Congruence distributivity implies bounded width

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Abstract

We show that a constraint language with compatible Jónnson terms (i.e. associated with an algebra generating a congruence distributive variety) defines a Constraint Satisfaction Problem solvable by the local consistency checking algorithm.

1 Introduction

The Constraint Satisfaction Problem (CSP) is one of few problems central to the development of theoretical computer science. An instance of CSP consists of variables and constraints and the aim is to determine whether variables can be evaluated in such a way that all the constraints are satisfied. CSP provides a common framework for many problems in various areas of computer science; some of the most interesting algorithmic questions in database theory [28], machine vision recognition [23], temporal and spatial reasoning [27], technical design [25], scheduling [22], natural language comprehension [1] and programming language comprehension [24] are examples of constraint satisfaction problems.

The problem of solving CSP with arbitrary constraints is NP-complete, and therefore the research in this area is focused on solving CSP's with constraints taken from a fixed, finite set. More precisely, for any finite set of constraints (i.e. finitary relations) over a finite set we seek to determine the complexity of solving CSP restricted to instances with constraints coming exclusively from this set. The Dichotomy Conjecture of Feder and Vardi [15] postulates that every problem in such a family is NP-complete or solvable in polynomial time.

The Dichotomy Conjecture has proved to be a challenging question and the advances using standard methods were slow. A breakthrough in the development occurred when Jeavons, Cohen and Gyssens [18] announced an algebraic approach to the problem. Their work, refined later by Bulatov, Jeavons and Krokhin [9, 5], showed that the complexity of any particular CSP is fully determined by a set of functions – *polymorphisms* of the constraints. This allowed a rephrasing of the problem in algebraic terms and provided tools necessary to deal with conjectures open for years (for example [2]). More importantly, the algebraic approach allowed to conjecture a structure of problems solvable in polynomial time [9] and pointed out classes of problems that have to be characterized before the Dichotomy Conjecture can be attacked.

A positive verification of the Dichotomy Conjecture requires a construction of an algorithm (or a class of algorithms) unifying all known algorithms. A characterization of applicability classes of existing algorithms is crucial for constructing such a unification. In particular the class of *problems of bounded width* i.e. problems solvable by the widest known algorithm – the *Local Consistency Checking* algorithm has to be described. The only plausible conjecture on a structural characterization of this class was proposed by Larose and Zádori in [21]. They conjectured that a problem has bounded width if and only if the algebra associated with it (where operations of the algebra are polymorphisms of the constraints) generates a congruence meet semi-distributive variety.

A part of this conjecture that received most attention in the past years states that if the algebra associated with a set of constraints generates a congruence distributive variety then the associated CSP has bounded width. This class of problems is a natural first step towards verifying the conjecture of Larose and Zádori. Algebras generating congruence distributive varieties are equivalently described as ones with a chain of Jónnson terms and the first attempts of an attack resulted in proving the conjecture for chains of three [20] and four [13] terms. We employ an approach, different from the one from [20] and [13], based on the global behavior of the algorithm and prove bounded width for algebras with an arbitrary chain of Jónsson terms. We believe that the methods developed in this paper are crucial to a potential verification of the conjecture of Larose and Zádori.

2 Algebraic preliminaries

We briefly recall universal algebra notions and results, which will be needed in this article. For a more in depth introduction to universal algebra we recommend [12].

An n-ary *relation* on a set A is a subset of A^n and an n-ary *operation* on A is a mapping $A^n \to A$.

2.1 Relational structures

A *relational structure* is a tuple $\mathbb{A} = (A, R_0, R_1, \dots)$, where A is a set and $R_0, R_1 \dots,$ are relations on A. Relational structures $\mathbb{A} = (A, R_0, R_1, \dots)$, $\mathbb{B} = (B, S_0, S_1 \dots)$ have the same *type*, if they have the same number of relations and the relation R_i has the same arity as S_i for every i (denoted by ar_i). In this situation, a mapping $f : A \to B$ is called *a homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if it preserves all the relations, that is for every i and every $(a_1, \dots, a_{\mathrm{ar}_i}) \in R_i$, we have $(f(a_1), \dots, f(a_{\mathrm{ar}_i})) \in S_i$. $\mathbb{A}$ is *homomorphic* to $\mathbb{B}$ if there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

A structure $\mathbb{A}$ is a *core*, if every homomorphism $\mathbb{A} \to \mathbb{A}$ is bijective.

We say that a structure $\mathbb{A}$ is an *induced substructure* of $\mathbb{B}$, if $A \subseteq B$ and $R_i = S_i \cap A^{\mathrm{ar}_i}$ for every i. A *partial homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a homomorphism from an induced substructure of $\mathbb{A}$ to $\mathbb{B}$.

Corollary 2.1. *Let $\mathbb{A}, \mathbb{B}$ be relational structures, let p be a number greater or equal to arity of every relation in $\mathbb{A}$ and let $f : A \to B$ be a mapping. If, for every $K \subseteq A$ with $|K| \leq p$, the mapping $f_{|K}$ is a partial homomorphism from $\mathbb{A}$ to $\mathbb{B}$, then f is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.*

All relational structures in this paper are assumed to be finite and with finite number of relations.

2.2 Algebras and basic constructions

An *algebra* is a tuple $\mathbf{A} = (A, t_0, t_1, \dots)$, where A is a set (called a *universe*) and $t_0, t_1, \dots$ are operations on A. Similarly as with relational structures, algebras $\mathbf{A}, \mathbf{B}$ are of the same *type* if they have the same number of operations and corresponding operations have equal arities. By abuse of notation we denote operations of two algebras of the same type by the same symbols.

A mapping $f : A \to B$ is a homomorphism, if it preserves all the operations, that is $f(t_i(a_1, \dots, a_{\mathrm{ar}_i})) = t_i(f(a_1), f(a_2), \dots, f(a_{\mathrm{ar}_i}))$ for any i and any $a_1, a_2, \dots \in A$. A bijective homomorphism is an *isomorphism*.

A set $B \subseteq A$ is a *subuniverse* of an algebra $\mathbf{A}$ if, for any i, the operation t_i restricted to B^{n_i} has all the results in B. For a subuniverse B of an algebra $\mathbf{A}$ the algebra $\mathbf{B} = (B, t'_0, \dots)$ (where t'_i is a restriction of t_i to B^{n_i}) is a *subalgebra* of $\mathbf{A}$. A *term function* of an algebra is any function that can be obtained as a composition using the operations of the algebra together with all the projections. A set $C \subseteq A$ *generates* a subuniverse B in an algebra $\mathbf{A}$ if B is the smallest subuniverse containing C – such a subuniverse always exists and can be obtained by applying all the term functions of the algebra $\mathbf{A}$ to all the choices of arguments coming from C.

Given algebras $\mathbf{A}, \mathbf{B}$ of the same type, a *product* $\mathbf{A} \times \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is the algebra with universe $A \times B$ and operations are computed coordinatewise. *Subdirect product* of $\mathbf{A}$ and $\mathbf{B}$ is a subalgebra $\mathbf{C}$ of $\mathbf{A} \times \mathbf{B}$ such that the projections of C to A and B are full. For a set H, an *H-power* $\mathbf{A}^H$ of an algebra $\mathbf{A}$ has a universe A^H (the set of mappings from H to A) and the operations are again computed coordinatewise (the algebra $\mathbf{A}^H$ is naturally isomorphic to $\mathbf{A} \times \dots \times \mathbf{A}$, where the product is taken $|H|$-times.)

An equivalence relation $\sim$ on A is called a *congruence* of an algebra $\mathbf{A}$ if $\sim$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$. An equivalence is a congruence iff it is a kernel of some homomorphisms from $\mathbf{A}$.

A *variety* is a class of algebras of the same type closed under forming of subalgebras, products and homomorphic images. The smallest variety containing an algebra $\mathbf{A}$ is a variety *generated* by $\mathbf{A}$.

2.3 Congruence (semi)distributivity

The set of all congruences of an algebra $\mathbf{A}$ with the inclusion relation forms a lattice, that is a partially ordered set such that all two-element subsets $\{x, y\}$ have supremum $x \vee y$ and infimum $x \wedge y$. A lattice $\mathbb{L}$ is

- *distributive*, if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$ (equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$);
- *join semi-distributive*, if $a \vee b = a \vee c$ implies $a \vee (b \wedge c) = a \vee b$;
- *meet semi-distributive*, if $a \wedge b = a \wedge c$ implies $a \wedge (b \vee c) = a \wedge b$.

A variety $\mathcal{V}$ is called congruence distributive (join semi-distributive, meet semi-distributive) if all the algebras in $\mathcal{V}$ have distributive (join semi-distributive, meet semi-distributive) congruence lattices. If a variety is congruence distributive then it is congruence join semi-distributive; and if it is congruence join semi-distributive, then also congruence meet semi-distributive (but the latter implication is not true for a single lattice).

Congruence properties of a variety can often be characterized by existence of certain term functions. In the case of congruence distributivity such a characterization was given by Jónsson [19].

Definition 2.2. *A sequence $t_0, t_1, \dots, t_s$ of ternary operations on a set A is called a* Jónsson chain*, if for every $a, b, c \in A$*

$$\begin{array}{llcll}
(J1) & t_0(a,b,c) & = & a & \\
(J2) & t_s(a,b,c) & = & c & \\
(J3) & t_r(a,b,a) & = & a & \text{for all } r \leq s \\
(J4) & t_r(a,a,b) & = & t_{r+1}(a,a,b) & \text{for all even } r < s \\
(J5) & t_r(a,b,b) & = & t_{r+1}(a,b,b) & \text{for all odd } r < s
\end{array}$$

An algebra $\mathbf{A} = (A, t_0, \dots, t_s)$, where $t_0, \dots, t_s$ is a Jónsson chain, will be called a $CD(s)$-algebra.

Theorem 2.3. *An algebra $\mathbf{A}$ has a Jónsson chain of term functions, if and only if the variety generated by $\mathbf{A}$ is congruence distributive.*

Similar conditions are available for join and meet semi-distributivity [17, 29] as well. In both cases the characterization is obtained by weakening the condition $(J3)$ from Definition 2.2.

3 CSP and polymorphisms

The Constraint Satisfaction Problem can be defined in several ways. In this paper we use a formulation using homomorphisms of relational structures. For different descriptions and more information about the algebraic approach to CSP we recommend [5, 11].

Let $\mathbb{A}$ be a relational structure (with finite universe and finite number of operations, each of a finite arity). The Constraint Satisfaction Problem with template $\mathbb{A}$, CSP($\mathbb{A}$), is the following decision problem:

INPUT:	A relational structure $\mathbb{X}$ of the same type
QUESTION:	Does there exist a homomorphism $\mathbb{X} \to \mathbb{A}$?

Using this definition we can formulate the central problem in this area as follows:

Conjecture 3.1 (The Dichotomy Conjecture of Feder and Vardi [15]). *For every relational structure $\mathbb{A}$,* CSP($\mathbb{A}$) *is NP-complete or solvable in polynomial time.*

To state the Algebraic Dichotomy Conjecture, we need to introduce the notion of compatible operation and polymorphism. An operation $f : A^m \to A$ is *compatible* with a relation $R \subseteq A^n$ if

$$(f(a_{11}, \dots, a_{1m}), \dots, f(a_{n1}, \dots, a_{nm})) \in R$$

whenever $(a_{11}, \dots, a_{n1})$, $\dots$, $(a_{1m}, \dots, a_{nm}) \in R$. An operation $f : A^m \to A$ is a *polymorphism* of a relational structure $\mathbb{A}$ if it is compatible with all the relations of $\mathbb{A}$.

To every relational structure $\mathbb{A}$ we associate an algebra $\mathbf{A}$ which operations are all the polymorphisms of $\mathbb{A}$ (in arbitrarily chosen order). It is easy to see that every projection is a polymorphism and polymorphisms are closed under composition, therefore every term function of $\mathbf{A}$ is an operation of $\mathbf{A}$.

One of the crucial observations in the development of the algebraic approach is that the complexity of CSP($\mathbb{A}$) depends on $\mathbf{A}$ only [18, 9]. "Nice" properties of $\mathbf{A}$ ensures tractability of CSP($\mathbb{A}$), while "bad" properties of $\mathbf{A}$ cause NP-completeness of CSP($\mathbf{A}$). Bulatov, Jeavons and Krokhin [9] proved that if $\mathbb{A}$ is a core and the variety generated by $\mathbf{A}$ contains a G-set (i.e. an at least two-element algebra which every operation is of the form $f(x_1, \ldots, x_n) = g(x_i)$ for some permutation g) then CSP($\mathbb{A}$) is NP-complete. They conjectured that otherwise CSP($\mathbb{A}$) is tractable.

Conjecture 3.2 (The Algebraic Dichotomy Conjecture). *Let $\mathbb{A}$ be a core relational structure and $\mathbf{A}$ be the algebra associated to it. Then* CSP($\mathbb{A}$) *is solvable in polynomial time if the variety generated by $\mathbf{A}$ doesn't contain a G-set. Otherwise,* CSP($\mathbb{A}$) *is NP-complete.*

Note that the assumption that $\mathbb{A}$ is a core is not restrictive at all as it is easy to see that CSP($\mathbb{A}$) is equivalent to CSP($\mathbb{A}'$) for some core $\mathbb{A}'$.

All known results about complexity of CSP agree with Conjecture 3.2. It holds when A is a three-element set [8] (which generalizes the result of Schaefer for two-element relational structures [26]), $\mathbf{A}$ is a conservative algebra [6], $\mathbf{A}$ has few subalgebras of powers [3] (which generalizes [4] and [14]) and $\mathbb{A}$ is a digraph with no sources or sinks [2] (which generalizes [16]).

4 Bounded width, main theorem

Bounded width can be introduced in a number of equivalent ways (using duality, infinitary logic, pebble games, Datalog programs, strategies), see [21, 10]. We define it using the notion of (k, l)-strategy:

Definition 4.1. *Let $\mathbb{X}$, $\mathbb{A}$ be relational structures of the same type and $k \leq l$ be natural numbers. A set $\mathcal{F}$ of partial homomorphisms from $\mathbb{X}$ to $\mathbb{A}$ is called a* (k, l)-strategy *for $(\mathbb{X}, \mathbb{A})$, if it satisfies the following:*

($S1$) $|\operatorname{dom}(f)| \leq l$, *for any $f \in \mathcal{F}$.*

($S2$) *For any $f \in \mathcal{F}$ and any $K \subseteq \operatorname{dom}(f)$ the function $f_{|K}$ belongs to $\mathcal{F}$.*

($S3$) *For any $K \subseteq L \subseteq A$ with $|K| \leq k, |L| \leq l$, and $f \in \mathcal{F}$ with $\operatorname{dom}(f) = K$, there exists $g \in \mathcal{F}$ such that $\operatorname{dom}(g) = L$ and $g_{|K} = f$.*

For $K \subseteq X$ with $|K| \leq l$ the set of all partial homomorphisms from $\mathcal{F}$ with domain K will be denoted by $\mathcal{F}_K$, that is $\mathcal{F}_K = \mathcal{F} \cap X^K$.

A standard procedure [15] called (k, l)-consistency checking, finds the greatest (with respect to inclusion) (k, l)-strategy $\mathcal{F}$ for $(\mathbb{X}, \mathbb{A})$. The algorithm starts by throwing initially in $\mathcal{F}$ all partial homomorphisms (from $\mathbb{X}$ to $\mathbb{A}$) with domain of size less than l. Then we remove from $\mathcal{F}$ all those mappings which falsify one of the conditions ($S2$), ($S3$) and we repeat this iterative process until it stabilizes. It is not difficult to see that this algorithm runs in polynomial time with respect to $|X|$.

Observe that for any homomorphism $f : \mathbb{X} \to \mathbb{A}$ and any $K \subseteq X$ with $|K| \leq l$, the partial homomorphism $f_{|K}$ belongs to the strategy returned by the (k,l)-consistency algorithm. Therefore if the algorithm returns $\mathcal{F} = \emptyset$ then there is certainly no homomorphism from $\mathbb{X}$ to $\mathbb{A}$. The structure $\mathbb{A}$ is of *width* (k,l) if the converse is also true:

Definition 4.2. *A relational structure $\mathbb{A}$ has* width (k,l) *if for every relational structure $\mathbb{X}$, if there exists a nonempty (k,l)-strategy for $(\mathbb{X}, \mathbb{A})$ then $\mathbb{X}$ is homomorphic to $\mathbb{A}$.*

$\mathbb{A}$ *is said to be of* width k*, if it has width (k,l) for some l, and to be of* bounded width *if it has width k for some k.*

In other words, a relational structure $\mathbb{A}$ has bounded width, if there exist k, l such that we can use the (k,l)-consistency checking algorithm to solve CSP($\mathbb{A}$). As noted above, this algorithm works in polynomial time, thus if $\mathbb{A}$ has bounded width then CSP($\mathbb{A}$) is tractable.

Larose and Zádori [21] proved that if a core $\mathbb{A}$ has bounded width, then the variety generated by $\mathbf{A}$ is congruence meet semi-distributive and conjectured the converse:

Conjecture 4.3 (The Bounded Width Conjecture). *A core relational structure $\mathbb{A}$ has bounded width if and only if the variety generated by $\mathbf{A}$ is congruence meet-semidistributive.*

The conjecture was verified in the case that $\mathbf{A}$ has a semilattice operation [15], a near-unanimity operation [15], a 2-semilattice operation [7] a short Jónsson chain ([20] for $CD(3)$ and [13] for $CD(4)$). Our main theorem verifies this conjecture in the case of $\mathbf{A}$ with Jónsson chain of an arbitrarily length.

Theorem 4.4. *Let $\mathbb{A}$ be a relational structure such that the variety generated by $\mathbf{A}$ is congruence distributive. Then $\mathbb{A}$ has width $(2\lceil \frac{p}{2} \rceil, 3\lceil \frac{p}{2} \rceil)$, where p is the maximal arity of a relation in $\mathbb{A}$.*

The most natural next step for future research seems to be to generalize this theorem to the join semi-distributive case.

5 Reduction to $(2,3)$-systems

We prove Theorem 4.4 using a variation of a definition of a $(2,3)$-strategy for binary relational structures:

Definition 5.1. *An* $(2,3)$-system *is a set of finite $CD(s)$ algebras $\{\mathbf{B}_i, \mathbf{B}_{i,j} |\ i,j < n\}$ such that for any $i,j,k < n$:*

($B1$) *$\mathbf{B}_{i,j}$ is a subdirect product of $\mathbf{B}_i$ and $\mathbf{B}_j$;*

($B2$) *$\mathbf{B}_{i,i}$ is a diagonal subalgebra i.e. $B_{i,i} = \{(a,a) |\ a \in B_i\}$ for any i;*

($B3$) *$(a,b) \in B_{i,j}$ if and only if $(b,a) \in B_{j,i}$;*

($B4$) *if $(a,b) \in B_{i,j}$ then there exists $c \in B_k$ such that $(a,c) \in B_{i,k}$ and $(b,c) \in B_{j,k}$.*

A solution *of such a system is a tuple* $(b_0,\dots,b_{n-1})$ *such that* $(b_i,b_j)\in B_{i,j}$ *for any* $i,j<n$.

The following theorem is the core result of this paper.

Theorem 5.2. *Every* $(2,3)$*-system has a solution.*

We present a proof Theorem 4.4 using Theorem 5.2 which we prove in the next section.

Proof of Theorem 4.4. Let $\mathbb{A}$ be a relational structure such that the variety generated by $\mathbf{A}$ is congruence distributive, let p be the maximal arity of a relation in $\mathbb{A}$ and let $q=\lceil\frac{p}{2}\rceil$. According to Theorem 2.3, there exists a number s and term functions $t_0,\dots,t_s$ of $\mathbf{A}$ (= polymorphisms of $\mathbb{A}$) which form a Jónsson chain. Let $\mathbf{A}'$ denote the $CD(s)$-algebra $(A,t_0,\dots,t_s)$.

Let $\mathbb{X}$ be a relational structure of the same type as $\mathbb{A}$ and let $\mathcal{F}$ be a nonempty $(2q,3q)$-strategy for ($\mathbb{X}$, $\mathbb{A}$). Let $\mathcal{G}_K$ be the subuniverse of $\mathbf{A}^K$ generated by $\mathcal{F}_K$ for every $K\subseteq X$, $|K|\le 3q$. It is easy to see (and widely known) that the family $\mathcal{G}=\cup_{|K|\le l}\mathcal{G}_K$ is a $(2q,3q)$-strategy again. Thus we can without loss of generality assume that $\mathcal{F}_K$ is a subuniverse of $\mathbf{A}^K$ for every K such that $|K|\le 3q$.

We aim to show that there exists a homomorphism $f:\mathbb{X}\to\mathbb{A}$. We can assume that $|X|\ge 3q$, since otherwise any $f\in\mathcal{F}_X$ is a homomorphism.

We define a $(2,3)$-system indexed by a q-element subsets of X (instead of natural numbers). Let $K,L\subseteq X$ be such that $|K|=|L|=q$. We define the universes of the algebras in the $(2,3)$-system to be:

- $B_K=\mathcal{F}_K$,
- $B_{K,L}=\{(f,g)\in B_K\times B_L:\exists h\in\mathcal{F}_{K\cup L}\ h_{|K}=f,\ h_{|L}=g\}$.

Since $\mathcal{F}_K$ is a subuniverse of $(\mathbf{A}')^K$ (it is even subuniverse of $\mathbf{A}^K$), we can define $\mathbf{B}_K$ to be the subalgebra of $(\mathbf{A}')^K$ with universe B_K. As $\mathcal{F}_{K\cup L}$ is a subuniverse of $\mathbf{A}^{K\cup L}$ it follows that $B_{K,L}$ is a subuniverse of $\mathbf{B}_K\times\mathbf{B}_L$.

To prove that the projection of $B_{K,L}$ to the first coordinate is full, consider an arbitrary $f\in\mathcal{F}_K$. The property $(S3)$ of the strategy $\mathcal{F}$ tells us that there exists $h\in\mathcal{F}_{K\cup L}$ such that $h_{|K}=f$. From the property $(S2)$ we get $h_{|L}\in\mathcal{F}$, hence $(f,h_{|L})\in B_{K,L}$. By an analogous argument, the projection of $B_{K,L}$ to the second coordinate is also full and the property $(B1)$ is proved. The property $(B4)$ can proved similarly and $(B2)$, $(B3)$ hold trivially.

By Theorem 5.2 there exists a solution ($f_K:K\subseteq X$ and $|K|=q$) of this $(2,3)$-system. Note that for any K and L if $i\in K\cap L$ then, since $(f_K,f_L)\in B_{K,L}$, we have $f_K(i)=f_L(i)$. Therefore there exists a (unique) function $f: X\to A$ such that $f_{|K}=f_K$ for any $K\subseteq X$, $|K|\le q$; and $f_{|K\cup L}$ is a partial homomorphism from $\mathbb{X}$ to $\mathbb{A}$ for any $K,L\subseteq X$ with $|K|,|L|\le q$. Now it is enough to apply Corollary 2.1. $\square$

6 Proof of Theorem 5.2

In this section we prove the core result 5.2 stating that every $(2,3)$-system $\{\mathbf{B}_i,\mathbf{B}_{i,j}|\ i,j<n\}$ has a solution. So, we assume that $\mathbf{B}_i=(B_i,t_0,\dots,t_s)$ and $\mathbf{B}_{i,j}=(B_{i,j},t_0,\dots,t_s)$ are $CD(s)$ algebras satisfying $(B1-4)$.

6.1 Patterns and realizations

We require the following definitions:

Definition 6.1. *A* pattern *is a finite sequence of natural numbers smaller than n. A concatenation of patterns is performed in a natural way: for patterns w, v we write wv for a pattern equal to concatenation of patterns w and v and w^k for a pattern equall to a k-ary concatenation of w with itself. We write w^{-1} for a pattern with reversed order and set $w^{-k} = (w^{-1})^k$ for any k.*

A sequence $a_0, \dots a_l$ is called a realization *of a pattern $w = (w_0, \dots, w_l)$, if $a_i \in B_{w_i}$ for all $i \le l$ and $(a_i, a_{i+1}) \in B_{w_i, w_{i+1}}$ for all $i < l$.*

We say that two elements $a \in B_i, b \in B_j$ are connected via a pattern $w = (i, \dots, j)$ if there exists a realization $a = a_0, a_1, \dots, a_l = b$ of the pattern w.

The following lemma is an easy consequence of the property $(B4)$.

Lemma 6.2. *Let $(a, b) \in B_{i,j}$ then a and b can be connected via any pattern beginning with i and ending with j.*

Proof. Let $(a, b) \in B_{i,j}$ and let $w = (i = w_0, \dots, w_m = j)$ be a pattern. Using $(B4)$ from the definition of $(2, 3)$-system to (a, b) and the coordinates i, j, w_1 we obtain $c_0 \in B_{w_1}$ such that $(a, c_0) \in B_{i,w_1}$ and $(c_0, b) \in B_{w_1,j}$. The element c_0 is second (after a) element of a realization of the pattern w. Continuing the reasoning we use $(B4)$ to $(c_0, b) \in B_{w_1,j}$ and the coordinates w_1, j, w_2 to obtain c_2 – the third element of a realization of w. Repeated application of this reasoning produce a realization of a pattern w connecting a to b. □

The lemma implies the following corollary.

Corollary 6.3. *Let w be a pattern starting and ending with i. If $a, b \in B_i$ can be connected via a pattern v starting and ending at i and using only numbers occurring in w then a and b can be connected via the pattern w^M for an appropriate large number M.*

Proof. Let $v = (i = v_0, \dots, v_l = i)$ and let $a = a_0, \dots, a_l = b$ be a realization of the pattern v. Since v_1 appears in w there exists an initial part of w, say w', starting with m and ending with v_1. Since $(a, a_1) \in B_{i,v_1}$ we use Lemma 6.2 to connect a to a_1 via w'. Since v_2 appears in w there exists w'' such that $w'w''$ is an initial part of w^2 and such that w'' ends in v_2. Since $(a_1, a_2) \in B_{v_1,v_2}$ we use Lemma 6.2 again to connect a_1 to a_2 via a pattern $v_1 w''$. Now a_0 and a_2 are connected via the pattern $w'w''$. By continuing this reasoning we obtain the pattern w^M (for some M) connecting a to b. □

6.2 Absorbing systems

Definition 6.4. *We say that $C \subseteq B_i$ absorbs B_i, if $t_r(c, b, c') \in C$ for any $c, c' \in C, b \in B_i$ and $r \le s$. (In particular, C_i is a subuniverse of $\mathbf{B}_i$.)*

For a subset C of B_i and $j < n$ we put

$$\gamma_{i,j}(C) = \{b \in B_j : \text{ there exists } a \in C \text{ such that } (a, b) \in B_{i,j}\}.$$

The following lemma lists some properties of absorbing sets.

Lemma 6.5. *Let $i, j < n$ and $C, D \subseteq B_i$.*

(i) The set $\{b\}$ absorbs B_i for any $b \in B_i$.

(ii) If C and D absorb B_i, then $C \cap D$ absorbs B_i as well.

(iii) If C absorbs B_i then $\gamma_{i,j}(C)$ absorbs B_j.

Proof.

(i) It follows from the property $(J3)$ of Jónsson chain.

(ii) Obvious.

(iii) Let $e, e' \in \gamma_{i,j}(C)$, $d \in B_j$ and $r \leq s$. As $B_{i,j}$ is subdirect (see $(B1)$), there exists $b \in B_i$ such that $(b, d) \in B_{i,j}$. Since $e, e' \in \gamma_{i,j}(C)$ there exist $c, c' \in C$ such that $(c, e) \in B_{i,j}$ and $(c', e') \in B_{i,j}$. The set $B_{i,j}$ is a subuniverse of $\mathbf{B}_i \times \mathbf{B}_j$ (see $(B1)$ again), therefore $(t_r(c, b, c'), t_r(e, d, e')) \in B_{i,j}$. Now $t_r(c, b, c') \in C$, because C absorbs B_i, hence $t_r(e, d, e') \in \gamma_{i,j}(C)$.

$\square$

Definition 6.6. *A sequence $C_0, C_1, \ldots, C_{n-1}$ is an* absorbing system *if for any numbers $i, j < n$*

$(A1)$ $\emptyset \neq C_i \subseteq B_i$,

$(A2)$ $\gamma_{i,j}(C_i) \supseteq C_j$ *and*

$(A3)$ C_i *absorbs* B_i.

For such an absorbing system we define

$$\delta_{i,j}(D) = \gamma_{i,j}(D) \cap C_j \quad \text{where } D \subseteq C_i$$

and similarly for any pattern $w = (w_0, \ldots, w_l)$ and any $D \subseteq C_{w_0}$ we write

$$\delta_w(D) = \delta_{w_{l-1},w_l}(\cdots \delta_{w_1,w_2}(\delta_{w_0,w_1}(D))) \cdots).$$

Observe that $B_0, B_1, \ldots, B_{n-1}$ is an absorbing system.

The following lemma states a crucial property of absorbing systems.

Lemma 6.7. *Let $C_0, \ldots, C_{n-1}$ be an absorbing system and let $D_0, \ldots, D_l$ and $m_0, \ldots, m_l$ be such that for any $0 \leq i < n$*

- $m_0 = m_l$ *and* $D_0 = D_l$,
- $D_i \subseteq C_{m_i}$
- $\delta_{m_i,m_{i+1}}(D_i) = D_{i+1}$;

then for any $i, j < n$

$$\delta_{m_i,m_j}(D_i) = D_j.$$

Proof. Let $w = (m_0, \dots, m_l)$ be the pattern derived from the sequence used in a statement of the lemma. Note that, under our assumptions, $\delta_w(D_0) = D_0$.

We will show that for any $i, j < n$ we have $\delta_{m_i,m_j}(D_i) \subseteq D_j$. Suppose, for a contradiction (choosing without loss of generality the coordinate m_0), that there exists $a_0 \in C_{m_0} \setminus D_0$ such that for some $a_1 \in D_k$ we have $(a_1, a_0) \in B_{m_k,m_0}$. Since $\gamma_{m_{l-1},m_0}(C_{m_{l-1}}) \subseteq C_{m_0}$ there exists $a_{-1} \in C_{m_{l-1}}$ such that $(a_{-1}, a_0) \in B_{m_{l-1},m_0}$ and, since $a_0 \notin D_0$ and $\delta_{m_{l-1},m_0}(D_{l-1}) = D_0$, we get $a_{-1} \in C_{m_{l-1}} \setminus D_{l-1}$. Repeating the same reasoning for a_{-1} instead of a_0 we obtain a_{-2} and further on we obtain an infinite sequence of a_i's (for negative i's) and as the set C_{m_0} is finite we get $a' \in C_{m_0} \setminus D_0$ such that a' can be connected to itself via a pattern w^{-M} (for some number M) realized *fully* inside the absorbing system and can be connected to a_0 via a pattern containing only numbers from the set $\{m_0, \dots, m_l\}$. Reversing a direction of a realization of the pattern w^{-M} we can connect a' to itself via the pattern w^M realized also fully in the absorbing system (and obviously M can be substituted by any multiple of M).

Moreover, since a_1 is in D_k and $\delta_{m_{k-1},m_k}(D_{k-1}) = D_k$ there exists an element $a_2 \in D_{k-1}$ such that $(a_2, a_1) \in B_{m_{k-1},m_k}$ and proceeding further as in a previous case we obtain an infinite sequence of a_i's (with positive i's this time) and then an element $a'' \in D_0$ connected to itself via pattern the w^N (for some number N) realized *fully* inside the absorbing system and connected to a_1 via a pattern containing only numbers from the set $\{m_0, \dots, m_l\}$

We know that a' is connected to a_0 via pattern containing only numbers from the set $\{m_0, \dots, m_l\}$ and a'' is connected to a_1 via a pattern containing only numbers from $\{m_0, \dots, m_l\}$. Since $(a_1, a_0) \in B_{m_k,m_0}$ we obtain a connection from a' to a'' containing only numbers from $\{m_0, \dots, m_l\}$. Using Lemma 6.2 to elements a', a'' and the pattern w^{MN} we immediately obtain two more numbers K and L such that a' can be connected to a'' via a pattern w^{MNK} and a'' can be connected to a' via a pattern w^{MNL} (none of the realizations of these patters has to be inside the absorbing system).

Let $a'' = b_0, b_1, \dots, b_{MNL} = a''$, $a' = c_0, c_1, \dots, c_{MNL} = a'$ and $a'' = d_0, d_1, \dots, d_{MNL} = a'$ be realizations of pattern w^{MNL}, where the elements $b_0, b_1, \dots, c_0, c_1, \dots$ are inside the absorbing system. From the property $(B1)$ of the $(2,3)$-system it follows that for any $r \leq s$

$$t_r(a'', a'', a') = t_r(b_0, c_0, d_0),\ t_r(b_1, c_1, d_1),\ \dots$$

$$\dots,\ t_r(b_{MNL}, c_{MNL}, d_{MNL}) = t_r(a'', a', a')$$

is a realization of the pattern w^{MNL}. The absorbing property $(A3)$ implies that this realization lies inside the absorbing system. Similarly, using a realization of a pattern connecting a'' to a' we infer that one can connect $t_r(a'', a', a')$ to $t_r(a'', a'', a')$ via a realization of a pattern w^{MNK} fully inside the absorbing system. By using the properties $(J1), (J2), (J4), (J5)$ of the Jónsson chain, we obtain a realization of a big power of a pattern w connecting a'' to a' fully inside the absorbing system — $a'' = t_0(a'', a'', a')$ is connected to $t_0(a'', a', a')$ via w^{MNL}, $t_0(a'', a', a') = t_1(a'', a', a')$ is connected to $t_1(a'', a'', a')$ via w^{MNK}, and so on. Since $\delta_w(D_0) = D_0$ and $a'' \in D_0$ this contradicts $a' \notin D_0$.

We have proved that $\delta_{m_i,m_j}(D_i) \subseteq D_j$ for any $i, j < n$. Since $\delta_{m_i,m_j}(D_i) \subseteq D_j$ then $\delta_{m_j,m_i}(D_j) \supseteq D_i$; on the other hand $\delta_{m_j,m_i}(D_j) \subseteq D_i$ and the lemma is proved. $\square$

We compare absorbing systems by inclusion on all of the coordinates i.e. an absorbing system $C_0, \ldots, C_{n-1}$ is *smaller* than an absorbing system $C'_0, \ldots, C'_{n-1}$ if, for all $i < n$, $C_i \subseteq C'_i$. Minimal elements of this ordering are the smallest possible:

Lemma 6.8. *If $C_0, \ldots, C_{n-1}$ is an absorbing system minimal under inclusion, then every set C_i contains exactly one element.*

Proof. Suppose for a contradiction that one of the sets in the system, say C_0, has more than one element and let $\{a\} \subsetneq C_0$. Let

$$Z = \{(E, i) : i < n,\ E \subsetneq C_i,\ E = \delta_w(\{a\}) \text{ for some pattern } w = (0, \ldots, i)\}.$$

Note that for any $(E, i) \in Z$ and any pattern w starting at i and ending at j, either $\delta_w(E) = C_j$ or $(\delta_w(E), j) \in Z$.

Let $(E, i) \in Z$ be arbitrary. The set E was obtained from $\{a\}$ by applying the operation γ and taking intersections with elements of the absorbing system $C_0, \ldots, C_{n-1}$. From Lemma 6.5 it follows that E absorbs B_i.

Consider the relation $\preceq$ on Z defined by

$$(E, i) \preceq (E', i') \text{ iff } E' = \delta_w(E) \text{ for some pattern } w = (i, \ldots, i')$$

Obviously, $\preceq$ is a quasiorder (i.e. a reflexive and transitive relation). Let $(E, k) \in Z$ be a maximal element of this quasiorder. From the maximality, we get that for any $j < n$ and any pattern $w = (k, \ldots, j)$, either $\delta_w(E) = C_j$ or there exists a pattern $v = (j, \ldots, k)$ such that $\delta_v(\delta_w(E)) = E$.

We will show that the sequence $E_0 = \delta_{k,0}(E), \ldots, E_{n-1} = \delta_{k,n-1}(E)$ is an absorbing system smaller than $C_0, \ldots, C_{n-1}$. As $E_k = E \subsetneq C_i$, the new system will be smaller. We already know that E_i absorbs B_i for every $k < n$, so it remains to prove the property $(A2)$ – for any $i, j < n$ we have to show that $\delta_{i,j}(E_i) \supseteq E_j$. If $\delta_{i,j}(E_i) = C_j$, then the inclusion holds trivially. As observed above, in the other case, there exists a pattern $v = (j = v_0, v_1, \ldots, v_l = k)$ such that

$$E = \delta_v(\delta_{i,j}(E_i)).$$

In such a case Lemma 6.7 used for the sequence

$$\begin{gathered} m_0 = k, D_0 = E, \quad m_1 = i, D_1 = E_i, \quad m_2 = j, D_2 = \delta_{i,j}(E_i), \\ m_3 = v_1, D_3 = \delta_{(i,j,v_1)}(E_i), \quad m_4 = v_1, D_4 = \delta_{(i,j,v_0,v_1)}(E_i), \quad \ldots \\ \ldots, m_{l+3} = v_l = k, D_{l+3} = \delta_{(i,j,v_0,\ldots,v_l)}(E_i) = D_0 \end{gathered}$$

provides $\delta_{k,j}(D_0) = D_2$. But $\delta_{k,j}(D_0) = E_j$ and $D_2 = \delta_{i,j}(E_i)$ and the proof is concluded.

□

6.3 Conclusion

We are ready to finish the proof of Theorem 5.2. Let us choose any minimal (with respect to inclusion) absorbing system $D_0, \ldots, D_{n-1}$. By Lemma 6.8 it consists of one element sets. Call b_j the unique element of D_j. The $(A2)$ property guarantees that $(b_j, b_k) \in B_{j,k}$ for all $j, k < n$ and therefore $b_0, \ldots, b_{n-1}$ constitutes a solution.

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