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The 2-Dimensional Attractor of a Differential Equation with State-Dependent Delay

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Abstract

The delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)),\ r = r(x(t))$$

with $\mu > 0$ and smooth real functions f, r satisfying $f(0) = 0$, $f' < 0$ and $r(0) = 1$ models a system governed by state-dependent delayed negative feedback and instantaneous damping. For a suitable $R \geq 1$ the solutions generate a semiflow F on a compact subset L_K of $C([-R, 0], \mathbb{R})$. F leaves invariant the subset S of $\phi \in L_K$ with at most one sign change on all subintervals of $[-R, 0]$ of length one. The induced semiflow on S has a global attractor $\mathcal{A}$. $\mathcal{A} \setminus \{0\}$ coincides with the set of segments of bounded globally defined slowly oscillating solutions. If $\mathcal{A} \neq \{0\}$ then $\mathcal{A}$ is homeomorphic to the closed unit disk, the unit circle corresponds to a periodic orbit.

Key words: state-dependent delay, negative feedback, slowly oscillating solutions, global attractor, discrete Lyapunov functional, asymptotic expansion, Poincaré–Bendixson type theorem

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1. Introduction

In this paper we study the state-dependent delay equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)),\ r = r(x(t)), \tag{1.1}$$

where $\mu > 0$, f and r are smooth real functions, $r(0) = 1$ and f satisfies the negative feedback condition $\xi f(\xi) < 0$ for all $\xi \neq 0$. Equation (1.1) with $r \equiv 1$ appears in several applications, see e.g. [15,30,34,36,37,40,52] and references therein. Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as in classical electrodynamics [18–22], in population models [7], in models of commodity price fluctuations [8,35] and in models of blood cell productions [38].

In case $r \equiv 1$ Eq. (1.1) generates a semiflow on the phase space $C([-1,0],\mathbb{R})$. Under the additional assumptions $f' < 0$, $\sup f < \infty$ or $\inf f > -\infty$, the semiflow leaves the subset T of elements $\phi \in C([-1,0],\mathbb{R})$ with at most one sign change invariant. A recent result of Mallet-Paret and Walther [46] shows that the domain of absorption into T is open and dense. Walther [49,50], Walther and Yebdri [51] described the global attractor A of the induced semiflow on T: either $A = \{0\}$ or A is a 2-dimensional C^1-smooth graph which is homeomorphic to the closed unit disk, the unit circle corresponds to a periodic orbit. A solution is called slowly oscillating if its zeros are spaced at distances larger than 1. A contains 0 and the segments $x(t+\cdot) \in C([-1,0],\mathbb{R})$ of all bounded slowly oscillating solutions $x : \mathbb{R} \to \mathbb{R}$.

Recent results of Mallet-Paret and Nussbaum [41,42], Mallet-Paret, Nussbaum and Paraskevopoulos [43], Kuang and Smith [33] and numerical studies suggest that the slowly oscillating solutions play an important role in the global dynamics of (1.1) also in the case $r \not\equiv 1$ for certain μ, f, r.

Our goal in this paper is to describe the asymptotic behavior of the slowly oscillating solutions of Eq. (1.1). The obtained results are in part analogous to that of Walther [50], but in the proofs a variety of new mathematical phenomena arise which are not present in the case $r \equiv 1$.

In addition to the above conditions on μ, f, r, we assume that $f \in C^1(\mathbb{R},\mathbb{R})$, $f' < 0$, and $\sup f < \infty$ provided $r(u) > 0$ for all $u \in \mathbb{R}$.

Some basic existence, uniqueness, continuation and continuous dependence results for differential equations with state-dependent delay are contained in [41,43]. The results of [41,43] are applicable to Eq. (1.1) and give existence, uniqueness, etc. for solutions having values in a certain compact interval. However, it is possible that there are slowly oscillating periodic solutions of the equation outside the region guaranteed by the results of [41,43]. In this paper we are interested in the asymptotic behavior of all slowly oscillating solutions of

Eq. (1.1). A slight modification of the technique of [41,43] gives the existence, uniqueness and continuous dependence results which are satisfactory for our purpose.

Let I_r denote the maximal subinterval of $\mathbb{R}$ with $0 \in I_r$ and $r(u) \geq 0$ for all $u \in I_r$. Our first result is that for every bounded continuous initial function $\phi : (-\infty, 0] \to I_r$, there is a solution $x : \mathbb{R} \to \mathbb{R}$ of equation (1.1) through ϕ, that is x is continuous on $\mathbb{R}$, continuously differentiable on $(0, \infty)$, $x|_{(-\infty,0]} = \phi$ and (1.1) holds for all $t > 0$. If ϕ is Lipschitz continuous then x is unique. Then we show the existence of positive constants A, B, R, K such that

$$0 < r(u) \leq R \quad \text{for all } u \in [-B, A], \qquad \max_{u,v \in [-B,A]} |-\mu u + f(v)| \leq K,$$

moreover for every solution $x : \mathbb{R} \to \mathbb{R}$ belonging to a bounded continuous initial function ϕ with $\phi((-\infty, 0]) \subset I_r$, there exists $s \geq 0$ such that

$$x(t) \in [-B, A] \qquad \text{for all } t \geq s.$$

Consequently, as we are interested in the asymptotic ($t \to \infty$) behavior of solutions, it suffices to consider only solutions with values in $[-B, A]$.

Let X denote the space of continuous real functions on $[-R, 0]$ equipped with the supremum-norm. The set

$$L_K = \left\{ \phi \in X : \phi([-R, 0]) \subset [-B, A], \ \left| \frac{\phi(t) - \phi(s)}{t - s} \right| \leq K \text{ for } -R \leq s < t \leq 0 \right\}$$

is a compact convex subset of X. For every $\phi \in L_K$ there is a unique continuous function $x^\phi : [-R, \infty) \to \mathbb{R}$ such that $x^\phi|_{[-R,0]} = \phi$, x^ϕ is continuously differentiable on $(0, \infty)$, and x^ϕ satisfies Eq. (1.1) for all $t > 0$. Then the relations

$$F(t, \phi) = x_t^\phi \text{ for } t \geq 0, \quad x_t^\phi(s) = x^\phi(t + s) \text{ for } -R \leq s \leq 0$$

define a semiflow F on L_K.

Motivated by the conjecture, which is true in the constant delay case [46], that the behavior of slowly oscillating solutions govern the typical long-term behavior of the solutions of Eq. (1.1), we consider the compact subset

$$S = \{ \phi \in L_K : \mathrm{sc}(\phi, [t - 1, t]) \leq 1 \text{ for all } t \in [-R + 1, 0] \}$$

of L_K, where $\mathrm{sc}(\phi, [t-1, t])$ denotes the number of sign changes of ϕ on the interval $[t-1, t]$. All segments x_t of slowly oscillating solutions x with values in $[-B, A]$ belong to S. The set S is positively invariant under the semiflow. The restriction of F to $\mathbb{R}^+ \times S$ defines a semiflow F_S. F_S has a global attractor $\mathcal{A}$ which is a subset of the global attractor of the full semiflow F. $\mathcal{A}$ consists of 0 and the segments x_t of the globally defined slowly oscillating solutions $x : \mathbb{R} \to [-B, A]$.

We prove a Poincaré–Bendixson type result on $\mathcal{A}$: the α- and ω-limit sets of phase curves in $\mathcal{A}$ are either $\{0\}$ or periodic orbits given by slowly oscillating periodic solutions. The second main result is that in the case $\mathcal{A} \neq \{0\}$, the set $\mathcal{A}$ is homeomorphic to the 2-dimensional closed unit disk so that the unit circle corresponds to a periodic orbit given by a slowly oscillating periodic solution.

The paper is organized as follows. Section 2 gives the appropriate framework for the study of the asymptotic behavior of solutions. An additional condition on r is introduced to guarantee that the function $t \mapsto t - r(x(t))$ is strictly increasing. For example, the smallness of r' or concavity of r are sufficient. This monotone property of $t \mapsto t - r(x(t))$ plays an important role in the proofs.

Section 3 contains results on the associated linear equation

$$\dot{y}(t) = -\mu y(t) + f'(0)y(t-1). \tag{1.2}$$

Although the map $X \ni \phi \mapsto -\mu\phi(0) + f(\phi(-r(\phi(0)))) \in \mathbb{R}$ is not, in general, differentiable, equation (1.2) can be considered as the linearization of (1.1) at 0 (see Cooke and Huang [14] and also [9,27]).

Section 4 introduces a discrete Lyapunov functional which counts the sign changes of solutions over intervals of the form $[t - r(x(t)), t]$. We need a modified version of the results of Mallet-Paret and Sell [44] on discrete Lyapunov functionals in order to handle the state-dependent delay case instead of the constant delay case. It seems to be crucial that the delay r depends only on $x(t)$ and not on x_t. We prove an analogue of the a priori estimate of Mallet-Paret [39], Cao [10], Arino [4] which can be used to show that slowly oscillating solutions do not decay faster than any exponential.

Section 5 introduces the set S, the global attractor $\mathcal{A}$ and intersection maps associated with the compact convex subset

$$U = \{\phi \in L_K : \phi(s) \geq 0 \text{ for all } s \in [-1, 0], \phi(0) = 0\}$$

of L_K. We find that $\mathcal{A} \cap U$ is connected, which is an essential step in the construction of a homeomorphism from $\mathcal{A}$ onto the closed unit disk.

Section 6 proves asymptotic expansion for slowly oscillating solutions converging to zero as $t \to -\infty$. The related result for the constant delay case is due to Cao [10].

Section 7 shows that if ϕ, ψ are different elements of $\mathcal{A}$ and $x^\phi, x^\psi : \mathbb{R} \to \mathbb{R}$ are the solutions through ϕ, ψ, respectively, then the difference $x^\phi - x^\psi$ has at most one sign change on the interval $[t - r(x^\phi(t)), t]$ for all $t \in \mathbb{R}$. This fact guarantees the injectivity of a map from $\mathcal{A}$ into $\mathbb{R}^2$ in Section 8. The proof uses, among others, properties of slowly oscillating periodic solutions obtained by Mallet-Paret and Nussbaum [41].

The last two sections contain the two main results with proofs.

We remark that the results can be easily modified to the case $\mu = 0$ and to the case when f is bounded below. Only the construction of the constants A, B, R, K in Section 2 is slightly different. So, Wright's equation [54] with state-dependent delay is a particular case.

We mention that related results on attractors for differential equations with constant delay are contained in [11,31–32]. For other results on functional differential equations with state-dependent delay we refer to [1,2,3,5,6,12,13,16,23,24,28,29,47,53,55,56].

Notation. The symbols $\mathbb{N}$ and $\mathbb{R}^+$ denote the nonnegative integers and reals, respectively. $\mathbb{R}$ and $\mathbb{Z}$ stand for the set of all reals and all integers, respectively.

An upper index tr denotes the transpose of a vector in $\mathbb{R}^n$.

A trajectory of a map $g : M \to N$, $M \subset N$, is a finite or infinite sequence $(x_j)_{j \in I \cap \mathbb{Z}}$, $I \subset \mathbb{R}$ an interval, in M with $x_{j+1} = g(x_j)$ for all $j \in I \cap \mathbb{Z}$ with $j + 1 \in I \cap \mathbb{Z}$.

A simple closed curve is a continuous map c from a compact interval $[a, b] \subset \mathbb{R}$, $a < b$, into $\mathbb{R}^n$ so that $c|_{[a,b)}$ is injective and $c(a) = c(b)$. The set of values of a simple closed curve c, or trace, is denoted by $|c|$. The Jordan curve theorem guarantees that the complement of the trace of a simple closed curve c in $\mathbb{R}^2$ consists of two nonempty connected open sets, one bounded and the other unbounded, and $|c|$ is the boundary of each of these components. We denote the bounded component by $\mathrm{int}(c)$ and the unbounded one by $\mathrm{ext}(c)$.

Spectra of continuous linear maps $T : E \to E$ are defined as spectra of their complexifications. If a decomposition

$$E = F \oplus G$$

into closed linear subspaces is given then $\mathrm{Pr}_F : E \to E$ and $\mathrm{Pr}_G : E \to E$ denote the associated projection operators along G onto F and along F onto G, respectively.

For given reals a, b with $a < b$, $C([a, b], \mathbb{R})$ denotes the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}$ with the norm given by

$$||\phi||_{C([a,b],\mathbb{R})} = \max_{a \le t \le b} |\phi(t)|.$$

$C^1([a, b], \mathbb{R})$ is the Banach space of all C^1-maps $\phi : [a, b] \to \mathbb{R}$, with the norm given by

$$||\phi||_{C^1([a,b],\mathbb{R})} = ||\phi||_{C([a,b],\mathbb{R})} + ||\dot{\phi}||_{C([a,b],\mathbb{R})}.$$

2. The equation and some basic properties

Consider the equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-r)), \quad r = r(x(t)), \tag{2.1}$$

under the hypotheses

$$\begin{cases} \mu > 0, \\ f \in C^2(\mathbb{R},\mathbb{R}), \ f(0)=0, \ f'(u) < 0 \text{ for all } u \in \mathbb{R}, \\ r \in C^1(\mathbb{R},\mathbb{R}) \text{ and } r(0)=1, \\ \sup\{f(u) : \ u \in \mathbb{R}\} < \infty \text{ if } r(u) > 0 \text{ for all } u \in \mathbb{R}. \end{cases} \tag{H1}$$

For intervals $I, J \subseteq \mathbb{R}$ with $I \subseteq J$, we say that x is a solution of Eq. (2.1) on (I, J) if $x : J \to \mathbb{R}$ is continuous, continuously differentiable on I, satisfies

$$t - r(x(t)) \in J \qquad \text{for all } t \in I,$$

and is such that (2.1) holds for all $t \in I$. (If t is an endpoint of I, then by $\dot{x}(t)$ we always mean the appropriate one-sided derivative.)

Let I_r be the maximal subinterval of $\mathbb{R}$ such that $0 \in I_r$ and $r(u) \geq 0$ for all $u \in I_r$.

Let $BC((-\infty,0], I_r)$ denote the set of bounded continuous functions on $(-\infty,0]$ with values in I_r.

The following results on the existence, uniqueness and continuous dependence of solutions can be obtained by using the technique of [41,43]. We need a slight modification of the results of [41,43] since we want to study the asymptotic behavior of all slowly oscillating solutions of Eq. (2.1).

Proposition 2.1.
(i) If $\phi \in BC((-\infty,0], I_r)$, then there exists a solution x of (2.1) on $([0,\infty),\mathbb{R})$ with $x|_{(-\infty,0]} = \phi$.
(ii) If $\phi \in BC((-\infty,0], I_r)$, $\beta \in (0,\infty]$ and x is a noncontinuable solution of (2.1) on $([0,\beta),(-\infty,\beta))$ with $x|_{(-\infty,0]} = \phi$, then $\beta = \infty$ and $x(t) \in I_r$ for all $t \in \mathbb{R}$.
(iii) If $\phi \in BC((-\infty,0], I_r)$ is Lipschitz continuous and $x, \bar{x}$ are solutions of (2.1) on $([0,\infty),\mathbb{R})$ with $x|_{(-\infty,0]} = \phi = \bar{x}|_{(-\infty,0]}$, then $x(t) = \bar{x}(t)$ for all $t \in \mathbb{R}$.

Proof. 1. Let $\phi \in BC((-\infty,0], I_r)$ be given. Define

$$m_\phi = \min\{0, \inf\{\phi(s) : \ s \leq 0\}\}, \quad M_\phi = \max\{0, \sup\{\phi(s) : \ s \leq 0\}\}.$$

First we determine two positive constants C_ϕ and D_ϕ such that $[m_\phi, M_\phi] \subseteq [-D_\phi, C_\phi] \subseteq I_r$ and any solution x of (2.1) on $([0,\beta), (-\infty,\beta))$ with $x|_{(-\infty,0]} = \phi$ satisfies

$$x(t) \in (-D_\phi, C_\phi) \quad \text{for all} \quad t \in (0,\beta). \tag{2.2}$$

Let $-b \in [-\infty, 0)$ and $a \in (0,\infty]$ denote the (possibly infinite) endpoints of I_r. Let I_r^+ denote the maximal subinterval of $\mathbb{R}$ such that $r(u) > 0$ for all $u \in I_r^+$. Choose $c, d \in (0,\infty]$ such that $I_r^+ = (-d, c)$. Clearly, $-\infty \le -b \le -d < 0 < c \le a \le \infty$ and $-b \le m_\phi \le 0 \le M_\phi \le a$. In the definition of C_ϕ and D_ϕ, we distinguish four cases.

Case 1: $c < \infty,\ d < \infty$. In this case we choose C_ϕ and D_ϕ such that

$$C_\phi = \min\left\{a, \max\left\{c, M_\phi, 1 + \frac{1}{\mu} f(m_\phi)\right\}\right\}$$

and

$$-D_\phi = \max\left\{-b, \min\left\{-d, m_\phi, -1 + \frac{1}{\mu} f(M_\phi)\right\}\right\}.$$

Case 2: $c = \infty,\ d < \infty$. In this case first we define D_ϕ such that

$$-D_\phi = \max\left\{-b, \min\left\{-d, m_\phi, -1 + \frac{1}{\mu} f(M_\phi)\right\}\right\}.$$

Then choose C_ϕ such that

$$C_\phi > \max\left\{M_\phi, \frac{1}{\mu} f(-D_\phi)\right\}.$$

Case 3: $c < \infty,\ d = \infty$. In this case first we define C_ϕ such that

$$C_\phi = \min\left\{a, \max\left\{c, M_\phi, 1 + \frac{1}{\mu} f(m_\phi)\right\}\right\}.$$

Then choose D_ϕ such that

$$-D_\phi < \min\left\{m_\phi, \frac{1}{\mu} f(C_\phi)\right\}.$$

Case 4: $c = d = \infty$. Then $I_r^+ = (-\infty, \infty)$ and, by (H1), $\sup f < \infty$. So we may choose C_ϕ such that

$$C_\phi > \max\left\{M_\phi, \frac{1}{\mu} \sup f\right\}$$

and then D_ϕ so that

$$-D_\phi < \min\left\{m_\phi, \frac{1}{\mu} f(C_\phi)\right\}.$$

Now we prove (2.2). First observe that $-b \le -D_\phi \le m_\phi \le M_\phi \le C_\phi \le a$ because of the definition of C_ϕ and D_ϕ. Therefore, $x(t) \in [-D_\phi, C_\phi]$ for all $t \le 0$.

Another observation, from Eq. (2.1) and (H1), is that

$$\begin{aligned} &t \in [0,\beta),\ x(t) > 0,\ r(x(t)) = 0 \ \text{ imply } \ \dot{x}(t) < 0, \\ &t \in [0,\beta),\ x(t) < 0,\ r(x(t)) = 0 \ \text{ imply } \ \dot{x}(t) > 0. \end{aligned} \tag{2.3}$$

If (2.2) is not true, then there exists $t_0 \in [0,\beta)$ such that $x(t) \in [-D_\phi, C_\phi]$ for all $t \le t_0$ and either $x(t_0) = C_\phi$, $\dot{x}(t_0) \ge 0$ or $x(t_0) = -D_\phi$, $\dot{x}(t_0) \le 0$.

Assume that $x(t) \in [-D_\phi, C_\phi]$ for all $t \le t_0$, $x(t_0) = C_\phi$ and $\dot{x}(t_0) \ge 0$. Then $r(x(t)) \ge 0$ for all $t \le t_0$ because of $[-D_\phi, C_\phi] \subseteq I_r$. From (2.3) it follows that $r(x(t_0)) > 0$.

In Case 1, the facts $r(a) = 0$ provided $a < \infty$, $r(c) = 0$, $r(x(t_0)) > 0$ and the definition of C_ϕ combined imply $c < C_\phi < a$. We also have $x(t) \neq 0$ for all $t \in [0, t_0)$, since $x(t_1) = 0$ for some $t_1 \in [0, t_0)$ and (2.3), $r(c) = 0$ together would imply $x(t) < c < C_\phi$ for all $t \in [t_1, \beta)$, contradicting $x(t_0) = C_\phi$. In particular, $x(t) \ge m_\phi$ for all $t \le t_0$. Then, using Eq. (2.1) and that $C_\phi > \frac{1}{\mu} f(m_\phi)$, we obtain

$$\dot{x}(t_0) \le -\mu C_\phi + f(m_\phi) < 0,$$

a contradiction.

In Case 2, from $-D_\phi \le x(t)$ for all $t \le t_0$, Eq. (2.1) and the definition of C_ϕ it follows that

$$\dot{x}(t_0) \le -\mu C_\phi + f(-D_\phi) < 0,$$

a contradiction.

In Case 3 the same proof works as in Case 1.

In Case 4, by Eq. (2.1) and the definition of C_ϕ, we obtain again the contradiction $\dot{x}(t_0) < 0$.

In the case when $x(t) \in [-D_\phi, C_\phi]$ for all $t \le t_0$, $x(t_0) = -D_\phi$ and $\dot{x}(t_0) \le 0$, we can get a contradiction in the same way as above. Therefore, (2.2) holds.

We modify the right hand side of Eq. (2.1) and r outside the sets $[-D_\phi, C_\phi] \times [-D_\phi, C_\phi]$ and $[-D_\phi, C_\phi]$, respectively. Let

$$g(x,y) = -\mu\kappa(x) + f(\kappa(y)), \quad \tilde{r} = r(\kappa(x)),$$

where

$$\kappa(x) = \begin{cases} -D_\phi & \text{if } x < -D_\phi, \\ x & \text{if } -D_\phi \le x \le C_\phi, \\ C_\phi & \text{if } x > C_\phi. \end{cases}$$

Consider the equation

$$\dot{x}(t) = g(x(t), x(t-\tilde{r})), \quad \tilde{r} = \tilde{r}(x(t)). \tag{2.4}$$

Let $\tilde{R} = \max\{r(u) : \ u \in [-D_\phi, C_\phi]\}$ and let $\tilde{C} = C([-\tilde{R}, 0], \mathbb{R})$ be the Banach space of continuous functions equipped with the maximum norm. It is easy to see that the mapping $\tilde{C} \ni \psi \mapsto g(\psi(0), \psi(-\tilde{r}(\psi(0)))) \in \mathbb{R}$ is continuous and there exists $c_1 > 0$ such that $|g(\psi)| \leq c_1 \max_{s\in[-\tilde{R},0]} |\psi(s)|$ for all $\psi \in \tilde{C}$. Therefore, the existence theorem of [26, Chapter 2] can be applied to Eq. (2.4). Let $\tilde{\phi} \in \tilde{C}$ be such that $\tilde{\phi} = \phi|_{[-\tilde{R},0]}$. Then Eq. (2.4) has a solution $\tilde{x} : [-\tilde{R}, \infty) \to \mathbb{R}$ with $\tilde{x}|_{[-\tilde{R},0]} = \tilde{\phi}$. Since the right hand sides of (2.1) and (2.4) are the same on $[-D_\phi, C_\phi] \times [-D_\phi, C_\phi]$, and the functions r and $\tilde{r}$ are the same on $[-D_\phi, C_\phi]$, the proof of (2.2) works also for $\tilde{x}$ to show that $\tilde{x}(t) \in (-D_\phi, C_\phi)$ for all $t > 0$. Then the extension x of $\tilde{x}$ to $\mathbb{R}$, such that $x|_{(-\infty,0]} = \phi$ and $x|_{[-\tilde{R},\infty)} = \tilde{x}$, is a solution of (2.1) on $([0,\infty), \mathbb{R}$ with $\tilde{x}|_{(-\infty,0]} = \phi$. This completes the proof of (i).

2. Now let x be a noncontinuable solution of (2.1) as in (ii). (2.2) holds for this x. Therefore, the restriction of x to the interval $[-\tilde{R}, \beta)$ is also a noncontinuable solution of Eq. (2.4) on $[-\tilde{R}, \beta)$. Since $|g(\psi)| \leq c_1 \max_{s\in[-\tilde{R},0]} |\psi(s)|$ for all $\psi \in \tilde{C}$, the continuation theorem of [26] gives $\beta = \infty$.

3. To prove the claim of uniqueness in (iii), assume that x and $\bar{x}$ are solutions of Eq. (2.1) on $([0,\infty), \mathbb{R})$ with $x|_{(-\infty,0]} = \phi = \bar{x}|_{(-\infty,0]}$. For both solutions x and $\bar{x}$, (2.2) is satisfied with $\beta = \infty$. Therefore, we may choose $M \geq \mu$ such that x and $\bar{x}$ are Lipschitz continuous on $\mathbb{R}$ and f, r are also Lipschitz continuous on $[-D_\phi, C_\phi]$ with Lipschitz constant M. Let $y(t) = x(t) - \bar{x}(t)$, $\eta(t) = t - r(x(t))$ and $\bar{\eta}(t) = t - r(\bar{x}(t))$. Then

$$\dot{y}(t) = -\mu y(t) + f(x(\eta(t))) - f(\bar{x}(\bar{\eta}(t)))$$

and

$$\begin{aligned} |\dot{y}(t)| &\leq \mu|y(t)| + |f(x(\eta(t))) - f(\bar{x}(\bar{\eta}(t)))| \\ &\leq M|y(t)| + M|x(\eta(t)) - \bar{x}(\eta(t))| + M|\bar{x}(\eta(t)) - \bar{x}(\bar{\eta}(t))| \\ &\leq M|y(t)| + M|y(\eta(t))| + M^3|y(t)|. \end{aligned}$$

Hence with $z(t) = \max_{s\in[0,t]} |y(s)|$,

$$\begin{aligned} |y(t)| &\leq \int_0^t (M^3 + 2M) z(s)\, ds \\ &\leq \int_0^\tau (M^3 + 2M) z(s)\, ds \qquad \text{for all } \ 0 \leq t \leq \tau. \end{aligned}$$

Then

$$z(\tau) \leq \int_0^\tau (M^3 + 2M) z(s)\, ds \qquad \text{for all } \tau \geq 0,$$

and the Gronwall lemma implies $z(\tau) = 0$ for all $\tau \geq 0$. This proves the uniqueness. $\square$

Now we need the following two simple observations about the asymptotic behavior of the solutions of (2.1).

Lemma 2.2.

(i) If $t_0 \in \mathbb{R}$ and x is a solution of Eq. (2.1) on $([t_0, \infty), \mathbb{R})$ with $x(\mathbb{R}) \subset I_r$ such that x has no zero on $[t_0, \infty)$, then $x(t) \to 0$ as $t \to \infty$.

(ii) If x is a bounded solution of Eq. (2.1) on $(\mathbb{R}, \mathbb{R})$ with $x(\mathbb{R}) \subset I_r$ then there is a sequence $(t_n)_0^\infty$ such that $t_n \to -\infty$ as $n \to \infty$ and $x(t_n) = 0$ for all $n \in \mathbb{N}$.

Proof. 1. The proof of (i). By the proof of Proposition 2.1, there are constants $C_0, D_0 \in (0, \infty)$, depending on $x|_{(-\infty, t_0]}$, such that $x(t) \in [-D_0, C_0]$ for all $t \in \mathbb{R}$. Let $R_0 = \max\{r(u) : \ u \in [-D_0, C_0]\}$. If $x(s) > 0$ for all $s \geq t_0$, and $t \geq t_0 + R_0$, then, from Eq. (2.1) and hypothesis (H1), it follows that $\dot{x}(t) < 0$. Therefore, $x(t)$ converges to some $\alpha \geq 0$ as $t \to \infty$. Suppose $\alpha > 0$. Then, by Eq. (2.1) and (H1), $\dot{x}(t) \to -\mu\alpha + f(\alpha) < 0$ as $t \to \infty$, a contradiction. The case, when $x(s) < 0$ for all $s \geq t_0$, is analogous.

2. The proof of (ii). By the boundedness of x, there are constants $C_0, D_0 \in (0, \infty)$ such that $x(t) \in [-D_0, C_0]$ for all $t \in \mathbb{R}$. Suppose that the statement is not true. Consider the case when $x(t) > 0$ for all $t \leq t_0$ for some $t_0 \in \mathbb{R}$. Then, by (H1),

$$\dot{x}(t) = -\mu x(t) + f(x(t - r(x(t)))) \leq -\mu x(t), \quad t \leq t_0.$$

Hence

$$0 < x(t_0) \leq x(t) e^{-\mu(t_0 - t)} \leq C_0 e^{-\mu(t_0 - t)}, \quad t \leq t_0.$$

Letting $t \to \infty$, we obtain that $x(t_0) = 0$, a contradiction. The case $x(t) < 0$ for all $t \leq t_0$, is analogous. $\square$

Now we show that all solutions of Eq. (2.1) with initial values in $BC((-\infty, 0], I_r)$ are eventually in a finite interval.

Proposition 2.3. *There exist positive constants A, B, R, K such that:*

(i)

$$\begin{aligned} &R \geq \max\{r(u) : \ u \in [-B, A]\}, \\ &\min\{r(u) : \ u \in [-B, A]\} > 0, \\ &K \geq \max\{|-\mu u + f(v)| : \ (u, v) \in [-B, A] \times [-B, A]\}. \end{aligned}$$

(ii) For each solution x of (2.1) on $([0, \infty), \mathbb{R})$ with $x|_{(-\infty, 0]} = \phi \in BC((-\infty, 0], I_r)$, there exists $s \geq 0$ such that

$$x(t) \in [-B, A] \qquad \text{for all } \ t \geq s.$$

(iii) If $\phi \in C([-R,0],[-B,A])$ is Lipschitz continuous with Lipschitz constant K, then there exists a unique solution x of (2.1) on $([0,\infty),[-R,\infty))$ with $x|_{[-R,0]} = \phi$, and this solution satisfies

$$x(t) \in [-B,A], \quad |\dot{x}(t)| \le K \qquad \textit{for all } t \ge 0.$$

(iv) If x is a bounded solution of (2.1) on $(\mathbb{R},\mathbb{R})$ with $x(\mathbb{R}) \subset I_r$, then $x(\mathbb{R}) \subset [-B,A]$.

Proof. 1. The proof of (i). First we define two constants $C > 0$ and $D > 0$. As in the proof of Proposition 2.1, $c, d \in (0,\infty]$ are chosen such that $r(u) > 0$ for all $u \in (-d,c)$ and $c < \infty$ implies $r(c) = 0$, $d < \infty$ implies $r(-d) = 0$. In order to define C and D, we distinguish four cases.

Case 1. If $c < \infty$ and $d < \infty$, then let $C = c$ and $D = d$.

Case 2. If $c = \infty$ and $d < \infty$, then let $D = d$ and choose C such that $C > \frac{1}{\mu} f(-D)$.

Case 3. If $c < \infty$ and $d = \infty$, then let $C = c$ and choose D such that $-D < \frac{1}{\mu} f(C)$.

Case 4. If $c = d = \infty$, then, by (H1), $\sup f < \infty$. Choose C such that $C > \frac{1}{\mu} \sup_{u \in \mathbb{R}} f(u)$ and D such that $-D < \frac{1}{\mu} f(C)$.

Set

$$K = \max\{|-\mu u + f(v)| : \ (u,v) \in [-D,C] \times [-D,C]\}$$

and let $L = \max\{|f'(u)| : u \in [-D,C]\}$.

Now we define the two positive constants A and B. Let

$$A = C \ \text{ if } \ c = \infty,$$

and

$$B = D \ \text{ if } \ d = \infty.$$

In case $c < \infty$, choose $A \in (0,c)$ such that

$$r(u) < \frac{\mu A}{2LK} \ \text{ for all } \ u \in [A,c].$$

If $d < \infty$, then choose $B \in (0,d)$ such that

$$r(u) < \frac{\mu B}{2LK} \ \text{ for all } \ u \in [-d,-B].$$

The existence of A and B in the cases $c < \infty$ and $d < \infty$ follows from the continuity of r and $r(c) = r(-d) = 0$.

Let

$$R = \max\{r(u) : \ u \in [-D,C]\},$$

$$r_0 = \min\{r(u):\ u \in [-B, A]\}.$$

Clearly, $r_0 > 0$ and (i) is satisfied.

2. The proof of (ii). By the proof of Proposition 2.1, there are constants $C_\phi, D_\phi \in (0, \infty)$ such that $x(t) \in [-D_\phi, C_\phi]$ for all $t \in \mathbb{R}$. Let $R_\phi = \max\{r(u):\ u \in [-D_\phi, C_\phi]\}$.

If there exists $t_0 \geq 0$ such that x has no zero on $[t_0, \infty)$, then $\lim_{t\to\infty} x(t) = 0$ because of Lemma 2.2(i). Therefore, $x(t) \in [-B, A]$ for all large t.

Assume that x has arbitrarily large zeros. Pick two zeros z_1, z_2 of x such that $z_2 \geq z_1 + R_\phi$, $z_1 \geq 0$. Then (2.3) can be used to get that

$$x(t) \in (-d, c) \qquad \text{for all } t \geq z_1. \tag{2.5}$$

We want to prove that

$$x(t) \in (-D, C) \qquad \text{for all } t \geq z_2. \tag{2.6}$$

We follow the four cases of the definition of C and D.

Case 1 is clear from (2.5).

In Case 2, (2.5) implies that $x(t) > -d = -D$ for all $t \geq z_1$. Thus, it suffices to show that $x(t) = C$ and $t \geq z_2$ imply $\dot{x}(t) < 0$. Indeed, this is the case by Eq. (2.1) and $C > \frac{1}{\mu} f(-D)$.

Case 3 is analogous to Case 2.

In Case 4, if $x(t) < C$ for all $t \geq z_1$ does not hold, then there is a smallest $t > z_1$ such that $x(t) = C$. We have $\dot{x}(t) \geq 0$ because of the definition of t. On the other hand, $x(t) = C$ and the definition of C imply $\dot{x}(t) \leq -\mu C + \sup f < 0$, a contradiction. Consequently, $x(t) < C$ for all $t \geq z_1$. If $t \geq z_2$ and $x(t) = D$, then, using the definition of D, we find $\dot{x}(t) \geq \mu D + f(C) > 0$. Therefore, $x(t) > -D$ can be obtained for all $t \geq z_2$. Thus (2.6) is proved. As a consequence, x is Lipschitz continuous on $[z_2 + R, \infty)$ with Lipschitz constant K.

In order to complete the proof of (ii), we need the following claim.

CLAIM. Assume that x is a solution of (2.1) on $([0, \infty), \mathbb{R})$ with $x|_{(-\infty,0]} = \phi \in BC((-\infty, 0], I_r)$, there exists $t_0 \geq 0$ such that $x(t) \in [-D, C]$ for all $t \in [t_0 - R, t_0]$ and x is Lipschitz continuous on $[t_0 - R, t_0]$ with Lipschitz constant K. Then $x(t) \in [-D, C]$ for all $t \geq t_0$, there exists $T > 0$ such that $x(t) \in [-B, A]$ for all $t \geq t_0 + T$, and

$$x(t_0) \in [-B, A] \ \text{ implies } \ x(t) \in [-B, A] \ \text{ for all } \ t \geq t_0.$$

Proof of the Claim. Assume that $x(t) \in [-D, C]$ for all $t \geq t_0$ does not hold. Then there exists $t \geq t_0$ such that $x(s) \in [-D, C]$ for all $s \in [t_0 - R, t]$ and either $x(t) = C$, $\dot{x}(t) \geq 0$ or $x(t) = -D$, $\dot{x}(t) \leq 0$. We can get a contradiction exactly in the same way as in the proof of (2.6). Therefore, $x(t) \in [-D, C]$ for all $t \geq t_0$. If $A = C$ and $B = D$, the proof is complete. Assume that $A < C$. Let $s \geq t_0$ be such that $x(s) \in [A, C]$. Using the definition of A, we have

$$\begin{aligned}
\dot{x}(s) &= -\mu x(s) + f(x(s - r(x(s)))) \\
&= -\mu x(s) + f(x(s)) + f(x(s - r(x(s)))) - f(x(s)) \\
&\leq -\mu A + |f(x(s - r(x(s)))) - f(x(s))| \\
&\leq -\mu A + L|x(s - r(x(s))) - x(s)| \\
&\leq -\mu A + LKr(x(s)) \\
&< -\mu A + \frac{\mu A}{2} = -\frac{\mu A}{2}.
\end{aligned}$$

Then it is easy to see that $x(t) \leq A$ for all $t \geq t_0 + 2(C - A)/(\mu A)$. Moreover, $x(t_0) \leq A$ implies $x(t) \leq A$ for all $t \geq t_0$. In the case $B < D$, we get analogously that $x(t) \geq -B$ for all $t \geq t_0 + 2(D - B)/(\mu B)$, and that $x(t_0) \geq -B$ implies $x(t) \geq -B$ for all $t \geq t_0$. This completes the proof of the claim.

Obviously, the Claim implies (ii).

3. The proof of (iii). Statement (iii) also follows from the above claim. Indeed, extending x to $\mathbb{R}$ with $x(t) = x(-R)$ for $t \leq -R$, we can apply the Claim with $t_0 = 0$ to get $x(t) \in [-B, A]$ for all $t \geq 0$. The estimation for $|\dot{x}(t)|$ is an obvious consequence. The uniqueness comes from Proposition 2.1.

4. The proof of (iv). If x is a bounded solution on $(\mathbb{R}, \mathbb{R})$ with values in I_r, then Lemma 2.2 (ii) implies that x has arbitrarily large negative zeros. Hence, in the same way as in the proof of (2.5) and (2.6), we get first that $x(t) \in (-d, c)$ and then that $x(t) \in [-D, C]$ for all $t \in \mathbb{R}$. Since the Claim can be applied with any $t_0 \in \mathbb{R}$, it is obtained that $x(t) \in [-B, A]$ for all $t \in \mathbb{R}$. $\square$

On the basis of Proposition 2.3, in the remaining part of the paper we consider only solutions with values in the interval $[-B, A]$. We define a suitable phase space and show that Eq. (2.1) generates a continuous semiflow on this phase space.

Let $X = C([-R, 0], \mathbb{R})$ denote the Banach space of continuous functions on $[-R, 0]$ with the maximum norm denoted by $|| \cdot ||$. Define

$$L_K = \{\phi \in X : \ -B \leq \phi(s) \leq A, \ |\phi(t) - \phi(s)| \leq K|t - s| \text{ for all } t, s \in [-R, 0]\}.$$

(The constants A, B, R, K are given in Proposition 2.3.) By the Arzèla–Ascoli theorem, L_K is a compact convex subset of X.

If $a > 0$, $x \in C([t_0 - R, t_0 + a), [-B, A])$ and x is Lipschitz continuous on $[t_0 - R, t_0 + a)$ with Lipschitz constant K, then, for $t \in [t_0, t_0 + a)$, $x_t \in L_K$ is defined by $x_t(s) = x(t + s)$,

$-R \le s \le 0$. In the followings, for given $\phi \in L_K$, $x^\phi : [-R, \infty) \to [-B, A]$ denotes the unique solution of (2.1) on $([0, \infty), [-R, \infty))$ with $x_0^\phi = \phi$ guaranteed by Proposition 2.3. Define

$$F : [0, \infty) \times L_K \ni (t, \phi) \mapsto x_t^\phi \in L_K.$$

Proposition 2.3 shows that F is well defined and maps $[0, \infty) \times L_K$ into L_K. It is easy to check that, for every $\phi \in L_K$, the function $[0, \infty) \ni t \mapsto F(t, \phi) \in L_K$ is continuous and $F(t + s, \phi) = F(t, F(s, \phi))$ for all $t, s \in [0, \infty)$. The continuity of F in ϕ and more are contained in the next lemma.

Lemma 2.4. *If $(\phi^n)_0^\infty$ is a sequence in L_K, $\phi \in L_K$, $||\phi^n - \phi|| \to 0$ as $n \to \infty$, and x^n, x denote the solutions of Eq. (2.1) on $([0, \infty), [-R, \infty))$ with $x_0^n = \phi^n, x_0 = \phi$, respectively, then for any $T > 0$,*

$$\begin{aligned} &x^n(t) \to x(t) && \text{as } n \to \infty \text{ uniformly in } t \in [-R, T], \\ &\dot{x}^n(t) \to \dot{x}(t) && \text{as } n \to \infty \text{ uniformly in } t \in [0, T]. \end{aligned}$$

Proof. If the first statement does not hold, then there exists $\delta > 0$ and a subsequence $(n_k)_0^\infty$ such that

$$\sup_{-R \le t \le T} |x^{n_k}(t) - x(t)| \ge \delta \qquad \text{for all } k \in \mathbb{N}.$$

By the Arzèla–Ascoli theorem, there is a subsequence $(n_{k_l})_{l=0}^\infty$ of $(n_k)_0^\infty$ with

$$x^{n_{k_l}}(t) \to y(t) \qquad \text{as } l \to \infty \text{ uniformly in } \; t \in [-R, T]$$

for some $y \in C([-R, T], [-B, A])$ which is also Lipschitz continuous with Lipschitz constant K and $y|_{[-R,T]} \ne x|_{[-R,T]}$. It is easy to see that y is a solution of Eq. (2.1) on $([0, T], [-R, T])$ with $y_0 = \phi$. This contradicts the uniqueness.

The second statement follows from the first one and

$$\begin{aligned} |\dot{x}^n(t) - \dot{x}(t)| \le& \mu|x^n(t) - x(t)| + |f(x^n(t - r(x^n(t)))) - f(x(t - r(x(t))))| \\ \le& \mu|x^n(t) - x(t)| + L_f|x^n(t - r(x^n(t))) - x(t - r(x^n(t)))| \\ &+ L_f|x(t - r(x^n(t))) - x(t - r(x(t)))| \\ \le& \mu|x^n(t) - x(t)| + L_f|x^n(t - r(x^n(t))) - x(t - r(x^n(t)))| \\ &+ L_f K|r(x^n(t)) - r(x(t))| \\ \le& \mu|x^n(t) - x(t)| + L_f|x^n(t - r(x^n(t))) - x(t - r(x^n(t)))| \\ &+ L_f K L_r|r(x^n(t)) - r(x(t))| \\ \le& (\mu + L_f + L_f K L_r) \max_{-R \le s \le T} |x^n(s) - x(s)|, \quad 0 \le t \le T, \end{aligned}$$

where L_f and L_r are Lipschitz constants for f and r on the interval $[-B, A]$. $\square$

As a consequence, we obtain that F is a continuous semiflow on the compact metric space L_K.

The increasing property of the function $t \mapsto \eta(t) = t - r(x(t))$, where x is a solution of Eq. (2.1) with values in $[-B, A]$, plays an important role in the theory. Either one of the following two hypotheses guarantees $\dot{\eta}(t) > 0$ for some interval.

$$|r'(u)| < \frac{1}{K} \qquad \text{for all } u \in [-B, A]. \tag{H2}$$

$$\begin{cases} r \in C^2([-B, A], \mathbb{R}) \text{ and there exists } a \in (0,1) \\ \text{with } r''(u) \le a\mu\big(r'(u)\big)^2 \text{ for all } u \in [-B, A]. \end{cases} \tag{H2'}$$

Condition (H2') was introduced by Mallet-Paret and Nussbaum [41]. The advantage of (H2') comparing to (H2) is that it is independent of f, and if it holds for some $\mu_0 > 0$, then it holds for all $\mu \ge \mu_0$. This was important in [41], where a singularly perturbed equation was considered.

In the remaining part of the paper we always assume that, in addition to (H1), either (H2) or (H2') holds.

Lemma 2.5. *Let $t_0 \in \mathbb{R}$ and let $x : [t_0 - R, \infty) \to [-B, A]$ be a solution of (2.1) on $([t_0, \infty), [t_0 - R, \infty))$. Suppose $\dot{x}(\rho) = 0$ for some $\rho \ge t_0$. Then $\frac{d}{dt}(t - r(x(t)) > 0$ for all $t \ge \rho$.*

Proof. Set $\eta : [t_0, \infty) \ni t \mapsto t - r(x(t)) \in \mathbb{R}$. If (H2) is assumed, then $\dot{\eta}(t) = 1 - r'(x(t))\dot{x}(t) \ge 1 - |r'(x(t))\dot{x}(t)| > 1 - \frac{1}{K}K = 0$ for all $t \ge t_0$.

Assume that (H2') holds and let $\rho \ge t_0$ with $\dot{x}(\rho) = 0$. Then $\dot{\eta}(\rho) = 1 - r'(x(\rho))\dot{x}(\rho) = 1$. We show that $\dot{\eta}(t) > 0$ for all $t \ge \rho$. If this is false, then define

$$t_1 = \inf\{t > \rho : \; \dot{\eta}(t) = 0\}.$$

At $t = t_1$ we have $\dot{\eta}(t_1) = 0$, $r'(x(t_1)\dot{x}(t_1) = 1$ and so

$$\begin{aligned} \frac{d^2}{dt^2}\eta(t_1) &= -r''(x(t_1))(\dot{x}(t_1))^2 - r'(x(t_1))\frac{d^2}{dt^2}x(t_1) \\ &= -r''(x(t_1))(r'(x(t_1)))^{-2} + \mu. \end{aligned}$$

The definition of t_1 implies $\frac{d^2}{dt^2}\eta(t_1) \le 0$. So, it follows that at $u = x(t_1)$ we have $r''(u) \ge \mu(r'(u))^2$, which is a contradiction since $r'(u) = r'(x(t_1)) \ne 0$. $\square$

The next lemma gives an equation for the difference of two solutions of Eq. (2.1). This fact enables us to define a discrete Lyapunov functional as a basic tool. The fact that the dependence of the delay on the state is of the form $r(x(t))$ seems to be crucial.

Lemma 2.6. *There are negative reals $\alpha_0 \leq \alpha_1$ with the following properties. For all solutions x, y of Eq. (2.1) on $(\mathbb{R}, \mathbb{R})$ with $x(\mathbb{R}) \subset [-B, A]$ and $y(\mathbb{R}) \subset [-B, A]$, there exist continuous functions $a : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(\mathbb{R}) \subset [\alpha_0, \alpha_1]$, a is bounded, and the function*

$$v : \mathbb{R} \ni t \mapsto [x(t) - y(t)] \exp\left(-\int_0^t a(s)\, ds\right) \in \mathbb{R}$$

satisfies

$$\dot{v}(t) = \alpha(t) v(t - r(x(t))) \qquad \textit{for all } t \in \mathbb{R}.$$

Proof. Define the real numbers a_0, b_0, b_1, α_0 and α_1 by

$$a_0 = \mu + K \max_{u \in [-B,A]} |f'(u)| \max_{u \in [-B,A]} |r'(u)|,$$

$$b_0 = \min_{u \in [-B,A]} f'(u), \qquad b_1 = \max_{u \in [-B,A]} f'(u),$$

$$\alpha_0 = b_0 e^{a_0 R}, \qquad \alpha_1 = b_1 e^{-a_0 R}.$$

Then $\alpha_0 \leq \alpha_1 < 0$.

Set

$$z : \mathbb{R} \ni t \mapsto x(t) - y(t) \in \mathbb{R},$$

$$\begin{aligned} a : \mathbb{R} \ni t \mapsto = -\mu - &\int_0^1 f'\left\{[1-s] y(t - r(y(t))) + s y(t - r(x(t)))\right\} ds \\ \times &\int_0^1 \dot{y}\left\{[1-s](t - r(y(t))) + s(t - r(x(t)))\right\} ds \\ \times &\int_0^1 r'\left\{\ [1-s] x(t) + s y(t)\right\} ds \in \mathbb{R}, \end{aligned}$$

$$b : \mathbb{R} \ni t \mapsto \int_0^1 f'\left\{[1-s] y(t - r(x(t))) + s x(t - r(x(t)))\right\} ds \in \mathbb{R}.$$

Clearly, z, a, b are continuous functions and

$$|a(t)| \leq a_0, \quad b_0 \leq b(t) \leq b_1 \qquad \text{for all } t \in \mathbb{R}.$$

It is not difficult to see that z is continuously differentiable and satisfies

$$\dot{z}(t) = a(t) z(t) + b(t) z(t - r(x(t))) \qquad \text{for all } t \in \mathbb{R}.$$

Setting

$$v : \mathbb{R} \ni t \mapsto z(t) \exp\left(-\int_0^t a(s)\, ds\right) \in \mathbb{R},$$

we obtain that v is continuously differentiable and

$$\dot{v}(t) = b(t) \exp\left(-\int_{t-r(x(t))}^{t} a(s)\, ds \right) v(t - r(x(t))) \qquad \text{for all } t \in \mathbb{R}.$$

Define

$$\alpha : \mathbb{R} \ni t \mapsto b(t) \exp\left(-\int_{t-r(x(t))}^{t} a(s)\, ds \right) \in \mathbb{R}.$$

Then α is continuous, and by using the bounds on a, b and the inequality $0 \le r(x(t)) \le R$, $t \in \mathbb{R}$, we conclude $\alpha(\mathbb{R}) \subset [\alpha_0, \alpha_1]$. $\square$

Backward uniqueness also holds for the solutions of Eq. (2.1) in the following sense.

Lemma 2.7. *If x, y are solutions of Eq. (2.1) on $(\mathbb{R}, \mathbb{R})$ with $x(\mathbb{R}) \subset [-B, A]$, $y(\mathbb{R}) \subset [-B, A]$, and $x_s = y_s$ for some $s \in \mathbb{R}$, then $x(t) = y(t)$ for all $t \in \mathbb{R}$.*

Proof. Clearly $x_t, y_t \in L_K$ for all $t \in \mathbb{R}$. Proposition 2.3 yields $x(t) = y(t)$ for all $t \ge s - R$. Let

$$t_0 = \inf\{t : \ x(u) = y(u) \text{ for all } u \ge t\}.$$

It is enough to show $t_0 = -\infty$. Suppose $t_0 > -\infty$. We apply Lemma 2.6. It follows that $v(t) = 0$ for all $t \ge t_0$. In particular $\dot{v}(t) = 0$ for all $t \ge t_0$. The differential equation for v and the fact $\alpha < 0$ combined yield

$$v(t - r(x(t))) = 0 \qquad \text{for all } t \ge t_0.$$

By Proposition 2.3(i) we have $r_0 = \min_{u \in [-B,A]} r(u) > 0$. Consequently, $v(t) = 0$ for all $t \ge t_0 - r_0$, which contradicts the definition of t_0. $\square$

Since F is a continuous semiflow on the compact metric space L_K, it follows from [25] that, for every $\phi \in L_K$, the solution $x^\phi : [-R, \infty) \to [-B, A]$ has a nonempty ω-limit set

$$\begin{aligned} \omega(\phi) = \{\psi \in L_K : & \text{ there is a sequence } (t_n)_0^\infty \text{ in } \mathbb{R}^+ \text{ such that} \\ & t_n \to \infty \text{ and } F(t_n, \phi) \to \psi \text{ as } n \to \infty\}, \end{aligned}$$

which is compact, connected and invariant. If $\phi \in L_K$ and there is a solution $x : \mathbb{R} \to [-B, A]$ of (2.1) on $(\mathbb{R}, \mathbb{R})$ such that $x_0 = \phi$, then Lemma 2.7 implies that x is unique. For such a $\phi \in L_K$, the α-limit set

$$\begin{aligned} \alpha(\phi) = \{\psi \in L_K : & \text{ there is a sequence } (t_n)_0^\infty \text{ in } (-\infty, 0] \text{ such that} \\ & t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to \infty\} \end{aligned}$$

is nonempty, compact, connected and invariant. By Lemma 2.7, the invariance of the α- and ω-limit sets means that, for each $\psi \in \alpha(\phi)$ ($\psi \in \omega(\phi)$), there is a unique solution $y : \mathbb{R} \to [-B, A]$ of (2.1) on $(\mathbb{R}, \mathbb{R})$ so that $y_0 = \psi$ and $y_t \in \alpha(\phi)$ ($y_t \in \omega(\phi)$) for all $t \in \mathbb{R}$.

In case $\phi \in L_K$ and there is a solution x on $(\mathbb{R}, \mathbb{R})$ with $x(\mathbb{R}) \subset [-B, A]$ and $x_0 = \phi$, we shall also use the symbol x^ϕ to denote such a solution.

3. The associated linear equation

The linear autonomous equation

$$\dot{x}(t) = -\mu x(t) + f'(0)x(t-1) \tag{3.1}$$

can be associated to solutions of Eq. (2.1) tending to zero as $t \to \infty$ or $t \to -\infty$. We recall some basic facts.

The phase space is $C = C([-1, 0], \mathbb{R})$ with the maximum norm $|| \cdot ||_C$. For each $\phi \in C$, there exists a unique solution of (3.1) starting from ϕ. Namely, there exists a unique continuous $x^\phi : [-1, \infty) \to \mathbb{R}$ such that $x^\phi|_{[-1,0]} = \phi$, and $x^\phi : [0, \infty) \to \mathbb{R}$ is differentiable and satisfies (3.1). Backward solutions, if exist, are also unique in the following sense: if I is an interval on $\mathbb{R}$ and x, y are continuous on $\cup_{t\in I}[t-1, t]$, are C^1 on I, satisfy (3.1) on I and $x(t+s) = y(t+s)$, $-1 \leq s \leq 0$, for some $t \in I$, then $x(u) = y(u)$ for all $u \in \cup_{t\in I}[t-1, t]$. For each $(t, \phi) \in [0, \infty) \times C$, defining $T(t)\phi = \psi$, where $\psi(s) = x^\phi(t+s)$, $-1 \leq s \leq 0$, $(T(t))_{t\geq 0}$ is a linear C_0-semigroup on C. $T(1)$ is a compact operator. The spectrum $\Sigma = \{\lambda \in \mathbb{C} : \ \lambda + \mu - f'(0)e^{-\lambda} = 0\}$ of the generator of $(T(t))_{t\geq 0}$ consists of complex conjugate pairs of eigenvalues in the double strips S_k given by

$$2k\pi < |\operatorname{Im}(\lambda)| < 2k\pi + \pi, \quad k = 1, 2, \dots ,$$

and at most two eigenvalues in the strip S_0 given by

$$|\operatorname{Im}(\lambda)| < \pi;$$

the total multiplicity of Σ in S_0 is 2.

We have

$$\max \operatorname{Re} \left(\cup_{k=1}^{\infty}(\Sigma \cap S_k)\right) < \min \operatorname{Re}(\Sigma \cap S_0).$$

Let L and Q denote the realified generalized eigenspaces associated with the spectral sets $\Sigma \cap S_0$ and $\cup_{k=1}^{\infty}(\Sigma \cap S_k)$, respectively. Then

$$C = L \oplus Q,$$

$\dim L = 2$, and both L and Q are positively invariant under the maps $T(t)$. Let $T_L(t)$ and $T_Q(t)$ denote the restrictions of $T(t)$ to L and Q, respectively. $T_L(t)$ can be defined for all $t \in \mathbb{R}$ so that T_L is a flow on L.

Let $u_0 = \max \operatorname{Re}(\Sigma \cap S_0)$. Define

$$v(\mu) \in (\frac{\pi}{2}, \pi) \text{ by } v(\mu) = -\mu \tan(v(\mu)).$$

Then

$$\begin{aligned} u_0 < 0 \qquad &\text{for } f'(0) > \frac{\mu}{\cos(v(\mu))}, \\ u_0 = 0 \qquad &\text{at } f'(0) = \frac{\mu}{\cos(v(\mu))}, \\ u_0 > 0 \qquad &\text{for } f'(0) < \frac{\mu}{\cos(v(\mu))}. \end{aligned}$$

If $u_0 \geq 0$ then $\Sigma \cap S_0$ consists of a complex conjugate pair $\{u_0 \pm iv_0\}$ with $v_0 \in (\frac{\pi}{2}, \pi)$.

The standard notation x_t is occupied to denote an element of $C([-R, 0], \mathbb{R})$. If x a solution of (3.1) on I and $[t-1, t] \subset I$, then $x_{t,C} \in C$ is defined by $x_{t,C}(s) = x(t+s)$, $-1 \leq s \leq 0$.

A solution of (3.1) is called slowly oscillating if for every pair of zeros $z' > z$, we have $z' - z > 1$.

Lemma 3.1.

(i) If $\phi \in L \setminus \{0\}$ then the unique solution $x^\phi : \mathbb{R} \to \mathbb{R}$ of (3.1) is slowly oscillating on $\mathbb{R}$.

(ii) If $u_0 < 0$ and $z : (-\infty, 0] \to \mathbb{R}$ is a solution of (3.1) with

$$||z_{t,C}||_C \leq ||z_{0,C}||_C \qquad \text{for all } t \leq 0,$$

then $z(t) = 0$ for all $t \leq 0$.

(iii) If $u_0 = 0$ and $z : (-\infty, 0] \to \mathbb{R}$ is a solution of (3.1) with

$$||z_{t,C}||_C \leq ||z_{0,C}||_C = 1 \qquad \text{for all } t \leq 0,$$

then z has at most one sign change on the intervals $[t-1, t]$ for all $t \leq 0$.

(iv) If $u_0 > 0$, $\epsilon > 0$ and $z : (-\infty, 0] \to \mathbb{R}$ is a solution of (3.1) with

$$||z_{t,C}||_C \leq e^{(u_0+\epsilon)t} ||z_{0,C}||_C \qquad \text{for all } t \leq 0,$$

then $z(t) = 0$ for all $t \leq 0$.

(v) If $u_0 > 0$ and $z : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating solution of (3.1) with

$$|z(t)| \leq k_1 e^{k_2 |t|} \qquad \text{for all } t \in \mathbb{R},$$

for some $k_1 > 0$ and $k_2 > 0$, then $z_{t,C} \in L$ for all $t \in \mathbb{R}$.

Proof. 1. The elementary proof of (i) can be found e.g. in [50].

2. The proof of (ii). There exist $K_1 > 0$ and $\delta > 0$ such that $u_0 + \delta < 0$ and

$$||T(t)|| \le K_1 e^{(u_0+\delta)t}, \quad t \ge 0.$$

For $\sigma \le t \le 0$, we have $z_{t,C} = T(t-\sigma) z_{\sigma,C}$ and thus

$$\begin{aligned} ||z_{t,C}||_C = ||T(t-\sigma) z_{\sigma,C}||_C &\le K_1 e^{(u_0+\delta)(t-\sigma)} ||z_{\sigma,C}||_C \\ &\le K_1 e^{(u_0+\delta)(t-\sigma)} ||z_{0,C}||_C \to 0 \end{aligned}$$

as $\sigma \to -\infty$. Therefore, $z_{t,C} = 0$ for all $t \le 0$.

3. The proof of (iii). There exist $K_2 > 0$ and $\delta > 0$ such that

$$||T_Q(t)|| \le K_2 e^{-\delta t}, \quad t \ge 0.$$

If $\sigma \le t \le 0$ and $z_{u,C} = z^Q_{u,C} + z^L_{u,C}$ with $z^Q_{u,C} \in Q$, $z^L_{u,C} \in L$, then

$$\begin{aligned} ||z^Q_{t,C}||_C &= ||T_Q(t-\sigma) z^Q_{\sigma,C}||_C \le ||T_Q(t-\sigma)|| ||z^Q_{\sigma,C}||_C \\ &\le K_2' ||T_Q(t-\sigma)|| ||z_{\sigma,C}||_C \le K_2' ||T_Q(t-\sigma)|| ||z_{0,C}||_C \le K_2' K_2 e^{-\delta(t-\sigma)} \to 0 \end{aligned}$$

as $\sigma \to -\infty$, where $K_2' > 0$ is a bound for the norm of the projection operator from C onto Q along L. It follows that $z_{t,C} \in L$ for all $t \le 0$. $z_{t,C} \ne 0$ since $||z_{0,C}|| = 1$. Thus (i) can be applied to get the statement.

4. The proof of (iv). There exist $\delta \in (0, \epsilon)$ and $K_3 > 0$ such that

$$||T(t)|| \le K_3 e^{(u_0+\delta)t}, \quad t \ge 0.$$

Then for $t \le 0$

$$\begin{aligned} ||z_{0,C}||_C = ||T(-t) z_{t,C}||_C &\le K_3 e^{(u_0+\delta)(-t)} ||z_{t,C}||_C \\ &\le K_3 e^{(u_0+\delta)(-t)} e^{(u_0+\epsilon)t} ||z_{0,C}||_C = K_3 e^{(\epsilon-\delta)t} ||z_{0,C}||. \end{aligned}$$

Hence, for sufficiently large negative t, $||z_{0,C}|| = 0$ follows. Then $z(t) = 0$ for all $t \le 0$.

5. The proof of (v). Consider another decomposition

$$C = \tilde{Q} \oplus \tilde{L}$$

of C into $T(t)$ positively invariant subspaces such that $\operatorname{Re} \lambda < -k_2$ for all $\lambda \in \Sigma$ associated with $\tilde{Q}$. Then there exists $\alpha > 0$ such that $\operatorname{Re} \lambda < -\alpha < -k_2$ for all $\lambda \in \Sigma$ associated with $\tilde{Q}$ and

$$||T_{\tilde{Q}}(t)|| \le K_4 e^{-\alpha t}, \quad t \ge 0.$$

We have $z_{t,C} = z^{\tilde{Q}}_{t,C} + z^{\tilde{L}}_{t,C}$ with $z^{\tilde{Q}}_{t,C} \in \tilde{Q}$, $z^{\tilde{L}}_{t,C} \in \tilde{L}$. Then, for $\sigma \leq t$,

$$\begin{aligned}||z^{\tilde{Q}}_{t,C}||_C &= ||T_{\tilde{Q}}(t-\sigma)z^{\tilde{Q}}_{\sigma,C}|| \leq K_4' K_4 e^{-\alpha(t-\sigma)} k_1 e^{k_2|\sigma|} \\ &= K_4' K_4 k_1 e^{-\alpha t} e^{(\alpha - k_2)\sigma} \to 0\end{aligned}$$

as $\sigma \to -\infty$, where $K_4' > 0$ is a bound for the norm of the projection operator from C onto $\tilde{Q}$ along $\tilde{L}$. Therefore, $z_{t,C} \in \tilde{L}$ for all $t \in \mathbb{R}$. Consequently,

$$z(t) = \sum_{k=0}^{N} a_k e^{u_k t} \sin(v_k t + b_k),$$

for some nonnegative integer N such that $a_N \neq 0$. For large negative t, the term with greatest index is dominant in this sum. Since $\sin(v_N t + b_N)$ has zeros at distances $\frac{\pi}{|v_N|} < 1$ for $N \geq 1$ and z is slowly oscillating, it follows that $N = 0$, and thus $z_{t,C} \in L$ for all $t \in \mathbb{R}$.

□

4. A discrete Lyapunov functional

In this section we define a discrete, integer-valued Lyapunov functional. For equations with constant delay Mallet-Paret [39] introduced a discrete Lyapunov functional. A more general version is contained in [44]. The state-dependent delay requires a modified version of the functional. We have to count sign changes of solutions x of Eq. (2.1) on intervals of the form $[t - r(x(t)), t]$ instead of on intervals with fixed length.

Let $[a, b]$ be an interval and ϕ be a real valued continuous function defined on an interval containing $[a, b]$ such that $\phi|_{[a,b]} \neq 0$. Then the number of sign changes $\mathrm{sc}(\phi, [a, b])$ of ϕ on $[a, b]$ is 0 if either $\phi(s) \geq 0$ for all $s \in [a, b]$ or $\phi(s) \leq 0$ for all $s \in [a, b]$, otherwise $\mathrm{sc}(\phi, [a, b])$ is given by

$$\begin{aligned}\mathrm{sc}(\phi, [a, b]) = \sup \big\{ k :\ & \text{there exists } s^0 < s^1 < \ldots < s^k \text{ such that } s^i \in [a, b] \text{ for} \\ & i = 0, 1, \ldots, k, \text{ and } \phi(s^i)\phi(s^{i+1}) < 0 \text{ for } i = 0, 1, \ldots, k-1 \big\}.\end{aligned}$$

Let

$$V(\phi, [a, b]) = \begin{cases} \mathrm{sc}(\phi, [a, b]) & \text{if } \mathrm{sc}(\phi, [a, b]) \text{ is odd or infinite,} \\ \mathrm{sc}(\phi, [a, b]) + 1 & \text{if } \mathrm{sc}(\phi, [a, b]) \text{ is even.} \end{cases}$$

Therefore $V(\phi, [a, b]) \in \{1, 3, \ldots\} \cup \{\infty\}$. Define

$$\begin{aligned}H_{[a,b]} = \big\{ \phi \in C^1([a, b], \mathbb{R}) : \ & \phi(b) \neq 0 \text{ or } \phi(a)\dot{\phi}(b) < 0, \\ & \phi(a) \neq 0 \text{ or } \dot{\phi}(a)\phi(b) > 0, \\ & \text{all zeros of } \phi \text{ in } (a, b) \text{ are simple} \big\}.\end{aligned}$$

$H_{[a,b]}$ is an open dense subset of $C^1([a,b],\mathbb{R})$.

Lemma 4.1.

(i) V is lower semi-continuous in the following sense. If ϕ, ϕ^n are nonzero continuous functions on the intervals $[a,b]$, $[a^n,b^n]$, respectively, and

$$\max_{s\in[a,b]\cap[a^n,b^n]} |\phi^n(s)-\phi(s)| \to 0,\ a^n \to a,\ b^n \to b \qquad \text{as } n\to\infty,$$

then

$$V(\phi,[a,b]) \le \liminf_{n\to\infty} V(\phi^n,[a^n,b^n]).$$

(ii) If $\phi \in H_{[a,b]}$ then $V(\phi,[a,b])<\infty$.

(iii) If $\phi \in C^1([a-\delta,b+\delta],\mathbb{R})$ for some $\delta>0$ and $\phi|_{[a,b]} \in H_{[a,b]}$, then there is $\gamma\in(0,\delta)$ such that

$$|a-c|<\gamma,\ |b-d|<\gamma,\ \psi\in C^1([c,d],\mathbb{R}),\ ||\psi-\phi||_{C^1([c,d],\mathbb{R})}<\gamma$$

imply

$$V(\psi,[c,d]) = V(\phi,[a,b]).$$

Proof. 1. The proof of (i). The cases $V(\phi,[a,b])=\infty$ and $a=b$ are clear. Assume that $a<b$ and $V(\phi,[a,b])<\infty$. Then there exists $\gamma\in(0,\frac{b-a}{4})$ such that ϕ does not change sign on the intervals $[a,a+2\gamma]$ and $[b-2\gamma,b]$. For large n, we have $[a^n,b^n]\supset[a+\gamma,b-\gamma]$. If $|\phi^n(s)-\phi(s)|$ is sufficiently small for all $s\in[a+\gamma,b-\gamma]$, which is the case for sufficiently large n, then obviously

$$V(\phi^n,[a^n,b^n]) \ge V(\phi^n,[a+\gamma,b-\gamma]) \ge V(\phi,[a+\gamma,b-\gamma]) = V(\phi,[a,b]).$$

2. The proof of (ii). $V(\phi,[a,b])=\infty$ implies the existence of an $s\in[a,b]$ with $\phi(s)=\dot\phi(s)=0$, a contradiction.

3. The proof of (iii). If $\phi(a)\neq 0$ and $\phi(b)\neq 0$, then clearly $\mathrm{sc}(\psi,[c,d])=\mathrm{sc}(\phi,[a,b])$ provided $|a-c|$, $|b-d|$ and $||\psi-\phi||_{C^1([c,d],\mathbb{R})}$ are small enough. In the case $\phi(b)=0$, $\phi(a)\dot\phi(b)<0$, the number of sign changes $\mathrm{sc}(\phi,[a,b])$ of ϕ on $[a,b]$ is an even number. If $|a-c|$, $|b-d|$ and $||\psi-\phi||_{C^1([c,d],\mathbb{R})}$ are sufficiently small, then

$$\mathrm{sc}(\phi,[a,b]) \le \mathrm{sc}(\psi,[c,d]) \le \mathrm{sc}(\phi,[a,b])+1,$$

that is $V(\psi,[c,d])=V(\phi,[a,b])$. The same works for the case $\phi(a)=0$, $\dot\phi(a)\phi(b)>0$. □

Let $I=[c,d]$ be an interval and let $\alpha: I\to\mathbb{R}$, $\tau: I\to\mathbb{R}$ be continuous functions such that $\alpha(t)<0$, $\tau(t)>0$ for all $t\in I$, and the function $\eta: I\ni t\mapsto t-\tau(t)\in\mathbb{R}$ is strictly increasing on I.

Let $k \in \mathbb{N} \setminus \{0,1\}$ be given. Assume that there exists a finite sequence $(c_j)_1^k$ in $[c,d]$ such that $c_1 = c$ and $\eta(c_j) = c_{j-1}$ for all $j \in \{2,\ldots,k\}$. Then we define the functions $\eta^0, \eta^1, \ldots, \eta^k$ by $\eta^0(t) = t$ for all $t \in [c,d]$, and

$$\eta^j : [c,d] \ni t \mapsto \eta(\eta^{j-1}(t)) \in \mathbb{R}$$

for $j \in \{1,2,\ldots,k\}$.

Set $J = \{t - \tau(t) : t \in I\} \cup I$. Let $v : J \to \mathbb{R}$ be a continuous function which is continuously differentiable on I and satisfies

$$\dot{v}(s) = \alpha(s)v(s-\tau(s)) \tag{4.1}$$

for all $s \in I$.

Lemma 4.2. *Assume that $I = [c,d]$, α, τ, η, v, k and $(c_j)_1^k$ are given as above, moreover, for all $t \in I$, $v|_{[\eta(t),t]}$ is not identically zero. Then*
(i) $t^1, t^2 \in I$, $t^1 < t^2$ imply $V(v,[\eta(t^1),t^1]) \geq V(v,[\eta(t^2),t^2])$;
(ii) $k \geq 3$, $t \in [c_3,d]$, $v(t) = v(\eta(t)) = 0$ imply that either $V(v,[\eta(t),t]) = \infty$ or $V(v,[\eta(t),t]) < V(v,[\eta^3(t),\eta^2(t)])$;
(iii) $k \geq 4$, $t \in [c_4,d]$ and $V(v,[\eta(t),t]) = V(v[\eta^4(t),\eta^3(t)]) < \infty$ imply $v|_{[\eta(t),t]} \in H_{[\eta(t),t]}$.

Proof. 1. The proof of (i). We claim that it suffices to show that for all $t \in I$ there exists $\epsilon^0 = \epsilon^0(t) > 0$ such that for all $\epsilon \in [0,\epsilon^0]$ with $t+\epsilon \in I$,

$$V(v,[\eta(t),t]) \geq V(\eta(t+\epsilon),t+\epsilon). \tag{4.2}$$

Indeed, let t^1, t^2 in I with $t^1 < t^2$ be given and assume that for every $t \in I$ there is $\epsilon^0 = \epsilon^0(t) > 0$ so that for all $\epsilon \in [0,\epsilon^0]$ with $t+\epsilon \in I$ we have (4.2). Define

$$t^* = \sup\{s \in [t^1,t^2] : V(v,[\eta(t^1),t^1]) \geq V(v,[\eta(u),u]) \text{ for all } t^1 \leq u \leq s\}.$$

Then $t^1 < t^* \leq t^2$. From the definition of t^* it follows that there is a sequence $(s^n)_0^\infty$ in $[t^1,t^*]$ so that $s^n \to t^*$ as $n \to \infty$ and $V(v,[\eta(t^1),t^1]) \geq V(v,[\eta(s^n),s^n])$ for all $n \in \mathbb{N}$. Clearly, $\eta(s^n) \to \eta(t^*)$ as $n \to \infty$. Then Lemma 4.1(i) yields $V(v,[\eta(t^1),t^1]) \geq V(v,[\eta(t^*),t^*])$. If $t^* < t^2$ then there is $\epsilon^0(t^*) \in (0,t^2-t^*]$ so that $V(v,[\eta(t^*),t^*]) \geq V(v,[\eta(t^*+\epsilon),t^*+\epsilon])$ for all $\epsilon \in [0,\epsilon^0(t^*)]$. This contradicts the definition of t^*. Consequently, $t^* = t^2$, and the claim holds.

If $V(v,[\eta(t^1),t^1]) = \infty$, then there is nothing to prove. Assume that $V(v,[\eta(t^1),t^1]) < \infty$. Again, the case $v(t^1) \neq 0$ is obvious by using the increasing property of η. Assume that $v(t^1) = 0$. From the finiteness of $V(v,[\eta(t^1),t^1])$, it follows that v does not change sign on $[\eta(t^1), \eta(t^1)+\delta]$ for some $\delta > 0$. Assume that $v(t) \geq 0$ on this interval. Since (4.1) is linear, the case $v(t) \leq 0$ is analogous. By the continuity and increasing property of η, there is $\epsilon^0 > 0$ such that $t \in [t^1, t^1+\epsilon^0]$ implies $\eta(t) \in [\eta(t^1), \eta(t^1)+\delta]$. Hence, using (4.1), $\dot{v}(t) \leq 0$ follows for $t \in [t^1, t^1+\epsilon^0]$. Since $v(t^1) = 0$, we obtain that $v(t) \leq 0$ for all $t \in [t^1, t^1+\epsilon^0]$. If $v(t) = 0$ for all $t \in [t^1, t^1+\epsilon^0]$, then (4.2) holds with equality for all $\epsilon \in [0, \epsilon^0]$. If $v(t) < 0$ for some $t \in [t^1, t^1+\epsilon^0]$, then, by (4.1) and $\alpha < 0$, we have $v(\eta(\bar{t})) > 0$ for some $\bar{t} \in (t^1, t)$ with $\eta(\bar{t}) \in [\eta(t^1), \eta(t^1)+\delta]$. Then there exists $\gamma \in (0, t^1 - \eta(t^1))$ such that v is not identically zero on $[t^1-\gamma, t^1]$ and either $v(t) \geq 0$ for all $t \in [t^1-\gamma, t^1]$ or $v(t) \leq 0$ for all $t \in [t^1-\gamma, t^1]$. If $v(t) \geq 0$ on $[t^1-\gamma, t^1]$, then

$$\mathrm{sc}(v, [\eta(t^1), t^1+\epsilon]) \leq \mathrm{sc}(v, [\eta(t^1), t^1]) + 1, \quad 0 \leq \epsilon \leq \epsilon^0.$$

But $\mathrm{sc}(v, [\eta(t^1), t^1])$ is even (since v has the same sign on the right of $\eta(t^1)$ and on the left of t^1), and thus (4.2) is satisfied for all $\epsilon \in [0, \epsilon^0]$. If $v(t) \leq 0$ on $[t^1-\gamma, t^1]$, then

$$\mathrm{sc}(v, [\eta(t^1), t^1+\epsilon^0]) = \mathrm{sc}(v, [\eta(t^1), t^1]),$$

and (4.2) holds again for all $\epsilon \in [0, \epsilon^0]$.

2. The proof of (ii). Assume $V(v, [\eta(t), t]) < \infty$, since there is nothing to prove if V is infinite. Let $k = \mathrm{sc}(v, [\eta(t), t])$. We can choose $(t^i)_{i=0}^{k+2}$ such that

$$\eta(t) = t^{k+2} < t^{k+1} < \ldots < t^1 < t^0 = t$$

and

$$v(t^i)v(t^{i+1}) < 0, \quad i = 1, 2, \ldots, k.$$

Applying the mean value theorem to each interval $[t^{i+1}, t^i]$ and using the facts that $v(t^0) = v(t^{k+2}) = 0$, that $\dot{v}(s)$ and $v(\eta(s))$ have different signs (if none of them is zero), and that η is increasing, we get a sequence $(\bar{t})_{i=0}^{k+1}$ such that

$$\eta^2(t) < \bar{t}^{k+1} < \bar{t}^k < \ldots < \bar{t}^1 < \bar{t}^0 < \eta(t)$$

and

$$v(\bar{t}^i)v(\bar{t}^{i+1}) < 0, \quad i = 0, 1, \ldots, k.$$

Therefore $\mathrm{sc}(v, [\eta^2(t), \eta(t)]) \geq k+1$, and thus, in case of odd k,

$$V(v, [\eta^2(t), \eta(t)]) \geq k+2 > k = V(v, [\eta(t), t]),$$

and the stated inequality follows from (i).

Assume that k is even. Then $v(\bar{t}^0)$ and $v(\bar{t}^{k+1})$ have different signs. Using that $v(\eta(t)) = 0$, we can choose $t^* \in (\bar{t}^0, \eta(t))$ such that $v(t^*)$ and $v(\bar{t}^0)$ have the same sign, and $\dot{v}(t^*)$ and $v(\bar{t}^0)$ have different signs. Then, since $\dot{v}(t^*)$ and $v(\eta(t^*))$ have different signs, we conclude that the signs of $v(\eta(t^*))$ and $v(\bar{t}^{k+1})$ are different. Consequently, $\mathrm{sc}(v, [\eta(t^*), t^*]) \geq k+2$ because of $\eta(t^*) < \eta^2(t) < \bar{t}^{k+1}$. Thus, from $\eta(t^*) > \eta^3(t)$ and statement (i),

$$V(v, [\eta^3(t), \eta^2(t)]) \geq V(v, [\eta(t^*), t^*]) \geq k+3 > k+1 = V(v, [\eta(t), t]).$$

3. The proof of (iii). Assume that $V(v, [\eta(t), t]) = V(v, [\eta^4(t), \eta^3(t)]) < \infty$. Then, for any $s \in [\eta(t), t]$, we have $\eta^3(t) \leq \eta^2(s) \leq s \leq t$. Consequently, by statement (i),

$$V(v, [\eta(s), s]) = V(v, [\eta^3(s), \eta^2(s)]) < \infty, \quad s \in [\eta(t), t].$$

From statement (ii) it follows that

$$\big(v(s), v(\eta(s))\big) \neq (0,0) \qquad \text{for all } s \in [\eta(t), t].$$

Using that $\dot{v}(s) = \alpha(s)v(\eta(s))$ and $\alpha(s) \neq 0$, we obtain

$$\big(v(s), \dot{v}(s)\big) \neq (0,0) \qquad \text{for all } s \in [\eta(t), t],$$

that is, the zeros of v on $[\eta(t), t]$ are simple. As a consequence, in case $v(t) = 0$ we get $0 \neq \dot{v}(t) = \alpha(t)v(\eta(t))$ and $\dot{v}(t)v(\eta(t)) < 0$ because of $\alpha(t) < 0$. Now assume $v(\eta(t)) = 0$. By statement (ii), $v(t) \neq 0$, $v(\eta^2(t)) \neq 0$, and thus $\dot{v}(\eta(t)) \neq 0$. Assume that $\dot{v}(\eta(t))v(t) < 0$. Then $\mathrm{sc}(v, [\eta(t), t])$ is an odd number k, and similarly to the proof of statement (ii), there is a sequence $(t^i)_{i=0}^{k+2}$ such that

$$\eta(t) = t^{k+1} < t^k < \dots t^1 < t^0 = t$$

and

$$v(t^i)v(t^{i+1}) < 0, \quad i = 1, 2, \dots, k.$$

Applying the mean value theorem and using $\dot{v}(\eta(t))v(t) < 0$, we get $k+1$ sign changes in the interval $[\eta^2(t), \eta(t)]$. This gives that

$$V(v, [\eta^2(t), \eta(t)]) \geq k+2 > k = \mathrm{sc}(v, [\eta(t), t]) = V(v, [\eta(t), t]),$$

a contradiction. Therefore, $v|_{[\eta(t),t]} \in H_{[\eta(t),t]}$. $\square$

The next result shows that the Lyapunov functional V can be effectively used to show that solutions of (4.1) can not decay too fast at ∞. For constant delay Mallet-Paret [39] Cao [10] and Arino [4] proved estimates of this type.

Lemma 4.3. *Assume that t', t are real numbers with $t' < t$, $\alpha : [t', t] \to \mathbb{R}$ and $\tau : [t', t] \to \mathbb{R}$ are continuous functions, and there are positive constants a_0, a_1, τ_0, L_τ such that*

$$-a_1 \leq \alpha(s) \leq -a_0 \qquad \textit{for all } s \in [t', t],$$

$$\tau_0 \leq \tau(s) \qquad \textit{for all } s \in [t', t],$$

$$|\tau(s^1) - \tau(s^2)| \leq L_\tau |s^1 - s^2| \qquad \textit{for all } s^1, s^2 \textit{ in } [t', t],$$

the function $\eta : [t', t] \ni s \mapsto s - \tau(s) \in \mathbb{R}$ is strictly increasing, and $t' = \eta^4(t)$. Let v be a continuous function on $[\eta^5(t), t]$ such that (4.1) holds for all $s \in [\eta^4(t), t]$ and $V(v, [\eta^5(t), \eta^4(t)]) = 1$.

Then there exists a constant $k = k(a_0, a_1, \tau_0, L_\tau) > 0$ such that

$$\max_{s \in [\eta^2(t), \eta(t)]} |v(s)| \leq k \max_{s \in [\eta(t), t]} |v(s)|. \tag{4.3}$$

Proof. First we prove the following claim.

CLAIM. For any $\delta \in (0, \tau_0)$ there exists $c = c(\delta, a_0, L_\tau) > 0$ such that for each interval $\Delta \subset [\eta^4(t), \eta(t)]$ with length δ, we have

$$\min_{s \in \Delta} |v(s)| \leq c \max_{s \in [\eta(t), t]} |v(s)|. \tag{4.4}$$

Proof of Claim. Let $\bar{v} = \max_{s \in [\eta(t), t]} |v(s)|$. First choose Δ in the interval $[\eta^2(t), \eta(t)]$, that is, $\Delta = [\eta(s^1), \eta(s^2)]$, $\delta = \eta(s^2) - \eta(s^1)$ and $\eta(t) \leq s^1 < s^2 \leq t$. Integrating (4.1) on $[s^1, s^2]$, we get

$$v(s^2) - v(s^1) = \int_{s^1}^{s^2} \alpha(u) v(\eta(u))\, du.$$

The length of $[s^1, s^2]$ can be estimated from

$$\delta = \eta(s^2) - \eta(s^1) \leq s^2 - s^1 + |\tau(s^2) - \tau(s^1)| \leq (1 + L_\tau)(s^2 - s^1).$$

Hence

$$\min_{s \in \Delta} |v(s)| \leq \frac{2(1 + L_\tau)}{a_0 \delta} \bar{v}.$$

Define $c_1 = c_1(\delta, a_0, L_\tau) = \frac{2(1+L_\tau)}{a_0 \delta}$.

Now consider any interval $\Delta \subset [\eta^3(t), \eta(t)]$ of length δ. If the length of $\Delta \cap [\eta^2(t), \eta(t)]$ is greater than or equal to $\frac{\delta}{2}$, then we choose $c = c_1(\frac{\delta}{2})$. Assume that $|\Delta \cap [\eta^3(t), \eta^2(t)]| > \frac{\delta}{2}$.

There are $t^1, t^2 \in [\eta^2(t), \eta(t)]$ such that $[\eta(t^1), \eta(t^2)] \subset \Delta$ and $\eta(t^2) - \eta(t^1) = \frac{\delta}{2}$. From the Lipschitz continuity of τ, we obtain

$$t^2 - t^1 \geq \frac{\delta}{2(1 + L_\tau)}.$$

Considering the intervals

$$[t^1, t^1 + \frac{\delta}{6(1 + L_\tau)}],\ [t^2 - \frac{\delta}{6(1 + L_\tau)}, t^2] \subset [\eta^2(t), \eta(t)]$$

of length $\bar{\delta} = \frac{\delta}{6(1+L_\tau)}$, the first part of the proof gives that

$$\min_{s \in [t^1, t^1 + \bar{\delta}]} |v(s)| \leq c_1(\bar{\delta}, a_0, L_\tau)\bar{v}, \quad \min_{s \in [t^2 - \bar{\delta}, t^2]} |v(s)| \leq c_1(\bar{\delta}, a_0, L_\tau)\bar{v}.$$

Applying the mean value theorem, we obtain a $t^* \in (t^1, t^2)$ such that

$$|\dot{v}(t^*)| \leq \frac{2c_1(\bar{\delta}, a_0, L_\tau)\bar{v}}{\bar{\delta}}.$$

Using equation (4.1),

$$|v(\eta(t^*))| \leq \frac{|\dot{v}(t^*)|}{a_0} \leq \frac{2c_1(\bar{\delta}, a_0, L_\tau)\bar{v}}{a_0\bar{\delta}}.$$

Since $\eta(t^*) \in \Delta$, it follows that

$$\min_{s \in \Delta} |v(s)| \leq \frac{2c_1(\bar{\delta}, a_0, L_\tau)}{a_0\bar{\delta}}\bar{v}.$$

Then, for any interval $\Delta \subset [\eta^3(t), \eta(t)]$ of length δ, (4.4) holds with $c = c_2$, where

$$c_2 = c_2(\delta, a_0, L_\tau) = \max\left\{c_1(\frac{\delta}{2}, a_0, L_\tau), 2\frac{c_1(\bar{\delta}, a_0, L_\tau)}{a_0\bar{\delta}}\right\}.$$

Repeating the above argument we obtain that

$$c = c(\delta, a_0, L_\tau) = \max\left\{c_2(\frac{\delta}{2}, a_0, L_\tau), \frac{c_2(\bar{\delta}), a_0, L_\tau}{a_0\bar{\delta}}\right\}$$

is an appropriate constant for any $\Delta \subset [\eta^4(t), \eta(t)]$. This completes the proof of the claim.

Now we prove Lemma 4.3. Choose $\delta > 0$ such that $2\delta(1 + L_\tau)(2 + L_\tau) \leq \tau_0$. By the above claim, there is a $c = c(\delta) > 0$ such that (4.4) holds. Clearly, $c > 1$ may be assumed. We prove that (4.3) is satisfied if $k > 0$ is chosen such that $\frac{k-c}{a_1\delta} > c$. Let

$\bar{v} = \max_{s\in[\eta(t),t]} |v(s)|$ and assume that (4.3) is not true. Then there is $t^* \in [\eta^2(t), \eta(t)]$ such that $|v(t^*)| > k\bar{v}$. By the above claim,

$$\min_{s\in[t^*-\delta,t^*]} |v(s)| \leq c\bar{v}, \quad \min_{s\in[t^*,t^*+\delta]} |v(s)| \leq c\bar{v}.$$

If $t^*+\delta > \eta(t)$, then the claim does not apply to get the second inequality. But in that case it clearly holds since $c > 1$. The mean value theorem implies the existence of $s^1 \in [t^*-\delta, t^*]$ and $s^2 \in [t^*, t^*+\delta]$ such that

$$|\dot{v}(s^i)| \geq \frac{(k-c)\bar{v}}{\delta}, \quad i = 1, 2,$$

moreover $\dot{v}(s^1)\dot{v}(s^2) < 0$. Hence it follows that

$$|v(\eta(s^i))| \geq \frac{|\dot{v}(s^i)|}{a_1} \geq \frac{(k-c)\bar{v}}{a_1\delta} > c\bar{v}$$

and $v(\eta(s^1))v(\eta(s^2)) < 0$. A second application of the claim gives

$$\min_{s\in[\eta(s^1)-\delta,\eta(s^1)]} |v(s)| \leq c\bar{v}, \quad \min_{s\in[\eta(s^2),\eta(s^2)+\delta]} |v(s)| \leq c\bar{v}.$$

We prove later that $\eta(s^1) - \delta \geq \eta^4(t)$, i.e., that the claim is applicable. Then, again by the mean value theorem, it is obtained that $\dot{v}$ has at least two sign changes on the interval $[\eta(s^1) - \delta, \eta(s^2) + \delta]$. Eq. (4.1) implies that then v also has at least two sign changes on $[\eta(\eta(s^1) - \delta), \eta(\eta(s^2) + \delta)]$. From the Lipschitz continuity of τ it follows that

$$|\eta(s^2) + \delta - (\eta(s^1) - \delta)| \leq |\eta(t^* + \delta) + \delta - (\eta(t^* - \delta) - \delta)| \leq 2\delta(2 + L_\tau) \leq \tau_0, \tag{4.5}$$

$$|\eta(\eta(s^2) + \delta) - \eta(\eta(s^1) - \delta)| \leq 2\delta(1 + L_\tau)(2 + L_\tau) \leq \tau_0. \tag{4.6}$$

From (4.5) $\eta(s^1) - \delta \geq \eta^4(t)$ follows, since $\eta(t^*) \geq \eta^3(t)$ and $\eta(t^*) \in (\eta(s^1) - \delta, \eta(s^2) + \delta)$. In addition, $\eta(\eta(s^1) - \delta) \geq \eta^5(t)$ is also obtained. Then (4.6), $\mathrm{sc}(v, [\eta(\eta(s^1) - \delta), \eta(\eta(s^2) + \delta)]) \geq 2$ and Lemma 4.2(i) combined imply

$$V(v, [\eta^5(t), \eta^4(t)]) > 1,$$

a contradiction. □

The next result gives a connection between the distances of consecutive zeros of solutions of (4.1) and the values of V.

Lemma 4.4. *Assume that $\alpha, \tau, v : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $\alpha(\mathbb{R}) \subset (-\infty, 0)$, $\tau(\mathbb{R}) \subset (0, \infty)$, the function $\mathbb{R} \ni t \mapsto t - \tau(t) \in \mathbb{R}$ is strictly increasing, $\tau(t) = 1$*

for all $t \in \mathbb{R}$ with $v(t) = 0$, v is continuously differentiable on $\mathbb{R}$ and satisfies (4.1) for all $s \in \mathbb{R}$.

Then the following statements are equivalent.

(i) $|z_1 - z_2| > 1$ holds for every pair of zeros $z_1 \neq z_2$ of v.

(ii) $v|_{[t-\tau(t),t]} \not\equiv 0$ and $V(v, [t - \tau(t), t]) = 1$ for all $t \in \mathbb{R}$.

Proof. 1. Assume (i). If v has no zero then (ii) holds. Suppose v has at least one zero. For a given zero z of v, define $z_+ = \infty$ if v has no zero on (z, ∞), otherwise $z_+ = \min\{t > z : v(t) = 0\}$. For every $t \in \mathbb{R}$ either there exists a zero z of v with $t \in [z, z_+)$ or $v(s) \neq 0$ for all $s \leq t$. In the latter case clearly $v|_{[t-\tau(t),t]} \not\equiv 0$ and $V(v, [t - \tau(t), t]) = 1$. Assume that $t \in [z, z_+)$ for some zero z of v. Then z is the only zero of v on $[z - 1, z_+)$. We also have $z - 1 = z - \tau(z) \leq t - \tau(t) < t < z_+$. Therefore $v|_{[t-\tau(t),t]} \not\equiv 0$ and $V(v, [t - \tau(t), t]) = 1$.

2. Assume (ii). Let z be a zero of v. Then $\tau(z) = 1$. By Lemma 4.2(iii), all zeros of v are simple on $[\eta(z), z] = [z - 1, z]$ and $v(z - 1)\dot{v}(z) < 0$. These facts and $\mathrm{sc}(v, [z - 1, z]) \leq V(v, [z - 1, z]) = 1$ combined yield $z(t) \neq 0$ for all $t \in [z - 1, z)$. ☐

Remark 4.5 Let x, y be solutions of Eq. (2.1) on $(\mathbb{R}, \mathbb{R})$ with $x(\mathbb{R}) \subset [-B, A]$ and $y(\mathbb{R}) \subset [-B, A]$. Lemma 2.2(ii) and Lemma 2.5 combined imply that the function $\mathbb{R} \ni t \mapsto t - r(x(t)) \in \mathbb{R}$ is strictly increasing. Defining v, α as in Lemma 2.6 and τ by $\tau(t) = t - r(x(t))$ we find that (4.1) holds for all $s \in \mathbb{R}$. Using the properties of α, v stated in Lemma 2.6, we see that Lemmas 4.2, 4.3 and 4.4 can be applied.

5. Slowly oscillating solutions

A solution x of Eq. (2.1) is called slowly oscillating if for every pair of zeros $z' > z$ of x,

$$z' - z > 1$$

holds. Our aim is to describe the set of globally defined slowly oscillating solutions with values in $[-B, A]$. Recall from $r(0) = 1$ and Proposition 2.3 that $R \geq 1$. Set

$$S = \{\phi \in L_K : \ \mathrm{sc}(\phi, [t - 1, t]) \leq 1 \text{ for all } t \in [-R + 1, 0]\},$$

$$S_0 = \{\phi \in S : \ \phi(s) = 0 \text{ for all } s \in [-1, 0]\}.$$

S is a closed subset of L_K, therefore it is compact. For each $t \in [-1, 1]$ and $\phi \in S$, clearly $t\phi \in S$. S is not convex. It is clear that, if $x : J \to [-B, A]$ is a slowly oscillating solution of Eq. (2.1) on (I, J), then its segments x_t belong to the set $S \setminus S_0$.

Define

$$U = \{\phi \in L_K : \ \phi(s) \geq 0 \text{ for all } s \in [-1, 0], \ \phi(0) = 0\},$$

$$U_0 = \{\phi \in L_K : \ \phi(s) = 0 \text{ for all } s \in [-1, 0]\}.$$

If $x : \mathbb{R} \to [-B, A]$ is a slowly oscillating solution of Eq. (2.1) and z is a zero with $\dot{x}(z) < 0$, then $x_z \in U \setminus U_0$. The set U is a compact convex subset of L_K. The next result of Mallet-Paret and Nussbaum [41] shows that, for any $\phi \in U \setminus U_0$, there is a sufficiently large t_0 such that x^ϕ is slowly oscillating on $[t_0, \infty)$.

Proposition 5.1. [41] *(i) If $\phi \in U_0$ then $x^\phi(t) = 0$ for all $t \geq 0$.*
(ii) If $\phi \in U \setminus U_0$ then define

$$q_0 = \sup\{t : \ x^\phi(s) = 0 \textit{ for } 0 \leq s \leq t\}.$$

Define

$$q_1 = \inf\{t > q_0 : \ x^\phi(t) = 0\}$$

and $q_1 = \infty$ if $x^\phi(t) < 0$ for all $t > q_0$. If q_k is finite, define

$$q_{k+1} = \inf\{t > q_k : \ x^\phi(t) = 0\}$$

and $q_{k+1} = \infty$ if $x(t) \neq 0$ for all $t > q_k$. If $q_k = \infty$ then define $q_{k+1} = \infty$. Then $q_0 < 1$, $q_1 - q_0 > 1$ and $q_{k+1} - q_k > 1$ for all k such that $q_k < \infty$. If $q_k = \infty$ for some k, then $\lim_{t\to\infty} x^\phi(t) = 0$.

Remark 5.2. Setting

$$\tilde{U} = \{\phi \in L_K : \ \phi(s) \leq 0 \text{ for all } s \in [-1, 0], \ \phi(0) = 0\},$$

for each $\phi \in \tilde{U} \setminus U_0$, an analogous statement to Proposition 5.1(ii) holds.

We shall make use of a return map on the compact convex set U. For every $k \in \mathbb{N} \setminus \{0\}$ Proposition 5.1 permits to define a map $P_k : U \to U$ by

$$P_k(\phi) = F(q_{2k}, \phi) \text{ if } \phi \in U \setminus U_0 \text{ and } q_{2k} < \infty,$$
$$P_k(\phi) = 0 \text{ if } \phi \in U_0 \text{ or } \phi \in U \setminus U_0 \text{ and } q_{2k} = \infty.$$

$P_k(\phi)$ is the k-th intersection of the trajectory x_t^ϕ with U provided $q_{2k}(\phi) < \infty$.

If R is large then P_k is not, in general, continuous at nonzero elements of U_0. Let us choose $l \in \mathbb{N}$ such that $2l \geq R$. Then we have the following result.

Proposition 5.3. *P_l is continuous.*

Proof. First we prove the following claim.

CLAIM. For every $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that if $\phi \in U \setminus U_0$ and $q_{2l} \geq 2lT$, then $||P_l(\phi)|| < \epsilon$.

Proof of the Claim. If $q_{2l} = \infty$ then $P_l(\phi) = 0$ and thus $||P_l(\phi)|| = 0$. So, it suffices to deal with those $\phi \in U \setminus U_0$ for which $q_{2l} < \infty$. Let $T > 2R$. If $q_{2l} \geq 2lT$, then there exists a $k \in \{1, 2, \dots, 2l\}$ such that $q_k - q_{k-1} \geq T$. Then from Eq. (2.1), $|x|$ is decreasing on $[q_{k-1} + R, q_k]$, and for all $t \in [q_{k-1} + R, q_k]$ we have

$$|\dot{x}(t)| \leq -\mu |x(t)|,$$

$$|x(t)| \leq |x(q_{k-1} + R)| e^{\mu(q_{k-1}+R-t)} \leq \max\{A, B\} e^{\mu(q_{k-1}+R-t)}.$$

It follows that

$$||x_{q_k}|| \leq \max\{A, B\} e^{\mu(2R-T)}.$$

Since $x \equiv 0$ is a solution of Eq. (2.1), by Lemma 2.4, for each $\gamma > 0$ there exists $\delta = \delta(\gamma) > 0$ such that $||\phi|| < \delta$ and $\phi \in L_K$ imply $||x_t^\phi|| < \gamma$ for all $t \in [0, R]$.

We assert that for every $\gamma > 0$, $||x_{q_j}|| < \delta(\gamma)$ implies $||x_{q_{j+1}}|| < \gamma$. Assume $||x_{q_j}|| < \delta(\gamma)$. The case $q_{j+1} - q_j \leq R$ is obvious. If $q_{j+1} - q_j > R$, then $|x|$ is decreasing on $[q_j + R, q_{j+1}]$. By the definition of $\delta(\gamma)$, $||x_{q_j+R}|| < \gamma$. Consequently,

$$||x_{q_{j+1}}|| \leq ||x_{q_j+R}|| < \gamma.$$

Therefore, it can be shown by induction that there exists $\eta = \eta(\epsilon) > 0$ such that

$$||x_{q_k}|| < \eta \quad \text{implies} \quad ||x_{q_{2l}}|| < \epsilon.$$

If

$$T > \frac{1}{\mu} \log \frac{\max\{A, B\}}{\eta(\epsilon)} + 2R,$$

then $||x_{q_k}|| < \eta$. This completes the proof of the claim.

We will now prove the proposition. Let a sequence $(\phi^n)_0^\infty$ in U and $\phi \in U$ be given with $\phi^n \to \phi$ as $n \to \infty$. Write $x^n = x^{\phi^n}$, $x = x^\phi$ and $q_k^n = q_k(\phi^n)$, $q_k = q_k(\phi)$ if $\phi^n, \phi \in U \setminus U_0$, respectively. We divide the proof into three cases.

Case 1: $\phi \in U \setminus U_0$ and $q_{2l} < \infty$. We have $\phi^n \in U \setminus U_0$ for all sufficiently large $n \in \mathbb{N}$, say for $n \geq n_0$. Proposition 5.1 implies $q_0 < 1$, $q_0^n < 1$, $q_1 - q_0 > 1$ and $q_1^n - q_0^n > 1$ for all $n \geq n_0$. It follows that $q_1^n > 1$ and $q_1 > 1$ for all $n \geq n_0$ and $x(1) < 0$. Lemma 2.4 implies that for any positive number $\epsilon > 0$ and any number T with $q_{2l} < T < q_{2l+1}$ there exists $n_1(\epsilon, T)$ with

$$\sup\{|x^n(t) - x(t)| : \ 1 \le t \le T\} < \epsilon \qquad \text{for all integers } n \ge n_1(\epsilon, T).$$

For a given positive number $\delta < \min\{\frac{1}{2}, T - q_{2l}\}$ we can assume (by taking ϵ as small as needed) that $|x(t)| \ge \epsilon$ for all $t \in [1, T] \cap_{j=1}^{2l} \{t \in \mathbb{R} : \ |t - q_j| \ge \delta\}$. For $n \ge n_1(\epsilon, T)$, it follows that $x^n(t) \ne 0$ for all $t \in [1, T] \cap_{j=1}^{2l} \{t \in \mathbb{R} : \ |t - q_j| \ge \delta\}$ and that $x^n(t)$ has a zero on the intervals $[q_j - \delta, q_j + \delta]$, $j = 1, 2, \ldots, 2l$. By Proposition 5.1, it follows that $|q_j^n - q_j| < \delta$ for all integers $n \ge n_1(\epsilon, T)$. Since $\delta > 0$ can be arbitrarily small, it follows that $q_j^n \to q_j$ as $n \to \infty$, $j = 1, 2, \ldots, 2l$. Then Lemma 2.4 implies $P_l(\phi^n) \to P_l(\phi)$ as $n \to \infty$.

Case 2: $\phi \in U \setminus U_0$ and $q_{2l} = \infty$. Then $P_l(\phi) = 0$. For each real $T > 0$ one can prove as in Case 1, by using Lemma 2.4 and Proposition 5.1, that $q_{2l}^n \ge T$ for all sufficiently large n. Then the Claim implies that, for any given $\epsilon > 0$, $||P_l(\phi^n)|| < \epsilon$ for all sufficiently large integers n.

Case 3: $\phi \in U_0$. Then $P_l(\phi) = 0$. We may assume that $\phi^n \in U \setminus U_0$ for all n, since $P_l(\psi) = 0$ for $\psi \in U_0$. Take $\epsilon > 0$ and applying the claim above select T such that $||P_l(\phi^n)|| < \epsilon$ for all n such that $q_{2l}^n \ge T$. By Lemma 2.4, there exists $n_2(\epsilon, T)$ such that if $n \ge n_2(\epsilon, T)$ then $\sup\{|x^{\phi^n}(t)| : \ 0 \le t \le T\} < \epsilon$. In the case $q_{2l}^n \le T$, from $q_k^n - q_{k-1}^n > 1$, $k = 1, 2, \ldots, 2l$, and $2l \ge R$, it follows that $q_{2l}^n > R$, and, consequently $||P_l(\phi^n)|| = ||F(q_{2l}^n, \phi^n)|| < \epsilon$ for all integers $n \ge n_2(\epsilon, T)$. Therefore, $||P_l(\phi^n)|| < \epsilon$ for both cases $q_{2l}^n \ge T$ and $q_{2l}^n < T$ provided $n \ge n_2(\epsilon, T)$. $\square$

The next result shows that the set S is positively invariant under the semiflow F.

Proposition 5.4. *$F(\mathbb{R}^+ \times S) \subset S$.*

Proof. Let $\phi \in S$ and write $x = x^\phi$.

Case 1: $\mathrm{sc}(\phi, [-1, 0]) = 0$. If $\phi \in (U \cup \tilde{U}) \cap S$, then Proposition 5.1 and Remark 5.2 can be used to conclude that $x_t \in S$ for all $t \ge 0$. If $\phi \notin U \cup \tilde{U}$ then $\phi(0) \ne 0$ and either $x(t) \ne 0$ for all $t \ge 0$ or there exists a smallest zero $z > 0$ of x. If $x(t) \ne 0$ for all $t \ge 0$, then $x_t \in S$ clearly holds for all $t \ge 0$. Otherwise, $x_z \in (U \cup \tilde{U}) \cap S$ and, by applying again Proposition 5.1 and Remark 5.2, we easily obtain that $x_t \in S$ for all $t \ge 0$.

Case 2: $\mathrm{sc}(\phi, [-1, 0]) = 1$. There exists $s_0, s_1 \in (-1, 0)$ with $s_0 < s_1$ such that either $\phi(s_0) < 0$ and $\phi(s_1) > 0$ or $\phi(s_0) > 0$ and $\phi(s_1) < 0$. We consider only the first possibility since the second one is analogous. Set

$$z_0 = \inf\{t : \ \phi(s) \ge 0 \text{ for } t \le s \le 0\},$$

$$z_1 = \sup\{t : \ x(s) \ge 0 \text{ for } 0 \le s \le t\}.$$

Clearly, $-1 < z_0 < 0$, $z_1 \in [0, \infty]$, $\phi(z_0) = 0$ and if $z_1 < \infty$ then $x(z_1) = 0$. Moreover,

$$\begin{aligned} \phi(s) &\le 0 \quad \text{for} \quad -1 \le s \le z_0, \\ \phi(s) &\ge 0 \quad \text{for} \quad z_0 \le s \le 0. \end{aligned}$$

If $z_1 = \infty$ then $x_t \in S$ follows for all $t \ge 0$. Assume $z_1 < \infty$. We have

$$\phi(s) \le 0 \qquad \text{for all} \quad s \in [\max\{-R, s_1 - 1\}, z_0],$$

since $\phi \in S$ and thus ϕ cannot have two sign changes on the interval $[\max\{-R, s_1 - 1\}, s_1]$. We assert that $z_1 - z_0 \ge 1$. If $z_1 - z_0 < 1$, then there exists $\epsilon > 0$ such that

$$\max\{-R, s_1 - 1\} \le t - r(x(t)) < z_0 \qquad \text{for all} \quad t \in (z_1, z_1 + \epsilon),$$

since $z_1 - r(x(z_1)) = z_1 - 1 \in [-1, z_0)$ and $t - r(x(t)) \ge -R$ for all $t \ge 0$. Hence, for $t \in (z_1, z_1 + \epsilon)$, from Eq. (2.1) we obtain

$$e^{\mu} x(t) = \int_{z_1}^{t} e^{\mu s} f(x(s - r(x(s))))\, ds \ge 0.$$

This contradicts the definition of z_1. Therefore $z_1 - z_0 \ge 1$. Thus $x_{z_1} \in U$ follows. Then Proposition 5.1 and the definition of z_1 combined imply that the distance of consecutive zeros of x in $[z_1, \infty)$ is greater than 1. Hence we conclude $x_t \in S$ for all $t \ge 0$. $\square$

Consider a complete metric space M, a semiflow $G : \mathbb{R}^+ \times M \to M$, and a subset $N \subset M$. The set N is called invariant if $G(t, N) = N$ for all $t \ge 0$. The set N is said to attract a set $N' \subset M$ if for every open set $O \subset M$ with $N \subset O$ there exists $t \ge 0$ such that $\{G(s, u) : \; u \in N'\} \subset O$ for all $s \ge t$. A global attractor is a compact invariant set which attracts every bounded subset of M. The bounded complete orbits, i.e. the sets $\{u(t) : t \in \mathbb{R}\}$ with $u : \mathbb{R} \to M$ satisfying $u(t) = G(t - s, u(s))$ for all reals $t \ge s$, with compact closures are contained in the global attractor.

Since L_K is a compact metric space, [25, Theorem 3.4.2] implies that the semiflow F has a global attractor $\mathcal{A}(F)$. By Proposition 5.4, the restriction of F to $\mathbb{R}^+ \times S$ defines a semiflow F_S on the compact metric space S. Define

$$\mathcal{A} = \mathcal{A}(F_S) = \cap_{t \ge 0} F(t, S).$$

Proposition 5.5.
(i) $\mathcal{A}$ *is the global attractor of the semiflow* F_S.
(ii) The map $F_{\mathcal{A}} : \mathbb{R} \times \mathcal{A} \ni (t, \phi) \mapsto x_t^{\phi} \in \mathcal{A}$ *is a continuous flow.*
(iii) $\mathcal{A}$ *is connected.*

(iv) The following statements are equivalent.

(a) $\phi \in \mathcal{A} \setminus \{0\}$.

(b) There is a solution $x : \mathbb{R} \to [-B, A]$ *with* $x_0 = \phi$ *and* $x_t \in S \setminus S_0$ *for all* $t \in \mathbb{R}$.

(c) There is a slowly oscillating solution $x : \mathbb{R} \to [-B, A]$ *with* $x_0 = \phi$.

(d) There is a nonzero solution $x : \mathbb{R} \to [-B, A]$ *such that* $x_0 = \phi$, $x_t|_{[-r(x(t)),0]} \neq 0$ *and* $V(x, [t - r(x(t)), t]) = 1$ *for all* $t \in \mathbb{R}$.

Proof. 1. By Proposition 5.4, the compact set S attracts all subsets of S. Therefore, [25, Theorem 3.4.2] implies that $\mathcal{A}$ is the global attractor of F_S.

2. It also follows from [25, Theorem 3.4.2] that $F_{\mathcal{A}}$ is a continuous flow provided $F(t, \cdot)$ is injective on $\mathcal{A}$ for all $t \geq 0$. Let $\phi, \psi \in \mathcal{A}$ and assume that $F(t, \phi) = F(t, \psi)$ for some $t \geq 0$. Since $\mathcal{A}$ is invariant, there exist solutions $x, y : \mathbb{R} \to [-B, A]$ with $x_0 = \phi$ and $y_0 = \psi$. Then Lemma 2.7 yields $\phi = \psi$. Thus, $F(t, \cdot) : \mathcal{A} \to \mathcal{A}$ is injective.

3. Suppose that $\mathcal{A}$ is not connected. Then there are open disjoint subsets V_1, V_2 of S such that $\mathcal{A} \subset V_1 \cup V_2$, $\mathcal{A} \cap V_1 \neq \emptyset$, $\mathcal{A} \cap V_2 \neq \emptyset$. We have $F(t, S) \supset F(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$ since $\mathcal{A}$ is invariant. As $\mathcal{A}$ attracts S, there exists $t \geq 0$ such that

$$\{F(t, \phi) : \ \phi \in S\} \subset V_1 \cup V_2.$$

Then

$$F(t, S) \cap V_1 \supset \mathcal{A} \cap V_1 \neq \emptyset \neq \mathcal{A} \cap V_2 \subset F(t, S) \cap V_2,$$

and hence it follows that $F(t, S)$ cannot be arcwise connected.

On the other hand, S is arcwise connected since $[-1, 1]S \subset S$. Then $F(t, S)$ is also arcwise connected as it is the continuous image of S. This is a contradiction.

4. Let $\phi \in \mathcal{A} \setminus \{0\}$. The facts that $\mathcal{A}$ is invariant, $F_{\mathcal{A}}$ is a flow on $\mathcal{A}$, and $F_{\mathcal{A}}(t, 0) = 0$ for all $t \in \mathbb{R}$ combined yield the existence of a unique solution $x : \mathbb{R} \to [-B, A]$ such that $x_0 = \phi$ and $x_t \in \mathcal{A} \setminus \{0\}$ for all $t \in \mathbb{R}$. The definition of $\mathcal{A}$ implies $\mathcal{A} \subset S$. Hence $x_t \in S$ follows for all $t \in \mathbb{R}$. If $x_s \in S_0$ for some $s \in \mathbb{R}$, then, by Proposition 5.1, $x_t = 0$ for all sufficiently large t, a contradiction. Therefore, (a)$\Rightarrow$(b).

Assume that (b) holds. We have to show that x is a slowly oscillating solution. Let $z' > z$ be two zeros of x. In order to show $z' - z > 1$, it suffices to find a $t_0 \leq z - 1$ such that $x_{t_0} \in U \setminus U_0$, since Proposition 5.1 applied to $\phi = x_{t_0} \in U \setminus U_0$ implies that $z' = q_j(x_{t_0})$ and $z = q_k(x_{t_0})$ for some integers $k > j \geq 1$, which gives $z' - z > 1$. From Lemma 2.2 it follows that x has arbitrarily large negative zeros. For every real T, x takes both positive and negative values in $(-\infty, T]$. Indeed, assuming the contrary, Eq. (2.1) implies that $|x|$ is decreasing on $(-\infty, T]$. This together with the existence of arbitrary large negative zeros implies that $x(s) = 0$ for all $s \leq T$, a contradiction. Select s_1 and s_2 such that $s_1 < s_2 < z - 1$ and $x(s_1) > 0$, $x(s_2) < 0$. Define

$$t_0 = \sup\{t : \ x(s) \geq 0 \text{ for all } s_1 \leq s \leq t\}.$$

We claim that $x_{t_0} \in U \setminus U_0$. Since $x_{t_0} \in S \setminus S_0$ and $x(t_0) = 0$ follows from the definition of t_0, it is enough to prove that $x(s) \geq 0$ for all $s \in [t_0 - 1, t_0]$. This is the case if $t_0 \geq s_1 + 1$. If $t_0 < s_1 + 1$ and $x(s) \geq 0$ for all $s \in [t_0 - 1, t_0]$ does not hold, then there exists $t_1 \in (t_0 - 1, s_1)$ such that $x(t_1) < 0$. Consider the sign changes of x on the interval $[t_1, t_1 + 1]$. The definition of t_0 implies that there is a positive sequence $(\delta_n)_0^\infty$ such that $\delta_n \to 0$ as $n \to \infty$ and $x(t_0 + \delta_n) < 0$ for all $n \in \mathbb{N}$. Hence and from $t_1 + 1 > t_0$, $x(t_1) < 0$ and $x(s_1) > 0$ it follows that $\mathrm{sc}(x, [t_1, t_1 + 1]) \geq 2$, a contradiction to $x_{t_1+1} \in S$. Thus, (b)$\Rightarrow$(c).

Assume that $x : \mathbb{R} \to [-B, A]$ is a slowly oscillating solution. Consider the interval $[t - r(x(t)), t]$. We claim that x cannot have more than one zero in $[t - r(x(t)), t]$. If $z_1 < z_2$ were two zeros in $[t - r(x(t)), t]$, then $z_2 - z_1 > 1$ and $r(x(z_2)) = r(0) = 1$ would imply $z_2 - r(x(z_2)) = z_2 - 1 > z_1$. Using that $\mathbb{R} \ni t \mapsto t - r(x(t)) \in \mathbb{R}$ is increasing by Lemmas 2.2 and 2.5, we obtain from $t \geq z_2$ that $t - r(x(t)) \geq z_2 - r(x(z_2)) > z_1$, a contradiction to $z_1 \geq t - r(x(t))$. Therefore, (c)$\Rightarrow$(d).

Let $x : \mathbb{R} \to [-B, A]$ be a nonzero solution such that $V(x, [t - r(x(t)), t]) = 1$ for all $t \in \mathbb{R}$. Lemmas 2.2 and 2.5 yield that the function $\mathbb{R} \ni t \mapsto t - r(x(t)) \in \mathbb{R}$ is strictly increasing. We assert first that the set $N' = \{x_t : t \in \mathbb{R}\}$ is a subset of S. We have to show that, for any $t \in \mathbb{R}$, x cannot have more than one sign change in $[t - 1, t]$. The case when x has no zero in $[t - 1, t]$ is obvious. Let s denote the largest zero of x in $[t - 1, t]$. Then $r(x(s)) = 1$. The case $s = t$ is again obvious since $\mathrm{sc}(x, [t - 1, t]) \leq V(x, [t - 1, t]) = 1$. If $s < t$, then by the monotonicity of $t - r(x(t))$ and $s - 1 = s - r(x(s)) < t - 1$, we find $t' \in (s, t)$ such that $[t - 1, s] \subset [t' - r(x(t')), t']$. By the definition of s, x has the same sign on $(s, t]$. Therefore, $\mathrm{sc}(x, [t - 1, t])= \mathrm{sc}(x, [t - 1, t'])\leq \mathrm{sc}(x, [t' - r(x(t')), t']) \leq V(x, [t' - r(x(t')), t']) = 1$. Thus, N' is a subset of S. The set N' satisfies $F(t, N') = N'$ for all $t \geq 0$. As $\mathcal{A}$ attracts N', it follows that $N' \subset \mathcal{A}$, and in particular $x_0 = \phi \in \mathcal{A}$. $x_0 = \phi = 0$ is impossible since x is a nonzero solution. So (d)$\Rightarrow$(a), and the proof is complete. $\square$

Corollary 5.6.

(i) If $(\phi^n)_0^\infty$ is a sequence in $\mathcal{A}$ and $\phi \in \mathcal{A}$ such that $\phi^n \to \phi$ as $n \to \infty$, then $x^{\phi^n}(t) \to x^\phi(t)$ as $n \to \infty$ uniformly on each compact subinterval of $\mathbb{R}$.

(ii) The topologies induced on $\mathcal{A}$ from $C([-R, 0], \mathbb{R})$ and $C^1([-R, 0], \mathbb{R})$ are equivalent.

Proof. 1. Let I be a compact interval. Choose $k \in \mathbb{N}$ such that $I \subset [-kR, kR]$. The continuity of $F_{\mathcal{A}}(jR, \cdot)$ by Proposition 5.5(ii) implies that for every given $\epsilon > 0$ there exist $\delta_j > 0$ such that from $||\phi^n - \phi|| < \delta_j$ it follows that $||F(jR, \phi^n) - F(jR, \phi)|| < \epsilon$. Taking $\delta = \min\{\delta_j : j \in \{-k, -k + 1, \dots, k\}\}$, it follows from $||\phi^n - \phi|| < \delta$ that $\sup\{|x^{\phi^n}(t) - x^\phi(t)| : t \in I\} < \epsilon$.

2. It suffices to show that if $\phi^n, \phi \in \mathcal{A}$ and $||\phi^n - \phi|| \to 0$ as $n \to \infty$, then $||\dot{\phi}^n - \dot{\phi}|| \to 0$. By statement (i), $||x^{\phi^n}_{-R} - x^{\phi}_{-R}|| \to 0$ follows. Then Lemma 2.2 implies $||\dot{\phi}^n - \dot{\phi}|| \to 0$ as $n \to \infty$. □

We turn to the map P_l. Since U is compact, it follows from [25, Theorem 2.4.2] that the set

$$\mathcal{A}(P_l) = \cap_{n=0}^{\infty} P_l^n(U)$$

is the global attractor of P_l, that is, it is a compact invariant subset of U attracting all bounded subsets (which in our case means all subsets) of U. Since U is compact and convex, the closed convex hull of $\mathcal{A}(P_l)$ is a subset of U. Then the arguments from the proof of [25, Lemma 2.4.1] show that $\mathcal{A}(P_l)$ is connected.

Proposition 5.7. $\mathcal{A} \cap U = \mathcal{A}(P_l)$.

Proof. It is clear that 0 is an element of $\mathcal{A} \cap U$ and $\mathcal{A}(P)$.

Let $\phi \in (\mathcal{A} \cap U) \setminus \{0\}$. Proposition 5.5 implies that there is a slowly oscillating solution $x : \mathbb{R} \to [-B, A]$ such that $x_0 = \phi$. Using Eq. (2.1) and $r(0) = 1$ we obtain that all zeros of x are simple. From Lemma 2.2 it follows that x has arbitrary large negative zeros. If $0 = z^0 > z^1 > \dots$ are the zeros of x in $(-\infty, 0]$, then from $x_0 \in U \setminus U_0$ it follows that $x_{z^{2ln}} \in U \setminus U_0$ and, by Proposition 5.1, $P_l(x_{z^{2l(n-1)}}) = x_{z^{2ln}}$ for all $n \in -\mathbb{N}$. Then $\phi = x_0 = P_l^n(x_{z^{-2ln}})$ for all $n \in -\mathbb{N}$, and thus $\phi \in \mathcal{A}(P_l)$.

Let $\phi \in \mathcal{A}(P_l) \setminus \{0\}$. Clearly, $\phi \in U \setminus U_0$. There is a trajectory $(\phi^n)_{-\infty}^0$ of the map P_l in U with $\phi^0 = \phi$. Clearly $\phi^n \in U \setminus U_0$ for all $n \in -\mathbb{N}$. Let $(q_j^n)_{j=0}^{\infty}$ denote the sequence associated with ϕ^n by Proposition 5.1(ii). The fact $\phi^n \neq 0$ for all $n \in -\mathbb{N}$ yields $q_{2l}^n < \infty$ for all $n \in -\mathbb{N}$. Then by the definition of P_l it is not difficult to see that $x : \mathbb{R} \to \mathbb{R}$ defined by

$$x(t) = F\left(t + \sum_{j=n}^{0} q_{2l}^j, \phi^n\right) \qquad \text{for } t \in [-\sum_{j=n}^{0} q_{2l}^j, \infty)$$

gives a solution of Eq. (2.1) with $x(\mathbb{R}) \subset [-B, A]$, and $x_{-\sum_{j=n}^{0} q_{2l}^j} = \phi^n$. Propositions 5.1(ii) and 5.5(iv) combined yield $\phi \in \mathcal{A} \setminus \{0\}$. The proof is complete. □

Corollary 5.8. *$\mathcal{A} \cap U$ is compact and connected.*

Using Proposition 5.1, the definition of P_1 and the fact that $\mathcal{A}$ is invariant under F, we obtain

$$P_1(\mathcal{A} \cap U) \subset \mathcal{A} \cap U.$$

Set

$$\mathcal{B} = \{\phi \in \mathcal{A} \cap U : P_1(\phi) \neq 0\}.$$

Define the map

$$P : \mathcal{B} \ni \phi \mapsto P_1(\phi) \in L_K.$$

Proposition 5.9. *$P_1|_{\mathcal{A}\cap U}$ is continuous. P is a homeomorphism from $\mathcal{B}$ onto $\mathcal{A}\cap U \setminus \{0\}$.*

Proof. The proof of the continuity of $P_1|_{\mathcal{A}\cap U}$ is essentially the same as that of P_l. We used only in Case 3 of the proof of Proposition 5.3 that $2l \geq R$. If $\phi \in U_0 \cap (\mathcal{A}\cap U)$, then $\phi = 0$ by Proposition 5.5. Let a sequence $(\phi^n)_0^\infty$ in $\mathcal{A}\cap(U \setminus U_0)$ be given so that $\phi^n \to 0$ as $n \to \infty$. Let $x^n = x^{\phi^n}$, $q_2^n = q_2(\phi^n)$. Let $\epsilon > 0$ be fixed. Analogously to the claim in the proof of Proposition 5.3, we find $T > 0$ such that $||P_1(\phi^n)|| < \epsilon$ for all $n \in \mathbb{N}$ with $q_2^n \geq T$. By Lemma 2.4 there exists n_0 such that $\sup\{|x^{\phi^n}(t)| : -R \leq t \leq T\} < \epsilon$ follows for all $n \geq n_0$. So, $||P_1(\phi^n)|| < \epsilon$ follows for all $n \geq n_0$. Therefore, $P_1|_{\mathcal{A}\cap U}$ is continuous.

The injectivity of P on $\mathcal{B}$ follows from the backward uniqueness of solutions on $\mathcal{A}$. If $\phi \in \mathcal{A}\cap U \setminus \{0\}$, then Proposition 5.5 implies that there is a unique slowly oscillating solution $x : \mathbb{R} \to [-B, A]$ with $x_0 = \phi$. Lemma 2.2 implies that x has arbitrary large negative zeros. We have $x(0) = 0$ since $x_0 = \phi \in U$. Let $z_{-3} < z_{-2} < z_{-1} < 0$ be defined such that $z_{-3}, z_{-2}, z_{-1}, 0$ are consecutive zeros of x. Then $z_{-2} - z_{-3} > 1$, $z_{-1} - z_{-2} > 1$ and $z_{-1} < -1$. From $x_0 = \phi \in U$, it follows that $x(s) > 0$ for all $s \in (z_{-1}, 0)$, and hence $\dot{x}(z_{-1}) \geq 0$. We also have $x(s) \neq 0$ for all $s \in (z_{-2}, z_{-1}) \cup (z_{-3}, z_{-2})$. Since $x(z_{-1}) = 0$, $r(x(z_{-1})) = 1$ and $z_{-1} - z_{-2} > 1$, we obtain $\dot{x}(z_{-1}) \neq 0$ from Eq. (2.1). Therefore $\dot{x}(z_{-1}) > 0$ and $x(s) < 0$ for all $s \in (z_{-2}, z_{-1})$. Continuing the same argument, we obtain that $\dot{x}(z_{-2}) < 0$ and $x(s) > 0$ for all $s \in (z_{-3}, z_{-2})$. Clearly $x_{z_{-2}} \in \mathcal{A}\cap U$ and $P_1(x_{z_{-2}}) = \phi$. Therefore $P(\mathcal{B}) = \mathcal{A}\cap U \setminus \{0\}$. As the inverse of P is given by

$$\mathcal{A}\cap U \setminus \{0\} \ni \phi \mapsto x_{z_{-2}} \in \mathcal{B},$$

the continuity of the inverse of P follows from Corollary 5.6. □

Let $P^{-1} : \mathcal{A}\cap U \setminus \{0\} \to \mathcal{B}$ denote the inverse of P.

Let $\phi \in \mathcal{A} \setminus \{0\}$ and $x = x^\phi$. Proposition 5.5 implies that x is slowly oscillating and $V(x, [t - r(x(t)), t]) = 1$ for all $t \in \mathbb{R}$. All zeros of x are simple since x is slowly oscillating. Let $z_{-2}(\phi)$ denote the largest negative zero of x^ϕ with $\dot{x}^\phi(z_{-2}(\phi)) < 0$. Using also Lemma 2.2 and Proposition 5.1 the next proposition easily follows and thus the proof is omitted.

Proposition 5.10.

(i) If $\phi \in \mathcal{A} \setminus \{0\}$ and $x = x^\phi$, then the zeroset of x is given by a sequence $(z_j(\phi))_{-\infty}^{J(\phi)}$, where $J(\phi) = \infty$ if the zeroset is unbounded from above, $J(\phi) \in \mathbb{Z}$ if the zeroset is bounded from above. Moreover, $z_{j-1}(\phi) < z_j(\phi) - 1$ and $\dot{x}(z_j(\phi)) \neq 0$ for all $j \in \mathbb{Z}$ with $j \leq J(\phi)$.

(ii) If $\phi \in \mathcal{A} \cap U \setminus \{0\}$ and $x = x^{\phi}$, then $J(\phi) \geq 0$ and

$$
\begin{aligned}
t \notin \{z_j(\phi) :\ j \in \mathbb{Z},\ j \leq J(\phi)\} &\implies x_t \notin U, \\
j \in \mathbb{Z},\ 2j \leq J(\phi) &\implies x_{z_{2j}(\phi)} \in \mathcal{A} \cap U \setminus \{0\}, \\
j \in \mathbb{Z},\ 2j \leq J(\phi) &\implies P(x_{z_{2j-2}(\phi)}) = x_{z_{2j}(\phi)}, \\
j \in \mathbb{Z},\ 2j \leq J(\phi) &\implies P^{-1}(x_{z_{2j}(\phi)}) = x_{z_{2j-2}(\phi)}.
\end{aligned}
$$

The next proposition contains information about slowly oscillating periodic solutions of (2.1).

Proposition 5.11. *Assume that $\phi \in \mathcal{A} \setminus \{0\}$ and $x = x^{\phi}$ is a periodic solution of Eq. (2.1) with minimal period $p > 0$. Then*
(i) $p = z_2(\phi) - z_0(\phi)$ and $\dot{x}$ has exactly one zero between two consecutive zeros of x,
(ii)

$$V(x_t - x_{t-\tau}, [-r(x(t)), 0]) = 1 \qquad \text{for all } \tau \in (0, p) \text{ and } t \in \mathbb{R}.$$

Proof. 1. (i) is contained in [41, Theorem 2.6].

2. The periodicity of x and Proposition 4.2 (i) imply that, for every fixed $\tau \in (0, p)$,

$$V(x_t - x_{t-\tau}, [-r(x(t)), 0])$$

is independent of t. Thus, it suffices to show that

$$V(x_{z_2(\phi)} - x_{z_2(\phi)-\tau}, [-1, 0]) = 1 \text{ for all } \tau \in (0, p).$$

This holds if we prove that, for every $\tau \in (0, p)$, $x_{z_2(\phi)} - x_{z_2(\phi)-\tau}$ has at most one zero in $[-1, 0]$. By way of contradiction, let $t_1, t_2 \in [z_2(\phi) - 1, z_2(\phi)]$ be such that $t_1 < t_2$ and

$$x(t_1) = x(t_1 - \tau), \quad x(t_2) = x(t_2 - \tau).$$

From (i) it follows that there is a unique $s \in (z_1(\phi), z_2(\phi))$ so that $\dot{x}(s) = 0$ and x is strictly monotone on the intervals $(z_1(\phi), s)$ and $(s, z_2(\phi))$. Using also the facts that the signs of x on $(z_0(\phi), z_1(\phi))$ and on $(z_1(\phi), z_2(\phi))$ are different and that x is p-periodic and $\tau \in (0, p)$, we conclude that

$$
\begin{aligned}
&s \leq t_1 < t_2 \Longrightarrow z_1(\phi) \leq t_2 - \tau < t_1 - \tau \leq s, \\
&t_1 < t_2 \leq s \Longrightarrow s - p \leq t_2 - \tau < t_1 - \tau \leq z_0(\phi), \\
&t_1 < s < t_2 \Longrightarrow s - p < t_1 - \tau \leq z_0(\phi),\ z_1(\phi) \leq t_2 - \tau < s.
\end{aligned}
$$

In the first two cases we get $t_2 < t_1$. In the third case $t_2 - t_1 \geq z_1(\phi) - z_0(\phi) > 1$ follows. This is a contradiction and the proof is complete. $\square$

6. Asymptotic expansion for slowly oscillating solutions

In this section we prove an asymptotic expansion for slowly oscillating solutions converging to zero as $t \to -\infty$. In the constant delay case, Cao [10] proved asymptotic expansions in more general situations than slowly oscillating solutions converging to zero. Our proof can be extended to get the same type of results for solutions of Eq. (2.1).

Recall that $u_0 = \max \mathrm{Re}(\Sigma)$ where Σ denotes the spectrum of the generator of the solution semiflow $(T(t))_{t\geq 0}$ of Eq. (3.1). If $u_0 > 0$ then Σ consists of a complex conjugate pair $\{u_0 \pm i v_0\}$ with $v_0 \in (\frac{\pi}{2}, \pi)$. Recall that Q and L are the realified generalized eigenspaces associated with the spectral sets $\cup_{k=1}^{\infty}(\Sigma \cap S_k)$ and $\Sigma \cap S_0$, respectively.

Observe that Q and L are also the realified generalized eigenspaces of $T(1)$ associated with the spectral sets $\{e^{\lambda} : \lambda \in \sum_{k=1}^{\infty}(\Sigma \cap S_k)\} \cup \{0\}$ and $\{e^{\lambda} : \lambda \in \Sigma \cap S_0\}$, respectively. Define $T_L(1) : L \ni \phi \mapsto T_L(1)\phi \in L$.

We want to apply the variation-of-constants formula from [17]. Let us recall a few basic facts about dual semigroups. It is convenient to denote dual spaces and adjoint operators by an asterisk in the sequel. The elements $\phi^{\odot} \in C^*$ for which the curve

$$\mathbb{R}^+ \ni t \mapsto T(t)^*\phi^{\odot} \in C^*$$

is continuous form a closed subspace $C^{\odot}$ which is positively invariant under the adjoints $T(t)^*$, $t \geq 0$. The operators

$$T^{\odot}(t) : C^{\odot} \ni \phi^{\odot} \mapsto T(t)^*\phi^{\odot} \in C^{\odot}, \quad t \geq 0,$$

constitute a strongly continuous semigroup on $C^{\odot}$. Repeating this process we obtain a subspace $C^{\odot\odot} \subset C^{\odot *}$. The original state space C is sun-reflexive in the sense that there exists a norm-preserving linear map $j : C \to C^{\odot *}$ with $jC = C^{\odot\odot}$.

For every continuous map $\tilde{g} : \mathbb{R} \to C^{\odot *}$ and reals $c \leq d$ the weak-star integral

$$\int_c^d T^{\odot}(d-t)^*\tilde{g}(t)\,dt \in C^{\odot *}$$

is defined by

$$\left(\int_c^d T^{\odot}(d-t)^*\tilde{g}(t)\,dt\right)(\phi^{\odot}) = \int_c^d \left(T^{\odot}(d-t)^*\tilde{g}(t)\right)(\phi^{\odot})\,dt$$

for all $\phi^\odot \in C^\odot$. One finds that all such weak-star integrals are elements of the subspace $C^{\odot\odot} = jC$.

There is an isomorphism $k : C^{\odot *} \to \mathbb{R} \times L^\infty(-1, 0; \mathbb{R})$. Set $r^{\odot *} = k^{-1}(1, 0)$.

If $g : \mathbb{R} \to \mathbb{R}$ is continuous and if $x : \mathbb{R} \to \mathbb{R}$ is a solution of the linear inhomogeneous equation

$$\dot{x}(t) = -\mu x(t) + f'(0)x(t-1) + g(t)$$

then the curve $u : \mathbb{R} \ni t \mapsto x_{t,C} \in C$ satisfies

$$ju(t) = jT(t-\sigma)u(\sigma) + \int_\sigma^t T^\odot(t-s)^* \left(g(s)r^{\odot *}\right) ds$$

for all reals t, σ with $t \geq \sigma$.

The spectra of the generators of the semigroups $(T(t))_{t\geq 0}$ and $(T^\odot(t))_{t\geq 0}$ coincide. Let Pr_L and $\mathrm{Pr}_L^\odot$ denote the spectral projection operators in $L(C, C)$ and $L(C^\odot, C^\odot)$ which are associated with the spectral set $\{u_0 \pm iv_0\}$. We have $\mathrm{Pr}_L C = L$. The adjoint operator $\mathrm{Pr}_L^{\odot *} \in L(C^{\odot *}, C^{\odot *})$ satisfies

$$\mathrm{Pr}_L^{\odot *} C^{\odot *} = jL, \quad \mathrm{Pr}_L^{\odot *} \circ j = j \circ \mathrm{Pr}_L,$$

and for g, x, u as before

$$\mathrm{Pr}_L^{\odot *} ju(t) = \mathrm{Pr}_L^{\odot *} jT(t-\sigma)u(\sigma) + \int_\sigma^t T^\odot(t-s)^* \, \mathrm{Pr}_L^{\odot *} \left(g(s)r^{\odot *}\right) ds \tag{6.1}$$

and

$$(\mathrm{id} - \mathrm{Pr}_L^{\odot *})ju(t) = (\mathrm{id} - \mathrm{Pr}_L^{\odot *})jT(t-\sigma)u(\sigma) + \int_\sigma^t T^\odot(t-s)^*(\mathrm{id} - \mathrm{Pr}_L^{\odot *}) \left(g(s)r^{\odot *}\right) ds \tag{6.2}$$

for all reals $t \geq \sigma$. $T^\odot(t)^*$ can be extended to a one-parameter group on $\mathrm{Pr}_L^{\odot *} C^{\odot *}$ and (6.1) is valid for all t, σ in $\mathbb{R}$.

There exists $K_0 > 0$ such that

$$||T^\odot(t)^* \, \mathrm{Pr}_L^{\odot *}|| \leq K_0 e^{(u_0+\delta)t}, \qquad t \geq 0, \tag{6.3}$$

$$||T^\odot(t)^* \, \mathrm{Pr}_L^{\odot *}|| \leq K_0 e^{(u_0-\delta)t}, \qquad t \leq 0, \tag{6.4}$$

$$||T^\odot(t)^*(\mathrm{id} - \mathrm{Pr}_L^{\odot *})|| \leq K_0 e^{(u_0-\delta)t}, \qquad t \geq 0. \tag{6.5}$$

Proposition 6.1. *Assume that $u_0 > 0$ and $x : \mathbb{R} \to [-B, A]$ is a slowly oscillating solution of Eq. (2.1) with $\lim_{t\to-\infty} x(t) = 0$. Then there exist real numbers $\epsilon > 0$ and a, b such that*

$$x(t) = e^{u_0 t}(a\cos(v_0 t) + b\sin(v_0 t)) + O(e^{(u_0+\epsilon)t}) \qquad \text{as } t \to -\infty.$$

Proof. Select real numbers β and δ such that

$$\beta \in (\max\{e^{u_1}, e^{u_0-\delta/2}\}, e^{u_0}),\ 2\delta < u_0,\ \delta < u_0 - u_1,$$

where $u_1 = \max \mathrm{Re}(\cup_{k=1}^{\infty}(\Sigma \cap S_k))$. There is a norm $|\cdot|$ on C which is equivalent to the supremum-norm $||\cdot||_C$ on C and

$$|(T_L(1))^{-1}| < \frac{1}{\beta}.$$

First we claim that there exists $K_1 > 0$ such that

$$||x_{t,C}||_C \leq K_1 e^{(\log\beta)t} \qquad \text{for all } t \leq 0. \tag{6.6}$$

It is easy to see that (6.6) follows from

$$\limsup_{t\to-\infty} \frac{|x_{t-1,C}|}{|x_{t,C}|} < \frac{1}{\beta}. \tag{6.7}$$

If (6.7) does not hold, then there exists $\gamma \geq \frac{1}{\beta}$ and a sequence $(t^n)_0^{\infty}$ in $(-\infty, 0]$ with $t^n \to -\infty$ and $|x_{t^n-1,C}|/|x_{t^n,C}| \to \gamma$ as $n \to \infty$. Define

$$z^n : \mathbb{R} \ni t \mapsto \frac{x(t^n+t)}{|x_{t^n,C}|} \in \mathbb{R}.$$

The function z^n satisfies

$$\dot z^n(t) = -\mu z^n(t) + \int_0^1 f'(sx(t^n + t - r(x(t^n+t))))\, ds\, z^n(t - r(x(t^n+t))), \quad t \in \mathbb{R},$$

and $|z^n_{0,C}| = 1$. Let $v^n : \mathbb{R} \ni t \mapsto e^{\mu t} z^n(t) \in \mathbb{R}$. Then

$$\dot v^n(t) = \int_0^1 f'(sx(t^n + t - r(x(t^n+t))))\, ds\, e^{\mu r(x(t^n+t))} v^n(t - r(x(t^n+t))) \tag{6.8}$$

for all $t \in \mathbb{R}$. From the fact that x is slowly oscillating, it follows that $V(x, [t^n+t-r(x(t^n+t)), t^n+t])$=$V(z^n, [t-r(x(t^n+t)), t])$= $V(v^n, [t-r(x(t^n+t)), t]) = 1$. We also have

$$|r(x(s_1)) - r(x(s_2))| \leq \max\{r'(u) :\ u \in [-B, A]\} K |s_1 - s_2|.$$

Using also $\min\{r(u) : u \in [-B, A]\} > 0$ and $f' < 0$, it is not difficult to see that Proposition 4.3 can be applied to get $K_1' > 0$ and $\alpha_1' > 0$ such that

$$|v^n(t)| \leq K_1' e^{\alpha_1'|t|} \qquad \text{for all } t \leq 0 \text{ and } n \in \mathbb{N}. \tag{6.9}$$

Using the facts $x(t) \in [-B, A]$ for all $t \in \mathbb{R}$, $0 < \inf_{t\in\mathbb{R}} r(x(t)) \le \sup_{t\in\mathbb{R}} r(x(t)) \le R$, (6.8), (6.9) and the method of steps, we find $K_1 > 0$ and $\alpha_1 > 0$ such that

$$|v^n(t)| \le K_1 e^{\alpha_1 |t|} \qquad \text{for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Hence we obtain an exponential bound also for z^n on $\mathbb{R}$ independently of n. The right hand sides of the differential equations for z^n are bounded on each compact subinterval of $\mathbb{R}$. Therefore $(z^n)_0^\infty$ is a uniformly bounded and equicontinuous sequence of functions on each compact subinterval of $\mathbb{R}$. By the Arzèla–Ascoli theorem and the diagonalization process, there is subsequence $(z^{n_k})_{k=0}^\infty$ and a continuous function $z : \mathbb{R} \to \mathbb{R}$ such that

$$z^{n_k}(t) \to z(t) \qquad \text{as } k \to \infty \text{ uniformly on compact subsets of } \mathbb{R}.$$

Using the differential equation for z^{n_k} we obtain that $(\dot{z}^{n_k})_{k=0}^\infty$ also converges uniformly on compact subsets of $\mathbb{R}$. Moreover, from $x(t) \to 0$ as $t \to -\infty$ it follows that

$$\int_0^1 f'(sx(t^n + t - r(x(t^n + t))))\, ds \to f'(0), \quad r(x(t^n + t)) \to 1 \ \text{ as } \ n \to \infty$$

uniformly on compact subsets of $\mathbb{R}$. Consequently, z is differentiable on $\mathbb{R}$ and satisfies $\dot{z}(t) = -\mu z(t) + f'(0)z(t-1)$ for all $t \in \mathbb{R}$, moreover $|z_{0,C}| = 1$, $|z_{-1,C}| = \gamma \ge \frac{1}{\beta}$. The fact that x is a slowly oscillating solution and Lemmas 2.6, 4.4 combined yield

$$V(x, [t^n + t - r(x(t^n + t)), t^n + t]) = 1 \qquad \text{for all } t \in \mathbb{R},\ n \in \mathbb{N}.$$

Then

$$V(z^n, [t - r(x(t^n + t)), t]) = 1 \qquad \text{for all } t \in \mathbb{R},\ n \in \mathbb{N}.$$

Applying also Lemma 4.1(i) and the facts that $z_{t,C} \ne 0$ for all $t \in \mathbb{R}$, $r(x(t^n + t)) \to 1$ as $n \to \infty$ for all $t \in \mathbb{R}$, and $z^{n_k} \to z$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$, we obtain

$$V(z, [t-1, t]) = 1 \qquad \text{for all } t \in \mathbb{R}.$$

Hence, by Lemmas 3.1(v) and 4.4, $z_{t,C} \in L$ for all $t \le 0$. Then from $|(T_L(1))^{-1}| < \frac{1}{\beta}$ it follows that

$$|z_{-1,C}| < \frac{1}{\beta}|z_{0,C}| = \frac{1}{\beta},$$

a contradiction. Therefore (6.7) and consequently (6.6) hold.

We want to apply the variation of constants formula from [17]. We may write

$$\dot{x}(t) = -\mu x(t) + f'(0)x(t-1) + h(t)$$

for all $t \in \mathbb{R}$, where

$$h : \mathbb{R} \ni t \mapsto f(x(t - r(x(t)))) - f'(0)x(t-1) \in \mathbb{R}$$

is a continuous function. Using assumption (H1), the Taylor formula and the mean value theorem, for every $t \in \mathbb{R}$, we find reals ξ, η, θ between $0, x(t - r(x(t)))$ and $1, r(x(t))$ and $0, x(t)$, respectively, so that

$$\begin{aligned} h(t) &= f(x(t - r(x(t)))) - f'(0)x(t - r(x(t))) + f'(0)[x(t - r(x(t))) - x(t-1)] \\ &= \frac{f''(\xi)}{2}x^2(t - r(x(t))) + f'(0)\dot{x}(t-\eta)[1 - r(x(t))] \\ &= \frac{f''(\xi)}{2}x^2(t - r(x(t))) - f'(0)\dot{x}(t-\eta)r'(\theta)x(t). \end{aligned}$$

From (H1), (6.6) and Eq. (2.1) it follows that there exists $K_2' > 0$ such that

$$|\dot{x}(t)| \le K_2' e^{(\log \beta)t} \qquad \text{for all } t \le 0.$$

Therefore, there exists $K_2 > 0$ such that

$$|h(t)| \le K_2 e^{2(\log \beta)t} \qquad \text{for all } t \le 0. \tag{6.10}$$

Applying (6.3) and (6.10), for all $\phi^{\odot} \in C^{\odot}$ with $||\phi^{\odot}|| \le 1$ and reals $s \le t \le 0$, we obtain

$$\begin{aligned} \left| \left[T^{\odot}(t-s)^* \operatorname{Pr}_L^{\odot *}(h(s) r^{\odot *}) \right] (\phi^{\odot}) \right| &\le ||T^{\odot}(t-s)^* \operatorname{Pr}_L^{\odot *}|| \, |h(s)| \, ||r^{\odot *}|| \\ &\le K_0 K_2 ||r^{\odot *}|| e^{(u_0+\delta)t} e^{(u_0 - 2\delta)s}. \end{aligned} \tag{6.11}$$

Analogously, from (6.5) and (6.10), for all $\phi^{\odot} \in C^{\odot}$ with $||\phi^{\odot}|| \le 1$ and reals $s \le t \le 0$, we get

$$\left| \left[T^{\odot}(t-s)^* \operatorname{Pr}_L^{\odot *}(h(s) r^{\odot *}) \right] (\phi^{\odot}) \right| \le K_0 K_2 ||r^{\odot *}|| e^{(u_0-\delta)t} e^{u_0 s}. \tag{6.12}$$

The last two inequalities, (6.4) and $u_0 > 2\delta > 0$ combined yield that, for every $t \le 0$ and $\sigma \le 0$, the weak-star integrals

$$\int_{-\infty}^{t} T^{\odot}(t-s)^* (\mathrm{id} - \operatorname{Pr}_L^{\odot *})(h(s) r^{\odot *}) \, ds, \quad \int_{-\infty}^{t} T^{\odot}(t-s)^* \operatorname{Pr}_L^{\odot *}(h(s) r^{\odot *}) \, ds,$$

$$\int_{-\infty}^{\sigma} T^{\odot}(t-s)^* \operatorname{Pr}_L^{\odot *}(h(s) r^{\odot *}) \, ds$$

exist. Moreover these integrals are elements of $C^{\odot\odot}$.

From (6.5), (6.6) and the choice of β, δ it follows that, for all $\sigma \le t \le 0$,

$$\begin{aligned}||(\mathrm{id} - \mathrm{Pr}_L^{\odot *}) jT(t-\sigma)u(\sigma)|| &= ||T^{\odot}(t-\sigma)^*(\mathrm{id} - \mathrm{Pr}_L^{\odot *}) ju(\sigma)|| \\ &\le K_0 K_1 e^{(u_0-\delta)(t-\sigma)} e^{(\log\beta)\delta} \\ &\le K_0 K_1 e^{(u_0-\delta)(t-\sigma)} e^{(u_0-\delta/2)\sigma} \\ &\le K_0 K_1 e^{(u_0-\delta)t} e^{(\delta/2)\sigma}.\end{aligned}$$

Consequently, letting $\sigma \to -\infty$ in (6.2) with $g = h$, we conclude

$$(\mathrm{id} - \mathrm{Pr}_L^{\odot *}) ju(t) = \int_{-\infty}^{t} T^{\odot}(t-s)^*(\mathrm{id} - \mathrm{Pr}_L^{\odot *})(h(s)r^{\odot *})\, ds, \qquad t \le 0.$$

Using the above equality, (6.11) and the definition of weak-star integrals, we find

$$||(\mathrm{id} - \mathrm{Pr}_L^{\odot *}) ju(t)|| \le \frac{K_0 K_2 ||r^{\odot *}||}{u_0} e^{(2u_0-\delta)t}, \qquad t \le 0.$$

For all $t \le 0$ we have

$$\begin{aligned}&\int_0^t T^{\odot}(t-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds \\ &\quad = \int_{-\infty}^{t} T^{\odot}(t-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds - \int_{-\infty}^{0} T^{\odot}(t-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds \\ &\quad = \int_{-\infty}^{t} T^{\odot}(t-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds - T^{\odot}(t)^* \int_{-\infty}^{0} T^{\odot}(-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds.\end{aligned}$$

The integral $\int_{-\infty}^{0} T^{\odot}(-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds$ is an element of $\mathrm{Pr}_L^{\odot *} C^{\odot *} = jL$. Set

$$\psi^{\odot\odot} = \int_{-\infty}^{0} T^{\odot}(-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds.$$

Inequality (6.12) and the definition of weak-star integrals yield

$$||\int_{-\infty}^{t} T^{\odot}(t-s)^* \,\mathrm{Pr}_L^{\odot *}(h(s)r^{\odot *})\, ds|| \le \frac{K_0 K_2 ||r^{\odot *}||}{u_0 - 2\delta} e^{(2u_0-\delta)t}, \quad t \le 0.$$

Therefore

$$ju(t) = \mathrm{Pr}_L^{\odot *} jT(t)u(0) - T^{\odot}(t)^* \psi^{\odot\odot} + O(e^{(2u_0-\delta)t}) \quad \text{as } t \to -\infty.$$

Using the relations $\mathrm{Pr}_L^{\odot *} jT(t) = jT(t)\,\mathrm{Pr}_L$, $j^{-1}T^{\odot}(t)^* = T(t)j^{-1}$, the fact that the term $O(e^{(2u_0-\delta)t})$ above is an element of $C^{\odot\odot}$ and applying j^{-1} we conclude

$$x_{t,C} = T(t)(\mathrm{Pr}_L\, x_{0,C} - j^{-1}\psi^{\odot\odot}) + O(e^{(2u_0-\delta)t}) \quad \text{as } t \to -\infty.$$

Since $\Pr_L x_{0,C} - j^{-1}\psi^{\odot\odot} \in L$, there exist reals a, b with

$$T(t)(\Pr_L x_{0,C} - j^{-1}\psi^{\odot\odot})(0) = e^{u_0 t}[a\cos(v_0 t) + b\sin(v_0 t)]$$

for all $t \in \mathbb{R}$. Consequently, the assertion holds with $\epsilon = u_0 - \delta$. $\square$

7. Sign changes for differences in $\mathcal{A}$

In this section we show that for two different elements ϕ and ψ of $\mathcal{A}$ and the corresponding solutions $x = x^\phi : \mathbb{R} \to [-B, A]$ and $y = y^\psi : \mathbb{R} \to [-B, A]$ we have

$$V(x - y, [t - r(x(t)), t]) = 1 \qquad \text{for all} \;\; t \in \mathbb{R}. \tag{7.1}$$

This fact is important in the proof of the injectivity of a map from $\mathcal{A}$ into $\mathbb{R}^2$ in Section 8.

We first remark that (7.1) implies

$$V(x - y, [t - r(y(t)), t]) = 1 \qquad \text{for all} \;\; t \in \mathbb{R}. \tag{7.2}$$

Indeed, if $x(t) - y(t) \neq 0$ for all large negative t, then $V(x-y, [t-r(y(t)), t]) = 1$ for all large negative t because of the definition of V. Then, by Lemma 2.5, the monotonicity property of V in Lemma 4.2(i) can be applied to get (7.2). If $x(t^n) - y(t^n) = 0$ for a sequence $\{t^n\}$ with $t^n \to -\infty$ as $n \to \infty$, then $r(x(t^n)) = r(y(t^n))$ and $V(x - y, [t^n - r(y(t^n)), t^n]) = 1$. Hence the monotonicity of V implies (7.2).

Proposition 7.1. $V(\phi - \psi, [-r(\phi(0)), 0]) = 1$ *for all* ϕ, ψ *in* $\mathcal{A}$ *with* $\phi \neq \psi$.

Proof. Let $\phi, \psi \in \mathcal{A}$ with $\phi \neq \psi$. Set $x = x^\phi$, $y = y^\psi$ and define $\eta : \mathbb{R} \ni t \mapsto t - r(x(t)) \in \mathbb{R}$. Recall, from Proposition 5.5, that: $x_t \neq y_t$ for all $t \in \mathbb{R}$, and x and y are either slowly oscillating or zero. It is also true that

$$(x - y)|_{[t-r(x(t)),t]} \not\equiv 0, \quad (x - y)|_{[t-r(y(t)),t]} \not\equiv 0 \qquad \text{for all } t \in \mathbb{R}.$$

Indeed, assume $(x - y)|_{[t-r(x(t)),t]} \equiv 0$ for some $t \in \mathbb{R}$. Let $t_0 = \inf\{s : \; x(u) = y(u) \text{ for all } s \le u \le t\}$. We have $t - R < t_0 \le t - r(x(t))$. Then $\dot{x}(s) + \mu x(s) = \dot{y}(s) + \mu y(s)$ and $r(x(s)) = r(y(s))$ for all $s \in [t_0, t]$. The equations for x, y and the injectivity of f imply

$$x(s - r(x(s))) = f^{-1}(\dot{x}(s) + \mu x(s)) = f^{-1}(\dot{y}(s) + \mu y(s)) = y(s - r(y(s))) = y(s - r(x(s)))$$

for all $s \in [t_0, t]$. Hence $x(u) = y(u)$ follows for all $u \in [\min\{s - r(x(s)) : t_0 \le s \le t\}, t]$. This contradicts the definition of t_0 since $\min\{r(u) : \; -R \le u \le 0\} > 0$.

In the remaining part of the proof we distinguish several cases and subcases.

Case 1: $\alpha(\phi) = \alpha(\psi) = \{0\}$. Either $\phi \neq 0$ or $\psi \neq 0$. We may assume $\phi \neq 0$ since, by the remark preceding the proposition, there is a symmetry in the role of ϕ and ψ. Then $x(t) \to 0$ as $t \to -\infty$. So, there exists a sequence $(t^n)_0^\infty$ in $(-\infty, 0]$ such that $t^n \to -\infty$ as $n \to \infty$ and

$$|x(t^n)| = \sup\{|x(t^n + t)| : \ t \leq 0\}.$$

Define

$$z^n : (-\infty, 0] \ni t \mapsto \frac{x(t^n + t)}{|x(t^n)|} \in \mathbb{R}.$$

The functions z^n satisfy

$$\dot{z}^n(t) = -\mu z^n(t) + \left(\int_0^1 f'(sx(\eta(t^n + t)))\, ds\right) \frac{x(\eta(t^n + t))}{|x(t^n)|} \quad \text{for all } t \leq 0,$$

and $|z^n(t)| \leq 1$ for all $t \leq 0$. There is a uniform bound for the right hand side of the differential equations for z^n, $n \in \mathbb{N}$, on $(-\infty, 0]$. Therefore $(z^n)_0^\infty$ is a uniformly bounded and equicontinuous sequence of functions on $(-\infty, 0]$. By the Arzèla–Ascoli theorem and the diagonalization process, there is subsequence $(z^{n_k})_{k=0}^\infty$ and a continuous function $z : (-\infty, 0] \to [-1, 1]$ such that

$$z^{n_k}(t) \to z(t) \ \text{ as } k \to \infty \text{ uniformly on compact subsets of } (-\infty, 0].$$

Using the differential equations for z^{n_k} we obtain that $(\dot{z}^{n_k})_{k=0}^\infty$ also converges to $\dot{z}$ uniformly on compact subsets of $(-\infty, 0]$. Moreover, from the fact $x(t) \to 0$ as $t \to -\infty$ it follows that

$$\int_0^1 f'(sx(\eta(t^n + t)))\, ds \to f'(0), \quad r(x(t^n + t)) \to 1 \ \text{ as } \ n \to \infty$$

uniformly in $(-\infty, 0]$. Consequently, z satisfies

$$\dot{z}(t) = -\mu z(t) + f'(0) z(t - 1) \qquad \text{for all } t \leq 0,$$

and $|z(0)| = 1$, $||z_t|| \leq 1$ for all $t \leq 0$. Then Lemma 3.1(ii) implies $u_0 \geq 0$, where u_0 denotes the maximum of the real parts of the points in the spectrum of the generator of the solution semiflow of Eq. (3.1).

Case 1.1: $u_0 = 0$. From $x(t) - y(t) \to 0$ as $t \to -\infty$, it follows that there exists a sequence $(t^n)_0^\infty$ in $(-\infty, 0]$ with $t^n \to -\infty$ as $n \to \infty$ and

$$|x(t^n) - y(t^n)| = \sup\{|x(t^n + t) - y(t^n + t)| : \ t \leq 0\}.$$

Define

$$z^n : (-\infty, 0] \ni t \mapsto \frac{x(t^n + t) - y(t^n + t)}{|x(t^n) - y(t^n)|} \in \mathbb{R}.$$

Then the functions z^n satisfy

$$\dot{z}^n(t) = a^n(t)z^n(t) + b^n(t)z^n(t - r(x(t^n + t))), \quad t \leq 0,$$

with

$$\begin{aligned} a^n :&(-\infty, 0] \ni t \mapsto \\ &- \mu - \int_0^1 f' \{[1-s]y(t^n + t - r(y(t^n + t))) + sy(t^n + t - r(x(t^n + t)))\} \, ds \\ &\times \int_0^1 \dot{y} \{[1-s](t^n + t - r(y(t^n + t))) + s(t^n + t - r(x(t^n + t)))\} \, ds \\ &\times \int_0^1 r' \{ \ [1-s]x(t^n + t) + sy(t^n + t)\} \, ds \in \mathbb{R}, \end{aligned}$$

$$\begin{aligned} b^n :&(-\infty, 0] \ni t \mapsto \\ &\int_0^1 f' \{[1-s]y(t^n + t - r(x(t^n + t))) + sx(t^n + t - r(x(t^n + t)))\} \, ds \in \mathbb{R}. \end{aligned}$$

From $x(t) \to 0, \ y(t) \to 0$ as $t \to -\infty$, it follows that

$$a^n(t) \to -\mu, \ b^n(t) \to f'(0), \ r(x(t^n + t)) \to 1 \ \text{ as } \ n \to \infty$$

uniformly in $(-\infty, 0]$. Then, in the same way as in Case 1, the Arzèla–Ascoli theorem can be applied to find a subsequence $(z^{n_k})_{k=0}^{\infty}$ of $(z^n)_0^{\infty}$ converging uniformly on compact subsets of $(-\infty, 0]$ to a continuously differentiable function $z : (-\infty, 0] \to \mathbb{R}$ satisfying

$$\begin{aligned} &\dot{z}(t) = -\mu z(t) + f'(0)z(t-1), \quad t \leq 0, \\ &||z_t|| \leq |z(0)| = 1, \quad t \leq 0. \end{aligned}$$

Moreover, $(\dot{z}^{n_k})_{k=0}^{\infty}$ converges uniformly on compact subsets of $(-\infty, 0]$ to $\dot{z}$. Then Lemma 3.1(iii) implies

$$V(z, [t-1, t]) = 1 \qquad \text{for all } t \leq 0.$$

Defining

$$w : (-\infty, 0] \ni t \mapsto e^{\mu t} z(t) \in \mathbb{R},$$

we have

$$\dot{w}(t) = e^{\mu} f'(0) w(t-1) \qquad \text{for all } t \leq 0,$$

and

$$V(w,[t-1,t]) = 1 \qquad \text{for all } t \le 0.$$

Hence Lemma 4.2(iii) implies $w|_{[t-1,t]} \in H|_{[t-1,t]}$ for all $t \le 0$. Then it follows easily that also $z|_{[t-1,t]} \in H|_{[t-1,t]}$ for all $t \le 0$. Thus Lemma 4.1(iii) can be used to get, for all sufficiently large $k \in \mathbb{N}$, that

$$V(z^{n_k}, [-r(x(t^{n_k})), 0]) = V(z,[-1,0]) = 1.$$

By Lemma 2.6, the differential equation for $x - y$ can be transformed to the form of Eq. (4.1), where $x(t) - y(t)$ and $v(t)$ have the same signs for all $t \le 0$. Hence, Lemma 4.2(i) yields

$$\begin{aligned} 1 &\le V(\phi - \psi, [-r(\phi(0)), 0]) \le V(x(t^{n_k} + \cdot) - y(t^{n_k} + \cdot), [-r(x(t^{n_k})), 0]) \\ &= V(z^{n_k}, [-r(x(t^{n_k})), 0]) = V(z, [-1, 0]) = 1 \end{aligned}$$

for all sufficiently large $k \in \mathbb{N}$. Thus, $V(\phi - \psi, [-r(\phi(0)), 0]) = 1$.

Case 1.2: $u_0 > 0$. In the case $\psi \ne 0$, also Propositions 5.5 and 6.1 imply that there exist real numbers $\epsilon_x > 0, \epsilon_y > 0, a, b, c, d$ such that with $\epsilon = \min\{\epsilon_x, \epsilon_y\}$ we have

$$\begin{aligned} x(t) &= e^{u_0 t}\left(a\cos(v_0 t) + b\sin(v_0 t)\right) + O(e^{(u_0+\epsilon)t}), \\ y(t) &= e^{u_0 t}\left(c\cos(v_0 t) + d\sin(v_0 t)\right) + O(e^{(u_0+\epsilon)t}) \end{aligned}$$

as $t \to -\infty$. In the case $\psi = 0$, $y \equiv 0$ and the above asymptotic expansions hold with $\epsilon = \epsilon_x$, $c = d = 0$, and a, b, ϵ_x given as above. If $(a,b) \ne (c,d)$, then

$$x(t) - y(t) = e^{u_0 t}\left(\sqrt{(a-c)^2 + (b-d)^2}\,\sin(v_0 t + \gamma) + O(e^{\epsilon t})\right)$$

for some $\gamma \in [-\pi, \pi]$ as $t \to -\infty$. For every integer k, defining

$$t^k = \frac{(k+\frac{1}{2})\pi - \gamma}{v_0} + \frac{1}{2},$$

we obtain

$$(k+\frac{1}{2})\pi - \frac{v_0}{2} \le v_0 t + \gamma \le (k+\frac{1}{2})\pi + \frac{v_0}{2} \qquad \text{for } t^k - 1 \le t \le t^k.$$

Using $v_0 \in (\frac{\pi}{2}, \pi)$, we find $\delta > 0$ such that, for every integer k,

$$\sqrt{(a-c)^2 + (b-d)^2}\,|\sin(v_0 t + \gamma)| \ge \delta, \qquad t^k - 1 - \delta \le t \le t^k.$$

If $t \to -\infty$, then $r(x(t)) \to 1$. So, for all sufficiently large negative integers k, $r(x(t^k)) < 1+\delta$ and

$$|x(t) - y(t)| > 0 \qquad \text{for } \ t^k - r(x(t^k)) \le t \le t^k.$$

Hence, the monotone property of V implies $V(\phi - \psi, [-r(\phi(0)), 0]) = 1$.

Now we show that the case $(a, b) = (c, d)$ is impossible. Assume $(a, b) = (c, d)$. Then there exists $K_0 > 0$ such that

$$|x(t) - y(t)| \le K_0 e^{(u_0+\epsilon)t}, \qquad t \le 0. \tag{7.3}$$

Then it is easy to see that there exists a sequence $(t^n)_0^\infty$ in $(-\infty, 0]$ such that $t^n \to -\infty$ as $n \to \infty$ and

$$|x(t^n + t) - y(t^n + t)| \le |x(t^n) - y(t^n)| e^{(u_0+\epsilon/2)t} \qquad \text{for all } t \le 0. \tag{7.4}$$

Define

$$z^n : (-\infty, 0] \ni t \mapsto \frac{x(t^n + t) - y(t^n + t)}{|x(t^n) - y(t^n)|} \in \mathbb{R}.$$

(7.4) implies that

$$|z^n(t)| \le e^{(u_0+\epsilon/2)t} \qquad \text{for all } t \le 0 \text{ and } n \in \mathbb{N}$$

and $|z^n(0)| = 1$. Similarly to Case 1.1, by the application of the Arzèla–Ascoli theorem, we find a subsequence of $(z^n)_0^\infty$ converging uniformly on compact subintervals of $(-\infty, 0]$ to a solution z of $\dot{z}(t) = -\mu z(t) + f'(0) z(t-1)$ with

$$|z(0)| = 1, \quad |z(t)| \le e^{(u_0+\epsilon/2)t} \qquad \text{for all } t \le 0.$$

This contradicts Lemma 3.1(iv), and the proof of Case 1.2 is completed.

Case 2: $\alpha(\phi) \ne \{0\}$. The compactness of $\mathcal{A}$ implies the existence of χ, ρ in $\mathcal{A}$ and a sequence $(t^n)_0^\infty$ in $(-\infty, 0]$ such that $\chi \ne 0$, $x_{t^n} \to \chi$, $y_{t^n} \to \rho$ and $t^n \to -\infty$ as $n \to \infty$. Let w, z denote the solutions of Eq. (2.1) with $w_0 = \chi$, $z_0 = \rho$.

If $\chi \ne \rho$, then it suffices to show that

$$V(w - z, [t - r(w(t)), t]) = 1$$

holds for some $t \le -4R$. Indeed, $V(w - z, [s - r(w(s)), s]) = 1$ for all $s \ge -4R$ implies, by Remark 4.5 and Lemma 4.2(i) and (iii), that

$$(w - z)|_{[-r(w(0)), 0]} \in H_{[-r(w(0)), 0]}.$$

Since the C and C^1 topologies in $\mathcal{A}$ are equivalent by Corollary 5.6(ii), it follows that

$$x_{t^n} - y_{t^n} \to w_0 - z_0 = \chi - \rho \qquad \text{as } \ n \to \infty$$

in the C^1-topology. Then Lemma 4.1(iii) implies

$$V(x - y, [t^n - r(x(t^n)), t^n]) = 1$$

for all sufficiently large $n \in \mathbb{N}$. Hence, by Lemma 4.2(i), $V(x - y, [t - r(x(t)), t]) = 1$ for all $t \in \mathbb{R}$, and in particular, $V(\phi - \psi, [-r(\phi(0)), 0]) = 1$.

Now we consider different subcases.

Case 2.1: $\chi \neq 0,\ \rho = 0$. Then $z \equiv 0$. From $\chi \neq 0$ it follows that $\chi \in \mathcal{A} \setminus \{0\}$, and thus Proposition 5.5 implies

$$V(w - z, [t - r(w(t)), t]) = V(w, [t - r(w(t)), t]) = 1, \quad t \in \mathbb{R}.$$

Case 2.2: $\chi \neq 0,\ \rho \neq 0,\ \chi \neq \rho$. By Lemma 2.2, w has arbitrarily large negative zeros. Lemma 4.2(iii), Remark 4.5 and Proposition 5.5 combined imply that w is slowly oscillating with simple zeros. If $t' < t'' \leq -4R$ are consecutive zeros of w and $w > 0$ in (t', t''), then $t'' - t' > 1$ and $r(w(t'')) = 1$. Then $w > 0$ on $[t - r(w(t)), t]$ for all $t < t''$ sufficiently close to t''. The function z is also slowly oscillating with arbitrarily large negative zeros. Consequently, there is $s < t'$ such that $z(u) < 0$ for all $[s - 1, s]$. In the case z has arbitrarily large zeros we find $\sigma > t''$ so that $z(u) < 0$ for all $[\sigma - 1, \sigma]$. Continuity of r, w, z allow to choose $t < t''$ sufficiently close to t'' so that the reals s, t, σ satisfy $s < t < -4R$, $t < \sigma$,

$$w(t + u) - z(s + u) > 0 \qquad \text{for all } u \in [-r(w(t)), 0], \tag{7.5}$$

and

$$w(t + u) - z(\sigma + u) > 0 \qquad \text{for all } u \in [-r(w(t)), 0]. \tag{7.6}$$

If $z(u) \neq 0$ for all large u, then $z(u) \to 0$ as $u \to \infty$ by Lemma 2.2. In this case fixing $t \in (t', t'')$ so that (7.5) and $w_t|_{[-r(w(t)),0]} > 0$ are satisfied, and choosing $\sigma > t$ with

$$\max_{u \in [-r(w(t)),0]} |z(\sigma + u)| < \min_{u \in [-r(w(t)),0]} w(t + u),$$

(7.6) holds.

Our aim is to show

$$V(w_t - z_t, [-r(w(t)), 0]) = 1.$$

Assume the contrary, that is

$$V(w_t - z_t, [-r(w(t)), 0]) > 1. \tag{7.7}$$

(7.5) and (7.6) imply

$$V(w_t - z_s, [-r(w(t)), 0]) = V(w_t - z_\sigma, [-r(w(t)), 0]) = 1.$$

Define

$$\begin{aligned}\epsilon_s = \sup\{\epsilon \geq 0 : &w_t|_{[-r(w(t)),0]} \neq z_{s+u}|_{[-r(w(t)),0]},\\ &V(w_t - z_{s+u}, [-r(w(t)), 0]) = 1 \text{ for all } 0 \leq u < \epsilon\}\end{aligned}$$

and

$$\begin{aligned}\epsilon_\sigma = \sup\{\epsilon \geq 0 : &w_t|_{[-r(w(t)),0]} \neq z_{\sigma-u}|_{[-r(w(t)),0]},\\ &V(w_t - z_{\sigma-u}, [-r(w(t)), 0]) = 1 \text{ for all } 0 \leq u < \epsilon\}.\end{aligned}$$

(7.5) and (7.6) imply $\epsilon_s > 0$ and $\epsilon_\sigma > 0$. We also have $\epsilon_s < t - s$ and $\epsilon_\sigma < \sigma - t$, since the set

$$\{u \in \mathbb{R} : w_t|_{[-r(w(t)),0]} \neq z_u|_{[-r(w(t)),0]},\ V(w_t - z_u, [-r(w(t)), 0]) > 1\}$$

is open by Lemma 4.1(i), and t belongs to this set by (7.7).

Case 2.2.1: $w_t|_{[-r(w(t)),0]} \neq z_{s+\epsilon_s}|_{[-r(w(t)),0]}$. Lemma 4.1(i) implies

$$V(w_t - z_{s+\epsilon_s}, [-r(w(t)), 0]) = 1.$$

Then, by Lemma 4.2(i),

$$V(w_{t+\tau} - z_{s+\epsilon_s+\tau}, [-r(w(t+\tau)), 0]) = 1 \qquad \text{for all } \tau \geq 0.$$

Fix $\tau \geq 4R$. Lemma 4.2(iii) yields

$$w_{t+\tau} - z_{s+\epsilon_s+\tau} \in H_{[-r(w(t+\tau)),0]}.$$

Using Lemma 4.2(iii), we find $\gamma > 0$ such that

$$c < 0, \quad |-r(w(t+\tau)) - c| < \gamma, \quad \kappa \in C^1([c, 0], \mathbb{R}),$$

$$\max_{u\in[c,0]} |w(t+\tau+u) - z(s+\epsilon_s+\tau+u) - \kappa(u)| + \max_{u\in[c,0} |\dot{w}(t+\tau+u) - \dot{z}(s+\epsilon_s+\tau+u) - \dot{\kappa}(u)| < \gamma$$

imply

$$V(\kappa, [c, 0]) = 1.$$

We claim that there exist $\beta_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$V(x_{t+\tau+t^n} - y_{s+\epsilon_s+\tau+\beta+t^n}, [-r(x(t+\tau+t^n)), 0]) = 1 \tag{7.8}$$

for all integers $n \geq n_1$ and $\beta \in [0, \beta_1]$.

Since $z : \mathbb{R} \to [-B, A]$ is a solution of Eq. (2.1), we obtain that $\dot{z}$ and $\ddot{z}$ are bounded functions on $\mathbb{R}$. Hence it follows that

$$\sup_{u\in\mathbb{R}} |z(u) - z(u+\beta)| \to 0, \qquad \sup_{u\in\mathbb{R}} |\dot{z}(u) - \dot{z}(u+\beta)| \to 0$$

as $\beta \to 0$. Choose $\beta_1 > 0$ such that

$$\sup_{u\in\mathbb{R}} |z(u) - z(u+\beta)| + \sup_{u\in\mathbb{R}} |\dot{z}(u) - \dot{z}(u+\beta)| < \frac{\gamma}{3}$$

and $w_t \neq z_{s+\epsilon_s+\beta}$ for all $\beta \in [0, \beta_1]$.

We have

$$\begin{aligned} &x(t^n+u) \to w(u),\ \dot{x}(t^n+u) \to \dot{w}(u) \qquad \text{as } n \to \infty,\\ &y(t^n+u) \to z(u),\ \dot{y}(t^n+u) \to \dot{z}(u) \qquad \text{as } n \to \infty \end{aligned}$$

uniformly on compact subsets of $\mathbb{R}$. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &||w_{t+\tau} - x_{t+\tau+t^n}||_{C^1} < \frac{\gamma}{3},\\ &||z_{s+\epsilon_s+\tau+\beta} - y_{s+\epsilon_s+\tau+\beta+t^n}||_{C^1} < \frac{\gamma}{3} \end{aligned}$$

for all integers $n \geq n_0$ and all $\beta \in [0, \beta_1]$. Let us choose $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and

$$|r(x(t+\tau+t^n)) - r(w(t+\tau))| < \gamma \qquad \text{for all integers } n \geq n_1.$$

Fixing $n \geq n_1$, $\beta \in [0, \beta_1]$, and choosing $c = -r(x(t+\tau+t^n))$ and $\kappa(u) = x(t+\tau+t^n+u) - y(s+\epsilon_s+\tau+\beta+t^n+u)$, we obtain $V(\kappa, [c, 0]) = 1$. Consequently, (7.8) holds for all integers $n \geq n_1$ and all $\beta \in [0, \beta_1]$.

Now, for any $n \geq n_1$, pick $k \in \mathbb{N}$ such that $k \geq n_1$ and $t^n - t^k - \tau \geq 0$. There is such a k since $t^k \to -\infty$ as $k \to \infty$. Then $t + \tau + t^k \leq t + t^n$. Therefore, using (7.8) and Lemma 4.2(i),

$$\begin{aligned} 1 &= V(x_{t+\tau+t^k} - y_{s+\epsilon_s+\tau+\beta+t^k}, [-r(x(t+\tau+t^k)), 0])\\ &\geq V(x_{t+t^n} - y_{s+\epsilon_s+\beta+t^n}, [-r(x(t+t^n)), 0]), \end{aligned}$$

for all integers $n \geq n_1$ and all $\beta \in [0, \beta_1]$. Hence Lemma 4.1(i) implies

$$V(w_t - z_{s+\epsilon_s+\beta}, [-r(w(t)), 0]) = 1$$

for all $0 \le \beta \le \beta_1$. This contradicts the definition of ϵ_s.

Case 2.2.2: $w_t|_{[-r(w(t)),0]} \neq z_{\sigma-\epsilon_\sigma}|_{[-r(w(t)),0]}$. We get a contradiction analogously to Case 2.2.1. We have

$$V(w_t - z_{\sigma-\epsilon_\sigma}, [-r(w(t)),0]) = 1$$

and for fixed $\tau \ge 4R$

$$w_{t+\tau} - z_{\sigma-\epsilon_\sigma+\tau} \in H_{[-r(w(t+\tau)),0])}.$$

The application of Lemma 4.1(iii) gives $\beta_2 > 0$ and $n_2 \in \mathbb{N}$ such that

$$V(x_{t+\tau+t^n} - y_{\sigma-\epsilon_\sigma+\tau+t^n-\beta}, [-r(x(t+\tau+t^n)),0]) = 1$$

for all $n \ge n_2$ and all $\beta \in [0,\beta_2]$. Hence, in the same way as in Case 2.2.1, the monotonicity of V implies

$$V(x_{t+t^n} - y_{\sigma-\epsilon_\sigma+t^n-\beta}, [-r(x(t+t^n)),0]) = 1$$

for all $n \ge n_2$ and all $\beta \in [0,\beta_2]$. The lower semicontinuity of V gives

$$V(w_t - z_{\sigma-\epsilon_\sigma-\beta}, [-r(w(t)),0]) = 1$$

for all $\beta \in [0,\beta_2]$, contradicting the definition of ϵ_σ.

Case 2.2.3: $w_t|_{[-r(w(t)),0]} = z_{s+\epsilon_s}|_{[-r(w(t)),0]} = z_{\sigma-\epsilon_\sigma}|_{[-r(w(t)),0]}$. In this case z is periodic since z is determined by $z_{s+\epsilon_s}|_{[-r(z(s+\epsilon_s)),0]} = z_{s+\epsilon_s}|_{[-r(w(t)),0]}$ and $s+\epsilon_s < t < \sigma-\epsilon_\sigma$. w is also periodic since it is a translate of z. Using also $w_t \neq z_t$, it follows that $z_t = w_{t-\tau}$ for some $\tau \in (0,p)$, where p is the minimal period of w. Proposition 5.11(ii) yields

$$V(w_t - z_t, [-r(w(t)),0]) = 1 \qquad \text{for all } t \in \mathbb{R}.$$

Case 2.3: $\chi \neq 0$, $\chi = \rho$. Then $w \not\equiv 0$ is a slowly oscillating solution. Either w is not periodic or w is periodic with minimal period $p > 0$. Let $\epsilon_0 > 0$ be arbitrary if w is not periodic, otherwise choose $\epsilon_0 \in (0,p)$. Then

$$w_\epsilon \neq w_0 \qquad \text{for } 0 < \epsilon < \epsilon_0.$$

For $0 < \epsilon < \epsilon_0$, define $x^\epsilon : \mathbb{R} \ni t \mapsto x(\epsilon+t) \in \mathbb{R}$. Then

$$x^\epsilon_{t^n} \to w_\epsilon, \qquad y_{t^n} \to z_0$$

as $n \to \infty$. We have

$$w_\epsilon \neq 0, \quad z_0 \neq 0, \quad w_\epsilon \neq z_0 \qquad \text{for } 0 < \epsilon < \epsilon_0.$$

Therefore, replacing x and χ with x^ϵ and w_ϵ, respectively, Case 2.2 can be applied to obtain

$$V(x^\epsilon - y, [t - r(x^\epsilon(t)), t]) = 1 \qquad \text{for all } t \in \mathbb{R}.$$

We have $\sup_{u\in\mathbb{R}} |x^\epsilon(u) - x(u)| \to 0$ as $\epsilon \to 0$. Thus, the lower semicontinuity of V in Lemma 4.1(i) implies

$$V(x - y, [t - r(x(t)), t]) = 1 \qquad \text{for all } t \in \mathbb{R},$$

and in particular

$$V(\phi - \psi, [-r(\phi(0)), 0]) = 1.$$

□

8. The Poincaré–Bendixson theorem on $\mathcal{A}$

Recall from Proposition 5.5 that for every $\phi \in \mathcal{A}$ there is a unique phase curve $\mathbb{R} \ni t \mapsto x_t^\phi \in L_K$ in $\mathcal{A}$ and $\omega(\phi)$, $\alpha(\phi)$ are nonempty compact connected and invariant subsets of $\mathcal{A}$.

Theorem 8.1. *For every $\phi \in \mathcal{A}$, either $\omega(\phi) = \{0\}$, or $0 \notin \omega(\phi)$ and $\omega(\phi)$ is a slowly oscillating periodic orbit; either $\alpha(\phi) = \{0\}$, or $0 \notin \alpha(\phi)$ and $\alpha(\phi)$ is a slowly oscillating periodic orbit. If $\phi \in \mathcal{A}$ and x^ϕ is neither identically zero nor periodic, then $\alpha(\phi) \cap \omega(\phi) = \emptyset$.*

Proof. Define

$$h : \mathcal{A} \ni \phi \mapsto \begin{pmatrix} \phi(0) \\ \phi(-r(\phi(0))) \end{pmatrix} \in \mathbb{R}^2.$$

Clearly, h is continuous. Let ϕ, ψ be in $\mathcal{A}$ with $\phi \neq \psi$, and set $x = x^\phi$, $y = y^\psi$. Then x_t, y_t are in $\mathcal{A}$ and $x_t \neq y_t$ for all $t \in \mathbb{R}$. Proposition 7.1 implies

$$V(x - y, [t - r(x(t)), t]) = 1 \qquad \text{for all } t \in \mathbb{R}.$$

Then it can be easily shown, by applying Lemma 4.2(ii), that

$$(x(t) - y(t), x(t - r(x(t))) - y(t - r(x(t)))) \neq (0, 0) \qquad \text{for all } t \in \mathbb{R}.$$

In particular

$$h(\phi) \neq \begin{pmatrix} \psi(0) \\ \psi(-r(\phi(0)) \end{pmatrix}.$$

If $\phi(0) \neq \psi(0)$, then $h(\phi) \neq h(\psi)$. If $\phi(0) = \psi(0)$, then $\psi(-r(\phi(0))) = \psi(-r(\psi(0)))$. Therefore,

$$h(\phi) \neq h(\psi).$$

So, h is injective. Since $\mathcal{A}$ is compact, it follows that $h(\mathcal{A})$ is also compact and h is a homeomorphism.

For each $\xi^0 \in h(\mathcal{A})$, there is a unique $\phi \in \mathcal{A}$ with $h(\phi) = \xi^0$. The unique solution x^ϕ of Eq. (2.1) gives the continuous curve $\xi : \mathbb{R} \ni t \mapsto h(x_t^\phi) \in h(\mathcal{A}) \subset \mathbb{R}^2$. We call ξ the canonical curve through ξ^0. The canonical curves are C^1-curves since the mapping

$$\mathbb{R} \ni t \mapsto \frac{d}{dt} h(x_t^\phi) = \begin{pmatrix} \dot{x}^\phi(t) \\ \dot{x}^\phi(t - r(x^\phi(t)))[1 - r'(x^\phi(t))\dot{x}^\phi(t)] \end{pmatrix} \in \mathbb{R}^2$$

is continuous.

Define

$$v_+ = \{(u, v)^{tr} \in \mathbb{R}^2 : \ u = 0, v > 0\}.$$

Let $\phi \in \mathcal{A}$, $x = x^\phi$ and assume that, for some $t \in \mathbb{R}$, $h(x_t) \in v_+$, that is, $x(t) = 0$ and $x(t - r(x(t))) = x(t-1) > 0$. Then $\phi \neq 0$ and Proposition 5.5 implies that x is a slowly oscillating solution. This fact, $x(t-1) > 0$ and $x(t) = 0$ combined imply $x_t \in \mathcal{A} \cap U \setminus \{0\}$. It also follows that $\dot{x}(t) < 0$. Hence we obtain that v_+ is transversal to the canonically determined curves in the following sense:

$$\langle (1, 0), \frac{d}{dt} h(x_t) \rangle = \dot{x}(t) < 0.$$

This implies that if $h(x_t^\phi) \in v_+$ for some $\phi \in \mathcal{A} \setminus \{0\}$ and $t \in \mathbb{R}$, then there exists $\epsilon > 0$ such that $h(x_s^\phi)$ belongs to the first quadrant of $\mathbb{R}^2$ for all $s \in (t - \epsilon, t)$ and to the second quadrant of $\mathbb{R}^2$ for all $s \in (t, t + \epsilon)$.

If $\phi = 0$ then $\alpha(\phi) = \omega(\phi) = \{0\}$. Let $\phi \in \mathcal{A} \setminus \{0\}$. Proposition 5.10 implies that there are $t \in \mathbb{R}$ and $\psi \in \mathcal{A} \cap U \setminus \{0\}$ such that $\psi = x_t^\phi$. Clearly, $\alpha(\phi) = \alpha(\psi)$ and $\omega(\phi) = \omega(\psi)$. Thus, it is enough to prove the statement of the theorem for $\phi \in \mathcal{A} \cap U \setminus \{0\}$.

Let $\phi \in \mathcal{A} \cap U \setminus \{0\}$ and $\xi^0 = h(\phi)$. Let ξ denote the canonical curve through ξ^0. As x^ϕ is a slowly oscillating solution by Proposition 5.5, we find

$$h(\mathcal{A} \cap U \setminus \{0\}) = v_+ \cap h(\mathcal{A}). \tag{8.1}$$

Hence it follows that $\xi(t) \in v_+$ if and only if $h^{-1}(\xi(t)) = x_t^\phi \in \mathcal{A} \cap U \setminus \{0\}$. Proposition 5.10 yields that, for each $t \in \mathbb{R}$,

$$\xi(t) \in v_+ \ \text{ if and only if } \ t \in \{z_{2j}(\phi) : j \in \mathbb{Z}, 2j \leq J(\phi)\},$$

where $(z_j(\phi))_{-\infty}^{J(\phi)}$ is the zeroset of x^ϕ. Let

$$\xi^j = \xi(z_{2j}(\phi)) = h(x^\phi_{z_{2j}(\phi)}) \ \text{ for } j \in \mathbb{Z}, \ 2j \leq J(\phi),$$

and define s^j so that $x^{\phi}_{s^j} = h^{-1}(\xi^j)$, or equivalently $\xi(s^j) = \xi^j$. $(\xi^j)_{-\infty}^{J^*}$ is a sequence in $h(\mathcal{A} \cap U \setminus \{0\}) \subset v_+$, where $J^* = \infty$ if $J(\phi) = \infty$, and $J^* \in \mathbb{Z}$ if $J(\phi) \in \mathbb{Z}$. Clearly, the sequence $(s^j)_{-\infty}^{J^*}$ is increasing.

The sequence $(\xi^j)_{-\infty}^{J^*}$ is monotone with respect to the natural ordering $<_v$ of $\{(u,v)^{tr} \in \mathbb{R}^2 : u = 0\}$. Indeed this follows from the Jordan curve theorem and the facts that $F_{\mathcal{A}}$ is a flow on $\mathcal{A}$ and h is a homeomorphism of $\mathcal{A}$ onto $h(\mathcal{A})$.

Define

$$\xi_{-\infty} = \lim_{j \to -\infty} \xi^j,$$

$$\xi_{\infty} = \begin{cases} \lim_{j\to\infty} \xi^j & \text{if } J^* = \infty, \\ (0,0)^{tr} & \text{if } J^* \in \mathbb{Z}. \end{cases}$$

$\xi_{-\infty}, \xi_{\infty} \in h(\mathcal{A} \cap U) \subset v_+ \cup \{(0,0)^{tr}\}$ since $h(\mathcal{A} \cap U)$ is a compact subset of $\mathbb{R}^2$. Now we need the following two claims.

CLAIM 1. (i) If $\xi^j \to \bar{\xi}$ as $j \to -\infty$ and $\bar{\xi} \in v_+$, then $\bar{\xi} \in h(\mathcal{A} \cap U) \setminus \{0\}$, $x^{h^{-1}(\bar{\xi})}$ is a slowly oscillating periodic solution of (2.1) and $\alpha(\phi) = \{x_t^{h^{-1}(\bar{\xi})} : t \in \mathbb{R}\}$.

(ii) If $J^* = \infty$ and $\xi^j \to \hat{\xi}$ as $j \to \infty$ and $\hat{\xi} \in v_+$, then $\hat{\xi} \in h(\mathcal{A} \cap U) \setminus \{0\}$, $x^{h^{-1}(\hat{\xi})}$ is a slowly oscillating periodic solution of (2.1) and $\omega(\phi) = \{x_t^{h^{-1}(\hat{\xi})} : t \in \mathbb{R}\}$.

Proof of Claim 1. Suppose $\xi^j \to \bar{\xi}$ as $j \to -\infty$ and $\bar{\xi} \in v_+$. We have $\xi^j, \bar{\xi} \in h(\mathcal{A} \cap U \setminus \{0\})$ for all integers $j \leq J^*$. Proposition 5.10 implies that

$$P^{-1}(h^{-1}(\xi^j)) = h^{-1}(\xi^{j-1}) \qquad \text{for all integers } j \leq J^*.$$

Using that $h^{-1}(\xi^j), h^{-1}(\bar{\xi}) \in \mathcal{A} \cap U \setminus \{0\}$ and that h^{-1} is continuous on $h(\mathcal{A})$ and P^{-1} is continuous on $\mathcal{A} \cap U \setminus \{0\}$, by letting $j \to -\infty$, we obtain

$$P^{-1}(h^{-1}(\bar{\xi})) = h^{-1}(\bar{\xi}).$$

Therefore, $h^{-1}(\bar{\xi}) = P(h^{-1}(\bar{\xi})) = F(q_2(h^{-1}(\bar{\xi})), h^{-1}(\bar{\xi}))$ and thus $x^{h^{-1}(\bar{\xi})}$ is $q_2(h^{-1}(\bar{\xi}))$-periodic. Proposition 5.1 implies that $q_2 = q_2(h^{-1}(\bar{\xi}))$ is the minimal period and $x^{h^{-1}(\bar{\xi})}$ is slowly oscillating. Let $O = \{x_t^{h^{-1}(\bar{\xi})} : 0 \leq t \leq q_2\}$. We have to show that $\operatorname{dist}(x_t^{\phi}, O) \to 0$ as $t \to -\infty$. Let $\epsilon > 0$ be given. From Lemma 2.4 and Proposition 5.1 it follows that there exists $\delta = \delta(\epsilon) > 0$ so that for every $\psi \in \mathcal{A} \cap U \setminus \{0\}$ with $||\psi - h^{-1}(\bar{\xi})|| < \delta$ we have

$$\operatorname{dist}(x_s^{\psi}, O) < \epsilon \qquad \text{for all } s \in [0, q_2 + 1]$$

and

$$|q_2(\psi) - q_2| < 1.$$

There is $j_0 \in \mathbb{Z}$ such that

$$||h^{-1}(\xi^j) - h^{-1}(\bar{\xi})|| < \delta \qquad \text{for all integers } j \leq j_0.$$

Let $t < s^{j_0}$. Choose $j_1 \in \mathbb{Z}$ so that $j_1 < j_0$ and $s^{j_1} \leq t < s^{j_1+1} \leq s^{j_0}$. By the choice of j_0, $||h^{-1}(\xi^{j_1}) - h^{-1}(\bar{\xi})|| < \delta$. Hence, using also $h^{-1}(\xi^{j_1}) \in \mathcal{A} \cap U \setminus \{0\}$, it follows that

$$q_2(h^{-1}\xi^{j_1})) < q_2 + 1$$

and

$$\mathrm{dist}(x_s^{h^{-1}(\xi^{j_1})}, O) < \epsilon \qquad \text{for all } s \in [0, q_2 + 1].$$

From $x^\phi_{s^{j_1+1}} = F(q_2(h^{-1}(\xi^{j_1})), x^\phi_{s^{j_1}})$, we obtain $t - s^{j_1} < q_2(h^{-1}(\xi^{j_1})) < q_2 + 1$. Consequently,

$$\mathrm{dist}(x_t^\phi, O) < \epsilon.$$

As $\epsilon > 0$ was arbitrary, $\alpha(\phi) = O$ follows, and the proof of assertion (i) in Claim 1 is complete. The proof of assertion (ii) is analogous.

CLAIM 2. (i) If $\xi^j \to (0,0)^{tr}$ as $j \to -\infty$, then $\alpha(\phi) = \{0\}$.
(ii) If $J^* = \infty$ and $\xi^j \to (0,0)^{tr}$ as $j \to \infty$, then $\omega(\phi) = \{0\}$.

Proof of Claim 2. Assume $\xi^j \to (0,0)^{tr}$ as $j \to -\infty$. By Lemma 2.4, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ so that $\psi \in L_K$ and $||\psi|| < \delta$ imply $||x_s^\psi|| < \epsilon$ for all $s \in [0, R]$. Let $\epsilon > 0$ be fixed. Choose $j_0 \in \mathbb{Z}$ such that $||h^{-1}(\xi^j)|| < \delta(\delta(\epsilon))$ for all integers $j \leq j_0$. Let $t < s^{j_0}$. Choose $j_1 \in \mathbb{Z}$ so that $j_1 < j_0$ and $s^{j_1} \leq t < s^{j_1+1} \leq s^{j_0}$. We have $||h^{-1}(\xi^{j_1})|| < \delta(\delta(\epsilon))$. It follows that $||x_s^{h^{-1}(\xi^{j_1})}|| < \delta(\epsilon)$ for $0 \leq s \leq R$. We state

$$||F(s, h^{-1}(\xi^{j_1}))|| < \delta(\epsilon) \qquad \text{for } 0 \leq s \leq q_1(h^{-1}(\xi^{j_1})). \tag{8.2}$$

The case $q_1(h^{-1}(\xi^{j_1})) \leq R$ is obvious. Assume $q_1(h^{-1}(\xi^{j_1})) > R$. Propositions 5.1 and 5.10 imply $F(s, h^{-1}(\xi^{j_1}))(0) < 0$ for all $s \in (0, q_1(h^{-1}(\xi^{j_1})))$. Using Eq. (2.1) and (H1), it follows that the function $[0,\infty) \ni s \mapsto F(s, h^{-1}(\xi^{j_1}))(0) \in \mathbb{R}$ is increasing on the interval $[R, q_1(h^{-1}(\xi^{j_1}))]$. Therefore (8.2) holds. (8.2) implies

$$||F(s, h^{-1}(\xi^{j_1}))|| < \epsilon \qquad \text{for } 0 \leq s \leq q_1(h^{-1}(\xi^{j_1})) + R.$$

Hence, similarly to the proof of (8.2), we obtain

$$||F(s, h^{-1}(\xi^{j_1}))|| < \epsilon \qquad \text{for } q_1(h^{-1}(\xi^{j_1})) \leq s \leq q_2(h^{-1}(\xi^{j_1})). \tag{8.3}$$

Since $\delta(\epsilon) \le \epsilon$, (8.2) and (8.3) yield

$$||F(s, h^{-1}(\xi^{j_1}))|| < \epsilon \qquad \text{for } 0 \le s \le q_2(h^{-1}(\xi^{j_1})).$$

Observing that $t - s^{j_1} < q_2(h^{-1}(\xi^{j_1}))$ follows from $s^{j_1} \le t < s^{j_1+1}$, we conclude

$$||x_t^{\phi}|| = ||F(t - s^{j_1}, h^{-1}(\xi^{j_1}))|| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $x_t^{\phi} \to 0$ as $t \to -\infty$, and thus $\alpha(\phi) = \{0\}$. The proof of assertion (ii) is analogous.

According to the relation between $\xi_{-\infty}$ and ξ_{∞}, we consider 6 cases.

Case 1: $\xi_{-\infty} = \xi_{\infty} = (0,0)^{tr}$. We show that this case cannot occur. Assume $J^* < \infty$. Lemma 2.2 implies $x^{\phi}(t) \to 0$ as $t \to \infty$. Hence $\xi(t) \to (0,0)^{tr}$ as $t \to \infty$. From this fact and the Jordan curve theorem it follows that $(\xi^j)_{-\infty}^{J^*}$ is strictly decreasing. Consequently, $(0,0)^{tr} <_v \xi_{-\infty}$, a contradiction. So, $J^* = \infty$. As $(\xi^j)_{-\infty}^{\infty}$ is a monotone sequence in v_+, $\xi_{-\infty} = \xi_{\infty} = (0,0)^{tr}$ is impossible. Therefore this case cannot occur.

Case 2: $(0,0)^{tr} = \xi_{\infty} \neq \xi_{-\infty}$. There are two subcases.

Case 2.1: $(0,0)^{tr} = \xi_{\infty} \neq \xi_{-\infty}$ and $J^* < \infty$. Claim 1 gives that $\alpha(\phi)$ is a slowly oscillating periodic orbit. $\omega(\phi) = \{0\}$ follows from Lemma 2.2.

Case 2.2: $(0,0)^{tr} = \xi_{\infty} \neq \xi_{-\infty}$ and $J^* = \infty$. Claims 1 and 2 imply that $\alpha(\phi)$ is a slowly oscillating periodic orbit and $\omega(\phi) = \{0\}$.

In the remaining cases $(0,0)^{tr} <_v \xi_{\infty}$ which implies $J^* = \infty$.

Case 3: $(0,0)^{tr} = \xi_{-\infty} <_v \xi_{\infty}$. Applying Claims 1 and 2, we get $\alpha(\phi) = \{0\}$ and that $\omega(\phi)$ is a slowly oscillating periodic orbit.

Case 4: $(0,0)^{tr} <_v \xi_{\infty} <_v \xi_{-\infty}$. In this case both $\alpha(\phi)$ and $\omega(\phi)$ are slowly oscillating periodic orbits by Claim 1. Proposition 5.11 implies that the intersection of a slowly oscillating periodic orbit with $\mathcal{A} \cap U$ is a single point. As $h^{-1}(\xi_{\infty})$ and $h^{-1}(\xi_{-\infty})$ are different points of $\mathcal{A} \cap U$, it follows that $\alpha(\phi) \cap \omega(\phi) = \emptyset$.

Case 5: $(0,0)^{tr} <_v \xi_{-\infty} <_v \xi_{\infty}$. Analogously to Case 4, $\alpha(\phi)$ and $\omega(\phi)$ are slowly oscillating periodic orbits with $\alpha(\phi) \cap \omega(\phi) = \emptyset$.

Case 6: $(0,0)^{tr} <_v \xi_{-\infty} = \xi_{\infty}$. In this case $\xi^j = \xi_{-\infty} = \xi_{\infty}$ for all $j \in \mathbb{Z}$. Claim 1 implies that x^{ϕ} is a slowly oscillating periodic solution.

Observe that, by the uniqueness of the zero solution in $\mathcal{A}$, a slowly oscillating periodic orbit does not contain 0. The proof is complete. □

9. $\mathcal{A}$ is homeomorphic to the closed unit disk

Finally, we prove a topological property of $\mathcal{A}$ provided $\mathcal{A}$ is different from $\{0\}$.

A sufficient (and by Theorem 8.1 also necessary) condition for $\mathcal{A} \neq \{0\}$ is the existence of a slowly oscillating periodic solution. From [41] it can be obtained that if

$$f'(0) < \frac{\mu}{\cos(v(\mu))},$$

where $v(\mu) \in (\frac{\pi}{2}, \pi)$ is the solution of $v = -\mu \tan v$, then Eq. (2.1) has a slowly oscillating periodic solution, and consequently, $\mathcal{A} \neq \{0\}$.

Theorem 9.1. *Assume that $\mathcal{A} \neq \{0\}$. Then there exists a slowly oscillating periodic solution y with minimal period $\tau > 0$ such that the simple closed curve $\eta : [0, \tau] \to y_t \in L_K$ with trace in $\mathcal{A}$ satisfies*

$$h(\mathcal{A}) = \overline{\operatorname{int}(h \circ \eta)}.$$

Consequently, $\mathcal{A}$ is homeomorphic to the 2-dimensional closed unit disk so that the unit circle corresponds to a slowly oscillating periodic orbit.

Proof. 1. From (8.1) and $h(0) = (0,0)^{tr}$, it follows that

$$h(\mathcal{A} \cap U) = (v_+ \cup \{(0,0)^{tr}\}) \cap h(\mathcal{A}). \tag{9.1}$$

Recall that $\mathcal{A} \cap U$ is a connected set by Corollary 5.8. From these facts and $\mathcal{A} \neq \{0\}$ we obtain the existence of $v^* \in v_+$ with

$$h(\mathcal{A} \cap U) = \{sv^* : 0 \leq s \leq 1\}. \tag{9.2}$$

Set $y = x^{h^{-1}(v^*)}$.

We claim that y is periodic. Let $(\xi^j)_{-\infty}^{J^*}$ be the monotone sequence of intersections with v_+ of the canonical curve ξ through v^* as defined in the proof of Theorem 8.1. We have $\xi^0 = v^*$. The sequence $(\xi^j)_{-\infty}^{J^*}$ is either constant or strictly increasing since $(1,\infty)v^* \cap h(\mathcal{A}) = \emptyset$ by (9.1) and (9.2). Define $\xi_{-\infty}$ and ξ_∞ as in the proof of Theorem 8.1. If $(\xi^j)_{-\infty}^{J^*}$ is strictly increasing, then necessarily $J^* < \infty$ and thus $\xi_\infty = (0,0)^{tr}$ since $\xi^0 = v^*$ and $(1,\infty)v^* \cap h(\mathcal{A}) = \emptyset$. On the other hand, by the increasing property of $(\xi^j)_{-\infty}^{J^*}$, we have $(0,0)^{tr} = \xi_\infty >_v \xi_{-\infty} \geq_v (0,0)^{tr}$, a contradiction. Therefore, $(\xi^j)_{-\infty}^{J^*}$ is a constant sequence and $J^* = \infty$. Analogously to Case 6 of the proof of Theorem 8.1, we conclude that y is a slowly oscillating periodic solution.

2. Let $\tau > 0$ denote the minimal period of y, and set $\eta : [0, \tau] \ni t \mapsto y_t \in L_K$. Propositions 5.10, 5.11 and the fact that h is a homeomorphism combined imply that the curve $h \circ \eta$ is simple closed and has values in $h(\mathcal{A}) \setminus \{(0,0)^{tr}\}$. Let $\operatorname{ext}(h \circ \eta)$ and $\operatorname{int}(h \circ \eta)$ denote the unbounded and bounded components of $\mathbb{R}^2 \setminus |h \circ \eta|$, respectively. Using that the only intersection of $h \circ \eta$ with $v_+ \cup \{(0,0)^{tr}\}$ is v^* and $(1,\infty)v^*$ is unbounded, it follows

that $(1,\infty)v^* \subset \operatorname{ext}(h \circ \eta)$. Moreover, since the intersection of $h \circ \eta$ with $v_+ \cup \{(0,0)\}$ is transversal at v^*, it also follows that

$$[0,1)v^* \subset \operatorname{int}(h \circ \eta).$$

In particular, $(0,0)^{tr} \in \operatorname{int}(h \circ \eta)$.

3. We claim that $\operatorname{ext}(h \circ \eta) \cap h(\mathcal{A}) = \emptyset$ Suppose that there exists a point $\chi \in h(\mathcal{A}) \cap \operatorname{ext}(h\circ\eta)$. Then $\chi \neq (0,0)^{tr}$ and $h^{-1}(\chi) \neq 0$. Proposition 5.10 implies that the phase curve $\mathbb{R} \ni t \mapsto x_t^{h^{-1}(\chi)}$ intersects $\mathcal{A} \cap U$. Then the canonical curve $\xi : \mathbb{R} \ni t \mapsto h(x_t^{h^{-1}(\chi)}) \in \mathbb{R}^2$ through χ intersects $h(\mathcal{A} \cap U) = \{sv^* : 0 \leq s \leq 1\} \subset \operatorname{int}(h \circ \eta) \cup |h \circ \eta|$. On the other hand, $\chi \in \operatorname{ext}(h \circ \eta)$ implies that $\xi(t) \in \operatorname{ext}(h \circ \eta)$ for all $t \in \mathbb{R}$. This is a contradiction.

4. We remark that $h(\mathcal{A}) \cap (\operatorname{int}(h \circ \eta) \setminus \{(0,0)^{tr}\})$ is closed in $\operatorname{int}(h \circ \eta) \setminus \{(0,0)^{tr}\}$ in the relative topology of this set.

We claim that $h(\mathcal{A}) \cap (\operatorname{int}(h \circ \eta) \setminus \{(0,0)^{tr}\})$ is also open in $\operatorname{int}(h \circ \eta) \setminus \{(0,0)^{tr}\}$ in the relative topology. Let $\chi \in h(\mathcal{A}) \cap (\operatorname{int}(h \circ \eta) \setminus \{(0,0)^{tr}\})$. We have to show that there is an open disk D in $\mathbb{R}^2$ containing χ so that $D \subset h(\mathcal{A}) \cap (\operatorname{int}(h \circ \eta) \setminus \{(0,0)^{tr}\})$. As $h^{-1}(\chi) \in \mathcal{A} \setminus \{0\}$, Proposition 5.10 implies that there is $\psi \in \mathcal{A} \cap U \setminus \{0\}$ and $T > 0$ so that $h^{-1}(\chi) = F(T, \psi)$. Then $h(\psi) \in h(\mathcal{A} \cap U \setminus \{0\}) = (0,1]v^*$. From $\chi \in \operatorname{int}(h \circ \eta)$ it follows that $h(\psi) \neq v^*$. So, $h(\psi) = s_0 v^*$ for some $s_0 \in (0,1)$. Let $\epsilon \in (0, \min\{s_0, 1 - s_0, T\})$. Consider the map

$$g : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \ni (t, s) \mapsto h(F_{\mathcal{A}}(T + t, h^{-1}((s_0 + s)v^*))) \in \mathbb{R}^2.$$

g is continuous since h is a homeomorphism, $F_{\mathcal{A}}$ is a flow on $\mathcal{A}$ and $(s_0 - \epsilon, s_0 + \epsilon)v^* \subset h(\mathcal{A})$. We want to show that g is also injective. Let $(t^1, s^1), (t^2, s^2) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ and assume that $g(t^1, s^1) = g(t^2, s^2)$. Without loss of generality we may assume $t^2 \geq t^1$. Since h is injective, it follows that

$$F_{\mathcal{A}}(T + t^1, h^{-1}((s_0 + s^1)v^*)) = F_{\mathcal{A}}(T + t^2, h^{-1}((s_0 + s^2)v^*)).$$

As $F_{\mathcal{A}}$ is a flow on $\mathcal{A}$, we obtain

$$h^{-1}((s_0 + s^2)v^*) = F_{\mathcal{A}}(t^2 - t^1, h^{-1}((s_0 + s^1)v^*)). \tag{9.3}$$

Assume that $t^1 \neq t^2$. (9.3) implies that $x = x^{h^{-1}((s_0+s^1)v^*)}$ is a $t^2 - t^1$-periodic solution. By Proposition 5.5, x is slowly oscillating. So, $t^2 - t^1 > 2$ follows. On the other hand, the choice of ϵ implies $t^2 - t^1 < 2\epsilon < 2$, a contradiction. Therefore $t^1 = t^2$. Then (9.3) implies $h^{-1}((s_0 + s^2)v^*) = h^{-1}((s_0 + s^1)v^*)$. Hence $s^1 = s^2$. Consequently, g is injective.

It follows that g is an open mapping. As $g(0,0) = \chi$, we obtain that $g((-\epsilon, \epsilon) \times (-\epsilon, \epsilon))$ is an open neighborhood of χ in $\mathbb{R}^2$. From $(s_0 - \epsilon, s_0 + \epsilon)v^* \subset h(\mathcal{A})$ it follows that

$g((-\epsilon,\epsilon)\times(-\epsilon,\epsilon)) \subset h(\mathcal{A})$. So, if we choose an open disk D in $\mathbb{R}^2$ with center at χ such that $D \subset g((-\epsilon,\epsilon)\times(-\epsilon,\epsilon))$ and $D \subset \mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}$, then $D \subset h(\mathcal{A})\cap(\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\})$.

5. $\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}$ is an open connected subset of $\mathbb{R}^2$. Therefore, the only nonempty subset of $\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}$, which is both closed and open in $\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}$ in the relative topology, is $\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}$ itself. Observe that $(0,1)v^* \subset h(\mathcal{A})\cap(\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\})$. This fact and the results of part 4 yield

$$h(\mathcal{A})\cap(\mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}) = \mathrm{int}(h\circ\eta)\setminus\{(0,0)^{tr}\}.$$

Using $(0,0)^{tr} \in h(\mathcal{A})$, $|h\circ\eta| \subset h(\mathcal{A})$ and the result of part 3, we conclude

$$h(\mathcal{A}) = \mathrm{int}(h\circ\eta)\cup|h\circ\eta| = \overline{\mathrm{int}(h\circ\eta)}.$$

The Scheonfliess theorem [48] gives that $\overline{\mathrm{int}(h\circ\eta)}$ is homeomorphic to the 2-dimensional closed unit disk so that $|h\circ\eta|$ corresponds to the unit circle. This completes the proof. □

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