Global Dynamics of Delay Differential Equations

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Dedicated to the memory of Miklós Farkas

Abstract

In this survey paper the delay differential equation $\dot{x}(t) = -\mu x(t) + g(x(t-1))$ is considered with $\mu \geq 0$ and a smooth real function g satisfying $g(0) = 0$. It is shown that the dynamics generated by this simple-looking equation can be very rich. The dynamics is completely understood only for a small class of nonlinearities. Open problems are formulated.

1. Introduction. Delay differential equations model phenomena in which the rate of change of the state at a given time depends on past states. At the 1908 International Congress of Mathematicians, Picard emphasized the importance of delayed effects in modeling of physical systems. The works of Volterra on predator-prey models and viscoelasticity, the great interest in delayed feedback mechanism in engineering problems and control theory contributed significantly to the rapid development of the theory. During the last 50 years the theory of delay differential equations has been developed extensively, see e.g., [2,5,7,8,9,12,30,39] as the main references. These types of equations were successfully applied in many areas such as viscoelasticity, mechanics, models for nuclear reactors, distributed networks, heat flow, neural networks, combustion theory, interaction of species, microbiology, learning models, epidemiology, physiology [11].

Despite of the wide variety of technical tools applied in the global study of nonlinear delay differential equations, the understanding even the simplest-looking equations has been slow.

In these notes we restrict our attention to the equation

$$\dot{x}(t) = -\mu x(t) + g(x(t-1)) \tag{1.1}$$

where $\mu \geq 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with $g(0) = 0$. Such equations appear in delayed feedback mechanisms. We show that the dynamics generated by eq. (1.1) can be very rich. In some cases a complete picture of the global dynamics is available, in other cases many questions are still open.

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First we consider the monotone feedback case, i.e., either $g' > 0$ or $g' < 0$. In these cases very much is known about the global dynamics. In the case of a nonmonotone feedback function g, the dynamics can be very complicated, and in general very little is known.

2. Preliminaries. Throughout this section we assume that $\mu \geq 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with $g(0) = 0$.

Let C denote the Banach space of continuous functions $\phi : [-1, 0] \to \mathbb{R}$ with the norm given by $||\phi|| = \max_{-1 \leq t \leq 0} |\phi(t)|$. If $I \subset \mathbb{R}$ is an interval, $x : I \to \mathbb{R}$ is a continuous map, $t \in \mathbb{R}$ and $t - 1 \in \mathbb{R}$, then the segment $x_t \in C$ is defined by $x_t(s) = x(t + s)$, $-1 \leq s \leq 0$.

Every $\phi \in C$ uniquely determines a solution $x = x^\phi : [-1, \infty) \to \mathbb{R}$ of eq. (1.1), i.e., a continuous function $x : [-1, \infty) \to \mathbb{R}$ such that x is differentiable on $(0, \infty)$, $x_0 = \phi$, and x satisfies eq. (1.1) for all $t > 0$. In fact, $x = x^\phi$ is easily found via the method of steps: if $n \geq 0$ is an integer and x is known for $t \in [-1, n]$, then, for $n \leq t \leq n + 1$, x is defined by

$$x(t) = e^{-\mu(t-n)} x(n) + \int_n^t e^{-\mu(t-s)} g(x(s-1))\, ds.$$

The map

$$F : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

is a continuous semiflow. The maps $F(t, \cdot) : C \to C$ are continuously differentiable for all $t \geq 0$, and compact for $t \geq 1$. If the nonlinearity g in eq. (1.1) is strictly monotone, then all maps $F(t, \cdot) : C \to C$, $t \geq 0$, are injective. In that case it follows that for every $\phi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}$ of eq. (1.1) with $x_0 = \phi$. We denote also by x^ϕ such a solution on $\mathbb{R}$ whenever it exists.

If $\phi \in C$ and x^ϕ is bounded on $[-1, \infty)$ then the ω-limit set

$$\begin{aligned}\omega(\phi) = \{\psi \in C : &\text{There is a sequence } (t_n)_0^\infty \text{ in } [0, \infty) \\ &\text{with } t_n \to \infty \text{ and } F(t_n, \phi) \to \psi \text{ as } n \to \infty\}\end{aligned}$$

is nonmempty, compact, connected and invariant. If $\phi \in C$ and there is a bounded solution $x : (-\infty, 0] \to \mathbb{R}$ with $x_0 = \phi$ then the α-limit set is

$$\begin{aligned}\alpha(x) = \{\psi \in C : &\text{There is a sequence } (t_n)_0^\infty \text{ in } (-\infty, 0] \\ &\text{with } t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to \infty\}.\end{aligned}$$

The global attractor of the semiflow F is a nonempty compact set $A \subset C$ which is invariant in the sense that $F(t, A) = A$ for all $t \geq 0$, and which attracts bounded sets in the sense that for every bounded set $B \subset C$ and for every open set $U \supset A$ there exists $t \geq 0$ with $F([t, \infty) \times B) \subset U$.

An easily verifiable sufficient condition for the existence of the global attractor A is, for example, $\mu > 0$ and $\limsup_{|x| \to \infty} |g(x)/x| < \mu$. Several other sufficient conditions for the existence of A are also known.

In case the global attractor A exists, the dynamical, geometrical and topological structure of A contains all important information about the long time ($t \to \infty$) behavior of the semiflow F. It is not difficult to show that

$$A = \{\phi \in C : \text{There is a bounded solution } x : \mathbb{R} \to \mathbb{R} \text{ of eq. (1.1) so that } x_0 = \phi\}.$$

All equilibria and periodic orbits of F belong to A. A $\xi \in C$ is an equilibrium point if and only if $\xi(\cdot) \equiv \xi_0$ for some $\xi_0 \in \mathbb{R}$ and $-\mu\xi_0 + g(\xi_0) = 0$ holds. Clearly $0 \in C$ is an equilibrium since $g(0) = 0$. The equilibrium points, in particular $0 \in C$, play an important role in the description of the global dynamics.

The linear variational equation along the zero solution of eq. (1.1) is

$$\dot{y}(t) = -\mu y(t) + \alpha y(t-1) \tag{2.1}$$

with $\alpha = g'(0)$. If $\psi \in C$ and $y^\psi : [-1, \infty) \to \mathbb{R}$ denotes the unique solution of (2.1) with $y_0 = \psi$, then we have $D_2F(t,0)\psi = y_t^\psi$, $t \geq 0$. The spectrum of the generator of the C_0-semigroup $D_2F(t,0)$, $t \geq 0$, consists of solutions $\lambda \in \mathbb{C}$ of the characteristic equation of (2.1)

$$\lambda + \mu - \alpha e^{-\lambda} = 0. \tag{2.2}$$

If $\alpha > 0$ then (2.2) has one real root λ_0, the other roots appear in complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_1^\infty$ with

$$\lambda_0 > \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \dots, \qquad 2j\pi - \pi < \operatorname{Im}\lambda_j < 2j\pi$$

for $1 \leq j \in \mathbb{N}$, and $\operatorname{Re}\lambda_j \to -\infty$ as $j \to \infty$ [5]. We have $\lambda_0 > 0$ if and only if $\alpha > \mu$.

If $\alpha < 0$ and $\alpha < -e^{-\mu-1}$, then all points in the spectrum form a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_1^\infty$ with

$$\operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \operatorname{Re}\lambda_2 > \dots, \qquad 2(j-1)\pi < \operatorname{Im}\lambda_j < (2j+1)\pi$$

for all $j \in \mathbb{N}$, and $\operatorname{Re}\lambda_j \to -\infty$ as $j \to \infty$ [5]. In particular, if the zero solution of (2.1) is unstable, then all points in the spectrum occur in complex conjugate pairs.

In both cases $\alpha > 0$ and $\alpha < 0$ we assume that there is a positive integer N such that

$$\operatorname{Re}\lambda_{N+1} \leq 0 < \operatorname{Re}\lambda_N. \tag{2.3}$$

Explicit conditions in terms of μ and α can be given for (2.3). For example, in case $\mu > 0$ and $\alpha > 0$, denoting by θ_N and θ_{N+1} the unique solutions of the equation $\theta = -\mu\tan\theta$ in $(2N\pi - \pi/2, 2N\pi)$ and $(2(N+1)\pi - \pi/2, 2(N+1)\pi)$, respectively, the inequality

$$\frac{\mu}{\cos\theta_N} \leq \alpha < \frac{\mu}{\cos\theta_{N+1}}$$

is equivalent to (2.3).

The oscillation frequency of solutions around zero also plays a role in the global results. For $\phi \in C$ let $\mathrm{sc}(\phi) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ denote the number of sign changes of ϕ. Define $V^{\pm} : C \setminus \{0\} \to \{0\} \cup \mathbb{N} \cup \{\infty\}$ as follows. If $\mathrm{sc}(\phi)$ is odd then set $V^{+}(\phi) = \mathrm{sc}(\phi) + 1$. If $\mathrm{sc}(\phi)$ is even then let $V^{-}(\phi) = \mathrm{sc}(\phi) + 1$. For all remaining cases we define $V^{\pm}(\phi) = \mathrm{sc}(\phi)$. V^{+} is a discrete Lyapunov functional for eq. (1.1), i.e., it is nonincreasing along solutions, provided the positive feedback condition

$$xg(x) > 0 \quad (x \neq 0)$$

holds. Under the negative feedback condition

$$xg(x) < 0 \quad (x \neq 0)$$

V^{-} is a discrete Lyapunov functional for eq. (1.1). See [26] for more details.

A solution x of eq. (1.1) is called slowly oscillatory if $|z_1 - z_2| > 1$ for each pair of zeros $z_1,\ z_2$ of the solution x.

Now we fix an integer $k \in \{1, \dots, N\}$. There exist a $\beta > 0$ (possibly different for $\alpha > 0$ and $\alpha < 0$) so that $\mathrm{Re}\,\lambda_j < \beta$ for all integers $j > k$, and $\mathrm{Re}\,\lambda_j > \beta$ for $j \in \{1, \dots, k\}$.

Let P_k denote the realified generalized eigenspace of the generator of the semigroup $D_2F(t,0)$, $t \geq 0$, associated with the spectral set

$$\{\lambda_0, \lambda_1, \overline{\lambda_1}, \dots, \lambda_k, \overline{\lambda_k}\} \quad \text{provided } \alpha > 0,$$

and associated with

$$\{\lambda_1, \overline{\lambda_1}, \dots, \lambda_k, \overline{\lambda_k}\} \quad \text{provided } \alpha < 0.$$

Let Q_k be the realified generalized eigenspace of the generator of the semigroup $D_2F(t,0)$, $t \geq 0$, associated with the remaining spectrum. Then we have the decomposition

$$C = P_k \oplus Q_k$$

with $\dim P_k = 2k+1$ if $\alpha > 0$, and $\dim P_k = 2k$ if $\alpha < 0$.

There exists a C^1-smooth local fast unstable manifold $W^u_{k,loc}(0)$ of 0 (see e.g. [18]) with $T_0 W^u_{k,loc}(0) = P_k$. $W^u_{k,loc}(0)$ consists of segments of solutions of eq. (1.1) on the interval $(-\infty, 0]$ approaching zero as $t \to -\infty$ faster than $t \mapsto e^{\beta t}$. We define the forward extension of this local fast unstable manifold:

$$W_k = F([0,\infty) \times W^u_{k,loc}(0)).$$

If the global attractor A exists then clearly $\overline{W_k} \subset A$.

Since there are differences between the results, in the sequel we use W_k^{+} and W_k^{-} for W_k to make a distinction between the cases $\alpha > 0$ and $\alpha < 0$.

In order to describe the structure of the global attractor, a first step can be to do that for the subsets $\overline{W_k^{\pm}}$, $k \in \{1, \ldots, N\}$, of A. This was successful for the monotone feedback case.

3. Monotone feedback. In this section we assume that either

$$g'(x) > 0 \qquad \text{for all } x \in \mathbb{R}, \tag{3.1}$$

that is we have a monotone positive feedback function g, or a monotone negative feedback function, i.e.,

$$g'(x) < 0 \qquad \text{for all } x \in \mathbb{R}. \tag{3.2}$$

We need the additional condition

$$g(x)/x < \mu \quad \text{for all } x \text{ so that } |x| \text{ is large enough} \tag{3.3}$$

in the positive feedback case, and the hypothesis

$$\text{either } \sup_{x\in\mathbb{R}} g(x) < \infty \text{ or } \inf_{x\in\mathbb{R}} g(x) > -\infty \tag{3.4}$$

in the negative feedback case.

In these cases very much is known about the dynamics. Based on the discrete Lyapunov functionals $V^{\pm}$, Mallet-Paret and Sell [27] proved that a Poincaré–Bendixson type result is true.

In case (3.2), (3.4) hold and $k = 1$, Walther [33] showed that $\overline{W_1^-}$ is a 2-dimensional C^1-smooth submanifold of C with boundary, and it is homeomorphic to the closed unit disk of $\mathbb{R}^2$. The manifold boundary $\overline{W_1^-} \setminus W_1^-$ of $\overline{W_1^-}$ is a slowly oscillating periodic orbit of eq. (1.1).

Defining the set

$$\begin{aligned} W_{so} = \{\phi \in C : &\text{There is a bounded slowly oscillating solution } x : \mathbb{R} \to \mathbb{R} \\ &\text{of eq. (1.1) with } x_0 = \phi\} \cup \{0\}, \end{aligned}$$

under conditions (3.2) and (3.4) Walther and Yebdri [35,37] proved more: There is a C^1-map $a : P_1 \supset \mathrm{dom}(a) \to Q_1$ such that

$$W_{so} = \{\phi + a(\phi) : \phi \in \mathrm{dom}(a)\},$$

and dom(a) is homeomorphic to the closed unit disk of $\mathbb{R}^2$ provided $W_{so} \neq \{0\}$. The manifold boundary of W_{so} is a slowly oscillating periodic orbit. Additional slowly oscillating periodic orbits may appear in W_{so}. In case of multiple periodic orbits, the periodic orbits are nested with the equilibrium point 0 in the interior of each of them. The nonperiodic orbits in $W_{so} \setminus \{0\}$ wind around 0, and form heteroclinic connections between periodic orbits, or between 0 and a periodic orbit.

The importance of the set W_{so} is that W_{so} attracts all solutions starting from an open dense subset of C [28].

For the positive feedback case, [18] described the structure of $\overline{W_1^+}$ under conditions (3.1) and (3.3). Assumption (3.3) combined with $g'(0) > \mu$ (which is a consequence of (2.3) with $N \geq 1$) implies the existence of a minimal $x^+ > 0$ and a maximal $x^- < 0$ with $-\mu x^\pm + g(x^\pm) = 0$. An additional condition is $g'(x^\pm) < \mu$, i.e., the equilibria $e^\pm$, defined by $e^\pm(s) = x^\pm$, $-1 \leq s \leq 0$, are hyperbolic. [18] shows that the set $\overline{W_1^+}$ consists of 3 equilibrium points $0, e^-, e^+$ and a unique periodic orbit $\mathcal{O}_1$. $\overline{W_1^+}$ is homeomorphic to the closed unit ball of $\mathbb{R}^3$, its manifold boundary $\overline{W_1^+} \setminus W_1^+$ is homeomorhic to the unit sphere of $\mathbb{R}^3$. W_1^+ is a C^1-smooth 3-dimensional submanifold of the phase space C. The set $\overline{W_1^+} \setminus (W_1^+ \cup \{e^-, e^+\})$, which is the manifold boundary of $\overline{W_1^+}$ without the equilibria e^-, e^+, is a 2-dimensional C^1-smooth submanifold of C, and it contains the periodic orbit $\mathcal{O}_1$ and heteroclinic connections from $\mathcal{O}_1$ to e^- and e^+. There exists a 2-dimensional smooth disk in $\overline{W_1^+}$ bordered by the periodic orbit $\mathcal{O}_1$. This disk contains 0, the periodic orbit $\mathcal{O}_1$ and heteroclinic connections from 0 to $\mathcal{O}_1$, moreover the disk separates $\overline{W_1^+}$ into two halves each of which belongs to the domain of attraction of e^- and e^+. The set $\overline{W_1^+}$ can be visualized as a solid spindle.

Now we turn to the general case. Assume that either (3.1) and (3.3) or (3.2) and (3.4) hold, and use either the $+$ or the $-$ sign as an upper index to make a disctinction between the two cases. The structure of the unstable sets $\overline{W_k^\pm}$ is described in [17].

$\overline{W_k^-}$ consists of 0, exactly k periodic orbits $\mathcal{O}_1, \dots, O_k$, and heteroclinic connections between 0 and the periodic orbits, and between certain periodic orbits. Introduce the connecting sets

$$\begin{aligned} C_j^0 = \{\phi \in \overline{W_k^-} : &\text{There is a solution } x : \mathbb{R} \to \mathbb{R} \\ &\text{of eq. (1.1) with } x_0 = \phi,\ \alpha(x) = \{0\},\ \omega(\phi) = \mathcal{O}_j\}, \end{aligned}$$

$$\begin{aligned} C_l^j = \{\phi \in \overline{W_k^-} : &\text{There is a solution } x : \mathbb{R} \to \mathbb{R} \\ &\text{of eq. (1.1) with } x_0 = \phi,\ \alpha(x) = \mathcal{O}_j,\ \omega(\phi) = \mathcal{O}_l\} \end{aligned}$$

for j and l in $\{1, \dots, k\}$. Then

$$\overline{W_k^-} = \{0\} \cup \left(\bigcup_{j=1}^{k} \mathcal{O}_j \right) \cup \left(\bigcup_{j=1}^{k} C_j^0 \right) \cup \left(\bigcup_{1 \leq l < j \leq k} C_l^j \right).$$

In addition [17] shows that the connecting sets are C^1-smooth submanifolds of the phase space C. For the periodic orbits we have $\mathcal{O}_j \subset (V^-)^{-1}(2j-1)$, that is the segments of the periodic orbit $\mathcal{O}_j$ have either $2j-2$ or $2j-1$ sign changes in $[-1, 0]$, $j \in \{1, \dots, k\}$.

The set $\overline{W_k^+}$ contains three equilibrium points 0, e^-, e^+ and k periodic orbits $\mathcal{O}_1, \dots, O_k$, and heteroclinic connections between 0 and e^-, e^+ and the periodic orbits, between the periodic orbits and e^-, e^+, and between certain periodic orbits. For this case, in addition to the connecting sets C_j^0 and C_l^j, which are defined analogously to the ones of the negative feedback case, we need the sets

$$C_\pm^0 = \{\phi \in \overline{W_k^+} : \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \\ \text{of eq. (1.1) with } x_0 = \phi,\ \alpha(x) = \{0\},\ \omega(\phi) = \{e^\pm\}\},$$

$$C_\pm^j = \{\phi \in \overline{W_k^+} : \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \\ \text{of eq. (1.1) with } x_0 = \phi,\ \alpha(x) = \mathcal{O}_j,\ \omega(\phi) = \{e^\pm\}\}.$$

for $j \in \{1, \ldots, k\}$. Then

$$\overline{W_k^+} = \{0, e^- e^+\} \cup \left(\bigcup_{j=1}^k \mathcal{O}_j\right) \cup \left(\bigcup_{j=1}^k C_j^0\right) \cup C_-^0 \cup C_+^0 \\ \cup \left(\bigcup_{1 \le l < j \le k} C_l^j\right) \cup \left(\bigcup_{j=1}^k C_-^j\right) \cup \left(\bigcup_{j=1}^k C_+^j\right).$$

The connecting sets are C^1-smooth submanifolds of the phase space C. For the periodic orbits we have $\mathcal{O}_j \subset (V^-)^{-1}(2j)$, that is the segments of the periodic orbit $\mathcal{O}_j$ have either $2j-1$ or $2j$ sign changes in $[-1, 0]$, $j \in \{1, \ldots, k\}$.

The proof of these results is similar to that of [18], however some parts require new ideas and methods. The main tools are invariant manifold theory, inclination lemmas, discrete Lyapunov functionals, Floquet multipliers, transversality. The importance of the monotonicity conditions (3.1) and (3.2) is that there exist discrete Lyapunov functionals also for weighted differences of solutions and for the solutions of linear variational equations. We emphasize that the hyperbolicity of the periodic orbits $\mathcal{O}_1, \ldots, O_k$ is not known for both cases, and this causes technical difficulties.

The above description of the sets $\overline{W_k^-}$ and $\overline{W_k^+}$ gives a Morse decomposition. Therefore the dynamics restricted to these sets is gradient like. This is true for more general equations also for the whole global attractor, see Fiedler and Mallet-Paret [6], McCord and Mischaikow [29], Polner [31]. However, the above results give more detailed information about the connecting sets.

In some cases it is expected that the sets $\overline{W_N^-}$ and $\overline{W_N^+}$ coincide with the corrseponding global attractors A^- and A^+. This is shown for odd feedback functions with a convexity property [13,14,15,16,34]. An example is

$$g(x) = a \tanh(bx) \quad \text{or} \quad g(x) = a \tan^{-1}(bx)$$

with $a \neq 0$ and $b > 0$. The proofs are nontrivial, the main step is the nonexistence of periodic orbits different from $\mathcal{O}_1, \ldots, O_N$.

Now we formulate some open problems.

It is expected that the equalities

$$A^- = \overline{W_N^-}, \qquad A^+ = \overline{W_N^+}$$

hold for more general feedback functions g. For example, not odd nonlinearities would be interesting.

Define the set

$$\Sigma = \{\phi \in C : x^\phi \text{ has arbitrarily large zeros in } [-1, \infty)\}.$$

Σ contains initial functions of oscillatory solutions. In the positive feedback case Σ is a codimension 1 closed Lipschitz submanifold of C, and the sets $\{e^-, e^+\}$, $C_\pm^0$, $C_\pm^j$ are disjoint from Σ [18]. The intersection

$$\overline{W_k^+} \cap \Sigma$$

and $\overline{W_k^-}$ seem to have the same structure. This motivates the next two problems.

It is suspected that the dynamics restricted to the sets $\overline{W_k^+} \cap \Sigma$ and $\overline{W_k^-}$ in the positive and negative feedback cases, respectively, are topologically equivalent.

Is it true for a given positive feedback equation with global attractor A^+ that there exists a negative feedback equation with global attractor A^- such that the dynamics restricted to $A^+ \cap \Sigma$ and A^-, respectively, are topologocally equivalent?

There are certain extensions of some of the above results to some particular systems, see e.g. [3,4]. It is expected that the same technique works for general cyclic feedback systems studied in [26,27]. However the description of the structure of the unstable sets and the global attractors can be interesting for each different system.

It seems that the above techniques do not work for equations with more delays or distributed delays. Only a very little is known in this direction about the global dynamics.

A particular example for monotone negative feedback is

$$\dot{x}(t) = -\alpha \left[e^{x(t-1} - 1 \right], \tag{3.5}$$

where $\alpha > 0$ is a parameter. This equation is also called Wright's equation. E.M. Wright [38] used it for a heuristic proof for the asymptotic distribution of prime numbers. He proved deep results on the asymptotic behavior of solutions. In particular, Wright showed that for $0 < \alpha \leq 3/2$ all solutions approach zero as $t \to \infty$, and conjectured that the same is true for all $\alpha \in (0, \pi/2)$ [38]. Using our notation this is equivalent to $A^{(3.5)} = \{0\}$ for $\alpha \in (0, \pi/2)$. The problem is still open. A more general problem is whether $\overline{W_N^{(3.5)}} = A^{(3.5)}$ holds. We call this the generalized Wright's conjecture.

Despite of the simplicity of eq. (3.5) and the variety of deep technical tools applied in its study, it seems that we are very far from the complete understanding of the dynamics.

4. Nonmonotone feedback. Now we assume that either $xg(x) < 0$ for all $x \neq 0$ (negative feedback) or $xg(x) > 0$ for all $x \neq 0$. Neither condition (3.1) nor (3.2) is required.

The unstable sets $W_k^{\pm}$ and their closures do not have so nice structures as described in Section 3 for the monotone feedback case.

Based on the existence of the discrete Lyapunov functions $V^{\pm}$ a Morse decomposition can be shown for the global attractor [6,25,29,31]. This means that there exists a finite ordered collection $S_1 < S_2 < \ldots < S_M$ of disjoint compact invariant subsets of the global attractor A such that for any $\phi \in A$ there are a bounded solution $x : \mathbb{R} \to \mathbb{R}$ with $x_0 = \phi$ and positive integers $j, l \in \{1, \ldots, M\}$ such that $j \geq l$, $\alpha(x) \subset S_j$, $\omega(\phi) \subset S_l$, and $j = l$ implies $\phi \in S_j$. The Morse sets $S_0, \ldots, S_M$ together with the connecting orbits between them give the global attractor A. However this does not mean that the dynamics is simple. In fact, there are examples where chaotic behavior appears within some Morse sets.

The first result about chaotic behavior for delay differential equations was given by an der Heiden and Walther [10]. They considered eq. (1.1) with a step function nonlinearity g which is discontinuous, and reduced the problem to an interval map having the Li–Yorke property for chaos.

Lani-Wayda and Walther [19,22,23] proved complicated behavior for equations of the form

$$\dot{x}(t) = g(x(t-1))$$

with a smooth nonlinearity g satisfying the negative feedback condition. In fact they started from an odd and monotone g, and found a Kaplan–Yorke type slowly oscillating periodic solution y of period 4 with the special symmetry

$$y(t) = -y(t-2) \qquad (t \in \mathbb{R}).$$

In additon they showed that the periodic orbit $\mathcal{O} = \{y_t : t \in [0,4]\}$ is hyperbolic, unstable, there is exactly one Floquet multiplier outside the unit circle. Then on a hyperplane of C which is transversal to the periodic orbit, a Poincaré return map can be constructed with a hyperbolic fixed point whose local unstable manifold W_u is one-dimensional. In the next step the nonlinearity g is modified outside the set $y(\mathbb{R})$ such that the periodic solution y is preserved, the local stable and unstable manifolds of the Poincaré map are also preserved, the solution of the new equation with initial value from W_u leads to a solution homoclinic to the periodic orbit $\mathcal{O}$, and in addition a transversality condition holds for the homoclinic orbit. Then the main result of Lani-Wayda and Walther states that there exists a maximal invariant set for the modified Poincaré map P so that the action of P on the trajectories in this set is conjugate to a symbol shift, and all of these trajectories of P correspond to slowly oscillating solutions. In [19] Lani-Wayda constructed a smooth g with the negative feedback condition so that there was an $x^* < 0$ with $g'(x) > 0$ for all $x < x^*$ and $g'(x) < 0$ for all $x > x^*$.

The results of Lani-Wayda and Walther construct equations where chaotic behavior appears. It is not clear how to apply their technique for some well known equations. Numerical results show complicated behavior for these equations for some parameter values. Chaos has not yet been proved for the Mackey–Glass equation [24]

$$\dot{x}(t) = -\mu x(t) + \frac{a x(t-1)}{1 + x^n(t-1)}$$

and for the Nicholson's blowflies equation

$$\dot{x}(t) = -\mu x(t) + ax(t-1)e^{-x(t-1)}$$

where $\mu > 0$, $a > 0$, and $n > 0$ is an integer.

The technique of Lani-Wayda and Walther can guarantee chaos only in a thin Cantor set in the state space C. Numerical results show that, for example, for the Mackey–Glass equations for some parameter values, almost all solutions are complicated. It would be interesting to develop methods for this type of complicated behavior as well.

Other results on chaos can be found e.g. in [20,21].

The result of Mallet-Paret and Walther [28] shows that for monotone negative feedback slow oscillation dominates the dynamics. Without the monotonicity this is not necessarily true. There are examples for stable rapidly oscillating periodic orbits for both the negative and positive feedback cases [1,32].

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