Boundedness and Periodic Solutions of Functional Differential Equations

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This dissertation concerns mainly finite or infinite delay differential equations of the form

$$x' = f(t, x_t).$$

First, we prove uniform boundedness and uniform ultimate boundedness for these equations; these properties are usually used for proving that there is a T-periodic solution. We also give examples illustrating the different properties and showing applications of our theorems. Later we present theorems showing dissipativeness of finite delay differential equations. We also restate a consequence of a (seemingly not very well known) result of Hale and Lopes and prove it in a short elementary way. This result shows that a dissipative T-periodic finite delay differential equation has a T-periodic solution. Since the dissipativeness is a weaker property than uniform ultimate boundedness, this last result is better than the one using uniform boundedness and uniform ultimate boundedness. Last, but not least, we consider linear differential and integral equations. For these special equations we need even less (the existence of a bounded (uniformly continuous) solution) to prove that there is a T-periodic solution. Examples are given to show how to find a bounded solution for these equations.

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0. Introduction

Differential equations are functional equations in which the derivative of the unknown function appears as well as the value of the function. Many phenomena in physics, biology and chemistry are described by differential equations. Differential equations can be used to study any time-dependent process for which the speed of change in the process's status can be written as a function of the current and any previous status. The above classical view of the investigators is coming from applications: the future behavior of an object is completely determined by its present and past history. From applications we also know that many processes are periodic: a running engine, the Earth going around the Sun, populations of predator and prey, for example. Thus, it is of prime importance to consider periodic differential equations and find conditions implying a periodic behavior of the solution. This dissertation is devoted to this subject.

Let us introduce our notation in view of the above remarks. Suppose that we can describe the status of a process by a d-dimensional real vector x, which is changing with time, usually denoted by t. So, the status of the process at a time t is $x(t)$, where x is a function mapping an interval of real numbers into the d-dimensional real vectors. The present and past of the process at time t is a function x_t defined by $x_t(s) := x(t+s)$ for $s \leq 0$ as long as $x(t+s)$ is defined. Thus, x_t is a function mapping an interval $[-h, 0]$ into $\mathbf{R}^d$, where h is a non-negative number or ∞. In the later case we naturally mean $[-h, 0] = (-\infty, 0]$. We separate the differential equations into two classes.

If the future status of the process depends only on the present and recent past of the process, then we say that the differential equation is of finite delay. In this case we denote by h the time-displacement so that if $t_0 \in \mathbf{R}$ and $t \geq t_0$ then $x(t)$ depends only on $x(s)$ for $s \in [t_0 - h, t_0]$. In this case $x_t(s) := x(t+s)$ for $s \in [-h, 0]$ and $x_t \in \mathcal{C} := \{\phi : [-h, 0] \to \mathbf{R}^d : \phi \text{ is continuous}\}$, the set of "past functions". The requirement, that ϕ must be continuous, is not really a restriction since we always use the derivative of x in the differential equation. To be able to talk about any kind of continuity we need a topology on $\mathcal{C}$; in this case we use the norm $\|\phi\| := \sup_{s\in[-h,0]} |\phi(s)|$, where $|\cdot|$ is an arbitrarily fixed norm in $\mathbf{R}^d$. Now we are ready to write down the finite delay differential equation

$$x'(t) = f(t, x_t)$$

where $'$ denotes the derivative with respect to t and f is a function mapping $\mathbf{R}\times\mathcal{C}$ into $\mathbf{R}^d$. In the unknown function x the argument t is usually suppressed, so the above equation is written as

$$x' = f(t, x_t). \tag{1}$$

The other case is when the future status of the process depends on the present and the whole past status of the process. In this case we are talking about infinite delay differential equations. Using a notation similar to the one above let $\mathcal{C} := \{\phi : (-\infty, 0] \to \mathbf{R}^d : \phi \text{ is bounded and continuous}\}$, $\|\phi\| := \sup_{s\leq 0} |\phi(s)|$ and $x_t \in \mathcal{C}$ is defined by $x_t(s) := x(t+s)$ for $s \leq 0$. The infinite delay differential equation

$$x' = f(t, x_t) \tag{2}$$

formally looks the same as the finite delay equation, but x_t has a different meaning in this case.

In both cases to start a process we need an initial time (denoted by t_0) and a "past", that is a function $\phi \in \mathcal{C}$. We are looking for a function x which satisfies the differential equation for $t > t_0$ and $x_{t_0} = \phi$. From the above inexact observations we expect a unique solution $x(t, t_0, \phi)$ defined for $t \in [t_0 - h, K]$ for some $K > t_0$ for the so called initial value problem

$$x' = f(t, x_t) \qquad x_{t_0} = \phi.$$

Mathematically, it has been proved [2] that if f is continuous (with the topology defined on $\mathcal{C}$) then $x(t, t_0, \phi)$ really exists for some $K > t_0$. Also, if f is locally Lipschitz in ϕ (i.e., if for all $(t, \phi) \in \mathbf{R} \times \mathcal{C}$ there are a $\delta > 0$ and an $L > 0$ such that if $t' \in \mathbf{R}$ and $\phi', \phi'' \in \mathcal{C}$ with $|t - t'| < \delta$, $\|\phi - \phi'\| < \delta$ and $\|\phi - \phi''\| < \delta$ then $|f(t', \phi') - f(t', \phi'')| \leq L\|\phi' - \phi''\|$) then $x(t, t_0, \phi)$ is unique and depends continuously on t_0 and ϕ (in our terminology, it depends continuously on the initial data). We always assume in the following that there is a unique solution of the initial value problem which depends continuously on the initial data.

We have mentioned before the importance of periodic processes. A periodic process in our terminology is a T-periodic solution of an initial value problem for some $T > 0$. Clearly, one cannot expect a T-periodic solution unless f is T-periodic in t. Thus, in most of the cases we assume that $f(t+T,\phi) = f(t,\phi)$.

We now summarize our results. We will give all the necessary definitions in the appropriate chapter.

In the first chapter we consider a conjecture coming from the theory of ordinary differential equations by which one hopes to prove uniform boundedness and uniform ultimate boundedness for functional differential equations. We are interested in uniform boundedness and uniform ultimate boundedness, because these conditions are often used to prove the existence of a T-periodic solution for a T-periodic differential equation (see e.g. [3, 7]). We show by an example that this conjecture is false, we cannot prove uniform ultimate boundedness. This example also solves an old problem on the asymptotic stability of functional differential equations. We also consider theorems generalizing a result of Burton [4]. We use Liapunov functionals to prove uniform boundedness and uniform ultimate boundedness at some point for general (non-periodic) finite and infinite delay differential equations. We see from [4] that this is enough to have a periodic solution. We also give examples illustrating the different properties and applications of our theorems.

The next chapter concerns T-periodic finite delay differential equations. The first two theorems show dissipativeness for these equations generalizing a result of Pliss [23]. In the rest of this chapter we restate a consequence of a (seemingly not very well known) result of Hale and Lopes [12] and prove it in a short elementary way. This result shows that a dissipative T-periodic finite delay differential equation has a T-periodic solution. Using this theorem we can prove theorems using Liapunov functionals with weaker conditions than the ones in the previous chapter, that show the existence of a periodic solution. We also give an example showing that dissipativeness is a weaker property than uniform ultimate boundedness.

In the last chapter we consider T-periodic linear differential equations with finite or infinite delay. We show that if there is a bounded solution of the equation, then there is a T-periodic solution as well. We will see that dissipativeness is a stronger property than the existence of a bounded solution and hence this result is better than the one in the previous chapter, but it works only for linear equations. We give a proof for finite delay differential equations, which is different from that of the result of Chow [8] and works for infinite delay differential equations as well. We also prove that for linear integral equations the existence of a uniformly continuous bounded solution implies the existence of a T-periodic solution. At the end examples are given showing how to prove the existence of a bounded solution (and hence a T-periodic solution) for ordinary and infinite delay differential equations.

1. Uniform boundedness and uniform ultimate boundedness

In this chapter we study some conditions in connection with uniform boundedness (UB) and uniform ultimate boundedness (UUB). We give the definitions of some properties (called UB and UUB at a time t_0), that are weaker than UB and UUB. Then we prove theorems for finite and infinite delay equations using Liapunov functionals, that ensure these properties. We also give examples to illustrate the connections and differences between the definitions. The importance of this work can be seen by noting that if solutions are UB and UUB at t_0 and the right-hand side of the equation is T-periodic in t, then there is a T-periodic solution (see e.g. [3] for the finite and [7] for the infinite delay case). Also, there are several spaces that can be considered in the infinite delay case, He, Huang and Wang [15] show some of them and prove theorems using Liapunov functionals to ensure UB and UUB. Hering [16, 17] worked with spaces and methods to prove UB and UUB similar to those we are concerned with here.

1.1. Definitions, preliminary remarks

The following definitions and theorems can be stated in the infinite delay case also; for simplicity we consider the finite delay case only.

First, we introduce our definitions of uniform boundedness and uniform ultimate boundedness. They can be found with $t_0 = 0$ in [3] but we need them for arbitrary t_0.

Definition. We say that the solutions of (1) are uniform bounded at t_0 (UB(t_0)) if for all $B_1 > 0$ there is a $B_2 > 0$ such that $t \geq t_0$ and $\phi \in \mathcal{C}$ with $\|\phi\| < B_1$ imply $|x(t, t_0, \phi)| < B_2$.

Definition. We say that the solutions of (1) are uniform ultimate bounded at t_0 (UUB(t_0)) if there is a $B > 0$ such that for all $B_3 > 0$ there is a $K > 0$ such that $t \geq t_0 + K$, $\phi \in \mathcal{C}$ with $\|\phi\| < B_3$ imply $|x(t, t_0, \phi)| < B$.

These conditions are clearly generalizations of the properties known as uniform boundedness and uniform ultimate boundedness, since in those we have the constants B_2, B and K independent of t_0 and the estimations for all t_0. With $h = 0$ these definitions work for ordinary differential equations as well.

We are also going to use a functional V mapping $\mathbf{R} \times \mathcal{C}$ into $\mathbf{R}$, which is usually called a Liapunov functional. For simplicity we always assume that V is continuous and locally Lipschitz in ϕ. We define the derivative of V with respect to the equation (1) or (2) by

$$V'(t, x_t) := \limsup_{\delta \to 0} \frac{1}{\delta} \left\{ V(t+\delta, x_{t+\delta}(\cdot, t, \phi)) - V(t, x_t(\cdot, t, \phi)) \right\},$$

where x is a solution of the appropriate equation. We note that if V has continuous first partial derivatives then we can compute V' by the chain rule for any fixed solution x of (1) or (2), because V' is actually the derivative of the function $v(t) := V(t, x_t)$ with respect to t. Although we do not know the solution we know that $x' = f(t, x_t)$ and we can use this in the chain rule. The beauty of this procedure is that V does not have anything to do with our equation, but V' does. The main idea of Liapunov's direct method we are using here is that using V as a measuring functional we can tell something about the solution. Generally we ask that V is non-negative and V' is (although not necessarily non-positive) "negative enough" to show that solutions are bounded, uniformly bounded, uniformly ultimately bounded, stable or asymptotically stable.

We also use non-negative, continuous and monotone increasing functions W with $W(0) = 0$. These functions are called wedges, the set of these functions is denoted by $\mathcal{K}$.

In this chapter we are working with Liapunov functionals to prove UB and UUB at some t_0. The following theorem gives the reason to do so.

THEOREM 1.1. Suppose that $f(t+T, \phi) = f(t, \phi)$ and for all $B > 0$ there is an $L > 0$ such that $(t, \phi) \in \mathbf{R} \times \mathcal{C}$ with $\|\phi\| \le B$ implies $|f(t, \phi)| \le L$. Also, assume that the solutions of (1) are UB and UUB at t_0. Then there is a T-periodic solution of (1).

PROOF. The proof is the same as that of Theorem 4.2.2 in [3], the only difference is that instead of starting the solutions at 0 we need to start them at t_0.

Actually, that theorem is the same as this one with $t_0 = 0$ and does not use the stronger UB or UUB.

One can state a similar theorem for infinite delay equations too. The condition on f we have in this theorem is called the boundedness condition. Note that if f is locally Lipschitz (in the sense of definition 3.5.3 in [3] and not our definition in the introduction), then we have this condition. To distinguish between the two definitions in the following we call our local Lipschitz condition weak and the other one strong. If we do not state which one we are using, we are using the weak one.

Another interesting connection between these properties is shown in the following

THEOREM 1.2. If the solutions of (1) are uniformly bounded, if they are uniformly ultimately bounded at $\bar{t}_0$, and if the right-hand side of (1) is T-periodic in t, then the solutions are uniformly ultimately bounded.

PROOF. Let $B > 0$ be that of uniform ultimate boundedness at $\bar{t}_0$. Since (1) is T-periodic, using a translation argument it is easy to prove that the solutions are uniformly ultimately bounded at any $\bar{t}_n = \bar{t}_0 + nT$ with the same bound B. Moreover, K is independent of n in the uniform ultimate boundedness. Let $B_3 > 0$ be given, define B_4 to be the B_2 of uniform boundedness for $B_1 = B_3$. For this B_4 we find a $K > 0$ from the uniform ultimate boundedness at $\bar{t}_n$ (for any n). Now, let $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ be given with $\|\phi\| < B_3$. Choose the smallest $\bar{t}_n$ with $t_0 \leq \bar{t}_n$. From the uniform boundedness we have $\|x_{\bar{t}_n}(\cdot, t_0, \phi)\| < B_4$ and hence from the uniform ultimate boundedness $\left|x(\bar{t}_n + t, t_0, \phi)\right| < B$ for $t \geq K$. This proves that if $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ with $\|\phi\| < B_3$ then $|x(t_0 + t, t_0, \phi)| < B$ for $t \geq K + T$, which was to be shown.

1.2. Examples

Using a similar result [Theorem 4.1.16, 3] of the theory of ordinary differential equations one can observe the following

CONJECTURE A. Suppose that there are a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, functions $W_i \in \mathcal{K}$ and a constant $U \geq 0$ such that

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'(t, \phi) \leq -W_3(|\phi(0)|)$ for $|\phi(0)| \geq U$ and

(iii) $W_1(r) \to \infty$ as $r \to \infty$.

Then the solutions of (1) are uniformly bounded and uniformly ultimately bounded.

Many authors are investigating how one can get uniform boundedness and uniform ultimate boundedness using Liapunov functionals; a good reference list can be found in [5]. We will prove that the conjecture is false; we cannot prove uniform ultimate boundedness.

We also study the asymptotic stability (see e.g. [3]) of the zero solution of equation (1) using Liapunov functionals. In this topic the following two theorems are basic :

THEOREM A (see [13, Theorem 4.1]). Suppose that for every $M > 0$ there is an $L(M) > 0$ such that

$$[t \in \mathbf{R},\ \phi \in \mathcal{C}_M := \{\phi \in \mathcal{C} :\ \|\phi\| \leq M\}] \Longrightarrow |f(t, \phi)| \leq L(M).$$

Also, assume that there are $M > 0$, a functional $V : \mathbf{R} \times \mathcal{C}_M \to \mathbf{R}$ and functions $W_1, W_2, W_3 \in \mathcal{K}$ such that

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'(t, \phi) \leq -W_3(|||\phi|||)$

for all $t \in \mathbf{R}$, $\phi \in \mathcal{C}_M$, where

$$|||\phi||| = \sqrt{\int_{-h}^{0} |\phi(s)|^2\, ds}.$$

Then the zero solution of equation (1) is uniformly asymptotically stable.

THEOREM B ([11, Theorem 5.2.1]). Suppose that all conditions but (ii) of Theorem A hold, and

(ii)' $V'(t, \phi) \leq -W_3(|\phi(0)|)$

holds. Then the zero solution of equation (1) is uniformly asymptotically stable.

The first not numbered condition of Theorem A is the boundedness condition. A classical question (see [2, p. 252]) asks whether the boundedness condition can be dropped from Theorems A and B. The first attempt to answer this question is by the author [21]. Kato answered the question, as it is stated, in one of his talks [20]. The answer is no; he gave an example satisfying conditions (i), (ii) and (ii)', but the zero solution of his equation is not asymptotically stable. Here we radically simplify his equation.

Let us define the right-hand side of the equation in the following way: Let $h = 1$ and $g : [0, \infty) \to (0, \infty)$ be a continuously differentiable function such that

1. $g(n) = 1 + \frac{1}{2^n}$ and $g'(n) = 0$;
2. g is monotone decreasing on the interval $[n, n + \frac{1}{2}]$ and monotone increasing on the interval $[n + \frac{1}{2}, n + 1]$;
3. $\int_n^{n+1} g(s)ds \leq \frac{1}{2^n}$;
4. $g(s) \leq 2$ for $s \in [0, \infty)$,

where $n = 0, 1, 2, \ldots$. Clearly, there is such a function g. Now let $\alpha_n = g(n + 1)/g(n)$ and $p_n = \alpha_n/(1 - \alpha_n)$ and define f on the interval $[n - \frac{1}{2}, n + \frac{1}{2}]$ by

$$f(t, \phi) := \begin{cases} \dfrac{g'(t)}{g(t)}\phi(0), \text{ if } 0 \leq \phi(0) \leq \alpha_n\|\phi\| \text{ or } t \in [n, n + \dfrac{1}{2}] \\ -p_n \dfrac{g'(t)}{g(t)}(\phi(0) - \|\phi\|), \text{ if } \phi(0) \geq \alpha_n\|\phi\| \text{ and } t \in [n - \dfrac{1}{2}, n] \\ -f(t, -\phi), \text{ if } \phi(0) < 0 \end{cases}$$

(I) First, we need to verify that f is a continuous functional; in particular we need to verify the continuity when we change definitions in the definition of f. If $\phi(0) = \alpha_n\|\phi\|$, then by the first and second part of the definition

$$\begin{aligned} f(t, \phi) = \frac{g'(t)}{g(t)}\phi(0) = \alpha_n \frac{g'(t)}{g(t)}\|\phi\| = -p_n \frac{g'(t)}{g(t)}(\alpha_n - 1)\|\phi\| = \\ - p_n \frac{g'(t)}{g(t)}(\phi(0) - \|\phi\|), \end{aligned}$$

so they give the same result. When $t = n$ then $g'(t) = 0$, so by either the first or second part of the definition $f(t, \phi) = 0$. Similarly, if $\phi(0) = 0$, then the first or third part gives $f(t, \phi) = 0$. Thus, f is continuous.

(II) It is clear that f satisfies a local Lipschitz condition in its second variable. This and (I) imply that there is a unique solution of the initial value problem and it depends continuously on the initial data.

(III) Let us examine the behavior of the solutions. We consider the case when $\phi(0) \geq 0$; the case for $\phi(0) < 0$ can be handled similarly. First, note that the supremum norm of a solution is monotone non-increasing because

(a) if $t \in [n, n+\frac{1}{2}]$ then $f(t,\phi) = (g'(t)/g(t))\phi(0) \leq 0$ because g is monotone decreasing on the interval $[n, n+\frac{1}{2}]$, or

(b) if $\phi(0)$ is near $\|\phi\|$ (the second case in the definition of f), then f becomes smaller and smaller as $\phi(0)$ goes to $\|\phi\|$ and $f(t,\phi) = 0$ when $\phi(0) = \|\phi\|$.

Next, note that if we start the solution at time t_0 with an initial function ϕ, for a time t with $t_0 \leq t < t_0 + 1$ and t an integer, the solution obeys the equation on $[t_0, t]$ and (by the definition of α_n) $x(s)$ will stay below $\alpha_n \|x_s\|$ as $s \in [n-1, n)$. Therefore the solution of our equation obeys the first part of the definition of f on the interval $[t, \infty)$, so $x(s) = cg(s)$, where c is a constant ($c = x(t)/g(t)$).

Now let us define the Liapunov functional

$$V(t,\phi) = \|\phi\| + \int_t^\infty |x(s)|\, ds + \int_t^\infty \int_{s-1}^s x^2(u)duds,$$

where $x(u) = x(u;t,\phi)$ is a solution of our equation with $x_t(\cdot;t,\phi) = \phi$. We need to check that the integrals exist in the definition of V. First, we use the facts that $\|x_s\|$ is monotone non-increasing and $x_t = \phi$ to obtain

$$\int_t^\infty |x(s)|\, ds \leq \int_t^{t+1} |x(s)|\, ds + \int_{t+1}^\infty |x(s)|\, ds \leq \|\phi\| + \int_{t+1}^\infty |x(s)|\, ds.$$

On the other hand, on the interval $[t+1, \infty)$ the solution is of the form $x(s) = cg(s)$, where $c < \|\phi\|$, because $\|g_s\| > 1$ for all $s \geq 0$ and $\|x_s\|$ is monotone non-increasing. Since $\int_{t+1}^\infty g(s)ds \leq \int_1^\infty g(s)ds = \sum_{i=1}^\infty \int_i^{i+1} g(s)ds \leq \sum_{i=1}^\infty \frac{1}{2^i} = 1$ we have

$$\int_t^\infty |x(s)|\, ds \leq \|\phi\| + \int_{t+1}^\infty |x(s)|\, ds = \|\phi\| + \int_{t+1}^\infty cg(s)ds \leq \|\phi\| + c \leq 2\|\phi\|.$$

Changing the order of integration in the double integral and using the boundedness of g, a similar argument shows that

$$\int_t^\infty \int_{s-1}^s x^2(u)duds \leq 4\|\phi\|^2.$$

Hence we have proved the existence of the integrals in V and that

$$\|\phi\| \leq V(t,\phi) \leq 3\|\phi\| + 4\|\phi\|^2,$$

so condition (i) of Theorems A and B and Conjecture A is satisfied. Also, condition (iii) of Conjecture A holds. A similar argument shows that V is continuous and satisfies a local Lipschitz condition in its second variable.

To prove conditions (ii) and (ii)' compute

$$V'(t,x_t) \leq \frac{d}{dt}(\|x_t\|) - |x(t)| - \int_{t-1}^t x^2(u)du \leq -|x(t)| - |||x_t|||^2,$$

because $\|x_t\|$ is monotone non-increasing.

Hence V satisfies all the conditions of Conjecture A and Theorems A and B. All that is left to prove is that the solutions are not uniformly ultimately bounded and the zero solution is not asymptotically stable.

Since $Mg(s)$ is a solution of our equation, $Mg(n) \geq M$ for all $n = 0, 1, 2, ...$ and M can be any large number we know that the solutions are not uniformly ultimately bounded even with $U = 0$ in Conjecture A. This proves that the conjecture is false.

Also, $\varepsilon g(s)$ is a solution starting at $t_0 = 1$ with initial function $\varepsilon g_1(s) = \varepsilon g(1+s)$. This solution satisfies $\|\varepsilon g_1\| = 2\varepsilon$ and it does not go to 0 because $\varepsilon g(n) \geq \varepsilon$. This shows that the boundedness condition cannot be dropped from Theorems A and B.

To see the point in defining UB and UUB at a time t_0 we give another example, where the solutions are not UB or UUB but they are UB and UUB at 0. To this end we need the function $g : \mathcal{C} \times [-h, 0] \times [-h, 0] \to \mathbf{R}$ defined by

$$g(\phi, \alpha, \beta) = \sup_{\alpha \leq s_0 \leq \ldots \leq s_n \leq \beta} \sum_{k=1}^n \max\{0, \phi(s_k) - \phi(s_{k-1}) - 1\}.$$

For this definition to make sense we always assume that $\alpha \le \beta$. Kato [19] defined a function g very similar to our function g and proved that it is continuous and locally Lipschitz in ϕ. It is also clear that g is continuous in α and β too. To clarify matters we note that g is a function, which gives the "large increasing variation" of ϕ on the interval $[\alpha, \beta]$. We also define a function, which is (more or less) a sign function:

$$f(s) := \begin{cases} 1, \text{ if } s \ge 1 \\ s, \text{ if } s \in [-1, 1] \\ -1, \text{ if } s \le -1 \end{cases}$$

Now we are ready to define the right-hand side of our equation. Let $t \in [2n, 2n + 2]$ for some integer n, then

$$F(t, \phi) := \begin{cases} \min\{g(\phi, 2n - t, 0) - 1, 4\,|\phi(0)|^2\} f(\phi(0)), \text{ if } t \in [2n, 2n + 1] \\ \min\{g(\phi, -1, 2n + 1 - t) - 1, 4\,|\phi(0)|^2\} f(\phi(0)), \text{ if } t \in [2n + 1, 2n + 2] \end{cases}$$

where $h = 1$ and $T = 2$. It is clear that F is continuous and locally Lipschitz, since g and f are. But, unfortunately, Kato used the weak local Lipschitz condition that does not imply that g (and hence F) satisfies the boundedness condition. To ensure that F satisfies the boundedness condition we put $4\,|\phi(0)|^2$ into the argument of the minimum in the definition of F. Since $4\,|\phi(0)|^2$ satisfies the boundedness condition, so does F. But $4\,|\phi(0)|^2$ is a good enough right-hand side for a differential equation to ensure that the solution blows up fast, so in our example we can be sure that the big influence of g will be in effect as we describe it in the following.

Let us consider how solutions of our example behave. First, note that if $x(t) = x(t, t_0, \phi)$ ever becomes 0, then it remains 0 forever after. Thus, solutions do not change sign. To be definite, consider a solution $x(t) = x(t, t_0, \phi)$ with $\phi(0) > 0$, so $x(t) \ge 0$ for all $t \ge t_0$. Since $g \ge 0$ and $f \le 1$, $x'(t) \ge -1$. Define n to be the integer with $t_0 \in (2n - 2, 2n]$. From the definition of F we see that $x'(2n) = -f(x(2n)) \le 0$ and hence $x(t)$ is monotone decreasing for $t \in [2n, \infty)$. Moreover, if $x(t) > 1$ then $x'(t) = -1$ and x is a linear function, if $x(t) \le 1$ then $x'(t) = -x(t)$ and x is an exponentially decreasing function. A similar proof works if $\phi(0) < 0$. Therefore, if we start our solution at $t_0 = 2n$ then the above consideration proves that the solutions are UB(t_0) (for every $B_1 > 0$ let $B_2 := B_1$) and UUB(t_0) (let $B := 1$, then for $B_3 > 0$ we may choose $K := B_3 - 1$).

But the solutions are not UB or UUB at any other point (we will prove it for only one), and hence they are not UB or UUB. To see this, we take a function ϕ which is oscillatory with amplitude greater than 1. If we start the solution at $t_0 = 2n + 1$, our function g in the definition of F is applied to a segment containing $[-1/2, 0]$ of ϕ while t is in $[2n + 1, 2n + (3/2)]$ and it gives larger and larger numbers as ϕ has higher and higher frequency. (Note that although g satisfies the weak local Lipschitz condition, ϕ may be bounded by 1, but g is not bounded for these ϕ's.) The minimum makes everything a little bit complicated, but one can argue as follows. From the above we know that $g(x_t, -1, 2n+1-t) \geq g(\phi, -1/2, 0)$ for $t \in [2n + 1, 2n + (3/2)]$, where $x(t) = x(t, t_0, \phi)$ is the solution. We can choose ϕ so that $g(\phi, -1/2, 0) > 1$ and $\phi(0) > 1$. From the definition of F it is clear that $x(t)$ is monotone increasing on the interval $[2n + 1, 2n + (3/2)]$. Thus, in the following, we do not have to worry about the function f, since it is 1 along this solution on the intervals considered here. We claim that the solution satisfies $x'(t) = g(x_t, -1, 2n + 1 - t) - 1$ for $t \in [2n + (5/4), 2n + (3/2)]$. If this is not true, then since $g(x_t, -1, 2n + 1 - t)$ is monotone decreasing, but $4\,|x(t)|^2$ is monotone increasing we must have $x'(t) = 4\,|x(t)|^2$ on the interval $[2n + 1, 2n + (5/4)]$. But the solution of this equation (with $t_0 = 2n + 1$ and $\phi(0) > 1$) blows up before $2n+(5/4)$, which is not possible, since $x' \leq g(x_t, -1, 2n+1-t)-1 \leq g(\phi, -1, 0)-1$ (a fixed constant for every ϕ) for $t \in [2n+1, 2n+(3/2)]$. This contradiction shows that $x'(t) = g(x_t, -1, 2n+1-t)-1 \geq g(\phi, -1/2, 0)-1$ for $t \in [2n+(5/4), 2n+(3/2)]$. Hence, the solution can be as large as one wants at $t = 2n$, because one can have any large number of the form $g(\phi, -1/2, 0)$ with bounded ϕ's. Therefore, the solutions are not UB(t_0). Also, since the solution is very large at $t = 2n$ and cannot decrease faster than a linear function with slope 1, we cannot find a suitable B in the definition of UUB(t_0).

The above example shows that there are differential equations with solutions that are not UB or UUB but UB(t_0) and UUB(t_0) for some t_0. Since we also have $|x(t, t_0, \phi)| \to 0$ for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$, the example shows that dissipativeness (see the next chapter) does not imply UB or UUB. With some changes we will show that dissipativeness does not imply UB or UUB at any t_0.

1.3. Finite delay differential equations

We see from the previous section that Conjecture A is false, and hence we need more conditions to prove uniform boundedness and uniform ultimate boundedness for (1). Also, we need only uniform boundedness at t_0 and uniform ultimate boundedness at t_0. In order to prove this we modify Theorem 4.2.11 in [3] to fit our needs.

THEOREM 1.3. Suppose that there are a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, a sequence $t_n \in \mathbf{R}$ with $h \le t_n - t_{n-1} \le k$ for some $k > 0$ and constants $a, b, M, U > 0$ such that

(i) $W_1(|\phi(0)|) \le V(t, \phi)$ and $V(t_n, \phi) \le W_2(\|\phi\|)$

(ii) $V'(t, x_t) \le M$

(iii) $V'(t, x_t) \le -a\,|x'(t)| - b$ for $|x(t)| \ge U$,

where $W_1(r) \to \infty$ as $r \to \infty$. Then the solutions of (1) are uniformly bounded and uniformly ultimately bounded at t_n for any n. Also, B_2 of the uniform boundedness and B, K of uniform ultimate boundedness does not depend on n.

PROOF. First, we prove uniform boundedness. Clearly, the proof is the same for any n, and we will see that B_2 does not depend on n, so we prove everything for t_0. Let $B_1 > 0$ be given; we need to find a $B_2 > 0$ such that $\phi \in \mathcal{C}$ with $\|\phi\| < B_1$ implies $|x(t, t_0, \phi)| < B_2$ for $t \ge t_0$. Define $L \ge B_1$ so that $-aL + Mk < -\alpha$ for some $\alpha \in (0, bh)$. Define B_2 by

$$B_2 := W_1^{-1}\left(W_2(U + L) + Mk\left(2 + \frac{W_2(U + L)}{\alpha}\right)\right)$$

and suppose for contradiction that $|x(t)| = |x(t, t_0, \phi)| \ge B_2$ for some $t \ge t_0$.

Using (i) we find that $V(t) := V(t, x_t) \geq W_1(|x(t)|) \geq W_1(B_2)$. Let

$$\beta := \frac{W_1(B_2) - W_2(U+L)}{M} = k\left(2 + \frac{W_2(U+L)}{\alpha}\right)$$

and integrate (ii) from $s \in [t-\beta, t]$ to t to get $V(t) - V(s) \leq M(t-s)$ or $V(s) \geq V(t) - M(t-s) \geq W_1(B_2) - M\beta = W_2(U+L)$. Thus, using (i) we obtain $\|x_{t_n}\| \geq W_2^{-1}(V(t_n)) \geq U + L$ for all $t_n \in [t-\beta, t]$. Define N to be the largest positive integer so that $t_N \leq t$. Since $\|x_{t_0}\| < B_1 \leq U + L$ there is an $\overline{N} > 0$ such that $\|x_{t_n}\| \geq U + L$ for $n = \overline{N}+1, ..., N$, but $\|x_{t_{\overline{N}}}\| < U + L$. Note that from the above observation we know that

$$N - \overline{N} \geq \frac{\beta}{k} - 2 = \frac{W_2(U+L)}{\alpha}. \tag{1.1}$$

Let $S_n = [t_{n-1}, t_n]$. Since $\|x_{t_n}\| \geq U + L$ for $n = \overline{N}+1, ..., N$ there is a $t_n' \in S_n$ such that $|x(t_n')| \geq U + L$. There are two cases.

Case 1. If $|x(s)| \geq U$ for $s \in S_n$ then integrating (iii) we obtain

$$V(t_n) - V(t_{n-1}) \leq \int_{t_{n-1}}^{t_n} [-a\,|x'(s)| - b]ds \leq -bh < -\alpha.$$

Case 2. If $|x(s)| = U$ for some $s \in [t_{n-1}, t_n]$ then without loss of generality we may assume that $|x(u)| \geq U$ for all u between s and t_n' and hence integrating (ii) and (iii) we have

$$V(t_n) - V(t_{n-1}) \leq -\left|\int_s^{t_n'} [a\,|x'(v)| + b]dv\right| + Mk \leq -aL + Mk \leq -\alpha$$

again.

In any case we find that $V(t_n) - V(t_{n-1}) \leq -\alpha$ for $n = \overline{N}+1, ..., N$. By the choice of $\overline{N}$ we have

$$\begin{aligned} V(t_N) &= V(t_{\overline{N}}) + \sum_{n=\overline{N}+1}^{N} (V(t_n) - V(t_{n-1})) \leq V(t_{\overline{N}}) - \alpha(N - \overline{N}) \\ &\leq W_2(\|x_{t_{\overline{N}}}\|) - \alpha(N - \overline{N}) < W_2(U+L) - \alpha(N - \overline{N}) \leq 0 \end{aligned}$$

by (1.1), which is a contradiction to the fact that $0 \leq V(t, x_t)$ and hence it proves that the solutions are uniformly bounded at t_0.

Consider now the uniform ultimate boundedness. Once again, we will prove everything for t_0 only. The definition of L is slightly different now, we do not have B_1 to ask for $L \geq B_1$. So, choose $L > 0$ large enough to have $-aL + Mk < -\alpha$ for some $\alpha \in (0, bh)$. Let $B > 0$ be the B_2 of the uniform boundedness (at t_n for any n) for $B_1 := U + L$. Let $B_3 > 0$ be given; we need to find a $K > 0$ such that if $\phi \in \mathcal{C}$, $\|\phi\| \leq B_3$ and $t \geq t_0 + K$ then $|x(t)| = |x(t, t_0, \phi)| < B$. In order to do this we need to prove only that $\|x_{t_n}\| < U + L$ for some $t_n \in [t_0, t_0 + K]$, since then from the uniform boundedness at t_n we obtain $|x(s)| < B$ for all $s \geq t_0 + K \geq t_n$. Define $K := (W_2(B_3)/\alpha + 1)k$ and suppose for contradiction that $\|x_{t_n}\| \geq U + L$ for all $t_n \in [t_0, t_0 + K]$. Let $S_n := [t_{n-1}, t_n]$ for $n = 1, 2, \ldots$. From here on, we do the same as we did in the proof of the uniform boundedness: find a $t'_n \in S_n$ with $|x(t'_n)| \geq U + L$ and using the two cases we prove that $V(t_n) - V(t_{n-1}) \leq -\alpha$ for $n = 1, 2, \ldots$ with $S_n \subset [t_0, t_0 + K]$. The last estimate works out with a slight difference:

$$V(t_N) = V(t_0) + \sum_{n=1}^{N} (V(t_n) - V(t_{n-1})) \leq W_2(B_3) - \alpha N < 0$$

if $N > W_2(B_3)/\alpha$, and we have this by our definition of K. This is a contradiction to the fact that $0 \leq V(t_N)$ and hence it proves that the solutions are uniformly ultimately bounded at t_0.

Remark. Note that conditions (ii) and (iii) of the theorem are equivalent with the following: There are constants $U, M > 0$ such that

(ii)' $V'(t, x_t) \leq -W_3(|x(t)|) + M$ and

(iii)' $V'(t, x_t) \leq -W_4(|x'(t)|) + M$ for $|x(t)| \geq U$,

where $W_3(U) > M$ and W_4 is convex downward.

Proof. Assume that (ii)' and (iii)' hold. Then for any $\alpha \in [0, 1]$ we have

$$V'(t, x_t) \leq -\alpha W_3(|x(t)|) - (1 - \alpha) W_4(|x'(t)|) + M.$$

As W_4 is convex downward there is a constant $c > 0$ such that $W_4(r) \geq cr - 1$. Substituting this into the above estimate, assuming that $|x(t)| \geq U$ and reordering we have

$$V'(t,x_t) \le (-\alpha W_3(|x(t)|) + M) - (1-\alpha)(c\,|x'(t)| - 1)$$
$$= -(1-\alpha)c\,|x'(t)| + (-\alpha W_3(U) + M + (1-\alpha)),$$

so taking α sufficiently close to 1 we have the desired (iii) estimate. Clearly, (ii) holds.

To prove the opposite direction note that (iii) implies (iii)'. For (ii)' define W_3 by $W_3(U) = 1$, $W_3(r) = M + b + 1$ for $r \ge U + 1$ and linear everywhere else. Then by (ii)

$$V'(t,x_t) \le M \le -W_3(|x(t)|) + (M+1)$$

for $|x(t)| \le U$. Similarly, by (iii)

$$V'(t,x_t) \le -b \le -W_3(|x(t)|) + (M+1)$$

for $|x(t)| > U$. Thus (ii)' is satisfied, and the proof is complete.

We now give an example for Theorem 1.3. Let $a, b, c : \mathbf{R} \to \mathbf{R}$ be continuous functions, $h > 0$, and consider

$$x'(t) = -a(t)x(t) + b(t)x(t-h) + c(t). \tag{1.2}$$

Suppose that there are constants $k > 0$, $B > 0$ and $\alpha > 0$ such that for $p := 1 + \alpha$ we have

(i) $-a(t) + p\,|b(t+h)| \le \min\{-\alpha, -\alpha a(t)\}$,

(ii) $|c(t)| \le B$ and

(iii) there is a sequence $t_n \in \mathbf{R}$ with $h \le t_n - t_{n-1} \le k$ with $\int_{t_n - h}^{t_n} |b(s+h)|\,ds \le B$.

Then the solutions of (1.2) are uniformly bounded and uniformly ultimately bounded at t_0.

Proof. Define

$$V(t,x_t) = |x(t)| + p\int_{t-h}^{t} |b(s+h)| \cdot |x(s)|\,ds.$$

Clearly, we have $|x(t)| \leq V(t, x_t)$ and

$$V(t_n, x_{t_n}) \leq \|x_{t_n}\| + p \int_{t_n - h}^{t_n} |b(s+h)| \, \|x_{t_n}\| ds \leq (1 + pB)\|x_{t_n}\|.$$

Also, differentiating V we find

$$\begin{aligned} V'(t, x_t) &\leq -a(t)\,|x(t)| + |b(t)|\,|x(t-h)| + |c(t)| + \\ &\quad p\,|b(t+h)|\,|x(t)| - p\,|b(t)|\,|x(t-h)| \\ &\leq [-a(t) + p\,|b(t+h)|]\,|x(t)| + (1-p)\,|b(t)|\,|x(t-h)| + B \\ &\leq -\alpha[a(t)\,|x(t)| + |b(t)|\,|x(t-h)|] + B. \end{aligned}$$

Using (i) from this last estimate we have $V' \leq -\alpha\,|x'(t)| + (1+\alpha)B$ and $V' \leq -\alpha\,|x(t)| + B$, which is enough for Theorem 1.3 by the remark. Thus, the solutions of (1.2) are uniformly bounded and uniformly ultimately bounded at t_0.

This result implies the existence of a T-periodic solution if the functions a, b and c are T-periodic. There are other ways to prove the existence of a periodic solution for this special kind of differential equation. We will consider again this equation in a more general form in the last chapter.

1.4. Infinite delay differential equations

Consider now (2), the functional differential equation with infinite delay. Let the function $g : (-\infty, 0] \to [1, \infty)$ be continuous and monotone decreasing with $g(s) \to \infty$ as $s \to -\infty$ and $g(0) = 1$. We define the norm $\|\phi\|_g := \sup_{s \leq 0} |\phi(s)| / g(s)$.

Definition. Solutions of (2) are uniformly bounded at t_0, if for all $B_1 > 0$ there is a $B_2 > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\| < B_1$ and $t \geq t_0$ imply $|x(t, t_0, \phi)| < B_2$.

Definition. Solutions of (2) are g-uniformly bounded at t_0, if for all $B_1 > 0$ there is a $B_2 > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\|_g < B_1$ and $t \geq t_0$ imply $|x(t, t_0, \phi)| < B_2$.

Definition. Solutions of (2) are uniformly ultimately bounded with bound B at t_0, if for all $B_3 > 0$ there is a $K > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\| < B_3$ and $t \geq t_0 + K$ imply $|x(t, t_0, \phi)| < B$.

DEFINITION. Solutions of (2) are g-uniformly ultimately bounded with bound B at t_0, if for all $B_3 > 0$ there is a $K > 0$ such that $\phi \in \mathcal{C}$, $\|\phi\|_g < B_3$ and $t \geq t_0 + K$ imply $|x(t, t_0, \phi)| < B$.

Since $\|\phi\|_g \leq \|\phi\|$, g-uniform boundedness is a stronger condition than uniform boundedness and the same is true for uniform ultimate boundedness. The next theorem proves uniform boundedness and uniform ultimate boundedness for infinite delay equations.

THEOREM 1.4. Suppose that there are a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, wedges W_i and constants $M, U, a, b > 0$ such that

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|_g)$

(ii) $V'(t, x_t) \leq -a\,|x'(t)| - b$ for $|x(t)| \geq U$

(iii) $V'(t, x_t) \leq M$,

where $W_1(r) \to \infty$ as $r \to \infty$. Also, assume that $g(s) \geq -\mu s$ for all $s \leq 0$ and some $\mu > 0$. Then the solutions of (2) are g-uniformly bounded and uniformly ultimately bounded.

PROOF. First, we prove the g-uniform boundedness. Let $B_1 > 0$ be arbitrary and $L \geq B_1$ be large enough to have

$$\mu\left(-a(U + L) + \frac{M}{\mu}\right) \leq -2b$$

and

$$\left(-a(U + L) + \frac{M}{\mu}\right) \leq -2aU.$$

The last estimation clearly implies $aU + M/\mu \leq aL$; we will need this later. Also, define B_2 by

$$\frac{b(W_1(B_2) - W_2(U + L))}{M} > W_2(U + L) + aL.$$

While these definitions are complicated, they will simplify later work. Suppose for contradiction that there is such a $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ such that $\|\phi\|_g < B_1$ and $|x(t, t_0, \phi)| \geq B_2$ for some $t \geq t_0$. Using the very same idea as we used in Theorem 1.3, we can prove that if

$$\beta := \frac{W_1(B_2) - W_2(U + L)}{M}$$

then $s \in (t-\beta, t]$ implies $\|x_s\| \geq \|x_s\|_g \geq W_2^{-1}(V(s)) > U + L$. Let $v_0 \in [t_0, t]$ be the largest number such that $\|x_{v_0}\|_g \leq U + L$ (since $\|x_{t_0}\|_g < B_1 \leq U + L$ we can find such a v_0). It is not difficult to see from the definition of v_0 and the monotonicity of g that $|x(v_0)| = U + L$. It is also clear by the above observation and the definition of B_2 that

$$t - v_0 \geq \beta > \frac{W_2(U+L) + aL}{b}. \tag{1.3}$$

Let $w_0 = u_0 = v_0$, $\eta_0 = 1$ and $V_0 := V(v_0) + aL\eta_0$. We now define the sequences η_n and $u_n \leq v_n \leq w_n$ by induction. The sequences will have the following property: either

(a) $v_n = t$ and $V(v_n) - V_0 \leq -b(v_n - v_0)$ or

(b) $\eta_n \geq 1$, $\eta_n(U+L)$ is the maximum of $|x(s)|$ on the interval $[u_n, v_n]$, $V(v_n) - V_0 \leq -b(v_n - v_0) - aL\eta_n$ and $\sup_{s \leq -(w_n - u_n)} |x(w_n + s)| / g(s) \leq U + L$.

Clearly, (b) holds for $n = 0$. In our induction step we prove that if (b) holds for η_n, u_n, v_n and w_n, then we can define $\eta_{n+1}, u_{n+1}, v_{n+1}$ and w_{n+1} so that either (a) or (b) holds. To define $u_{n+1}, v_{n+1}, w_{n+1}$ we consider two cases.

Case 1. If $|x(s)| \geq U$ for all $s \in [v_n, t]$ then define $v_{n+1} = t$. Clearly, (ii) implies $V(v_{n+1}) - V(v_n) \leq -b(v_{n+1} - v_n)$ and hence (a) holds for v_{n+1}.

Case 2. If there is a $u_{n+1} \in (v_n, t)$ such that $|x(u_{n+1})| = U$ and $|x(s)| \geq U$ for $s \in [v_n, u_{n+1}]$, then let $\eta(U + L)$ be the maximum of $|x(s)|$ on the interval $[v_n, u_{n+1}]$ and $w_{n+1} := u_{n+1} + (\eta + \eta_n)/\mu$.

Case 2/a. If $w_{n+1} \geq t$ then we define $v_{n+1} = t$.

Case 2/b. If $w_{n+1} < t$ then $\|x_{w_{n+1}}\|_g > U + L$. By the above definitions we obtain the followings:

$$\sup_{s \leq -(w_{n+1} - u_n)} \frac{|x(w_{n+1} + s)|}{g(s)} \leq \sup_{s \leq -(w_n - u_n)} \frac{|x(w_n + s)|}{g(s)} \leq U + L$$

$$\sup_{-(w_{n+1} - u_n) < s \leq -(w_{n+1} - v_n)} \frac{|x(w_{n+1} + s)|}{g(s)} \leq \frac{\eta_n(U+L)}{\eta + \eta_n} \leq U + L$$

$$\sup_{-(w_{n+1} - v_n) < s \leq -(w_{n+1} - u_{n+1})} \frac{|x(w_{n+1} + s)|}{g(s)} \leq \frac{\eta(U+L)}{\eta + \eta_n} \leq U + L$$

because $w_{n+1} - v_n \geq w_{n+1} - u_{n+1} = (\eta + \eta_n)/\mu$ and hence $g(s) \geq \eta + \eta_n$ in the last two estimates. Now putting these together we find

$$\begin{aligned} U + L < \|x_{w_n}\|_g &= \sup_{-(w_{n+1}-u_{n+1})<s\leq 0} \frac{|x(w_{n+1}+s)|}{g(s)} \\ &\leq \sup_{-(w_{n+1}-u_{n+1})<s\leq 0} |x(w_{n+1}+s)|, \end{aligned}$$

and hence we can find a $v_{n+1} \in (u_{n+1}, w_{n+1}]$ such that $|x(v_{n+1})| > U + L$.

Now, in both Case 2/a and 2/b we estimate $V(v_{n+1}) - V_0$. In the following we will use several times that $v_{n+1} - u_{n+1} \leq w_{n+1} - u_{n+1} = (\eta + \eta_n)/\mu$ without even mentioning it. Define $|x(s_1)| = \eta_{n+1}(U + L)$ to be the maximum of $|x(s)|$ on the interval $[u_{n+1}, v_{n+1}]$, and let $s_2 \in [u_{n+1}, s_1]$ be the largest number with $|x(s_2)| \leq U$ (in Case 2/a we do not have to do this, in this case we must drop the integral from s_2 to s_1 and $-aL\eta_{n+1}$ from the following estimations). Integrating (ii) and (iii) we have

$$\begin{aligned} V(v_{n+1}) - V_0 &\leq V(v_{n+1}) - V(v_n) + V(v_n) - V_0 \\ &\leq \int_{v_n}^{u_{n+1}} -a\,|x'(s)| - b ds + \int_{s_2}^{s_1} -a\,|x'(s)| - b ds + \int_{u_{n+1}}^{v_{n+1}} M ds \\ &\quad - b(v_n - v_0) - aL\eta_n \\ &\leq -a(\eta(U+L) - U) - b(u_{n+1} - v_0) + \frac{M(\eta + \eta_n)}{\mu} \\ &\quad - aL\eta_n - aL\eta_{n+1} \\ &\leq aU - b(u_{n+1} - v_0) + \eta\left(-a(U+L) + \frac{M}{\mu}\right) + \left(\frac{M}{\mu} - aL\right)\eta_n \\ &\quad - aL\eta_{n+1} \\ &\leq aU + \frac{1}{2}\left(-a(U+L) + \frac{M}{\mu}\right) + \left(\frac{M}{\mu} - aL\right)\eta_n - b(u_{n+1} - v_0) + \\ &\quad \frac{(v_{n+1} - u_{n+1})\mu - \eta_n}{2}\left(-a(U+L) + \frac{M}{\mu}\right) - aL\eta_{n+1} \\ &\leq -b(v_{n+1} - v_0) - aL\eta_{n+1} + \frac{1}{2}\left(\frac{M}{\mu} + aU - aL\right)\eta_n \\ &\leq -b(v_{n+1} - v_0) - aL\eta_{n+1} \end{aligned}$$

by our definition of L. Therefore, we proved that in Case 2/a v_{n+1} satisfies (a), and in Case 2/b $\eta_{n+1}, u_{n+1}, v_{n+1}, w_{n+1}$ satisfy (b).

This finishes our induction step. Continuing this process we either end up in Case 1 or Case 2/a, or we have sequences u_n, v_n and w_n for all $n \geq 0$. Also, by (i) $V_0 \leq W_2(\|x_{v_0}\|_g) + aL\eta_0 \leq W_2(U+L) + aL$.

Case A. If we end up in Case 1 or in Case 2/a when we started our procedure from v_n, then v_{n+1} satisfies (a) and we have $v_{n+1} = t$ and $V(v_{n+1}) - V_0 \leq -b(v_{n+1} - v_0)$. Using these and (1.3) we obtain

$$V(t) \leq V_0 - b(t - v_0) \leq W_2(U+L) + aL - b\beta < 0,$$

which is a contradiction.

Case B. If we have the sequences u_n, v_n and w_n for all $n \geq 0$, then $v_n \to v \leq t$ as $n \to \infty$ for some v. Then integrating (ii) and (iii) from v_0 to v_n we have

$$\begin{aligned} V(v_n) - V(v_0) &\leq M(v_n - v_0) - \sum_{i=0}^{n-1} \int_{v_i}^{u_{i+1}} [a\,|x'(s)| + b]ds \\ &\leq M(t - v_0) - \sum_{i=0}^{n-1} aL = M(t - v_0) - anL \end{aligned}$$

a contradiction for large n to $0 \leq V(v_n)$.

This finishes the proof of the g-uniform boundedness.

For the uniform ultimate boundedness we define $L > 0$ as we did in the proof of g-uniform boundedness. Let $B := B_2$ for $B_1 := U + L + 1$ in the g-uniform boundedness. In order to prove uniform ultimate boundedness it is sufficient to prove that for all $B_3 > 0$ there is a $K > 0$ such that for every (t_0, ϕ) with $\|\phi\| < B_3$ there is an $s \in [t_0, t_0 + K]$ with $\|x_s(\cdot, t_0, \phi)\|_g \leq U + L$; then g-uniform boundedness implies the necessary inequality. Let $B_3 > 0$ be arbitrary, $D := B_3/(\mu(U+L))$, $w_0 := t_0 + D$ and $\eta_0(U+L)$ to be the maximum of $|x(s)|$ on the interval $(-\infty, w_0]$. As x is bounded by B_3 on $(-\infty, t_0]$ and $V(s) \leq V(t_0) + M(s - t_0) \leq W_2(B_3) + MD$, we have $|x(s)| \leq \max\{B_3, W_1^{-1}(W_2(B_3) + MD)\} =: \overline{\eta}(U+L)$ for $s \leq w_0$ and hence $\eta_0 \leq \overline{\eta}$, a bound independent of t_0 and ϕ. Define $K := D + (W_2(B_3) + DM + aL\overline{\eta})/b + 1$ and $t := t_0 + K$. Suppose for contradiction that there is a (t_0, ϕ) such that

$\|x_s\|_g = \|x_s(\cdot, t_0, \phi)\|_g > U + L$ for all $s \in [t_0, t_0 + K]$. Then using the below estimation on g we find

$$\begin{aligned} U + L < \|x_{w_0}\|_g &= \sup_{s\le 0} \frac{|x(w_0+s)|}{g(s)} \\ &\le \max\left\{ \sup_{s\le -D} \frac{|x(w_0+s)|}{g(s)}, \sup_{-D\le s\le 0} \frac{|x(w_0+s)|}{g(s)} \right\} \\ &\le \max\left\{ \frac{B_3}{\mu D}, \sup_{-D\le s\le 0} \frac{|x(w_0+s)|}{g(s)} \right\} \\ &\le \max\left\{ U+L, \sup_{-D\le s\le 0} |x(w_0+s)| \right\}, \end{aligned}$$

and hence we find a $v_0 \in [t_0, w_0]$ with $|x(v_0)| > U + L$. Let $u_0 = -\infty$. From here on the proof is exactly the same as we did before. Note that $V(t_0) \le W_2(\|\phi\|_g) \le W_2(\|\phi\|) \le W_2(B_3)$ and hence $V(v_0) \le W_2(B_3) + (v_0 - t_0)M \le W_2(B_3) + DM$. We have Cases A and B again.

Case A. If $v_{n+1} = t$, then from $v_0 \le t_0 + D$ we have $t - v_0 \ge K - D > (W_2(B_3) + DM + aL\overline{\eta})/b$ and hence $V(v_{n+1}) - V_0 \le -b(v_{n+1} - v_0)$ implies

$$V(t) \le V_0 - b(t - v_0) < W_2(B_3) + DM + aL\eta_0 - b\frac{W_2(B_3) + DM + aL\overline{\eta}}{b} \le 0$$

a contradiction.

Case B. We get a contradiction as before.

This proves the uniform ultimate boundedness and the proof is complete.

Note that with a little more care one can prove the exact counterpart of Theorem 1.3 for infinite delay equations too. The theorem would be the following.

THEOREM 1.5. Suppose that there are a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, wedges W_i, constants $M, U, a, b, k > 0$ and a sequence $t_n \in \mathbf{R}$ with $1 \le t_n - t_{n-1} \le k$ such that

(i) $W_1(|\phi(0)|) \le V(t, \phi)$ and $V(t_n, \phi) \le W_2(\|\phi\|_g)$

(ii) $V'(t, x_t) \le -a\,|x'(t)| - b$ for $|x(t)| \ge U$

(iii) $V'(t, x_t) \le M$,

where $W_1(r) \to \infty$ as $r \to \infty$. Also, assume that $g(s) \ge -\mu s$ for all $s \le 0$ and some $\mu > 0$. Then the solutions of (2) are g-uniformly bounded and uniformly ultimately bounded at t_n. Also, B_2 of the uniform boundedness and B, K of uniform ultimate boundedness does not depend on n.

PROOF. We only mention the changes one needs to make on the proof of Theorem 1.4. We need to modify the definition of w_{n+1} (starting with w_0) so that w_{n+1} is one of the t_i's in the interval of length k with the left-end point being the previous definition of w_{n+1}, and hence we have $V(w_{n+1}) \leq W_2(\|x_{w_{n+1}}\|_g)$. Also, because we have $w_{n+1} \leq u_{n+1}+(\eta+\eta_n)/\mu+k$ only, we need to do the estimation on $V(v_{n+1})-V_0$ more carefully, we will have some terms containing k too. To get the same estimate as before, instead of cutting the term $\eta(-a(U+L)+M/\mu)$ into two equal pieces we need to use an $\alpha \in (0,1)$ to cut it. According to this α we need to modify the definition of L too.

2. Dissipative systems

In this chapter we consider the finite delay differential equation (1) with the assumption that $f(t+T,\phi) = f(t,\phi)$. The solutions of an ordinary differential equation are called dissipative if there is a bounded neighborhood of 0 in $\mathbf{R}^d$ such that all the solutions go into this neighborhood at some time and stay there forever after. Pliss [23, Theorem 2.1] showed that solutions of a T-periodic ordinary differential equation are dissipative if and only if there is an $r > 0$ such that for each (t_0, x_0) there is a $\tau > t_0$ with $|x(\tau, t_0, x_0)| < r$.

It is a beautiful result and an exact counterpart of it would have great application in functional differential equations. However, we show that an exact counterpart is false, but that a variation of it is true. Hale and Lopes [12] proved that dissipativeness implies the existence of a periodic solution. We will give a simpler proof of their more general result for this special case. We also use Liapunov functionals to prove dissipativeness.

2.1. Main theorems on dissipativeness

DEFINITION. Equation (1) is said to be dissipative or point dissipative, if there is an $r > 0$ such that

$$\limsup_{t\to\infty} |x(t, t_0, \phi)| < r$$

for all t_0 and $\phi \in \mathcal{C}$.

THEOREM 2.1. The solutions of equation (1) are dissipative iff there is an $r > 0$ such that for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a $\tau \geq t_0 + h$ such that $\|x(\cdot, t_0, \phi)\|^{[\tau-2h,\tau]} < r$.

PROOF. This proof is very similar to that of Theorem 2.1 in [23]. The major difference is the way we ensure compactness for the functions in the proof. That is why we need the solution to be "small" on an interval of length twice the delay.

The necessity is clear. We prove the sufficiency by way of contradiction. Suppose that there are functions $\overline{\phi}_1, \overline{\phi}_2, \ldots \in \mathcal{C}$, constants $t_0^{(1)}, t_0^{(2)}, \ldots \in \mathbf{R}$ and

$r_1, r_2, \ldots \to \infty$ such that

$$\limsup_{t\to\infty} \left| x(t, t_0^{(k)}, \overline{\phi}_k) \right| > r_k.$$

We may assume that $r_k > r$. By the given condition there is a $\tau_k \geq t_0^{(k)} + h$ such that $\|x(\cdot, t_0^{(k)}, \overline{\phi}_k)\|^{[\tau_k - 2h, \tau_k]} < r$. From the above assumption there is a $\bar{t}_k > \tau_k$ such that $\left| x(\bar{t}_k, t_0^{(k)}, \overline{\phi}_k) \right| > r_k$ and hence there exists a $\overline{\Theta}_k \in [\tau_k, \bar{t}_k]$ such that

$$\|x(\cdot, t_0^{(k)}, \overline{\phi}_k)\|^{[\overline{\Theta}_k - 2h, \overline{\Theta}_k]} = r \text{ and } \|x(\cdot, t_0^{(k)}, \overline{\phi}_k)\|^{[t-2h,t]} > r \text{ for all } t \in (\overline{\Theta}_k, \bar{t}_k).$$

From the periodicity of f in t there are Θ_k, t_k, ϕ_k such that $0 \leq \Theta_k < T$, $t_k > \Theta_K$, $\phi_k = x_{\overline{\Theta}_k}(t_0^{(k)}, \overline{\phi}_k)$ with $\|\phi_k\|^{[-2h,0]} = r$, $|x(t_k, \Theta_k, \phi_k)| > r_k$ and $\|x(\cdot, \Theta_k, \phi_k)\|^{[t-2h,t]} > r$ for all $t \in (\Theta_k, t_k)$. Since $\|x(\cdot, t_0^{(k)}, \overline{\phi}_k)\|^{[\overline{\Theta}_k - 2h, \overline{\Theta}_k]} = r$ and f is bounded for ϕ bounded we have $|\phi_k'| \leq M$, where M depends only on r and f. Hence $\phi_k \in \{\phi \in \mathcal{C} : \|\phi\| = r, |\phi(u) - \phi(v)| \leq M\,|u - v|\}$. Thus, by compactness (using Ascoli's Theorem) without loss of generality we may assume that $\Theta_k \to \Theta$ and $\phi_k \to \phi$. Now for this Θ and ϕ there is a $\tau \geq \Theta + h$ such that $\|x(\cdot, \Theta, \phi)\|^{[\tau - 2h, \tau]} < r$. Let $R > r$ be such that $|x(t, \Theta, \phi)| < R$ for all $t \in [\Theta, \tau]$. By the continuous dependence of the solutions on the initial conditions there is a k_0 such that for all $k \geq k_0$ we have $|x(t, \Theta_k, \phi_k)| < R$ for all $t \in [\Theta_k, \tau]$ and $\|x(\cdot, \Theta_k, \phi_k)\|^{[\tau-2h,\tau]} < r$. Let $k \geq k_0$ be fixed so that $r_k > R$, then $|x(t_k, \Theta_k, \phi_k)| > r_k$ implies $t_k > \tau$. Therefore, $\|x(\cdot, \Theta_k, \phi_k)\|^{[t-2h,t]} > r$ for all $t \in (\Theta_k, t_k]$ and this contradicts $\|x(\cdot, \Theta_k, \phi_k)\|^{[\tau-2h,\tau]} < r$. The proof is complete.

Remark. Note that the same proof works for any set of solutions of (1); in particular we can prove the following

Proposition. The solution $x : [t_0 - h, \infty) \to \mathbf{R}^d$ of (1) is bounded iff there is an $r > 0$ such that for all $K \geq t_0 + h$ there is a $\tau \geq K$ such that $\|x(\cdot)\|^{[\tau-2h,\tau]} < r$.

We do not use the proposition in the following, but this statement is interesting as it is. It says that either the solution is bounded or $\lim_{t\to\infty} \|x(\cdot)\|^{[t-2h,t]} = \infty$. A very similar proof shows in the case of ordinary differential equations that the solution is either bounded or $\lim_{t\to\infty} |x(t)| = \infty$.

We now give an example which shows that the solutions of a general differential equation can be zero on an interval of length less than the delay, but they are not dissipative.

Let $\varepsilon > 0$ be given and $h = 1$. Consider a function $\phi : [0,1] \to \mathbf{R}$ such that $\phi(0) = 1$, $\phi(t) = 0$ on the interval $[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}]$ and ϕ has a smooth 1-periodic extension to the interval $[0, \infty)$. Consider the differential equation

$$x'(t) = x([t])\phi'(\{t\}),$$

where $[\cdot]$ is the greatest integer function and $\{t\} = t - [t]$. Clearly, if $x(t)$ is a solution, then $x(t+1)$ is a solution too ($T = 1$). It is also clear that the solutions of this equation are not dissipative; to see this, if we start the solution at $t = 1$ with the initial function $M\phi$ for an arbitrary $M > 0$ we get the solution as the 1-periodic extension of $M\phi$ to the interval $[0, \infty)$. Hence we are unable to find a number r independent of the solutions, such that every solution tends to the r-ball. It is also obvious that for every solution there is an interval of length $1 - \varepsilon$ (say the interval $[[t_0] + 1 + \frac{\varepsilon}{2}, [t_0] + 2 - \frac{\varepsilon}{2}]$) on which the solution is zero. This finishes our example.

One can construct an example so that every solution becomes bounded by 2 on an interval of length less than twice the delay, but the equation is not dissipative. This example is much too complicated to present here. It involves the function g defined by Kato in [19] we used before. We use that function at the end of this chapter to give another example.

We know then that in the very general case we cannot expect to relax the condition in Theorem 2.1. However, if we have a growth condition on f we can prove dissipativeness from a weaker condition. To do this we need the following

Generalized Bellman's Lemma. Let $Y(t)$, $H(t)$ be positive, continuous functions on the interval $[a,b]$, $k \geq 0$ and $\omega : \mathbf{R} \to \mathbf{R}$ be a positive, continuous, non-decreasing function. Then the inequality

$$Y(t) \leq k + \int_a^t H(s)\omega(Y(s))ds \qquad (a \leq t \leq b)$$

implies the inequality

$$Y(t) \leq \Omega^{-1}\left(\Omega(k) + \int_a^t H(s)ds\right) \qquad (a \leq t \leq b' \leq b)$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{ds}{\omega(s)} \qquad (u_0 > 0, u \geq 0)$$

and Ω^{-1} means the inverse function of $\Omega(u)$ ($\Omega^{-1}(u)$ certainly exists owing to the monotonicity of $\Omega(u)$). (Of course, t must be in a subinterval (a,b') of (a,b) so that the argument $\Omega(k) + \int_a^t H(s)ds$ will be within the domain of definition of Ω^{-1}.)

The proof of this lemma can be found in [1].

THEOREM 2.2. Consider again equation (1) and suppose that there are non-negative, T-periodic functions H and G which are integrable over one period, together with a positive, continuous and non-decreasing function ω such that $\int^{\infty}(1/\omega(s))ds = \infty$ and $|f(t,\phi)| \leq H(t)\omega(\|\phi\|) + G(t)$. Then the solutions of equation (1) are dissipative iff there is an $r > 0$ such that for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a $\tau > t_0$ such that $\|x_\tau(\cdot, t_0, \phi)\| < r$.

PROOF. The necessity is clear. Let us integrate equation (1) from τ to t ($t \geq \tau$):

$$x(t) - x(\tau) = \int_\tau^t f(s, x_s)ds,$$

and hence

$$|x(t)| \leq |x(\tau)| + \int_\tau^t |f(s,x_s)|\, ds \leq r + \int_\tau^t H(s)\omega(\|x_s\|)ds + \int_\tau^t G(s)ds.$$

Taking the supremum of both sides in t on the interval $[\tau, p]$ and using the property that $\|x_\tau\| < r$ we have

$$\begin{aligned}\|x_p\| &\leq r + \int_\tau^p H(s)\omega(\|x_s\|)ds + \int_\tau^p G(s)ds \\ &\leq \left[r + \int_\tau^{\tau+h} G(s)ds\right] + \int_\tau^p H(s)\omega(\|x_s\|)ds,\end{aligned}$$

if $p \in [\tau, \tau + h]$. Now using the generalized version of Bellman's Lemma we get

$$\|x_p\| \leq \Omega^{-1}\left(\Omega\left(r + \int_{\tau}^{\tau+h} G(s)ds\right) + \int_{\tau}^{p} H(s)ds\right).$$

Applying this inequality for $p = \tau + h$ we find

$$\|x_{\tau+h}\| \leq \Omega^{-1}\left(\Omega\left(r + \int_{\tau}^{\tau+h} G(s)ds\right) + \int_{\tau}^{\tau+h} H(s)ds\right).$$

Using the periodicity of H and G and the fact that they are integrable on one period we can find an $M > 0$ such that $\int_t^{t+h} G(s)ds \leq M$ and $\int_t^{t+h} H(s)ds \leq M$ for all t. Taking into account the inequality $\|x_\tau\| < r$ we arrive at

$$\|x(\cdot)\|^{[\tau-h,\tau+h]} \leq \Omega^{-1}(\Omega(r + M) + M)$$

and the conditions of Theorem 2.1 are satisfied. This completes the proof.

REMARK. Once again we note that the same proof works to establish the following

PROPOSITION. Suppose that there are non-negative, T-periodic functions H and G which are integrable over one period, together with a positive, continuous and non-decreasing function ω such that $\int^{\infty}(1/\omega(s))ds = \infty$ and $|f(t,\phi)| \leq H(t)\omega(\|\phi\|) + G(t)$. The solution $x : [t_0 - h, \infty) \to \mathbf{R}^d$ of (1) is bounded iff there is an $r > 0$ such that for all $K \geq t_0$ there is a $\tau \geq K$ such that $\|x_\tau\| < r$.

Also, observe that Example 1 shows that we cannot expect a better condition in the theorem's statement.

2.2. Periodic solutions of dissipative equations

We now show how to use the dissipativeness of the solutions in proving the existence of a periodic solution. In the following we restate some results of Hale and Lopes [12]. Their results are more general than necessary for our purposes. The author was surprised to discover (again) that dissipativeness itself is enough to prove the existence of a T-periodic solution. We introduce the following properties.

DEFINITION. Equation (1) is said to be point dissipative (at $t = 0$), if there is an $r > 0$ such that

$$\limsup_{t\to\infty} |x(t, 0, \phi)| < r$$

for all $\phi \in \mathcal{C}$.

DEFINITION. Equation (1) is compact dissipative (at $t = 0$), if there is an $r > 0$ such that for each compact subset K of $\mathcal{C}$ there is a $P > 0$ such that $|x(t, 0, \phi)| < r$ for all $\phi \in K$ and $t \geq P$.

DEFINITION. Equation (1) is compact uniform bounded (at $t = 0$), if for each compact subset K of $\mathcal{C}$ there is a $B > 0$ such that $|x(t, 0, \phi)| < B$ for all $\phi \in K$ and $t \geq 0$.

In the following we assume that all solutions can be continued for all future times. The property, that (1) is point dissipative, implicitly implies this condition, but we want to be clear on this point.

LEMMA. If (1) is point dissipative at $t = 0$, then $\limsup_{t\to\infty} |x(t, t_0, \phi)| < r$ for all $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$, so (1) is dissipative in the sense of the first definition of dissipativeness.

PROOF. For a given (t_0, ϕ), let $mT \in [t_0, t_0 + T)$ and $\psi := x_{mT}(\cdot, t_0, \phi)$. Then $x(t, t_0, \phi) = x(t, mT, \psi) = x(t - mT, 0, \psi)$ and hence

$$\limsup_{t\to\infty} |x(t, t_0, \phi)| = \limsup_{t\to\infty} |x(t - mT, 0, \psi)| < r$$

from point dissipativeness at $t = 0$, and the proof is complete.

THEOREM 2.3. If (1) is dissipative (point dissipative), then it is compact uniform bounded.

PROOF. Suppose that this is not the case. Then we find a compact set K in $\mathcal{C}$ and sequences $\phi_n \in K$ and $t_n \geq 0$ such that $|x(t_n, 0, \phi_n)| \to \infty$. Let $r > 0$ be the number in the definition of point dissipativeness and $L > 0$ be such that $|f(t, \phi)| \leq L$ for $\|\phi\| \leq r$. Without loss of generality, we may adjoin to K the compact set $\{\phi \in \mathcal{C} : \|\phi\| \leq r, |\phi(u) - \phi(v)| \leq L\,|u - v|\}$, and still call it K.

We can also assume that n is so large that $x_{t_n}(\cdot, 0, \phi_n)$ is not in K, because K is bounded and $|x(t_n, 0, \phi_n)| \to \infty$. Since $\phi_n = x_0(\cdot, 0, \phi_n) \in K$, we can define τ_n so that $\psi_n := x_{\tau_n}(\cdot, 0, \phi_n) \in K$, but $x_t(\cdot, 0, \phi_n) \notin K$ for $t \in (\tau_n, t_n]$. Using a translation argument we find a $\bar{t}_n \in [0, T]$ such that $x(t, \bar{t}_n, \psi_n) = x(t+(\tau_n - \bar{t}_n), \tau_n, \psi_n) = x(t + (\tau_n - \bar{t}_n), 0, \phi_n)$. Since K and $[0, T]$ are compact, there are subsequences, say ψ_n and $\bar{t}_n$ again, such that $\psi_n \to \psi$ and $\bar{t}_n \to \bar{t}$. Using the lemma for this $\bar{t}$ and ψ we find a $P \geq \bar{t}$ such that $\left|x(t, \bar{t}, \psi)\right| < r$ for $t \geq P$. Let $M > 0$ be a number with $\left|x(t, \bar{t}, \psi)\right| < M$ on the interval $[\bar{t}, P]$. Take any n large enough to have $\left|x(t, \bar{t}_n, \psi_n)\right| < M$ for $t \in [\bar{t}_n, P]$, $|x(t_n, 0, \phi_n)| = \left|x(t_n - (\tau_n - \bar{t}_n), \bar{t}_n, \psi_n)\right| \geq M$ and $\left|x(t, \bar{t}_n, \psi_n)\right| < r$ for $t \in [P, P+2h]$. Then we must have $t_n - (\tau_n - \bar{t}_n) > P + 2h$ and also $x_{P+2h}(\cdot, \bar{t}_n, \psi_n) \in \{\phi \in \mathcal{C} : \|\phi\| \leq r, |\phi(u) - \phi(v)| \leq L\,|u - v|\} \subset K$. But this is a contradiction to the choice of τ_n, because we must have $x_t(\cdot, \bar{t}_n, \psi_n) \notin K$ for $t \in [\bar{t}_n, t_n - (\tau_n - \bar{t}_n)]$. This contradiction shows the required compact uniform boundedness.

THEOREM 2.4. If (1) is dissipative (point dissipative), then it is compact dissipative.

PROOF. Let $r > 0$ be the number from point dissipativeness. From the previous theorem we find an $R > 0$ such that $\phi \in \{\psi \in \mathcal{C} : \|\psi\| \leq r : |\psi(u) - \psi(v)| \leq L\,|u - v|\}$ implies $|x(t, 0, \phi)| < R$, where L is from the proof of the previous theorem. We claim that (1) is compact dissipative with R. Suppose for contradiction that there is a compact set K and sequences $\phi_n \in K$ and $t_n \to \infty$ such that $|x(t_n, 0, \phi_n)| \geq R$. As K is compact, there is a subsequence of ϕ_n, say ϕ_n again, such that $\phi_n \to \phi \in K$. For this ϕ we find an $m > 0$ such that $|x(t, 0, \phi)| < r$ for $t \in [mT - 2h, mT]$. Then, using the continuous dependence of the solutions on the initial data take n large enough to have $|x(t, 0, \phi_n)| < r$ for $t \in [mT - 2h, mT]$ and $t_n > mT$. Using the compact uniform boundedness we find that $|x(t, 0, \phi_n)| = |x(t, mT, x_{mT}(\cdot, 0, \phi_n))| = |x(t - mT, 0, x_{mT}(\cdot, 0, \phi_n))| < R$ for all $t \geq mT$, which is a contradiction to $t_n > mT$ and $|x(t_n, 0, \phi_n)| \geq R$. This contradiction shows the compact dissipativeness.

THEOREM 2.5. If (1) is point dissipative, then it has a T-periodic solution.

PROOF. From Theorems 2.3 and 2.4 we know that (1) is compact uniform bounded and compact dissipative. Since from here on the proof of this theorem is very similar to the usual proof of the existence of a T-periodic solution assuming uniform boundedness and uniform ultimate boundedness (see [3, Theorem 4.2.2]), we will give only a sketch of the proof. Let $r > 0$ be the number from compact dissipativeness and $|f(t,\phi)| \le L$ for $\|\phi\| \le r$. Let

$$S_0 := \{\phi \in \mathcal{C} : \|\phi\| \le r, |\phi(u) - \phi(v)| \le L\,|u - v|\}$$

and define $P : \mathcal{C} \to \mathcal{C}$ by $P\phi := x_T(\cdot, 0, \phi)$. From the compact uniform boundedness we find $B_1 > 0$ such that $\|P^n(S_0)\| < B_1$ for $n \ge 0$. Define $L_1 > 0$ such that $\|\phi\| \le B_1$ implies $|f(t,\phi)| \le L_1$ and let

$$S_1 := \{\phi \in \mathcal{C} : \|\phi\| < B_1, |\phi(u) - \phi(v)| \le L_1\,|u - v|\}.$$

Once again using the compact uniform boundedness we define B_2 and L_2 such that if

$$S_2 := \{\phi \in \mathcal{C} : \|\phi\| \le B_2, |\phi(u) - \phi(v)| \le L_2\,|u - v|\}$$

then $P^n(S_1) \subset S_2$ for all $n \ge 0$. Also, from compact dissipativeness we find an $m > 0$ such that $P^n(S_1) \subset S_0$ for $n \ge m$. Now all the conditions of Horn's fixed-point theorem (see [18] or [3, Section 3.4]) are satisfied, and hence there is a fixed point of P, which is (of course) a T-periodic solution of (1). The proof is complete.

To satisfy the condition of this theorem we do not need the dissipativeness as it is stated at the beginning of this chapter, we need only point dissipativeness at $t = 0$. In the very same way as one proves the propositions stated after Theorems 2.1 and 2.2, one can prove that

THEOREM 2.6. The solutions of equation (1) are point dissipative at $t = 0$ iff there is an $r > 0$ such that for all $\phi \in \mathcal{C}$ and $K \ge h$ there is a $\tau \ge K$ such that $\|x(\cdot, 0, \phi)\|^{[\tau - 2h, \tau]} < r$.

Also, if we have the growth condition on f used in Theorem 2.2, we can prove a similar result with $\|x_\tau(\cdot, 0, \phi)\| < r$ in it.

We now review our Theorem 1.3. In view of Theorem 2.5 we do not need uniform boundedness and uniform ultimate boundedness at some t_0 to prove the existence of a T-periodic solution.

THEOREM 2.7. Suppose that there are a functional $V : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$ and constants $a, b, M, U > 0$ such that

(i) $0 \le V(t, \phi)$,

(ii) $V'(t, x_t) \le M$ and

(iii) $V'(t, x_t) \le -a\,|x'(t)| - b$ for $|x(t)| \ge U$.

Then the solutions of (1) are dissipative.

PROOF. By Theorem 2.1 we need only to prove that there is an $r > 0$ such that for every $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ there is a $\tau \ge t_0 + h$ such that $\|x(\cdot, t_0, \phi)\|^{[\tau-2h,\tau]} < r$. Define $L > 0$ such that $aL - 2Mh > 0$, and let $r := U + L$. Suppose for contradiction that there is $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ such that $\|x(\cdot, t_0, \phi)\|^{[t-2h,t]} \ge r$ for all $t \ge t_0 + h$. Define $t_n = t_0 + 2nh$ and $S_n := [t_{n-1}, t_n]$. From our assumption we can find a $t'_n \in S_n$ such that $|x(t'_n)| = |x(t'_n, t_0, \phi)| \ge r = U + L$. We have two cases.

Case 1. If $|x(t)| > U$ for all $t \in S_n$, then using (iii)

$$V(t_n) - V(t_{n-1}) \le \int_{t_{n-1}}^{t_n} V'(s)ds \le -b(t_n - t_{n-1}) = -2bh.$$

Case 2. If there is a $t \in S_n$ such that $|x(t)| = U$ then we may assume that $|x(s)| \ge U$ for s between t and t'_n and hence by (ii) and (iii)

$$V(t_n) - V(t_{n-1}) \le M(t_n - t_{n-1}) - \left| \int_t^{t'_n} a\,|x'(s)|\,ds \right| \le 2Mh - aL < 0.$$

In both cases we have proved that $V(t_n) - V(t_{n-1}) \le -\alpha < 0$ for some $\alpha > 0$. Therefore, $V(t_n) \le V(t_0) - n\alpha < 0$ for a large enough n which is a contradiction to (i). This contradiction shows that there is an $n > 0$ with $\|x(\cdot, t_0, \phi)\|^{[t_n-2h,t_n]} < r$. The proof is complete.

Note that we can replace b by a function $b : \mathbf{R} \to \mathbf{R}$ integrable on any finite interval with $\lim_{t\to\infty} \int_0^t b(s)ds = \infty$ and we do not need to change much in the proof. In this case we argue that we cannot have Case 2 infinitely many times, and hence there is an $N > 0$ such that Case 1 holds for $n \geq N$ and so $V(t_n) - V(t_N) \leq -\int_{t_N}^{t_n} b(s)ds$, a contradiction for large n.

If we consider this theorem purely as a tool to prove the existence of a T-periodic solution, then this theorem is better than Theorem 1.3 and better than Theorem 4.2.11 in [3]. Although we do not work with ordinary differential equations we mention that one can prove a theorem similar to Theorem 4.1.16 in [3]. For this theorem we really need that the solutions can be continued for all future times.

THEOREM 2.8. Suppose that there is a functional $V : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ such that

(i) $0 \leq V(t,x)$,

(ii) $V'(t,x_t) \leq -W(|x(t)|)$ for $|x(t)| \geq U$.

Then the solutions of the ordinary differential equation, say $x' = f(t,x)$, are dissipative.

This theorem (considering it again as a tool for proving the existence of a T-periodic solution) is better than Theorem 4.1.16 in [3].

2.3. A dissipative but not uniformly ultimately bounded equation

Now we give an example, which is point dissipative, and hence compact dissipative and compact uniformly bounded, but not uniformly bounded or uniformly ultimately bounded. We are using a function g similar to that defined by Kato [19]. Let $h \geq 2$,

$$g(\phi) = \sup_{-h \leq t_0 \leq \ldots \leq t_n \leq 0} \sum_{k=1}^{n} \max\{0, \phi(t_k) - \phi(t_{k-1}) - 1\}$$

and

$$f(s) := \begin{cases} 1, \text{ if } s \geq 1 \\ s, \text{ if } s \in [-1,1] \\ -1, \text{ if } s \leq -1 \end{cases}$$

We define $\alpha > 0$ so that $\alpha h \leq 1$, $\beta > 0$ and let the right-hand side of the equation be

$$F(t,\phi) := \min\{\alpha g(\phi) - \beta, |\phi(0)|^2\} f(\phi(0)).$$

Clearly, F is independent of t and hence T-periodic for any T. We could have a T-periodic perturbation in F (and define α smaller and β larger), but it would make things much too complicated.

Kato proved that g is continuous and locally Lipschitz in ϕ. But (as we mentioned earlier) he uses the weak local Lipschitz condition that does not imply that g (and hence F) takes bounded sets into bounded sets. To ensure that F takes bounded sets into bounded sets we put $|\phi(0)|^2$ into the argument of the minimum in the definition of F. It is also clear that F is continuous. Now we look at the solutions' behavior, which will show that the equation is point dissipative but not uniformly bounded or uniformly ultimately bounded.

Let $\phi \in \mathcal{C}$ be given and consider the solution $x(t) := x(t, 0, \phi)$. We show that this equation is point dissipative with $r = 1$. Clearly, if $x(t)$ ever becomes 0, then it will remain 0 forever after. Therefore, the solution does not change sign for $t \geq 0$. Suppose that $\phi(0) > 0$, the other case can be handled similarly.

We claim that $x(t)$ will become monotone decreasing in a neighborhood of a point $t_0 \geq 0$. Suppose for contradiction that $x(t)$ is monotone non-decreasing for $t \geq 0$. If $g(x_t)$ ever becomes 0 for some $t \geq 0$ then $x'(t)$ is negative and hence x is monotone decreasing in a neighborhood of t. Therefore, we must have $g(x_t) > 0$ for all $t \geq 0$ and hence $g(x_t) = x(t) - x(t-h) - 1$ and $F(t, x_t) = (\alpha(x(t) - x(t - h) - 1) - \beta)f(x(t))$. From $g(x_t) > 0$ it is clear that $x(h) - x(0) - 1 > 0$ and hence $x(h) > 1$. Then for $t \geq h$ we have

$$F(t, x_t) = \alpha(x(t) - x(t-h) - 1) - \beta = \alpha \int_{t-h}^{t} x'(s)ds - \alpha - \beta.$$

So our equation $x'(t) = F(t, x_t)$ became a very simple integral equation in $x'(t)$. Choosing $\alpha h \leq 1$ we see that $x'(t)$ is always less than or equal to the average of $x'(s)$ on the interval $[t-h, t]$ minus $\alpha+\beta$. Therefore $x'(t)$ cannot stay non-negative for all $t \geq h$, which is a contradiction to our indirect assumption. This proves that there is a neighborhood of a point t_0 in which x is monotone decreasing.

Since g is independent of the decrease in ϕ, in that neighborhood of t_0 $g(x_t)$ is also monotone decreasing, $F(t, x_t)$ is monotone decreasing and $F(t_0, x_{t_0}) < 0$, and hence $x(t)$ remains monotone decreasing for all $t \geq t_0$. Also, if the solution is monotone decreasing on an interval $[t-h, t]$ then $g(x_t) = 0$ and hence $x'(t) = -\beta f(x(t))$. Therefore, x decreases linearly with slope $-\beta$ as long as $x(t) \geq 1$ and it decreases exponentially for $x(t) < 1$. Consequently, $x(t) \to 0$ as $t \to \infty$. This proves that our equation is point dissipative with $r = 1$.

But the solutions are not uniformly bounded or uniformly ultimately bounded. To see this, take a ϕ oscillating rapidly between -1 and 1 on the interval $[-h, 0]$ and let $\phi(0) = 1$. Then $g(\phi)$ is a large number and $g(x_t)$ remains large for $t \in [0, h/2]$, because of the oscillation of ϕ. Also, the higher the frequency of ϕ, the larger $g(x_t)$ is. Suppose that $g(x_t) \geq (M+\beta)/\alpha$ for $t \in [0, h/2]$. Then, as long as $x^2(t) \leq M$ we have $F(t, x_t) = x^2(t)$ by the definition of F. Since $h \geq 2$, and the solution of the equation $x' = x^2$ with $x(0) = \phi(0) = 1$ blows up at $t = 1$, we must have a point $t \in [0, h/2]$ such that $x^2(t) > M$ or $x(t) > \sqrt{M}$. Since we can make M to be any large number we want, we can make $x(t)$ be as large as we want at a point in $[0, h/2]$.

Thus, although ϕ is chosen to be bounded by 1, we do not have a bound on x, that does not depend on ϕ. Hence, the solutions are not uniformly bounded. Also, since $x(t)$ is as large as we want at a point $t \in [0, h/2]$ and $x'(s) \geq -\beta$, we cannot find a time so that by that time all the solutions with an initial function bounded by 1 go below 1. So, the solutions are not uniformly ultimately bounded. This finishes our example.

It seems to be a good idea to summarize the relations among the different properties we used in here. We proved the following implications:

UB $\Rightarrow$ UB(0) $\Rightarrow$ compact UB(0),

UUB $\Rightarrow$ UUB(0) $\Rightarrow$ compact dissipativeness at 0 $\Rightarrow$ point dissipativeness at 0,

point dissipativeness at 0 $\Rightarrow$ compact dissipativeness at 0, and

point dissipativeness at 0 $\Rightarrow$ compact UB(0).

We gave examples showing that some of the implications cannot be reversed, i. e.

point dissipativeness at 0 $\not\Rightarrow$ UB(0),

$$\text{point dissipativeness at } 0 \not\Rightarrow \text{UUB}(0),$$
$$\text{UB}(0) \not\Rightarrow \text{UB, and}$$
$$\text{UUB}(0) \not\Rightarrow \text{UUB}.$$

Also, it is clear that

$$\text{UB} \not\Rightarrow \text{point dissipativeness at } 0.$$

We also proved that point dissipativeness itself is enough to prove that a T-periodic finite delay differential equation has a T-periodic solution.

3. Periodic solutions of linear equations

In this chapter we consider the functional differential equations (1) and (2) with periodic right-hand side. For one or two dimensional ordinary differential equations Massera [22] proved that if there is a bounded solution of the differential equation and the solutions can be continued for all future times then there is a T-periodic solution. He also proved that a linear ordinary differential system has a T-periodic solution if it has a bounded solution. Chow [8] proved a similar result for linear functional differential equations with finite delay.

We generalize Chow's Theorem for functional differential equations with infinite delay and for integral equations. Note that his theorems require a condition (the existence of a bounded solution) minimally expected for the existence of a periodic solution, since a periodic solution itself is a bounded solution. From the proof of our theorems it is obvious that we cannot drop or weaken (by taking a growth condition on the right-hand side of the equation, for example) this condition, since we use it to prove the convexity of a set. But our proof uses the linearity only at that particular point, so one might be able to find some conditions to ensure convexity in another way. Next, we are interested in finding a bounded solution for linear differential and integral equations. There are several theorems showing that all solutions are bounded (see e.g. [2]). Also, uniform boundedness, uniform ultimate boundedness or dissipativeness of the solutions implies the existence of a bounded solution. In our examples we show some techniques for finding a bounded solution for an ordinary differential equation and an equation with infinite delay when the equation does not satisfy the above conditions.

There are some related results the reader might want to look at. Yoshizawa [24, Theorem 15.10] proved for ordinary differential systems that if a bounded solution exists with a certain stability property, then there is an mT-periodic solution for some $m > 0$ integer. To prove the existence of a T-periodic solution using Liapunov functionals see [9]. For results on infinite delay equations see [6] and [14].

3.1. Finite delay equations

We consider equation (1) again. Note that the linearity of f we require later is enough to have a global Lipschitz condition on f. Throughout the proof of the following theorem we insist on not using the linearity of f if we do not have to. One feels that if this theorem is true for linear equations it would be true for "almost linear" equations also, so our work should make these studies easier. Note that the continuity and periodicity of f imply that f takes bounded sets of ϕ into bounded sets (the boundedness condition).

THEOREM 3.1. Suppose that $f(t, \phi)$ is a linear functional in ϕ, so that if $x(t)$ and $y(t)$ are solutions of (1) and $0 \leq \alpha \leq 1$ is a constant, then $\alpha x(t)+(1-\alpha)y(t)$ is also a solution of (1). If there is a bounded solution of (1) defined on an interval $[\beta, \infty)$, then there is a T-periodic solution.

PROOF. Let M be the maximum of h and T and define S as the Banach space of continuous functions mapping $[-M, 0]$ into $\mathbf{R}^d$ with the supremum norm. In the following we will write $x(t, t_0, \psi)$ for the solution of (1) starting at t_0 with the initial function $\psi \in S$ with the understanding, that if $M > h$ then we use only the values of ψ on the interval $[-h, 0]$. Consider the subset

$$
\begin{aligned}
U := \{\psi \in S : |\psi(s)| \leq r \text{ for } s \in [-M, 0], \qquad & (a)\\
|\psi(s_1) - \psi(s_2)| \leq K\,|s_1 - s_2| \text{ for } s_1, s_2 \in [-M, 0], \qquad & (b)\\
|x(t, 0, \psi)| \leq r \text{ for } t \geq 0\}, \qquad & (c)
\end{aligned}
$$

where r is the bound for the existing bounded solution and $|f(t, \phi)| \leq K$ for $\|\phi\| \leq r$ (from the boundedness condition on f). First of all, we need to show that this subset is non-empty. Consider the bounded solution. Since f is a T-periodic function in t, we may assume (by a translation argument) that the bounded solution $x(t)$ is defined on the interval $[-M - h, \infty)$ and it satisfies the equation on $[-M, \infty)$. Define $\psi_0(s) := x(s)$ for $s \in [-M, 0]$ and $\overline{\psi}_0(s) := x(s - M)$ for $s \in [-h, 0]$. By definition $x(t) = x(t, 0, \psi_0) = x(t, -M, \overline{\psi}_0)$ and hence ψ_0 satisfies (a) and (c) in the definition of U. For condition (b) we use the boundedness condition of f to get

$$\begin{aligned}|(\psi_0)(s_1)-(\psi_0)(s_2)| &= \left|x(s_1,-M,\overline{\psi}_0)-x(s_2,-M,\overline{\psi}_0)\right| \\ &= \left|\int_{s_2}^{s_1} x'(s,-M,\overline{\psi}_0)ds\right| = \left|\int_{s_2}^{s_1} f(s,x_s(\cdot,-M,\overline{\psi}_0))ds\right| \\ &\leq \left|\int_{s_2}^{s_1} Kds\right| \leq K\left|s_1-s_2\right|,\end{aligned}$$

where we used the boundedness of x. This proves that ψ_0 satisfies condition (b) as well as (a) and (c) in the definition of U, so $\psi_0 \in U$, and hence U is not empty. Next, we prove that this set is convex. If $\psi_1, \psi_2 \in U$ then

$$|\alpha\psi_1(s)+(1-\alpha)\psi_2(s)| \leq \alpha\left|\psi_1(s)\right| + (1-\alpha)\left|\psi_2(s)\right| \leq r,$$

and

$$\begin{gathered}|(\alpha\psi_1(s_1)+(1-\alpha)\psi_2(s_1))-(\alpha\psi_1(s_2)+(1-\alpha)\psi_2(s_2))| \leq \\ \alpha\left|\psi_1(s_1)-\psi_1(s_2)\right| + (1-\alpha)\left|\psi_2(s_1)-\psi_2(s_2)\right| \leq K\left|s_1-s_2\right|,\end{gathered}$$

so conditions (a) and (b) are satisfied for the convex linear combination. For condition (c) we note that by the linearity of the equation (this is the only place we really need it), if $\psi_1, \psi_2 \in U$ then

$$\begin{aligned}|x(t,0,\alpha\psi_1+(1-\alpha)\psi_2)| &= |\alpha x(t,0,\psi_1)+(1-\alpha)x(t,0,\psi_2)| \\ &\leq \alpha\left|x(t,0,\psi_1)\right| + (1-\alpha)\left|x(t,0,\psi_2)\right| \leq r\end{aligned}$$

using condition (c) for ψ_1 and ψ_2, and hence condition (c) is also satisfied for the convex linear combination. Also, this set is precompact by Ascoli's Theorem. Moreover, it is compact, because it is closed as we now prove. If $\psi_n \to \psi$ in the supremum norm and $\psi_n \in U$, then ψ clearly satisfies (a) and (b) in the definition of U. To prove condition (c) suppose for contradiction that $|x(t,0,\psi)| > r$ for some $t > 0$. By the continuous dependence of the solution on the initial function we find an $n > 0$ such that $|x(t,0,\psi_n)| > r$, a contradiction to the fact that $\psi_n \in U$, so ψ_n satisfies (c). This proves that ψ satisfies (c) and hence $\psi \in U$. Therefore, U is closed, so it is compact. Now let us define the map $P: U \to U$ by

$$(P\psi)(s) := x(s+T,0,\psi) \quad \text{for } -M \leq s \leq 0.$$

First, we prove that P maps into U. By (c) for ψ we find that $P\psi$ is bounded by r and hence $P\psi$ satisfies (a). For condition (c) on $P\psi$ we note that by the T-periodicity of f and the uniqueness of the solution we find that

$$|x(t,0,P\psi)| = |x(t,0,x_T(\cdot,0,\psi))| = |x(t+T,T,x_T(\cdot,0,\psi))| = |x(t+T,0,\psi)| \leq r.$$

To prove (b) for $P\psi$ we do exactly the same as we did proving that $\psi_0 \in U$. Hence, $P\psi$ satisfies the conditions (a), (b) and (c). Thus, P indeed maps U into U. Moreover, P is continuous from the continuous dependence of the solution on the initial function. By Schauder's fixed point theorem P has a fixed point, i.e. there is a $\psi \in U$ with $P\psi = \psi$. This means that $x(s+T,0,\psi) = \psi(s)$ for $s \in [-h,0]$ and hence by the periodicity of f and the uniqueness of the solution we find that $x(s+T,0,\psi) = x(s,0,\psi)$ for all $s \geq 0$. Therefore, $x(t,0,\psi)$ is a T-periodic solution of (1) and the proof is complete.

Note that this proof does not require that solutions can be continued for all future times, although the linearity implies that. Also, observe that the periodic solution we find in the proof is bounded by the bound of the bounded solution.

3.2. Infinite delay equations

We now consider equation (2), the functional differential equation with infinite delay. We need a Lipschitz-type condition on f to prove uniqueness and continuous dependence in the g-norm. To this end consider the following

LEMMA. Suppose that there is a monotone decreasing function $g : (-\infty,0] \to [1,\infty)$ with $g(s) \to \infty$ as $s \to -\infty$ such that whenever an $r > 0$ is given there is a $K > 0$ such that

$$|f(t,\psi_1) - f(t,\psi_2)| \leq K \|\psi_1 - \psi_2\|_g$$

for $\|\psi_i\|_g \leq r$ $(i = 1,2)$, where

$$\|\phi\|_g := \sup_{s\in(-\infty,0]} \frac{|\phi(s)|}{g(s)}.$$

Then the solution of an initial value problem is unique and for all $\varepsilon > 0$ and $M > 0$ there is a $\delta > 0$ such that if $\|\phi_1 - \phi_2\|_g < \delta$ then $\|x_M(\cdot, 0, \phi_1) - x_M(\cdot, 0, \phi_2)\|_g < \varepsilon$, where $x(\cdot, 0, \phi)$ is the solution of (2) starting at 0 with the initial function ϕ.

PROOF. We prove only the last part of the statement (the continuous dependence of the solution on the initial data), the uniqueness easily follows from that. Integrating the equation we have

$$x(t) - x(0) = \int_0^t f(s, x_s) ds.$$

Applying this equality for a solution y and subtracting these we find

$$x(t) - y(t) = x(0) - y(0) + \int_0^t f(s, x_s) - f(s, y_s) ds.$$

Thus, using the Lipschitz condition on f

$$\begin{aligned} |x(t) - y(t)| &\leq |x(0) - y(0)| + \int_0^t |f(s, x_s) - f(s, y_s)| \, ds \\ &\leq g(0)\|x_0 - y_0\|_g + \int_0^t K\|x_s - y_s\|_g ds. \end{aligned}$$

Taking the supremum norm on the interval $[0, t]$ on both sides we obtain

$$\sup_{s\in[0,t]} |x(s) - y(s)| \leq g(0)\|x_0 - y_0\|_g + \int_0^t K\|x_s - y_s\|_g ds.$$

Because g is monotone decreasing we find that

$$g(0) \sup_{s\in[0,t]} \frac{|x(s) - y(s)|}{g(s-t)} \leq \sup_{s\in[0,t]} |x(s) - y(s)|$$

and

$$g(0) \sup_{s\leq 0} \frac{|x(s) - y(s)|}{g(s-t)} \leq g(0) \sup_{s\leq 0} \frac{|x(s) - y(s)|}{g(s)} = g(0)\|x_0 - y_0\|_g,$$

therefore, putting together the last three inequalities we arrive at

$$g(0)\|x_t - y_t\|_g \leq g(0)\|x_0 - y_0\|_g + \int_0^t K\|x_s - y_s\|_g ds.$$

Applying Gronwall's inequality we obtain

$$\|x_t - y_t\|_g \leq \|x_0 - y_0\|_g e^{Kt/g(0)},$$

which proves the continuous dependence. This completes the proof of the lemma.

THEOREM 3.2. Suppose that $f(t, \phi)$ is a linear functional in ϕ, so that if $x(t)$ and $y(t)$ are solutions of (2) and $0 \le \alpha \le 1$ is a constant, then $\alpha x(t)+(1-\alpha)y(t)$ is also a solution of (2). If there is a bounded solution of (2) defined on the interval $(-\infty, \infty)$, then there is a T-periodic solution.

PROOF. We do exactly the same as in Theorem 3.1, with only slight changes. Define the Banach space S of continuous functions defined on the interval $(-\infty, 0]$ with the norm $\|\cdot\|_g$. Define the subset U exactly the same way as we did before. The rest of the proof is very similar to that of Theorem 3.1 with $M = -\infty$, we emphasize only some points, where more severe modifications are needed. To prove that U is precompact we cannot use Ascoli's Theorem, because we have an infinite interval. Instead, consider a sequence $\{\phi_n\} \subset U$. Using Ascoli's Theorem for the interval $[-1, 0]$ we find a subsequence $\{\phi_n^{(1)}\}$ converging in the supremum norm. For this sequence we find a subsequence $\{\phi_n^{(2)}\}$ converging on $[-2, 0]$, etc. Let $\psi_n := \phi_n^{(n)}$, then ψ_n converges uniformly on any finite subinterval of $(-\infty, 0]$, in particular, it converges pointwise to a function ψ. Since $\psi_n \in U$, ψ_n is bounded by r, and hence ψ is bounded by r, so it is in S. Moreover,

$$\begin{aligned}\|\psi_n - \psi\|_g &\le \max\left\{\sup_{s\le -L} \frac{|\psi_n(s)| + |\psi(s)|}{g(s)}, \sup_{s\in[-L,0]} \frac{|\psi_n(s) - \psi(s)|}{g(s)}\right\} \\ &\le \max\left\{\sup_{s\le -L} \frac{2r}{g(s)}, \sup_{s\in[-L,0]} |\psi_n(s) - \psi(s)|\right\} \le \varepsilon,\end{aligned}$$

where L is chosen large enough so that $g(s) \ge 2r/\varepsilon$ if $s \le -L$, and n large enough so that $\sup_{s\in[-L,0]} |\psi_n(s) - \psi(s)| \le \varepsilon$. So, this proves that $\psi_n \to \psi$ in $\|\cdot\|_g$ norm. Therefore, we have found a subsequence of $\{\phi_n\}$ that converges in S, and hence U is precompact. Since U is closed (can be proved as before), U is compact. Defining P and proving everything about it is the same as before; we can find a T-periodic solution.

Note that g can be a bit more general than it is stated in the lemma. With a little care one can modify the proof of the lemma and Theorem 3.2 so that g satisfies only the following conditions: $g(s) \to \infty$ as $s \to -\infty$ and

$\sup_{0\le v\le B}\sup_{s\le 0} g(s)/g(s-v) < \infty$ for all $B > 0$. Naturally, if g is monotone decreasing, then $\sup_{0\le v\le B}\sup_{s\le 0} g(s)/g(s-v) \le 1$ for all $B > 0$.

3.3. Integral equations

We now consider the integral equation

$$x(t) = F(t, x_t) \tag{3.1}$$

with infinite delay, i.e., $F : \mathbf{R} \times \mathcal{C} \to \mathbf{R}^d$ is continuous and T-periodic in t, where $\mathcal{C}$ is the Banach space of bounded functions $\phi : (-\infty, 0] \to \mathbf{R}^d$ with the supremum norm. Note that we do not require ϕ to be continuous; solutions of integral equations can have a discontinuity at the initial time (and any time after that) even if the initial function is "nice". Also, this is the reason why we need to ask in our theorems that there exists a bounded continuous solution. The finite delay case is a special case of (3.1), we emphasize only the differences in the following. We assume that (3.1) has a unique solution $x(t, t_0, \phi)$ for any $(t_0, \phi) \in \mathbf{R} \times \mathcal{C}$ which depends continuously on the initial data (using the $\|\cdot\|_g$ norm as before, or the $\|\cdot\|$ norm for the finite delay case), see e.g. [10, Chapters 11-13]. We also need a condition that replaces the boundedness condition on f in Theorems 3.1 and 3.2.

THEOREM 3.3. Suppose that for all $r > 0$ there is a wedge W such that $|F(t_1, x_{t_1}) - F(t_2, x_{t_2})| \le W(|t_1 - t_2|)$ for all $t_1, t_2 \in \mathbf{R}$ and function x with $|x(t)| \le r$ (we may call $F(t, x_t)$ equi-continuous in t uniformly for x with $|x(t)| \le r$). Also, assume that $F(t, \phi)$ is a linear functional in ϕ, so that if $x(t)$ and $y(t)$ are solutions of (3.1) and $0 \le \alpha \le 1$ is a constant, then $\alpha x(t) + (1-\alpha) y(t)$ is also a solution of (3.1). If there is a bounded solution of (3.1), then there is a T-periodic solution.

PROOF. We do exactly the same as in the proofs of Theorems 3.1 and 3.2 with only small changes. Let r be the bound for the existing bounded solution and W the function defined above for this r. We define U a little bit differently this time:

$$U := \{\psi \in S : |\psi(s)| \le r \text{ for } s \in [-M, 0], \tag{a}$$

$$|\psi(s_1) - \psi(s_2)| \leq W(|s_1 - s_2|) \text{ for } s_1, s_2 \in [-M, 0], \tag{b}$$

$$|x(t, 0, \psi)| \leq r \text{ for } t \geq 0\}. \tag{c}$$

For the rest of the proof we need to be careful only when we use the boundedness condition on f in the proof of Theorem 3.1 or 3.2. Instead, we say (when proving that U is not empty, for example) that

$$\begin{aligned}
|(\psi_0)(s_1) - (\psi_0)(s_2)| &= \left|x(s_1, -M, \overline{\psi}_0) - x(s_2, -M, \overline{\psi}_0)\right| \\
&= \left|F(s_1, x_{s_1}(\cdot, -M, \overline{\psi}_0)) - F(s_2, x_{s_2}(\cdot, -M, \overline{\psi}_0))\right| \\
&\leq W(|s_1 - s_2|).
\end{aligned}$$

Also, Ascoli's theorem can be applied, since condition (b) implies the equi-continuity of the functions in U. The remainder of the proof is exactly the same as before.

Although the equi-continuity of F in t uniformly in x may seem to be a strong condition, we show that the continuity, periodicity and linearity of F imply this condition, if F is really an integral form. Consider

$$x(t) = \int_{-\infty}^{t} c(t, s)x(s)ds + f(t),$$

where c and f are continuous, T-periodic functions (for c it means $c(t+T, s+T) = c(t, s)$), and $\int_{-\infty}^{t} |c(t, s)|\, ds$ is continuous and finite for $t \in \mathbf{R}$. Let $r > 0$ be given, x be any function with $|x(t)| \leq r$, suppose that $t_1 < t_2$ and consider

$$|F(t_1, x_{t_1}) - F(t_2, x_{t_2})| \leq r \int_{t_1}^{t_2} |c(t_2, s)|\, ds + r \int_{-\infty}^{t_1} |c(t_1, s) - c(t_2, s)|\, ds + \\ |f(t_1) - f(t_2)|\,.$$

Clearly, it is enough to prove the upper bound on $|F(t_1, x_{t_1}) - F(t_2, x_{t_2})|$ for small $|t_1 - t_2|$, since for larger numbers it follows from that. Let us look closely at the terms in the upper bound. Since $c(t, s)$ is T-periodic and continuous, $|c(t, s)| \leq C$ for all $t \in \mathbf{R}$, $s \in [t-1, t]$ and some $C > 0$. Consequently, $\int_{t_1}^{t_2} |c(t_2, s)|\, ds \leq C\, |t_1 - t_2|$ for $|t_1 - t_2| \leq 1$. To prove the bound on the second term, we only need to find a $\delta > 0$ for each $\varepsilon > 0$ such that if $|t_1 - t_2| < \delta$ then

$\int_{-\infty}^{t_1} |c(t_1,s) - c(t_2,s)|\, ds < \varepsilon$ (we can then define a W by $W(\delta) := \varepsilon$). Since c is T-periodic, so is $\int_{-\infty}^{t} |c(t,s)|\, ds$. Also, $\int_{-\infty}^{t} |c(t,s)|\, ds$ is continuous and finite and hence we can find a $K > 0$ (independent of t) such that $\int_{-\infty}^{t-K} |c(t,s)|\, ds < \varepsilon/4$ for all t. Then the periodicity and continuity of c imply that for some small enough $\delta > 0$ if $|t_1 - t_2| < \delta$ then we have $|c(t_1,s) - c(t_2,s)| < \varepsilon/2K$ for $s \in [t_1 - K, t_1]$. Putting together these remarks we arrive at

$$\begin{aligned}\int_{-\infty}^{t_1} |c(t_1,s) - c(t_2,s)|\, ds &\le \int_{-\infty}^{t_1-K} |c(t_1,s)| + |c(t_2,s)|\, ds + \\ &\quad \int_{t_1-K}^{t_1} |c(t_1,s) - c(t_2,s)|\, ds \\ &< 2\frac{\varepsilon}{4} + \int_{t_1-K}^{t_1} \frac{\varepsilon}{2K} ds = \varepsilon.\end{aligned}$$

This proves the bound on the second term. Clearly, f is continuous and T-periodic, so it is uniformly continuous and hence the third term has the necessary upper bound too. Thus, we see that we do not need any extra conditions on this equation to show that $F(t, x_t)$ is equi-continuous for $|x(t)| \le r$.

But there are some quite innocent-looking examples (the linear difference equation $x(t) = a(t) + b(t)x(t-h)$, for example), for which this condition is not satisfied. For these equations we can prove the following

THEOREM 3.4. Suppose that $F(t,\phi)$ is a linear functional in ϕ, so that if $x(t)$ and $y(t)$ are solutions of (3.1) and $0 \le \alpha \le 1$ is a constant, then $\alpha x(t) + (1-\alpha)y(t)$ is also a solution of (3.1). If there is a bounded, uniformly continuous solution $\overline{x}$ of (3.1), then there is a T-periodic solution.

PROOF. We need a definition for the proof.

DEFINITION. We say that the function $y : \mathbf{R} \to \mathbf{R}^d$ is in the equi-continuity class of the uniformly continuous function $x : \mathbf{R} \to \mathbf{R}^d$, if y is also uniformly continuous and whenever $\delta > 0$ is of the uniform continuity for $\varepsilon > 0$ for x then the same δ works for y in the definition of uniform continuity for ε.

Once again, we define U in a different way. Let r be the bound of the uniformly continuous solution $\overline{x}(t)$, and define

$$U := \{\psi \in S : |\psi(s)| \le r \text{ for } s \in [-M, 0], \qquad (a)$$

the solution $x(t, 0, \psi)$ is in the equi-continuity class of $\overline{x}$, $\quad (b)$

$|x(t, 0, \psi)| \leq r$ for $t \geq 0\}$. $\quad (c)$

Clearly, the functions in U are equi-continuous. The rest of the proof is the same as that of Theorem 3.3.

3.4. Examples

Since either uniform boundedness, or uniform ultimate boundedness, or dissipativeness, or boundedness of all solutions implies the existence of a bounded solution, we shall give examples not satisfying these properties. Moreover, we demonstrate how our theorem works for functional differential equations with infinite delay.

Let us consider the system

$$x'(t) = A(t)x(t) + f(t),$$

where A is a $d \times d$ matrix of continuous, T-periodic functions and f is a d dimensional vector of continuous, T-periodic functions. Suppose that all eigenvalues of $A(t) + A^T(t)$ have positive real parts for all $t \in \mathbf{R}$. Then there is a T-periodic solution of this equation.

Proof. As A is continuous and T-periodic we find an $\alpha > 0$ such that $x^T A(t)x \geq \alpha x^T x = \alpha |x|^2$. Also, we can find a $B > 0$ such that $|f(t)| \leq B$. Thus, there is an $M > 0$ such that if $|x| \geq M$ then $x^T(A(t)x + f(t)) \geq \alpha |x|^2 - B|x| > 0$, i.e., if a solution of the differential equation ever gets outside the M-ball it stays outside the M-ball. Let us pick an $x_0 \in \mathbf{R}^d$ on the surface of the M-ball, and consider the solution $x : (-\infty, \infty) \to \mathbf{R}^d$ with $x(0) = x_0$. Thus, as $x(0)$ is on the boundary of the M-ball, $x(s)$ must be in the M-ball for $s \leq 0$. The compactness of the M-ball implies that the set $\{x(-mT) : m \geq 0\}$ has a limit point x_1 in the M-ball. Consider the solution $x(t, 0, x_1)$. We claim that this solution remains in the M-ball. Suppose for contradiction that $|x(K, 0, x_1)| > M$ for some $K > 0$.

Let $\varepsilon > 0$ be small enough so that $|x(K, 0, x_1)| > M + \varepsilon$. By the continuous dependence of the solution on the initial data we find a $\delta > 0$ such that $|x_1 - y| < \delta$ implies $|x(K, 0, x_1) - x(K, 0, y)| < \varepsilon$ and hence $|x(K, 0, y)| > M$. As x_1 is the limit point of the set $\{x(-mT) : m \geq 0\}$ we find an $m \geq 0$ such that $mT > K$ and $|x_1 - x(-mT, 0, x_0)| < \delta$. Thus, we must have $|x(K, 0, x(-mT, 0, x_0))| > M$. On the other hand, the periodicity of the equation implies that

$$x_0 = x(0, -mT, x(-mT, 0, x_0)) = x(mT, 0, x(-mT, 0, x_0)),$$

and hence $x(s, 0, x(-mT, 0, x_0))$ remains in the M-ball for $s \in [0, mT]$. This is a contradiction to the fact that $|x(K, 0, x(-mT, 0, x_0))| > M$, which proves that $x(t, 0, x_1)$ is bounded by M for all $t \geq 0$. Since we have a bounded solution, by Theorem 3.1 we have a T-periodic solution as well.

Note that we cannot prove that the solution $x(t, 0, x_1)$ itself is T-periodic or even periodic of some period. Also, note that if the eigenvalues of $A(t) + A^T(t)$ have negative real parts, then we can prove uniform boundedness and uniform ultimate boundedness, which implies the existence of a T-periodic solution.

As our next example consider the functional differential equation with infinite delay

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - h_i(t)) + \int_{-\infty}^{t} c(t, s)x(s)ds + f(t),$$

where a, b_i, h_i, c and f are continuous and T-periodic functions, $h_i(t) \geq 0$ and $\int_{-\infty}^{t} |c(t, s)|\, ds < \infty$ for all $t \in \mathbf{R}$. Note that these conditions are enough to ensure the existence of a g satisfying the conditions of the lemma, so we have uniqueness and continuous dependence of the solutions in the initial data in the $\|\cdot\|_g$ norm. Suppose that $a(t) - \sum_{i=1}^{n} |b_i(t)| - \int_{-\infty}^{t} |c(t, s)|\, ds \geq \alpha$ for all $t \in \mathbf{R}$ and some $\alpha > 0$. Then there is a T-periodic solution of this equation.

Proof. From continuity and periodicity we find a $B > 0$ such that $|f(t)| \leq B$ for all t. This implies that if $\phi(0) \geq M := 1 + (B/\alpha)$ and $\|\phi\| \leq \phi(0)$, then

$$a(t)\phi(0) + \sum_{i=1}^{n} b_i(t)\phi(-h_i(t)) + \int_{-\infty}^{t} c(t, s)\phi(s - t)ds + f(t) \geq$$

$$\left(a(t) - \sum_{i=1}^{n} |b_i(t)| - \int_{-\infty}^{t} |c(t, s)|\, ds\right) \phi(0) - B \geq \alpha M - B = \alpha > 0.$$

Similarly, if $\phi(0) \leq -M$ and $\|\phi\| \leq |\phi(0)|$, then

$$a(t)\phi(0) + \sum_{i=1}^{n} b_i(t)\phi(-h_i(t)) + \int_{-\infty}^{t} c(t,s)\phi(s-t)ds + f(t) \leq -\alpha < 0.$$

Consider the set

$$S := \{\phi : (-\infty, 0] \to \mathbf{R} : \phi \text{ is a constant function}\}.$$

Denote by

$$S^+ := \{\phi \in S : x(t,0,\phi) > M \text{ for some } t \geq 0, \text{ and } x(s,0,\phi) > -M \text{ for } s \in [0,t]\}$$

and

$$S^- := \{\phi \in S : x(t,0,\phi) < -M \text{ for some } t \geq 0, \text{ and } x(s,0,\phi) < M \text{ for } s \in [0,t]\}.$$

By the continuous dependence of the solution on the initial condition if $\phi \in S^+$, $\psi \in S$ and $\|\phi - \psi\|_g = |\phi(0) - \psi(0)| / g(0)$ is small enough, then $|x(t,0,\phi) - x(t,0,\psi)|$ is arbitrarily small on a compact t-interval. Since for some $t \geq 0$ we have $x(t,0,\phi) > M$, for small enough $\|\phi - \psi\|_g$ we obtain $x(t,0,\psi) > M$. Similarly, $x(s,0,\psi) > -M$ for $s \in [0,t]$ too, so $\psi \in S^+$. This proves that S^+ is an open set in S. A similar proof gives that S^- is also open in S. Clearly, S^+ and S^- are disjoint and from the above estimations $S^+ \supset \{\phi \in S : \phi \geq M\}$ and $S^- \supset \{\phi \in S : \phi \leq -M\}$, so S^+ and S^- are not empty. This implies that $S^+ \cup S^-$ does not cover all S and hence there is a $\phi \in S$ with $\|\phi\| = |\phi(0)| < M$ and $\phi \notin S^+ \cup S^-$. From our first note in this proof it can be seen that if $x(t,0,\phi)$ is ever equal to M ($-M$ respectively) for the first time then $x(t,0,\phi)$ becomes strictly monotone increasing (decreasing) and hence $\phi \in S^+$ ($\phi \in S^-$). Therefore, we must have $|x(t,0,\phi)| < M$ for all $t \geq 0$, so we have found a bounded solution. Using Theorem 3.2 we prove the existence of a T-periodic solution.

Note that as the function $a(t) - \sum_{i=1}^{n} |b_i(t)| - \int_{-\infty}^{t} |c(t,s)|\, ds$ is continuous and T-periodic, if it is greater than zero, then we can find an $\alpha > 0$ bounding it away from zero. Also, if we had $a(t) + \sum_{i=1}^{n} |b_i(t)| + \int_{-\infty}^{t} |c(t,s)|\, ds \leq -\alpha < 0$, we could prove uniform boundedness and uniform ultimate boundedness, which, of course, implies the existence of a T-periodic solution. Also, observe that for a class of continuous functions $\phi : (-\infty, 0] \to \mathbf{R}$ we can use the above proof to find a $\beta \in \mathbf{R}$ such that $x(t, 0, \beta\phi)$ is a bounded solution. For example, if $|\phi(0)| \geq \|\phi\|$ then we can do this. So far this last note does not have any practical use, but we are in a new territory searching for just one bounded solution, and it may be important later.

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