Problems of the Miklós Schweitzer Memorial Competition, 2012.

1. s there any real number α for which there exist two functions $f, g : \mathbb{N} \to \mathbb{N}$ such that

$$\alpha = \lim_{n\to\infty} \frac{f(n)}{g(n)},$$

but the function which associates to n the n-th decimal digit of α is not recursive?

2. Call a subset A of the cyclic group $(\mathbb{Z}_n, +)$ rich if for all $x, y \in \mathbb{Z}_n$ there exists $r \in \mathbb{Z}_n$ such that $x-r, x+r, y-r, y+r$ are all in A. For what α is there a constant $C_\alpha > 0$ such that for each odd positive integer n, every rich subset $A \subset \mathbb{Z}_n$ has at least $C_\alpha n^\alpha$ elements?

3. There is a simple graph which chromatic number is equal to k. We painted all of the edges of graph using two colors. Prove that there exist a monochromatic tree with k vertices.

4. Let K be a convex shape in the n dimensional space, having unit volume. Let $S \subset K$ be a Lebesgue measurable set with measure at least $1-\varepsilon$, where $0 < \varepsilon < 1/3$. Prove that dilating K from its centroid by the ratio of $2\varepsilon \ln(1/\varepsilon)$, the shape obtained contains the centroid of S.

5. Let V_1, V_2, V_3, V_4 be four dimensional linear subspaces in $\mathbb{R}^8$ such that the intersection of any two contains only the zero vector. Prove that there exists a linear four dimensional subspace W in $\mathbb{R}^8$ such that all four vector spaces $W \cap V_i$ are two dimensional.

6. Let A, B, C be matrices with complex elements such that $[A, B] = C, [B, C] = A$ and $[C, A] = B$, where $[X, Y]$ denotes the $XY - YX$ commutator of the matrices. Prove that $e^{4\pi A}$ is the identity matrix.

7. Let Γ be a simple curve, lying inside a circle of radius r, rectifiable and of length ℓ. Prove that if $\ell > kr\pi$, then there exists a circle of radius r which intersects Γ in at least $k+1$ distinct points.

8. For any function $f : \mathbb{R}^2 \to \mathbb{R}$ consider the function $\Phi_f : \mathbb{R}^2 \to [-\infty, \infty]$ for which $\Phi_f(x, y) = \limsup_{z\to y} f(x, z)$ for any $(x, y) \in \mathbb{R}^2$.

(a) Is it true that if f is Lebesgue measurable then Φ_f is also Lebesgue measurable?

(b) Is it true that if f is Borel measurable then Φ_f is also Borel measurable?

9. Let D be the complex unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, and $0 < a < 1$ a real number. Suppose that $f : D \to \mathbb{C} \setminus \{0\}$ is a holomorphic function such that $f(a) = 1$ and $f(-a) = -1$. Prove that

$$\sup_{z\in D} |f(z)| \geqslant \exp\left(\frac{1-a^2}{4a}\pi\right).$$

10. Let K be a knot in the 3-dimensional space (that is a differentiable injection of a circle into $\mathbb{R}^3$, and D be the diagram of the knot (that is the projection of it to a plane so that in exception of the transversal double points it is also an injection of a circle). Let us color the complement of D in black and color the diagram D in a chessboard pattern, black and white so that zones which share an arc in common are coloured differently. Define $\Gamma_B(D)$ the black graph of the diagram, a graph which has vertices in the black areas and every two vertices which correspond to black areas which have a common point have an edge connecting them.

(a) Determine all knots having a diagram D such that $\Gamma_B(D)$ has at most 3 spanning trees. (Two knots are not considered distinct if they can be moved into each other with a one parameter set of the injection of the circle)

(b) Prove that for any knot and any diagram D, $\Gamma_B(D)$ has an odd number of spanning trees.

11. Let $X_1, X_2, ..$ be independent random variables with the same distribution, and let $S_n = X_1 + X_2 + ... + X_n, n = 1, 2, ...$ For what real numbers c is the following statement true:

$$P\left(\left|\frac{S_{2n}}{2n} - c\right| \leqslant \left|\frac{S_n}{n} - c\right|\right) \geqslant \frac{1}{2}.$$