Problems of the 2014 Miklós Schweitzer Memorial Competition in Mathematics

22 October — 3 November

1. Let n be a positive integer. Let $\mathcal{F}$ be a family of sets that contains more than half of all subsets of an n-element set X. Prove that from $\mathcal{F}$ we can select $\lceil \log_2 n \rceil + 1$ sets that form a separating family on X, i.e., for any two distinct elements of X there is a selected set containing exactly one of the two elements.

2. Let $k \geq 1$ and let $I_1, \ldots, I_k$ be non-degenerate subintervals of the interval $[0, 1]$. Prove

$$\sum \frac{1}{|I_i \cup I_j|} \geq k^2,$$

where the summation is over all pairs (i, j) of indices such that I_i and I_j are not disjoint.

3. We have $4n + 5$ points on the plane, no three of them are collinear. The points are colored with two colors. Prove that from the points we can form n empty triangles (they have no colored points in their interiors) with pairwise disjoint interiors, such that all points occurring as vertices of the n triangles have the same color.

4. For a positive integer n, let $f(n)$ be the number of sequences $a_1, \ldots, a_k$ of positive integers such that $a_i \geq 2$ and $a_1 \ldots a_k = n$; $k \geq 0$ is arbitrary. ($f(1)=1$.) Let α be the unique real number greater than 1 such that $\sum_{n=1}^{\infty} n^{-\alpha} = 2$. Prove that

   (a)

$$\sum_{k=1}^{n} f(k) = O(n^{\alpha})$$

   and

   (b) there exists no number $\beta < \alpha$ such that $f(n) = O(n^{\beta})$.

5. Let α be a non-real algebraic integer of degree two, and let P be the set of irreducible elements of the ring $\mathbb{Z}[\alpha]$. Prove that

$$\sum_{p \in P} \frac{1}{|p|^2} = \infty.$$

6. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite p-group G over a field of characteristic p. Prove that if the restriction of the linear map $\sum_{g \in G} \rho(g)$ to a finite dimensional subspace W of V is injective, then the subspace spanned by the subspaces $\rho(g)W$ ($g \in G$) is the direct sum of these subspaces.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g : \mathbb{R} \to \mathbb{R}$ be arbitrary. Suppose that the Minkowski sum of the graph of f and the graph of g (i.e., the set $\{(x+y, f(x)+g(y)) : x, y \in \mathbb{R}\}$) has Lebesgue measure zero. Does it follow then that the function f is of the form $f(x) = ax + b$ with suitable constants $a, b \in \mathbb{R}$?

8. Let $n \geq 1$ be a fixed integer. Calculate the distance

$$\inf_{p,f} \max_{0 \leq x \leq 1} |f(x) - p(x)|,$$

where p runs over polynomials of degree less than n with real coefficients and f runs over functions

$$f(x) = \sum_{k=n}^{\infty} c_k x^k$$

defined on the closed interval $[0,1]$, where $c_k \geq 0$ and $\sum_{k=n}^{\infty} c_k = 1$.

9. Let $\rho\colon \mathbb{R}^n \to \mathbb{R}$, $\rho(\mathbf{x}) = e^{-||\mathbf{x}||^2}$, and let $K \subset \mathbb{R}^n$ be a convex body, i.e., a compact convex set with nonempty interior. Define the barycenter $\mathbf{s}_K$ of the body K with respect to the weight function ρ by the usual formula

$$\mathbf{s}_K = \frac{\int_K \rho(\mathbf{x})\mathbf{x} d\mathbf{x}}{\int_K \rho(\mathbf{x}) d\mathbf{x}}.$$

Prove that the translates of the body K have pairwise distinct barycenters with respect to ρ.

10. To each vertex of a given triangulation of the two-dimensional sphere, we assign a convex subset of the plane. Assume that the three convex sets corresponding to the three vertices of any two-dimensional face of the triangulation have at least one point in common. Show that there exist four vertices such that the corresponding convex sets have at least one point in common.

11. Let U be a random variable that is uniformly distributed on the interval $[0,1]$, and let

$$S_n = 2\sum_{k=1}^{n} \sin(2kU\pi).$$

Show that, as $n \to \infty$, the limit distribution of S_n is the Cauchy distribution with density function $f(x) = \frac{1}{\pi(1+x^2)}$.

Rules of the competition stipulate that solutions must be written in Hungarian. However, foreign students studying in Hungary are invited to submit solutions in English. These will be informally evaluated but will not be officially part of the competition.

Separate sheets of paper should be used for each problem. Solutions should be handed in no later than 12:00 on 3 November 2014 to the János Bolyai Mathematical Society (1055 Budapest, Falk Miksa u. 12. I/4.), or to the office of the Mathematics Institute at ELTE TTK (1117 Budapest, Pázmány P. sétány 1/C., 3rd floor, office 510), or posted by ordinary mail, by the same deadline, to

Frenkel Péter, ELTE TTK Mat. Intézet, Algebra és Számelmélet Tanszék,
1117 Budapest, Pázmány P. stny. 1/C.,

or by email, in PDF format, to `frenkelp265@gmail.com`. Please write your name on each sheet and your affiliation (e.g. CEU programme or BSM), postal address and email address on one of the sheets.