi}{2} > 0$ and $f(\xi)$ is monotone decreasing on $(-1, 0]$, $f(\xi) > \frac{\pi}{2} > 0$ on $(-1, 0)$ and the aforesaid real part is positive for any $w \in (-1, 0)$. Hence $\delta(\gamma_0) \cong -g'''(0)$. $\square$

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