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Unstable Sets of Periodic Orbits and the Global Attractor for Delayed Feedback

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Abstract. The differential equation $\dot{x}(t) = -\mu x(t) + f(x(t-1))$ with $\mu > 0$ and a C^1-smooth real function f satisfying $f(0) = 0$ and $f' > 0$ models a system with instantaneous friction and delayed feedback. For a set of parameters μ and nonlinearities f, which include examples from neural network theory, we show that there is a global attractor A, A contains exactly 3 stationary points and N periodic orbits, and A is the union of 2 stable stationary points and the strong unstable sets of the unstable stationary point 0 and of the N periodic orbits.

1 Introduction

We study the class of delay differential equations

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

with parameter $\mu > 0$ and C^1-smooth nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(0) = 0 \qquad \text{and} \quad f'(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Eq. (1.1) models a system governed by delayed monotone positive feedback and instantaneous damping. Specific applications occur e.g. in neural network theory, for

$$f(\xi) = \alpha \tanh(\beta\xi)$$

with parameters $\alpha > 0$ and $\beta > 0$ (see e.g. Herz [**13**], Pakdaman, Malta, Grotta-Ragazzo and Vibert [**23**], Wu [**30**] and references therein).

Every element ϕ of the Banach space C of continuous real functions on the initial interval $[-1, 0]$ determines a solution $x^\phi : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1), i.e., a

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continuous function which is differentiable on $(0,\infty)$ and satisfies Eq. (1.1) for all $t>0$. The relations

$$F(t,\phi)=x_t,\ x=x^{\phi},\ x_t(s)=x(t+s),\ s\in[-1,0]$$

define a continuous semiflow $F:\mathbb{R}^+\times C\to C$ such that all maps $F(t,\cdot)$, $t\geq 0$, are injective and continuously differentiable, and F is monotone with respect to the pointwise ordering on C. The derivatives $D_2F(t,0)$, $t\geq 0$, form a strongly continuous semigroup, and the spectrum of the generator of the semigroup consists of simple eigenvalues which coincide with the zeros of the characteristic function

$$\mathbb{C}\ni\lambda\mapsto\lambda+\mu-f'(0)e^{-\lambda}\in\mathbb{C}.$$

There is one real eigenvalue λ_0, and the others form a sequence of complex conjugate pairs $(\lambda_j,\overline{\lambda_j})$ with

$$\operatorname{Re}\lambda_{j+1}<\operatorname{Re}\lambda_j<\lambda_0\quad\text{and}\quad(2j-1)\pi<\operatorname{Im}\lambda_j<2j\pi$$

for all integers $j\geq 1$, and $\operatorname{Re}\lambda_j\to-\infty$ as $j\to\infty$. The number of eigenvalues in the open right halfplane depends on μ and $f'(0)$.

The forward extension of a local unstable manifold of the stationary point 0 associated with the eigenvalues of the generator of the semigroup $(D_2F(t,0))_{t\geq 0}$ with positive real part is called the strong unstable set $W^u_{str}(0)$ of 0. The unstable set $W^u(0)$ of 0 is the set of $\phi\in C$ such that there is a solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) so that x is bounded on $(-\infty,0]$, $x_0=\phi$ and $\alpha(x)=\{0\}$. In general, $W^u_{str}(0)\subset W^u(0)$ holds. If 0 is a hyperbolic stationary point then $W^u_{str}(0)=W^u(0)$.

For a nontrivial periodic orbit $\mathcal{O}$ of Eq. (1.1), the Floquet multipliers of $\mathcal{O}$ outside the unit circle determine a local unstable manifold of $\mathcal{O}$. The forward extension of such a local unstable manifold is called the strong unstable set $W^u_{str}(\mathcal{O})$ of $\mathcal{O}$. The unstable set $W^u(\mathcal{O})$ of the periodic orbit $\mathcal{O}$ contains those elements ϕ in C for which a solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) exists such that x is bounded on $(-\infty,0]$, $x_0=\phi$ and $\alpha(x)=\mathcal{O}$. The inclusion $W^u_{str}(\mathcal{O})\subset W^u(\mathcal{O})$ is always satisfied. If the periodic orbit $\mathcal{O}$ is hyperbolic then equality holds.

In the description of the long term behaviour of the solutions of Eq. (1.1) a natural object to study is the global attractor of the semiflow, i.e., a compact set $A\subset C$ which is invariant and attracts every bounded subset of C (see Hale [**10**]).

In Krisztin, Walther and Wu [**16**] we described the closure $\overline{W}$ of the forward extension of a 3-dimensional local unstable manifold of the stationary point 0 associated with the 3 leading eigenvalues $\lambda_0,\lambda_1,\overline{\lambda_1}$ with positive real part. The set $\overline{W}$ consisted of 3 stationary points, a periodic orbit $\mathcal{O}$, and some orbits connecting the stationary point 0 to the nonzero stationary points, 0 to the periodic orbit $\mathcal{O}$, and $\mathcal{O}$ to the nonzero stationary points. In Krisztin and Walther [**15**], for a set of parameters $\mu>0$ and nonlinearities f, we proved that the set $\overline{W}$ coincides with the global attractor A. In particular, $\operatorname{Re}\lambda_2<0<\operatorname{Re}\lambda_1$ was assumed in [**15**]. The main steps toward the equality $A=\overline{W}$ were a uniqueness result on periodic orbits and that the unstable set $W^u(\mathcal{O})$ of the periodic orbit $\mathcal{O}$ is equal to the strong unstable set $W^u_{str}(\mathcal{O})$ of the periodic orbit $\mathcal{O}$. The result of [**15**] can also be stated as

$$A=\{\xi_-,\xi_+\}\cup W^u_{str}(0)\cup W^u_{str}(\mathcal{O}).$$

The purpose of this paper is to show a result of the above type on the structure of the global attractor of the semiflow F in more general situations. We suppose

$$\operatorname{Re}\lambda_{N+1}<0<\operatorname{Re}\lambda_N$$

for some integer $N > 0$, which can be guaranteed by an explicit condition on $f'(0)$ and μ. In addition, assuming oddness and a convexity condition on f, and that $\frac{f(\xi)}{\xi} < \mu$ outside a bounded neighbourhood of 0, we find that the semiflow F has exactly 3 stationary points $0, \xi_-, \xi_+$; 0 is unstable, ξ_- and ξ_+ are locally asymptotically stable; moreover, results of [**15**] and [**17**] give that F has exactly N periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$. The main result of this paper is that

$$A = \{\xi_-, \xi_+\} \cup W^u_{str}(0) \cup \left(\bigcup_{k=1}^{N} W^u_{str}(\mathcal{O}_k) \right). \tag{1.2}$$

We emphasize that the above equality is valid without assuming hyperbolicity of the periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$. It is shown in Krisztin and Wu [**17**] that $W^u_{str}(0)$ is a $(2N+1)$-dimensional C^1 submanifold of the phase space C. As $F(t, \cdot)$, $D_2F(t, \cdot)$, $t \geq 0$, are injective maps, it follows that the strong unstable sets $W^u_{str}(\mathcal{O}_1)$,$W^u_{str}(\mathcal{O}_2)$,..., $W^u_{str}(\mathcal{O}_N)$ are C^1 immersed submanifolds of C. In a subsequent paper we shall prove that these strong unstable sets are also C^1 submanifolds of C.

The sets

$$S_0 = \{\xi_-, \xi_+\},\ S_{2N+1} = \{0\},\ S_{2k} = \mathcal{O}_k \quad \text{for all } k \in \{1, 2, \ldots, N\}$$

define a Morse decomposition of the global attractor A (see Conley [**6**]), which means that $S_0, S_2, \ldots, S_{2N}, S_{2N+1}$ are disjoint, compact invariant subsets of A, and on $A \setminus (S_0 \cup S_2 \cup \ldots \cup S_{2N} \cup S_{2N+1})$ the semiflow F is gradient-like, i.e., for every $\phi \in A \setminus (S_0 \cup S_2 \cup \ldots \cup S_{2N} \cup S_{2N+1})$ and for the unique solution $x^\phi : \mathbb{R} \to \mathbb{R}$ there exist $k, l \in \{0, 2, 4, \ldots, 2N, 2N+1\}$ so that $k > l$ and $\alpha(x^\phi) \in S_k$ and $\omega(\phi) \in S_l$. Equality (1.2) with the proof that the strong unstable sets $W^u_{str}(\mathcal{O}_k)$ are also C^1 submanifolds of C will show that $A \setminus (S_0 \cup S_2 \cup \ldots \cup S_{2N} \cup S_{2N+1})$ is a finite disjoint union of C^1 submanifolds of C.

Let us mention that a Morse decomposition is known to exist under weaker conditions than ours both for the negative and the positive feedback cases (see Mallet-Paret [**18**] and Polner [**24**]). In addition, there are some results on the connecting sets for the negative feedback case in Fiedler and Mallet-Paret [**9**] and in McCord and Mischaikow [**22**]. Our hypotheses are more restrictive, but we get a finer and more detailed description of the global attractor.

The main tool, which was introduced by Mallet-Paret, is a discrete Lyapunov functional counting sign changes of elements $\phi \in C \setminus \{0\}$ (see [**18**] and [**19**]). We apply a Poincaré–Bendixson theorem of Mallet-Paret and Sell [**20**]. Results about the Floquet multipliers of periodic orbits are also important [**16**], [**17**], [**19**]. The basic idea of the proof of the equality

$$W^u(\mathcal{O}) = W^u_{str}(\mathcal{O})$$

for a periodic orbit $\mathcal{O}$ is very simple. Let $p : \mathbb{R} \to \mathbb{R}$ be a periodic solution of Eq. (1.1) with minimal period $\omega > 0$ so that $\mathcal{O} = \{p_t : t \in [0, \omega]\}$. We construct two solutions $x : [-1, \infty) \to \mathbb{R}$ and $y : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) such that in the plane $\mathbb{R}^2$ the curve

$$X : [0, \infty) \ni t \mapsto \begin{pmatrix} x(t) \\ x(t-1) \end{pmatrix} \in \mathbb{R}^2$$

spirals toward the trace $|P|$ of the simple closed curve

$$P : [0, \omega] \ni t \mapsto \begin{pmatrix} p(t) \\ p(t-1) \end{pmatrix} \in \mathbb{R}^2$$

in the interior of P as $t \to \infty$, while the curve

$$Y : [0,\infty) \ni t \mapsto \begin{pmatrix} y(t) \\ y(t-1) \end{pmatrix} \in \mathbb{R}^2$$

spirals toward $|P|$ in the exterior of P as $t \to \infty$. If $W^u(\mathcal{O}) \neq W^u_{str}(\mathcal{O})$ then there is a solution $z : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) such that the curve

$$Z : (-\infty,0] \ni t \mapsto \begin{pmatrix} z(t) \\ z(t-1) \end{pmatrix} \in \mathbb{R}^2$$

does not intersect the curves P, X, Y, and $Z(t)$ spirals toward $|P|$ as $t \to -\infty$. A planar argument applying the Jordan curve theorem leads to a contradiction. A solution x with the above property is given in Krisztin and Wu [**17**]. The existence of the solution y is shown by using homotopy methods and the Brouwer degree. The construction of z requires some information about the Floquet multipliers of the periodic orbit $\mathcal{O}$. We remark that, in [**15**] for the proof of $W^u(\mathcal{O}) = W^u_{str}(\mathcal{O})$ in a particular case, we used a different proof.

We mention that results on attractors of delay differential equations related to ours can be found in the works of Walther [**27**], [**28**], Walther and Yebdri [**29**], Mallet-Paret and Walther [**21**], Chen and Wu [**5**], Chen, Krisztin and Wu [**4**], Krisztin and Arino [**14**].

The organization of the paper is as follows: Section 2 contains some preliminary results on a discrete Lyapunov functional, periodic orbits, Floquet multipliers, and unstable manifolds. We prove the existence of a solution y with the above properties in Section 3. The equality $W^u(\mathcal{O}) = W^u_{str}(\mathcal{O})$ for periodic orbits $\mathcal{O}$ is shown in Section 4. In the last section we conclude the paper by proving equality (1.2) for the global attractor.

Notation. $\mathbb{N}$ and $\mathbb{R}^+$ stand for the nonnegative integers and reals, respectively. $S^1_{\mathbb{C}}$ is the unit circle in $\mathbb{C}$. An upper index tr denotes the transpose of a row vector.

Simple closed curves are continuous maps c from a compact interval $[a,b] \subset \mathbb{R}$, $a < b$, into $\mathbb{R}^n$ so that $c|_{[a,b)}$ is injective and $c(a) = c(b)$. The set of values of a simple closed curve c, or trace, is denoted by $|c|$. The Jordan curve theorem guarantees that the complement of the trace of a simple closed curve c in $\mathbb{R}^2$ consists of two nonempty connected open sets, one bounded and the other unbounded, and $|c|$ is the boundary of each of these components. We denote the bounded one by $\mathrm{int}(c)$ and the unbounded one by $\mathrm{ext}(c)$.

A trajectory of a map $g : M \to N$ is a finite or infinite sequence $(x_j)_{j \in I \cap \mathbb{Z}}$, $I \subset \mathbb{R}$ an interval, in M with $x_{j+1} = g(x_j)$ for all $j \in I \cap \mathbb{Z}$ with $j + 1 \in I \cap \mathbb{Z}$.

For a Banach space E and $r > 0$ we set

$$E_r = \{x \in E : ||x|| < r\}.$$

Spectra of continuous linear maps $T : E \to E$ are defined as spectra of their complexifications.

For a given continuous $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, solutions $x : \mathbb{R} \to \mathbb{R}$ of the equation

$$\dot{x}(t) = g(t, x(t), x(t-1)) \tag{1.3}$$

are differentiable functions which satisfy Eq. (1.3) everywhere. If $I \subset \mathbb{R}$ is an interval and if $t_0 \in I$ is given with $t_0 - 1 = \min I$ and $t_0 < \sup I \leq \infty$, and if a continuous function $g : (I \cap [t_0, \infty)) \times \mathbb{R}^2 \to \mathbb{R}$ is given, then a continuous function $x : I \to \mathbb{R}$ is a solution of Eq. (1.3) if x is differentiable on $I \cap (t_0, \infty)$ and satisfies

Eq. (1.3) for all $t \in I \cap (t_0, \infty)$. It is then clear how to define complex-valued solutions of equations given by functions of the form

$$g(t, x, y) = a(t)x + b(t)y.$$

For a map $x : D \to M$, and $t \in \mathbb{R}$ so that $[t-1, t] \subset D$, the segment $x_t : [-1, 0] \to M$ is defined by $x_t(s) = x(t+s)$ for $-1 \le s \le 0$.

C denotes the Banach space of continuous functions $\phi : [-1, 0] \to \mathbb{R}$, with the norm given by

$$||\phi|| = \max_{-1 \le t \le 0} |\phi(t)|.$$

C^1 is the Banach space of all C^1-maps $\phi : [-1, 0] \to \mathbb{R}$, with the norm given by

$$||\phi||_1 = ||\phi|| + ||\dot{\phi}||.$$

C^2 is defined analogously.

2 Preliminary results

Consider the delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

where

(H0): $\mu > 0$, $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, and $f(0) = 0$.

A growth bound on f is also required:

(H1): $|f(\xi)| < \mu|\xi|$ outside a bounded neighbourhood of 0.

Let ξ^- denote the minimal zero of $f - \mu\,\mathrm{id}$, and let ξ^+ denote the maximal zero of $f - \mu\,\mathrm{id}$. Then $\xi^- \le 0 \le \xi^+$.

We recall some basic facts. Every $\phi \in C$ uniquely determines a solution $x^\phi : [-1, \infty) \to \mathbb{R}$ with $x_0^\phi = \phi$. Any two solutions on a common domain are equal whenever they coincide on an interval of length one. The set of values of constant solutions coincides with the zeroset of $f - \mu\,\mathrm{id}$. We have continuous dependence on initial data in the sense that given $\phi \in C$, $t \ge 0$, $\epsilon > 0$ there exists $\delta > 0$ so that $|x^\psi(s) - x^\phi(s)| < \epsilon$ for all $s \in [-1, 0]$ and all $\psi \in C$ with $||\psi - \phi|| < \delta$.

The map

$$F : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

is a continuous semiflow. $0, \xi_-, \xi_+$ are stationary points of F, where $\xi_-(s) = \xi^-$ and $\xi_+(s) = \xi^+$ for all $s \in [-1, 0]$. All maps $F(t, \cdot) : C \to C$, $t \ge 0$, are injective. It follows that for every $\phi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0 = \phi$. We denote also by x^ϕ such a solution on $\mathbb{R}$ whenever it exists. The maps $F(t, \cdot)$, $t \ge 0$, are monotone with respect to the pointwise ordering on C given by the cone

$$K = \{\phi \in C : \phi(s) \ge 0 \text{ for all } s \in [-1, 0]\}.$$

All maps $F(t, \cdot)$, $t \ge 1$, are compact (i.e., send bounded sets into relative compact sets), and all maps

$$C \ni \phi \mapsto F(t, \phi) \in C^1, \qquad t \ge 1,$$

are continuous.

For reals a, b with $a < b$ set

$$C_{a,b} = \{\phi \in C : a < \phi(s) < b \text{ for all } s \in [-1, 0]\}.$$

Concerning boundedness properties, we have the following result.

Proposition 2.1 *Assume that hypotheses (H0) and (H1) hold. For every* $a, b \in \mathbb{R}$ *with* $a < \xi^-$, $b > \xi^+$,

$$F(\mathbb{R}^+ \times C_{a,b}) \subset C_{a,b},$$

and for every $\phi \in C$ *there exists* $t \geq 0$ *so that*

$$F(s, \phi) \in C_{a,b} \qquad \text{for all } s \geq t.$$

The proof is similar to that of Proposition 2.1 in [**15**], so it is omitted.

Using the Arzela–Ascoli theorem, Eq. (1.1) and boundedness of solutions on $[-1, \infty)$, we obtain that for every $\phi \in C$ the ω-limit set

$$\omega(\phi) = \{\psi \in C : \text{There exists a sequence } (t_n)_0^\infty \text{ in } \mathbb{R}^+ \text{ with } t_n \to \infty \text{ and } F(t_n, \phi) \to \psi \text{ as } n \to \infty\}$$

is nonempty. ω-limit sets are compact, connected, and invariant in the sense that for every $\psi \in \omega(\phi)$ there is a solution $x : \mathbb{R} \to \mathbb{R}$ with $x_0 = \psi$ and $x_t \in \omega(\phi)$ for all $t \in \mathbb{R}$. For bounded solutions $x : \mathbb{R} \to \mathbb{R}$, the α-limit set

$$\alpha(x) = \{\psi \in C : \text{There exists a sequence } (t_n)_0^\infty \text{ in } \mathbb{R} \text{ with } t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to \infty\}$$

is nonempty, compact, connected, and invariant.

Under hypotheses (H0) and (H1) Proposition 2.1 and arguments as in Chapter 17 of [**16**], or in [**10**], yield the existence of a global attractor of the semiflow F, i.e., of a nonempty compact set $A \subset C$ which is invariant in the sense that

$$F(t, A) = A \qquad \text{for all } t \geq 0,$$

and which attracts bounded sets in the sense that for every bounded set $B \subset C$ and for every open set $U \supset A$ there exists $t \geq 0$ with

$$F([t, \infty) \times B) \subset U.$$

Global attractors are uniquely determined.

It is shown in [**15**] that

$$A = \{\phi \in C : \text{There is a bounded solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1.1) and } t \in \mathbb{R} \text{ so that } \phi = x_t\}.$$

It is easy to obtain from Proposition 2.1 that

$$A \subset \{\phi \in C : \xi^- \leq \phi(s) \leq \xi^+,\ s \in [-1, 0]\}.$$

The compactness of A, its invariance property and the injectivity of the maps $F(t, \cdot)$, $t \geq 0$, combined permit to show that the map

$$[0, \infty) \times A \ni (t, \phi) \mapsto F(t, \phi) \in A$$

extends to a continuous flow

$$F_A : \mathbb{R} \times A \to A;$$

for every $\phi \in A$ and for all $t \in \mathbb{R}$ we have

$$F_A(t, \phi) = x_t$$

with the uniquely determined solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) satisfying $x_0 = \phi$.

Note that we have

$$A = F(1, A) \subset C^1;$$

A is a closed subset of C^1. Using the flow F_A and the continuity of the map

$$C \ni \phi \mapsto F(1, \phi) \in C^1$$

one obtains that C and C^1 define the same topology on A.

Now we linearize the semiflow F at its stationary point 0. The smoothness of f implies that each map $F(t, \cdot)$, $t \geq 0$, is continuously differentiable. For all ϕ, ψ in C and $t \geq 0$ we have

$$D_2F(t, \phi)\psi = v_t$$

with the solution $v : [-1, \infty) \to \mathbb{R}$ of the linear variational equation

$$\dot{v}(s) = -\mu v(s) + f'(x^{\phi}(s-1))v(s-1)$$

along x^{ϕ} which is given by $v_0 = \psi$. The operators $D_2F(t, 0)$, $t \geq 0$, form a strongly continuous semigroup; for $\phi = 0$ the linear variational equation is

$$\dot{v}(t) = -\mu v(t) + f'(0)v(t-1). \tag{2.1}$$

The spectrum of the generator of the semigroup $(D_2F(t, 0))_{t \geq 0}$ consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation

$$\lambda + \mu - f'(0)e^{-\lambda} = 0. \tag{2.2}$$

There is exactly one real λ_0 in the spectrum, the remaining points in the spectrum are given by a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_1^{\infty}$ with

$$\lambda_0 > \operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2 > \ldots , \quad 2j\pi - \pi < \operatorname{Im} \lambda_j < 2j\pi$$

for $1 \leq j \in \mathbb{N}$, and $\operatorname{Re} \lambda_j \to -\infty$ as $j \to \infty$. It is easy to see that $\lambda_0 > 0$ if and only if $f'(0) > \mu$.

Assume that there exists $N \in \mathbb{N}$ so that

$$\operatorname{Re} \lambda_{N+1} \leq 0 < \operatorname{Re} \lambda_N.$$

Let P denote the realified generalized eigenspace of the generator associated with the spectral set $\{\lambda_0, \lambda_1, \overline{\lambda_1}, \ldots , \lambda_N, \overline{\lambda_N}\}$. Let Q denote the realified generalized eigenspace given by the spectral set of all $\lambda_k, \overline{\lambda_k}$ with $k \geq N+1$. Then $C = P \oplus Q$. The spaces P and Q are also realified generalized eigenspaces of $D_2F(1, 0)$ given by the spectral sets $\{e^{\lambda_0}, e^{\lambda_1}, e^{\overline{\lambda_1}}, \ldots , e^{\lambda_N}, e^{\overline{\lambda_N}}\}$ and $\{e^{\lambda_k} : k \geq N+1\} \cup \{e^{\overline{\lambda_k}} : k \geq N+1\}$, respectively.

Choose $\beta > 1$ with $\beta < e^{\operatorname{Re} \lambda_N}$. According to Theorem I.3 in [**16**] there exist convex open neighbourhoods N_Q, N_P of Q, P, respectively, and a C^1-map $w_u : N_P \to Q$ with $W_u(N_P) \subset N_Q$, $w_u(0) = 0$, $Dw_u(0) = 0$ so that the strong unstable manifold of the fixed point 0 of $F(1, \cdot)$ in $N_Q + N_P$, namely

$$\begin{aligned} W^u(0, F(1, \cdot), N_Q + N_P) = \{\phi \in N_Q + N_P : & \text{ There is a trajectory } (\phi_n)_{-\infty}^0 \\ & \text{of } F(1, \cdot) \text{ with } \phi_0 = \phi,\ \phi_n \beta^{-n} \in N_Q + N_P \text{ for all } n \in -\mathbb{N}, \\ & \text{and } \phi_n \beta^{-n} \to 0 \text{ as } n \to -\infty\} \end{aligned}$$

coincides with the graph $\{\chi + w_u(\chi) : \chi \in N_P\}$. It is easy to show that every $\phi \in W^u(0, F(1, \cdot), N_Q + N_P)$ uniquely determines a solution $x^{\phi} : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1), and for this solution $x^{\phi}(t) \to 0$ as $t \to -\infty$ holds, moreover there exists $t \in \mathbb{R}$ with $x_s^{\phi} \in W^u(0, F(1, \cdot), N_Q + N_P)$ for all $s \leq t$.

We call the forward extension

$$W^u_{str}(0) = F(\mathbb{R}^+ \times W^u(0, F(1, \cdot), N_Q + N_P))$$

the strong unstable set of 0. The unstable set of 0 is defined by

$$W^u(0) = \{\phi \in C : \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1.1)} \text{ with } x_0 = \phi \text{ and } x_t \to 0 \text{ as } t \to -\infty\}.$$

If $\operatorname{Re}\lambda_{N+1} < 0 < \operatorname{Re}\lambda_N$ holds, then 0 is hyperbolic and

$$W^u(0) = W^u_{str}(0).$$

The following explicit condition in terms of μ and $f'(0)$ for the location of the solutions of (2.2) can be found e.g. in [**8**] or [**15**].

Proposition 2.2 *Let $\mu > 0$, $N \in \mathbb{N} \setminus \{0\}$, and let θ_N and θ_{N+1} denote the unique solution of the equation $\theta = -\mu \tan\theta$ in*

$$(2N\pi - \pi/2, 2N\pi) \quad \text{and} \quad (2(N+1)\pi - \pi/2, 2(N+1)\pi),$$

respectively. If

$$\frac{\mu}{\cos\theta_N} < f'(0) < \frac{\mu}{\cos\theta_{N+1}}$$

then

$$\operatorname{Re}\lambda_{N+1} < 0 < \operatorname{Re}\lambda_N.$$

We recall the definition and some properties of a discrete Lyapunov functional

$$V : C \setminus \{0\} \to 2\mathbb{N} \cup \{\infty\}$$

which goes back to the work of Mallet-Paret [**18**]. The version which we use was introduced in Mallet-Paret and Sell [**19**].

The definition is as follows. First, set

$$\operatorname{sc}(\phi) = \sup\big\{k \in \mathbb{N} \setminus \{0\} : \text{There is a strictly increasing finite sequence } (s^i)_0^k \text{ in } [-1,0] \text{ with } \phi(s^{i-1})\phi(s^i) < 0 \text{ for all } i \in \{1,2,\dots,k\}\big\} \le \infty$$

for $\phi \in C \setminus (K \cup (-K))$, and $\operatorname{sc}(\phi) = 0$ for $0 \neq \phi \in K \cup (-K)$. Then, define

$$V(\phi) = \begin{cases} \operatorname{sc}(\phi) & \text{if } \operatorname{sc}(\phi) \in 2\mathbb{N} \cup \{\infty\}, \\ \operatorname{sc}(\phi) + 1 & \text{if } \operatorname{sc}(\phi) \in 2\mathbb{N} + 1. \end{cases}$$

Set

$$R = \{\phi \in C^1 : \phi(0) \neq 0 \text{ or } \dot\phi(0)\phi(-1) > 0, \ \phi(-1) \neq 0 \text{ or } \dot\phi(-1)\phi(0) < 0, \ \text{all zeros of } \phi \text{ in } (-1,0) \text{ are simple}\}.$$

The next lemma lists basic properties of V. For a proof see e.g. [**19**] or [**16**].

Lemma 2.3 (i) *For every $\phi \in C \setminus \{0\}$ and for every sequence $(\phi_n)_0^\infty$ in $C \setminus \{0\}$ with $\phi_n \to \phi$ as $n \to \infty$,*

$$V(\phi) \le \liminf_{n\to\infty} V(\phi_n).$$

(ii) *For every $\phi \in R$ and for every sequence $(\phi_n)_0^\infty$ in $C^1 \setminus \{0\}$ with $||\phi_n - \phi||_1 \to 0$ as $n \to \infty$,*

$$V(\phi) = \lim_{n\to\infty} V(\phi_n) < \infty.$$

(iii) *Let an interval* $I \subset \mathbb{R}$*, a real* $\nu \geq 0$*, and continuous functions* $b : I \to (0, \infty)$ *and* $z : I + [-1, 0] \to \mathbb{R}$ *be given so that* $z|_I$ *is differentiable with*

$$\dot{z}(t) = -\nu z(t) + b(t)z(t-1) \tag{2.3}$$

for $\inf I < t \in I$*, and* $z(t) \neq 0$ *for some* $t \in I + [-1, 0]$*. Then the map* $I \ni t \mapsto V(z_t) \in 2\mathbb{N} \cup \{\infty\}$ *is decreasing. If* $t \in I$*,* $t - 2 \in I$ *and* $z(t) = 0 = z(t-1)$*, then* $V(z_t) = \infty$ *or* $V(z_{t-2}) > V(z_t)$*. For all* $t \in I$ *with* $t - 3 \in I$ *and* $V(z_{t-3}) = V(z_t) < \infty$*, we have* $z_t \in R$*.*

(iv) *If* $\nu \geq 0$*,* $b : \mathbb{R} \to (0, \infty)$ *is continuous and bounded,* $z : \mathbb{R} \to \mathbb{R}$ *is differentiable and bounded,* z *satisfies (2.3) for all* $t \in \mathbb{R}$*, and* $z(t) \neq 0$ *for some* $t \in \mathbb{R}$*, then* $V(z_t) < \infty$ *for all* $t \in \mathbb{R}$*.*

Observe that linear variational equations

$$\dot{v}(t) = -\mu v(t) + f'(x(t-1))v(t-1)$$

along solutions of Eq. (1.1) are of the form considered in statements (iii) and (iv), as well as the equation satisfied by weighted differences $y = (x - \hat{x})/c$, $c \neq 0$, of solutions $x, \hat{x}$ of Eq. (1.1) on a common domain,

$$\dot{y}(t) = -\mu y(t) + \left(\int_0^1 f'((1-s)\hat{x}(t-1) + sx(t-1))\, ds \right) y(t-1).$$

The next a-priori estimate is a special case of a result which says that solutions with finite oscillation frequency do not decay too fast as t increases. Estimates of this type go back to Walther [**25**] and Mallet-Paret [**18**], see also Arino [**2**] and Cao [**3**] and [**16**].

Lemma 2.4 *For every* $\nu > 0$*,* $l \in \mathbb{N}$*,* $b_0 > 0$ *and* $b_1 \geq b_0$ *there are* $k > 0$ *and an integer* $L > 0$ *so that for every* $t_0 \in \mathbb{R}$*, and for every continuous function* $b : [t_0 - L, t_0] \to \mathbb{R}$ *with range in* $[b_0, b_1]$*, and for every solution* $z : [t_0 - L - 1, t_0] \to \mathbb{R}$ *of Eq. (2.3) with* $z_{t_0 - L} \neq 0$ *and* $V(z_{t_0 - L}) \leq 2l$*, we have*

$$||z_{t_0 - 1}|| \leq k||z_{t_0}||.$$

For a $k \in \mathbb{N} \setminus \{0\}$ define the continuous mapping

$$\Pi_{k+1} : C \ni \phi \mapsto (\phi(-1), \phi(-1 + 1/k), \ldots, \phi(-1/k), \phi(0))^{tr} \in \mathbb{R}^{k+1}.$$

In case $k = 0$ we set $\Pi_1 \phi = \phi(0)$ for all $\phi \in C$.

The following lemma is shown in [**17**].

Lemma 2.5 *Let* $t_0 \in \mathbb{R}$*,* $k \in \mathbb{N}$*,* $\nu \geq 0$ *and the continuous functions* $b : [t_0 - 3 - 3k, t_0] \to (0, \infty)$*,* $z : [t_0 - 4 - 3k, t_0] \to \mathbb{R}$ *be given such that* z *is differentiable on* $(t_0 - 3 - 3k, t_0]$*,* $z_{t_0} \neq 0$*,* z *satisfies (2.3) for all* $t \in (t_0 - 3 - 3k, t_0]$*, and*

$$V(z_{t_0 - 3 - 3k}) \leq 2k.$$

Then

$$\Pi_{2k+1} z_{t_0} \neq 0.$$

We need the following corollary of a general Poincaré–Bendixson theorem for monotone cyclic feedback systems due to Mallet-Paret and Sell [**20**].

Proposition 2.6 *Assume that (H0) and (H1) hold.*

(i) *Let* $x : \mathbb{R} \to \mathbb{R}$ *be a bounded solution of Eq. (1.1). Then* $\alpha(x)$ *is either the orbit of a nonconstant periodic solution of Eq. (1.1), or for every solution* $y : \mathbb{R} \to \mathbb{R}$ *of Eq. (1.1) with* $y_0 \in \alpha(x)$ *the sets* $\alpha(y)$ *and* $\omega(y_0)$ *consist of stationary points of* F*.*

(ii) *For every $\phi \in C$, $\omega(\phi)$ is either the orbit of a nonconstant periodic solution of Eq. (1.1), or for every solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 \in \omega(\phi)$ the sets $\alpha(y)$ and $\omega(y_0)$ consist of stationary points of F.*

We introduce an additional hypothesis on f:

(H2): $f(\xi) = -f(-\xi)$ for all $\xi \in \mathbb{R}$, and
the function $(0, \infty) \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)} \in \mathbb{R}$ is strictly decreasing.

From Lemma 2.3(iii) and (iv) it follows that for any nonconstant periodic solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) there exists $k \in \mathbb{N}$ so that $V(x_t) = 2k$ and $x_t \in R$ for all $t \in \mathbb{R}$. In addition, for the derivative $\dot{x}$ of the nonconstant periodic solution we also find $l \in \mathbb{N}$ with $V(\dot{x}_t) = 2l$ and $\dot{x}_t \in R$ for all $t \in \mathbb{R}$. For $k \in \mathbb{N}$, we say that Eq. (1.1) has a periodic orbit in $V^{-1}(2k)$ if it has a nonconstant periodic solution $x : \mathbb{R} \to \mathbb{R}$ with $V(x_t) = 2k$ for all $t \in \mathbb{R}$. One of the main results of [**15**] is

Proposition 2.7 *Assume that hypotheses (H0), (H1) and (H2) are satisfied.*

(i) *For every $k \in \mathbb{N} \setminus \{0\}$, Eq. (1.1) has at most one periodic orbit in $V^{-1}(2k)$.*
(ii) *Eq. (1.1) has no periodic orbit in $V^{-1}(2k)$ if either $k = 0$ or $k \in \mathbb{N} \setminus \{0\}$ and $\operatorname{Re} \lambda_k \le 0$.*

The next result of [**17**] guarantees the existence of periodic orbits and an orbit connecting the stationary point 0 and the periodic orbit with a given oscillation frequency.

Proposition 2.8 *Assume that hypotheses (H0) and (H1) hold. If $N \in \mathbb{N} \setminus \{0\}$ and $\operatorname{Re} \lambda_N > 0$, then Eq. (1.1) has a periodic orbit $\mathcal{O}_N$ in $V^{-1}(2N)$, and Eq. (1.1) has a solution $x : \mathbb{R} \to \mathbb{R}$ with $\alpha(x) = \{0\}$, $\omega(x_0) = \mathcal{O}_N$, and $x_t \in R$, $x_t - \psi \in R$, $V(x_t) = V(x_t - \psi) = 2N$ for all $t \in \mathbb{R}$ and $\psi \in \mathcal{O}_N$.*

The following a-priori result on periodic solutions of Eq. (1.1) follows from general results in Mallet-Paret and Sell [**20**] for certain systems of delay differential equations.

Proposition 2.9 *Assume that hypothesis (H0) holds. If $p : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of Eq. (1.1) with minimal period $\omega > 0$, then there are $t^0 \in \mathbb{R}$ and $t^1 \in (t^0, t^0 + \omega)$ with $p(t^0) < 0 < p(t^1)$, $p(\mathbb{R}) = [p(t^0), p(t^1)]$, $0 < \dot{p}(t)$ for $t^0 < t < t^1$, and $\dot{p}(t) < 0$ for $t^1 < t < t^0 + \omega$. In particular, it follows that $[0, \omega] \ni t \mapsto \Pi_2 p_t \in \mathbb{R}^2$ is a simple closed curve, and if z denotes the unique zero of p in (t^0, t^1), then*

$$\{(0, v)^{tr} \in \mathbb{R}^2 : 0 \le v < p(z - 1)\} \subset \operatorname{int}(\Pi_2 \{p_t : t \in [0, \omega]\})$$

The next result on the sign changes of differences of elements of periodic orbits is shown in [**17**].

Proposition 2.10 *Assume that hypothesis (H0) holds. Let $N \in \mathbb{N} \setminus \{0\}$ and nonconstant periodic solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ be given with $V(p_t) = V(q_t) = 2N$ for all $t \in \mathbb{R}$. Then*

$$V(p_t - q_s) \ge 2N \qquad \text{for all } t, s \text{ in } \mathbb{R} \text{ with } p_t \ne q_s,$$

and

$$V(p_t - p_s) = 2N \qquad \text{for all } t, s \text{ in } \mathbb{R} \text{ with } p_t \ne p_s.$$

For a given $N \in \mathbb{N} \setminus \{0\}$, let $p : \mathbb{R} \to \mathbb{R}$ denote the periodic solution guaranteed by Proposition 2.8 and normalized so that $p(0) = 0$ and $p(-1) > 0$. Then $\mathcal{O}_N = \{p_t : t \in \mathbb{R}\}$. By Proposition 2.9, three consecutive zeros of p determine the minimal period ω of p. All zeros of p are simple since $p_t \in R$ for all $t \in \mathbb{R}$ by Lemma 2.3(iii). Then the definition of V and the fact $V(p_t) = 2N$ for all $t \in \mathbb{R}$ combined yield $N\omega \geq 1$. Define the monodromy operator

$$M = D_2F(\omega, p_0).$$

For every $\phi \in C$, we have $M\phi = v_\omega$, where $v : [-1, \infty) \to \mathbb{R}$ is the solution of the variational equation

$$\dot{v}(t) = -\mu v(t) + f'(p(t-1))v(t-1) \tag{2.4}$$

subject to the initial condition $v_0 = \phi$. The operator M^N is compact since $\omega \geq 1/N$. We then have that the spectrum σ of M contains 0, and that every point $\lambda \in \sigma \setminus \{0\}$ is an eigenvalue of M of finite multiplicity, and is isolated in σ. These eigenvalues in $\sigma \setminus \{0\}$ are called Floquet multipliers.

For $0 \neq \lambda \in \sigma$ with $\operatorname{Im}\lambda \geq 0$, let $G_{\mathbb{R}}(\lambda)$ stand for the realified generalized eigenspace of the eigenvalue λ of M. If $r > 0$ and $\{\lambda \in \sigma : r < |\lambda|\} \neq \emptyset$, then we use $C_{\leq r}$ and $C_{r<}$ to denote the realified generalized eigenspaces of M associated with the nonempty disjoint spectral sets $\{\lambda \in \sigma : |\lambda| \leq r\}$ and $\{\lambda \in \sigma : r < |\lambda|\}$, respectively. Then

$$C = C_{\leq r} \oplus C_{r<}, \qquad C_{r<} = \bigoplus_{\lambda\in\sigma, r<|\lambda|, \operatorname{Im}\lambda\geq 0} G_{\mathbb{R}}(\lambda).$$

Similarly, we can define $C_{<r}$ and $C_{r\leq}$.

The following result on the Floquet multipliers of the periodic orbit $\mathcal{O}_N$ can be found in Krisztin and Wu [**17**].

Proposition 2.11 (i) *There exists $r_M \in (0,1)$ such that*

$$C_{\leq r_M} \cap V^{-1}(\{0, 2, \ldots, 2N\}) = \emptyset, \quad C_{r_M<} \cap C_{\leq 1} \subset V^{-1}(2N) \cup \{0\},$$

$$\dim C_{r_M<} \cap C_{\leq 1} = 2.$$

(ii) $1 \leq \dim C_{1<} \leq 2N - 1$.

(iii) *If $v : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (2.4) with $v_0 \neq 0$ and $V(v_t) \leq 2N - 2$ for all $t \in \mathbb{R}$, then $v_0 \in C_{1<}$.*

Choose $\lambda \in (0,1)$ so that

$$\lambda > \max\left\{\max_{\zeta\in\sigma,|\zeta|>1} \frac{1}{|\zeta|}, \max_{\zeta\in\sigma,|\zeta|<1} |\zeta|\right\}.$$

Theorem I.3 in [**16**] guarantees the existence of a local strong unstable manifold of the period-ω map $F(\omega, \cdot)$ at its fixed point p_0; namely, there are convex open neighbourhoods $N_{1<}$ of 0 in $C_{1<}$ and $N_{\leq 1}$ of 0 in $C_{\leq 1}$, a C^1-map $w^u : N_{1<} \to C_{\leq 1}$ so that $w^u(0) = 0$, $Dw^u(0) = 0$, $w^u(N_{1<}) \subset N_{\leq 1}$, and with $N^u = N_{\leq 1} + N_{1<}$ the shifted graph

$$W^u(p_0, F(\omega, \cdot), N^u) = \{p_0 + \chi + w^u(\chi) : \chi \in N_{1<}\}$$

is equal to the set

$$\{\chi \in p_0 + N^u : \text{There is a trajectory } (\chi^n)_{-\infty}^0 \text{ of } F(\omega, \cdot) \text{ with } \chi^0 = \chi,$$
$$\lambda^n(\chi^n - p_0) \in N^u \text{ for all } n \in -\mathbb{N}, \text{ and } \lambda^n(\chi^n - p_0) \to 0 \text{ as } n \to -\infty\}.$$

The C^1-submanifold $W^u(p_0, F(\omega, \cdot), N^u)$ of C is called a local strong unstable manifold of $F(\omega, \cdot)$ at p_0.

The strong unstable set $W^u_{str}(\mathcal{O}_N)$ of the periodic orbit $\mathcal{O}_N$ is defined by

$$W^u_{str}(\mathcal{O}_N) = F(\mathbb{R}^+ \times W^u(p_0, F(\omega,\cdot), N^u)).$$

The unstable set $W^u(\mathcal{O}_N)$ of the periodic orbit $\mathcal{O}_N$ is given by

$$W^u(\mathcal{O}_N) = \{\phi \in C : \text{There exists a solution } x : \mathbb{R} \to \mathbb{R} \text{ so that } x_0 = \phi \text{ and } \operatorname{dist}(x_t, \mathcal{O}_N) \to 0 \text{ as } t \to -\infty\}.$$

It is not difficult to show that

$$W^u_{str}(\mathcal{O}_N) \subset W^u(\mathcal{O}_N).$$

If $\mathcal{O}_N$ is hyperbolic, i.e., $\sigma \cap S^1_{\mathbb{C}} = \{1\}$ and the generalized eigenspace of M associated with 1 is 1-dimensional, then the equality $W^u_{str}(\mathcal{O}_N) = W^u(\mathcal{O}_N)$ is satisfied. For a nonhyperbolic $\mathcal{O}_N$, in general, we do not have equality. The main purpose of this paper is to show that under hypotheses (H0), (H1) and (H2)

$$W^u_{str}(\mathcal{O}_N) = W^u(\mathcal{O}_N).$$

holds without assuming hyperbolicity of $\mathcal{O}_N$.

We need a result from Polner [**24**] which estimates the number of sign changes for segments of solutions tending to 0 as $t \to \infty$ or $t \to -\infty$.

Proposition 2.12 *Assume that (H0) holds and $N \in \mathbb{N}$ with $\operatorname{Re}\lambda_{N+1} \le 0 < \operatorname{Re}\lambda_N$.*

(i) *If $\phi \in C \setminus \{0\}$ with $\omega(\phi) = \{0\}$, then $V(\phi) \ge 2N + 2$.*

(ii) *If $x : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (1.1) so that x is bounded on $(-\infty, 0]$ and $\alpha(x) = \{0\}$, then $V(x_0) \le 2N + 2$. If $\operatorname{Re}\lambda_{N+1} < 0$ then $V(x_0) \le 2N$.*

Finally we prove a result on the number of sign changes of elements of limit sets.

Proposition 2.13 *Assume that (H0) holds and $N \in \mathbb{N}$.*

(i) *If $x : [-1, \infty) \to \mathbb{R}$ is a bounded solution of Eq. (1.1) with $\lim_{t\to\infty} V(x_t) = 2N$, then $\omega(x_0) \subset V^{-1}(2N) \cup \{0\}$.*

(ii) *If $x : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (1.1) which is bounded on $(-\infty, 0]$ and $\lim_{t\to-\infty} V(x_t) = 2N$, then $\alpha(x) \subset V^{-1}(2N) \cup \{0\}$.*

Proof Let $N \in \mathbb{N}$ and let $x : \mathbb{R} \to \mathbb{R}$ be a bounded solution of Eq. (1.1) with $\lim_{t\to\infty} V(x_t) = 2N$. Let $\psi \in \omega(x_0) \setminus \{0\}$. There is a solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 = \psi$ and $y_t \in \omega(x_0)$, $t \in \mathbb{R}$. Lemma 2.3(i) and the definition of $\omega(x_0)$ yield $V(y_t) \le 2N$ for all $t \in \mathbb{R}$. By Lemma 2.3(iii) there exist $T \in \mathbb{R}$ and $k \in \{0, 1, \dots, N\}$ such that $y_t \in R$, $V(y_t) = 2k$ for all $t \ge T$. There is a sequence $(t_n)_0^\infty$ in $\mathbb{R}^+$ so that $t_n \to \infty$ and $x_{t_n} \to y_T$ as $n \to \infty$. Eq. (1.1) and continuous dependence on initial data give that $||x_{t_n+1} - y_{T+1}||_1 \to 0$ as $n \to \infty$. Then Lemma 2.3(ii) implies $\lim_{n\to\infty} V(x_{t_n+1}) = V(y_{T+1}) = 2k$. This fact and the monotonicity of V show $k = N$. Then $V(y_t) = 2N$ for all $t \in \mathbb{R}$, by the monotonicity of V. Consequently, $\omega(x_0) \subset V^{-1}(2N) \cup \{0\}$. The proof of assertion (ii) is analogous. □

3 Existence of a large orbit in $V^{-1}(2N)$

In this section we show that, for every integer $N > 0$ and for every periodic orbit $\mathcal{O}$ of Eq. (1.1) in $V^{-1}(2N)$, there exists a solution $y : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) such that $y_t \in V^{-1}(2N)$ and $\Pi_2 y_t \in \operatorname{ext}(\Pi_2\mathcal{O})$ for all $t \ge 0$. The last property is why we call $\{y_t : t \ge 0\}$ a large orbit.

For technical reasons we consider also the equation

$$\dot{x}(t) = -\mu x(t) + g(x(t-1)) \tag{3.1}$$

with $g \in C^1(\mathbb{R}, \mathbb{R})$, $g(0) = 0$ and $g'(\xi) > 0$ for all $\xi \in \mathbb{R}$. First we introduce a set of integrable functions, and then associate solutions with these functions.

Let $L^1 = L^1([-1, 0], \mathbb{R})$ denote the space of Lebesgue integrable functions $\phi : [-1, 0] \to \mathbb{R}$ with norm $|\phi|_1 = \int_{-1}^0 |\phi(s)|\, ds$. We do not make distinction between elements of L^1, i.e., equivalence classes of integrable functions $\phi : [-1, 0] \to \mathbb{R}$, and representatives of these classes. This should not cause confusion.

For each $r > 0$ and for all integers $n > 0$ introduce the sets

$$\begin{aligned} X_r^n = \{\phi \in L^1 : &\text{There exist } s_0, s_1, \ldots, s_n \in [-1, 0] \text{ with} \\ &-1 = s_0 \le s_1 \le \ldots \le s_n = 0 \text{ such that for each } i \in \{1, 2, \ldots, n\} \\ &\text{either } \phi(s) = r \text{ for all } s \in (s_{i-1}, s_i) \text{ or } \phi(s) = -r \text{ for all } s \in (s_{i-1}, s_i)\}. \end{aligned}$$

Set

$$S^n = \{(a_0, a_1, \ldots, a_n)^{tr} \in \mathbb{R}^{n+1} : a_0^2 + a_1^2 + \ldots + a_n^2 = 1\}.$$

For $n \in \mathbb{N}$, let the function

$$\kappa_n : S^n \to X_r^{n+1}$$

be defined by

$$\kappa_n((a_0, a_1, \ldots, a_n)^{tr}) = \phi$$

where

$$s_0 = -1,\ s_i = -1 + \sum_{j=0}^{i-1} a_j^2 \qquad \text{for } i \in \{1, 2, \ldots, n+1\},$$

and, for every $i \in \{0, 1, \ldots, n\}$,

$$\phi(s) = r \operatorname{sign} a_i \qquad \text{for all } s \in (s_i, s_{i+1}).$$

It is easy to see that κ_n is continuous. As S^n is compact, and $\kappa_n(S^n) = X_r^{n+1}$, we conclude that X_r^{n+1} is also compact.

For every $\phi \in L^1$, there exists a unique continuous function $x : [0, \infty) \to \mathbb{R}$ so that

$$x(t) = e^{-\mu t} \int_0^t e^{\mu s} g(\phi(s-1))\, ds \qquad \text{for all } t \in [0, 1],$$

x is differentiable on $(1, \infty)$, and

$$\dot{x}(t) = -\mu x(t) + g(x(t-1)) \qquad \text{for all } t > 1.$$

We use $x(\phi)$ to denote this unique function x. Observe that for $\phi \in C$ with $\phi(0) = 0$ we have $x(\phi) = x^\phi|_{[0,\infty)}$, where $x^\phi : [-1, \infty) \to \mathbb{R}$ is the solution of Eq. (3.1) with $x_0^\phi = \phi$. It is easy to see that $x_1(\phi) \ne 0$ for all $\phi \in X_r^n$ with $r > 0$ and $n \in \mathbb{N} \setminus \{0\}$.

Proposition 3.1 *Assume that $g \in C^1(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, $g'(\xi) > 0$ for all $\xi \in \mathbb{R}$, and $m = \inf_{\xi \in \mathbb{R}} g'(\xi) > 0$. Let $N \in \mathbb{N} \setminus \{0\}$. Then for every $r > 0$ there exists $\phi \in X_r^{2N}$ so that for the function $x(\phi) : [0, \infty) \to \mathbb{R}$ we have*

$$V(x_t(\phi)) = 2N \qquad \textit{for all } t \ge 1,$$

and

$$||x_4(\phi)|| \ge \frac{m^4 e^{-\mu} r}{2^{24} N^4}.$$

Proof Let $r > 0$ be fixed.

1. We claim that, for every $k \in \mathbb{N} \setminus \{0\}$ and for every $\phi \in X_r^k$,

$$V(x_t(\phi)) \leq 2\left[\frac{k}{2}\right] \qquad \text{for all } t \geq 1.$$

Let $k \in \mathbb{N} \setminus \{0\}$, $\phi \in X_r^k$ and $t \geq 1$ be given. It is not difficult to show that there exists a sequence $(\phi^n)_0^\infty$ in $C \setminus \{0\}$ so that

$$\phi^n(0) = 0 \qquad \text{for all } n \in \mathbb{N},$$

$$\phi^n \to \phi \qquad \text{as } n \to \infty \text{ almost everywhere in } [-1, 0],$$

$$|\phi^n(s)| \leq r \qquad \text{for all } n \in \mathbb{N} \text{ and } s \in [-1, 0],$$

$$V(\phi^n) \leq 2\left[\frac{k}{2}\right] \qquad \text{for all } n \in \mathbb{N}.$$

Then Lebesgue's dominated convergence theorem yields

$$\int_0^1 e^{\mu s}|g(\phi^n(s-1)) - g(\phi(s-1))|\, ds \to 0 \qquad \text{as } n \to \infty.$$

Let $x^n = x^{\phi^n}$ denote the solution of Eq. (3.1) with $x_0^{\phi^n} = \phi^n$. It follows that

$$||x_1^n - x_1(\phi)|| \to 0 \qquad \text{as } n \to \infty.$$

By the continuous dependence on initial data of solutions of Eq. (3.1) we find

$$||x_t^n - x_t(\phi)|| \to 0 \qquad \text{as } n \to \infty.$$

The monotonicity of V gives $V(x_t^n) \leq 2[k/2]$ for all $n \in \mathbb{N}$. Using the lower semicontinuity of V in Lemma 2.3(i), we obtain

$$V(x_t(\phi)) \leq \liminf_{n\to\infty} V(x_t^n) \leq 2\left[\frac{k}{2}\right].$$

2. We show that there exists $\phi \in X_r^{2N}$ with

$$V(x_t(\phi)) = 2N \qquad \text{for all } t \geq 1.$$

Assume that this assertion fails, i.e., there is no $\phi \in X_r^{2N}$ with $V(x_t(\phi)) = 2N$ for all $t \geq 1$.

2.1. We claim that there exists $T > 3N + 1$ so that

$$\Pi_{2N-1}x_t(\phi) \neq 0 \qquad \text{for all } t \geq T \text{ and } \phi \in X_r^{2N}.$$

If this claim is not true then there exist a sequence $(\phi^n)_0^\infty$ in X_r^{2N} and a sequence $(t_n)_0^\infty$ in $(3N + 1, \infty)$ such that $t_n \to \infty$ as $n \to \infty$, and

$$\Pi_{2N-1}x_{t_n}(\phi^n) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

The result of part 1 and Lemma 2.5 with $k = N - 1$ combined imply that

$$V(x_{t_n-3N}(\phi^n)) = 2N \qquad \text{for all } n \in \mathbb{N}.$$

By the compactness of X_r^{2N}, without loss of generality we may assume

$$\phi^n \to \phi \in X_r^{2N} \qquad \text{as } n \to \infty$$

in the L^1-norm. Part 1 shows

$$V(x_t(\phi)) \leq 2N \qquad \text{for all } t \geq 1.$$

Lemma 2.3(iii) yields $t^* \geq 4$ so that

$$x_t(\phi) \in R \qquad \text{for all } t \geq t^*.$$

As in part 1, we obtain

$$||x_t(\phi^n) - x_t(\phi)|| \to 0 \qquad \text{as } n \to \infty$$

for all $t \geq 1$. Using Eq. (3.1) we conclude that

$$||x_t(\phi^n) - x_t(\phi)||_1 \to 0 \qquad \text{as } n \to \infty$$

for all $t \geq 2$. Then Lemma 2.3(ii) gives that

$$\lim_{n\to\infty} V(x_t(\phi^n)) = V(x_t(\phi))$$

for all $t \geq t^*$. Fix $t \geq t^*$. The monotonicity of V, $V(x_{t_n-3N}(\phi^n)) = 2N$ and $t_n \to \infty$ combined imply

$$V(x_t(\phi^n)) = 2N$$

for all sufficiently large n. Therefore,

$$V(x_t(\phi)) = 2N.$$

As $t \geq t^*$ was arbitrary and V is monotone,

$$V(x_t(\phi)) = 2N \qquad \text{for all } t \geq 1$$

follows. This contradiction justifies the claim.

2.2. Part 2.1 shows that the map

$$Y : S^{2N-1} \ni a \mapsto \frac{\Pi_{2N-1} x_T(\kappa_{2N-1}(a))}{||\Pi_{2N-1} x_T(\kappa_{2N-1}(a))||_{\mathbb{R}^{2N-1}}} \in S^{2N-2}$$

is well defined. Clearly, Y is continuous.

2.3. For $\alpha \in [0,1]$ and $\phi \in L^1$ we define a unique continuous function $x = x(\alpha,\phi) : [0,\infty) \to \mathbb{R}$ such that

$$x(t) = e^{-\mu t}\int_0^t e^{\mu s}\big((1-\alpha)g(\phi(s-1)) + \alpha g'(0)\phi(s-1)\big)\,ds \quad \text{for all } t \in [0,1],$$

x is differentiable on $(1,\infty)$, and

$$\dot{x}(t) = -\mu x(t) + (1-\alpha)g(x(t-1)) + \alpha g'(0)x(t-1) \qquad \text{for all } t > 1.$$

Clearly, $x(0,\phi) = x(\phi)$. It is not difficult to show that, for every $t \geq 1$ and $n \in \mathbb{N}\setminus\{0\}$, the map $[0,1]\times X_r^n \ni (\alpha,\phi) \mapsto x_t(\alpha,\phi) \in C$ is continuous. Applying the result of part 1 for the nonlinearity $\mathbb{R} \ni \xi \mapsto (1-\alpha)g(\xi) + g'(0)\xi \in \mathbb{R}$ instead of g we obtain

$$V(x_t(\alpha,\phi)) \leq 2N-2 \qquad \text{for all } t \geq 1,\ \alpha \in [0,1],\ \phi \in X_r^{2N-1}.$$

Then Lemma 2.5 yields

$$\Pi_{2N-1}(x_T(\alpha,\phi)) \neq 0 \qquad \text{for all } \alpha \in [0,1],\ \phi \in X_r^{2N-1}.$$

This fact and the continuity of $\kappa_{2N-2} : S^{2N-2} \to X_r^{2N-1}$ imply that the map

$$Z : [0,1]\times S^{2N-2} \ni (\alpha,a) \mapsto \frac{\Pi_{2N-1} x_T(\alpha,\kappa_{2N-2}(a))}{||\Pi_{2N-1} x_T(\alpha,\kappa_{2N-2}(a))||_{\mathbb{R}^{2N-1}}} \in S^{2N-2}$$

is well defined and continuous.

2.4. Setting

$$i : S^{2N-2} \ni (a_0,a_1,\ldots,a_{2N-2})^{tr} \mapsto (a_0,a_1,\ldots,a_{2N-2},0)^{tr} \in S^{2N-1},$$

we have $Y \circ i = Z(0, \cdot)$. Considering the map

$$h : [0,1] \times S^{2N-2} \ni (\beta, (a_0, a_1, \dots, a_{2N-2})^{tr}) \mapsto Y\left(\beta a_0, \beta a_1, \dots, \beta a_{2N-2}, \sqrt{1-\beta^2}\right)^{tr} \in S^{2N-2},$$

we see that $Y \circ i = Z(0, \cdot)$ is homotopic to a constant map. Then $Z(1, \cdot)$ is also homotopic to a constant map. Now we extend $Z(1, \cdot)$ from S^{2N-2} to $\overline{\mathbb{R}_1^{2N-1}}$ (the $(2N-1)$-dimensional closed unit ball) by

$$Z_1(\tau a) = \tau Z(1, a) \qquad \text{for all } \tau \in [0,1] \text{ and } a \in S^{2N-2}.$$

The fact that $Z(1, \cdot)$ is homotopic to a constant map easily implies that the Brouwer degree

$$\deg(Z_1, \mathbb{R}_1^{2N-1}, 0)$$

is zero. On the other hand, Z_1 is odd, and thus, by Borsuk's theorem [**7**],

$$\deg(Z_1, \mathbb{R}_1^{2N-1}, 0) \neq 0,$$

a contradiction. This completes the proof of the existence of a $\phi \in X_r^{2N}$ with $V(x_t(\phi)) = 2N$ for all $t \geq 1$.

3. Let $\phi \in X_r^{2N}$ be given so that for the function $x = x(\phi)$, $V(x_t) = 2N$, $t \geq 1$, holds. We show that

$$||x_4|| \geq \frac{m^4 e^{-\mu} r}{2^{24} N^4}.$$

3.1. First we prove the following

CLAIM. Let the positive numbers α, β and an open interval I of length $|I| = \beta$ be given. If $u : I \to \mathbb{R}$ is a continuously differentiable function with $|\dot{u}(t)| \geq \alpha$ for all $t \in I$, then there exists a subinterval J of I such that $|J| = \beta/4$ and

$$|u(t)| \geq \frac{\alpha\beta}{4} \qquad \text{for all } t \in J.$$

Proof of the claim. Let $I = (t_0, t_0 + \beta)$. Assume $\dot{u}(t) \geq \alpha$ for all $t \in I$. (The case $\dot{u}(t) \leq -\alpha$, $t \in I$, is analogous.) Then

$$u\left(t_0 + \frac{3\beta}{4}\right) - u\left(t_0 + \frac{\beta}{4}\right) = \int_{t_0+\beta/4}^{t_0+3\beta/4} \dot{u}(t)\, dt \geq \frac{\alpha\beta}{2}.$$

Hence $u(t_0 + 3\beta/4) \geq \alpha\beta/4$ or $u(t_0 + \beta/4) \leq -\alpha\beta/4$ follows. In case $u(t_0 + 3\beta/4) \geq \alpha\beta/4$ setting $J = (t_0 + 3\beta/4, t_0 + \beta)$, by the monotonicity of u, we have $u(t) \geq \alpha\beta/4$ for all $t \in J$. Otherwise, choosing $J = (t_0, t_0 + \beta/4)$, we obtain $u(t) \leq -\alpha\beta/4$ for all $t \in J$. This completes the proof of the claim.

3.2. Let $(s_j)_0^{2N}$ be the sequence in the definition of $\phi \in X_r^{2N}$. There exists $i \in \{1, 2, \dots, 2N\}$ so that

$$s_i - s_{i-1} \geq \frac{1}{2N}.$$

Then there is an open interval $I_0 \subset (s_{i-1}, s_i) \subset (-1, 0)$ such that $|I_0| = \frac{1}{2N}$, and either $\phi(s) = r$ for all $s \in I_0$ or $\phi(s) = -r$ for all $s \in I_0$. In either case, x is continuously differentiable on $I_1 = I_0 + 1 \subset (0, 1)$, and

$$\dot{x}(t) = -\mu x(t) + g(\phi(t-1)) \qquad \text{for all } t \in I_1.$$

Defining $y(t) = e^{\mu t} x(t)$, $t \geq 0$, we have

$$\dot{y}(t) = e^{\mu t} g(\phi(t-1)) \qquad \text{for all } t \in I_1,$$

and

$$\dot{y}(t) = e^{\mu t} g(x(t-1)) \qquad \text{for all } t > 1.$$

From $|\phi(s)| = r$, $s \in I_0$, it follows that

$$|g(\phi(t-1))| \geq mr \qquad \text{for all } t \in I_1.$$

Thus

$$|\dot{y}(t)| \geq mr \qquad \text{for all } t \in I_1,$$

and $|I_1| = \frac{1}{2N}$. Applying the claim of part 3.1 we get an open interval $J_1 \subset I_1 \subset (0,1)$ so that

$$|y(t)| \geq \frac{mr}{2^3 N} \qquad \text{for all } t \in J_1,$$

and $|J_1| = \frac{1}{2^3 N}$.

For all $t > 1$,

$$|\dot{y}(t)| = e^{\mu t}|g(x(t-1))| \geq e^{\mu t} m|x(t-1)| = me^{\mu}|y(t-1)|$$

holds. Setting $I_2 = J_1 + 1 \subset (1,2)$, one obtains

$$|\dot{y}(t)| \geq \frac{m^2 e^{\mu} r}{2^3 N} \qquad \text{for all } t \in I_2,$$

and

$$|I_2| = \frac{1}{2^3 N}.$$

The claim of part 3.1 gives an open interval $J_2 \subset I_2 \subset (1,2)$ so that

$$|y(t)| \geq \frac{m^2 e^{\mu} r}{2^8 N^2} \qquad \text{for all } t \in J_2,$$

and

$$|J_2| = \frac{1}{2^5 N}.$$

Repeating the above argument twice, we find an open interval $J_4 \subset (3,4)$ such that $|J_4| = \frac{1}{4^2}|J_2| = \frac{1}{2^9 N}$ and

$$|y(t)| \geq \frac{m^4 e^{3\mu} r}{2^{24} N^4} \qquad \text{for all } t \in J_4.$$

Using $x(t) = e^{-\mu t} y(t)$, $t \geq 0$, we conclude

$$|x(t)| \geq e^{-4\mu}|y(t)| \geq \frac{m^4 e^{-\mu} r}{2^{24} N^4} \qquad \text{for all } t \in J_4.$$

Consequently,

$$||x_4|| \geq \frac{m^4 e^{-\mu} r}{2^{24} N^4}.$$

□

Theorem 3.2 *Assume that hypotheses (H0) and (H1) hold, $N \in \mathbb{N} \setminus \{0\}$ and Eq. (1.1) has a periodic orbit $\mathcal{O}$ in $V^{-1}(2N)$. Then Eq. (1.1) has a solution $y : [-1,\infty) \to \mathbb{R}$ such that*

$$y_t \in R,\ y_t - \psi \in R,\ V(y_t) = V(y_t - \psi) = 2N \qquad \textit{for all } t \geq 0 \textit{ and } \psi \in \mathcal{O},$$

and

$$\Pi_2 y_t \in \text{ext}(\Pi_2 \mathcal{O}) \qquad \textit{for all } t \geq 0.$$

Proof Let $p:\mathbb{R}\to\mathbb{R}$ be a periodic solution of Eq. (1.1) such that $\mathcal{O}=\{p_t : t\in\mathbb{R}\}$. From Proposition 2.1 it follows that

$$\xi^- \le p(t) \le \xi^+ \qquad \text{for all } t\in\mathbb{R}.$$

The definition of ξ^-, ξ^+ yields $f(\xi^-)=\mu\xi^-$, $f(\xi^+)=\mu\xi^+$ and $f(\xi)>\mu\xi$ for $-\infty<\xi<\xi^-$, $f(\xi)<\mu\xi$ for $\xi^+<\xi<\infty$. Then it is easy to find $\eta^-\in(-\infty,\xi^-)$ and $\eta^+\in(\xi^+,\infty)$ such that

$$f'(\eta^-)<\mu, \quad f'(\eta^+)<\mu.$$

Let the function $g:\mathbb{R}\to\mathbb{R}$ be defined so that $g(\xi)=f(\xi)$ for $\eta^-\le\xi\le\eta^+$, $g(\xi)=f(\eta^-)+f'(\eta^-)(\xi-\eta^-)$ for $-\infty<\xi<\eta^-$, and $g(\xi)=f(\eta^+)+f'(\eta^+)(\xi-\eta^+)$ for $\eta^+<\xi<\infty$. Then g satisfies the conditions of Proposition 3.1. Clearly, $\mathcal{O}$ is a periodic orbit of Eq. (3.1) as well.

Choose $a>0$ so that

$$\Pi_2\mathcal{O}\subset\mathbb{R}^2_a.$$

Then, in particular, $a>\max_{t\in\mathbb{R}}|p(t)|$ follows. Set

$$r=\left(\frac{2^{24}N^4e^{\mu}}{m^4}+1\right)a.$$

By Proposition 3.1 we find $\phi\in X_r^{2N}$ such that the function $x=x(\phi):[0,\infty)\to\mathbb{R}$ satisfies

$$V(x_t)=2N \qquad \text{for all } t\ge 1,$$

$$||x_4||\ge a.$$

Let $(s_j)_0^{2N}$ be the sequence associated with $\phi\in X_r^{2N}$ by its definition. We can select $\epsilon>0$ so that

$$\epsilon<\frac{1}{2}\min\{s_i-s_{i-1} : i\in\{1,2,\dots,N\} \text{ and } s_i-s_{i-1}>0\}.$$

Choose $n_0\in\mathbb{N}$ with $n_0>r/\epsilon$. For each integer $n\ge n_0$ define the function $\phi^n:[-1,0]\to\mathbb{R}$ as follows. Let $\phi^n(-1)=\phi(s_0+\epsilon)$. If $s_i\in(-1,0)$ and $\operatorname{sign}\phi(s_i-\epsilon)=\operatorname{sign}\phi(s_i+\epsilon)$, then let $\phi^n(s_i)=\phi(s_i+\epsilon)$. If $s_i\in(-1,0)$ and $\operatorname{sign}\phi(s_i-\epsilon)\ne\operatorname{sign}\phi(s_i+\epsilon)$, then let

$$\phi^n(s)=n(s-s_i)\operatorname{sign}\phi(s_i+\epsilon) \qquad \text{for all } s\in\left(s_i-\frac{r}{n}, s_i+\frac{r}{n}\right).$$

For $-r/n<s\le 0$, let $\phi^n(s)=-ns\operatorname{sign}\phi(-\epsilon)$. Otherwise, set $\phi^n(s)=\phi(s)$. Then $\phi^n\in C$, $\phi^n\to\phi$ almost everywhere in $[-1,0]$ as $n\to\infty$, $|\phi^n-\phi|_{L^1}\to 0$ as $n\to\infty$. It also follows that

$$x^{\phi^n}|_{[0,\infty)}\to x(\phi) \qquad \text{as } n\to\infty$$

uniformly on compact subsets of $[0,\infty)$, where x^{ϕ^n} denotes the solution of Eq. (3.1) with $x_0^{\phi^n}=\phi^n$. Using Eq. (3.1) we find that

$$\dot{x}^{\phi^n}|_{[1,\infty)}\to\dot{x}(\phi)|_{[1,\infty)} \qquad \text{as } n\to\infty$$

uniformly on compact subsets of $[1,\infty)$.

Let $t\ge 1$ and $s\in\mathbb{R}$ be fixed. Select an integer $n_1\ge n_0$ with $n_1>\max_{t\in\mathbb{R}}|\dot{p}(t)|$. Then $r>a>\max_{t\in\mathbb{R}}|p(t)|$ and the choice of n_1 combined yield

$$V(\phi^n-p_\tau)\le 2N \qquad \text{for all } n\ge n_1 \text{ and } \tau\in\mathbb{R}.$$

The monotonicity of V implies

$$V(x_t^{\phi^n} - p_\tau) \leq 2N \qquad \text{for all } n \geq n_1 \text{ and } \tau \in \mathbb{R}.$$

Using $||x_t^{\phi^n} - x_t(\phi)|| \to 0$ as $n \to \infty$ and the lower semicontinuity of V, we conclude

$$V(x_t(\phi) - p_s) \leq 2N.$$

Lemma 2.3(iii) gives $u_0 \geq 3$ and $k \in \{0, 1, \ldots, N\}$ so that

$$V(x_{t+u}(\phi) - p_{s+u}) = 2k \qquad \text{for all } u \geq u_0$$

and

$$x_{t+u}(\phi) - p_{s+u} \in R \qquad \text{for all } u \geq u_0.$$

Using these facts, Lemma 2.3(ii) yields a $\delta_0 > 0$ such that

$$V(x_{t+u_0}(\phi) - p_{s+u_0+\delta}) = 2k \qquad \text{for all } \delta \in [0, \delta_0).$$

Hence

$$V(x_{t+u}(\phi) - p_{s+u+\delta}) \leq 2k$$

follows for all $\delta \in [0, \delta_0)$ and $u \geq u_0$.

Consider the ω-limit set $\omega(x_1(\phi))$ of the solution $x(\phi) : [0, \infty) \to \mathbb{R}$ of Eq. (3.1). Proposition 2.6 can be applied. Assume that $\omega(x_1(\phi))$ is not a periodic orbit of Eq. (3.1). As $V(x_t(\phi)) = 2N$ for all $t \geq 1$, $\omega(x_1(\phi))$ cannot contain a nonzero stationary point by Proposition 2.13. Therefore, $0 \in \omega(x_1(\phi))$. Then there is a sequence $(s_n)_0^\infty$ in $(0, \infty)$ with $s_n \to \infty$ and $x_{t+s_n}(\phi) \to 0$ as $n \to \infty$. Without loss of generality we may assume $p_{s+s_n} \to p_\tau$ as $n \to \infty$ for some $\tau \in \mathbb{R}$. Lemma 2.3(i) yields

$$2N = V(p_\tau) \leq \liminf_{n\to\infty} V(x_{t+s_n}(\phi) - p_{s+s_n}) = 2k.$$

Thus $k = N$ and by the monotonicity of V, we conclude

$$V(x_t(\phi) - p_s) = 2N.$$

Now assume that $\omega(x_1(\phi)) = \{q_t : t \in \mathbb{R}\}$, where $q : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of Eq. (3.1). By Proposition 2.13, $V(q_t) = 2N$ for all $t \in \mathbb{R}$. Then there exist a sequence $(u_n)_0^\infty$ in $(0, \infty)$ and reals τ_1, τ_2 such that $u_n \to \infty$ and

$$x_{t+u_n}(\phi) \to q_{\tau_1}, \qquad p_{s+\delta+u_n} \to p_{\tau_2+\delta}$$

for all $\delta \in [0, \delta_0)$ as $n \to \infty$. Fix $\delta \in (0, \delta_0)$ with $p_{\tau_2+\delta} \neq q_{\tau_1}$. Hence the lower semicontinuity of V yields

$$V(q_{\tau_1} - p_{\tau_2+\delta}) \leq 2k.$$

Lemma 2.10 gives $k \geq N$. Thus $k = N$ and $V(x_t(\phi) - p_s) = 2N$.

As $t \geq 1$ and $s \in \mathbb{R}$ were arbitrary, we obtain

$$V(x_t(\phi) - p_s) = 2N \qquad \text{for all } t \geq 1 \text{ and } s \in \mathbb{R}.$$

Applying Lemma 2.3(iii) it follows that

$$x_t(\phi) \in R,\ x_t(\phi) - p_s \in R \qquad \text{for all } t \geq 4 \text{ and } s \in \mathbb{R}.$$

Set

$$z : [-1, \infty) \ni t \mapsto x(\phi)(t+4) \in \mathbb{R}.$$

Then

$$z_t \in R,\ z_t - \psi \in R,\ V(z_t) = V(z_t - \psi) = 2N \qquad \text{for all } t \geq 0,\ \psi \in \mathcal{O}.$$

Consequently, $\Pi_2 z_t \notin \Pi_2\mathcal{O}$ for all $t \geq 0$. Using $||x_4(\phi)|| \geq a$, we find $s \in [3,4]$ with $|x(\phi)(s)| \geq a$. Then

$$\Pi_2 z_{s-3} = \Pi_2 x_{s+1}(\phi) = (x(\phi)(s+1), x(\phi)(s))^{tr} \notin \mathbb{R}^2_a.$$

By the choice of a, we have $\Pi_2\mathcal{O} \subset \mathbb{R}^2_a$. These facts yield $\Pi_2 z_{s-3} \in \mathrm{ext}(\Pi_2\mathcal{O})$. Consequently,

$$\Pi_2 z_t \in \mathrm{ext}\,(\Pi_2\mathcal{O}) \qquad \text{for all } t \geq 0.$$

Proposition 2.1 can be applied to obtain a $T \geq 0$ so that $z(t) \in (\eta^-, \eta^+)$ for all $t \geq T-1$. Then the function

$$y : [-1,\infty) \ni t \mapsto z(t+T) \in \mathbb{R}$$

is a solution of Eq. (1.1) with the desired properties. □

4 Unstable sets of periodic orbits

In this section we give sufficient conditions for the equality $W^u(\mathcal{O}) = W^u_{str}(\mathcal{O})$ for a periodic orbit $\mathcal{O}$ guaranteed by Proposition 2.8. The first result excludes the existence of two solutions $x : \mathbb{R} \to \mathbb{R}$ and $y : [-1,\infty) \to \mathbb{R}$ with $\alpha(x) = \omega(y_0) = \mathcal{O}$ and with certain additional properties.

Proposition 4.1 *Assume that hypotheses (H0) and (H1) hold. Let $N \in \mathbb{N}\backslash\{0\}$ and let $\mathcal{O}$ be a periodic orbit of Eq. (1.1) in $V^{-1}(2N)$. Then Eq. (1.1) does not have two solutions $x : \mathbb{R} \to \mathbb{R}$ and $y : [-1,\infty) \to \mathbb{R}$ so that*

$$\alpha(x) = \omega(y_0) = \mathcal{O},$$

$$V(x_t) = V(y_s) = V(x_t - \psi) = V(y_s - \psi) = 2N \qquad \textit{for all } t \leq 0,\ s \geq 0,\ \psi \in \mathcal{O},$$

$$x_t \in R,\ y_s \in R,\ x_t - \psi \in R,\ y_s - \psi \in R \qquad \textit{for all } t \leq 0,\ s \geq 0,\ \psi \in \mathcal{O},$$

and $\{\Pi_2 x_t : t \leq 0\}$, $\{\Pi_2 y_s : s \geq 0\}$ belong to the same open connected component of $\mathbb{R}^2 \setminus \Pi_2\mathcal{O}$.

Proof 1. Assume that Eq. (1.1) has two solutions $x : \mathbb{R} \to \mathbb{R}$ and $y : [-1,\infty) \to \mathbb{R}$ with the stated properties. We want to get a contradiction.

We claim that there are $t_1 \leq 0$ and $s_1 \geq 0$ with

$$V(x_t - y_s) = 2N \qquad \text{for all } t \leq t_1,\ s \geq s_1.$$

Let $p : \mathbb{R} \to \mathbb{R}$ be the periodic solution with minimal period $\omega > 0$ such that $\mathcal{O} = \{p_t : t \in \mathbb{R}\}$ and $p(0) = 0$, $p(-1) > 0$. Consider the closed curves

$$c : [0,\omega] \ni s \mapsto p_s - y_0 \in C^1,$$
$$d : [0,\omega] \ni s \mapsto p_s - x_0 \in C^1.$$

By assumption, $|c| \subset R$ and $|d| \subset R$. The traces $|c|$ and $|d|$ are compact subsets of C^1. There exist $\epsilon > 0$ and ϵ-neighbourhoods $N_{c,\epsilon}$, $N_{d,\epsilon}$ of $|c|$, $|d|$, respectively, in C^1 such that

$$V(\eta) = 2N \qquad \text{for all } \eta \in N_{c,\epsilon} \cup N_{d,\epsilon}.$$

The sets $N_{c,\epsilon} + y_0$ and $N_{d,\epsilon} + x_0$ are C^1-neighbourhoods of $\mathcal{O}$. Using Eq. (1.1) and the assumptions $\mathrm{dist}(x_t, \mathcal{O}) \to 0$ as $t \to -\infty$, $\mathrm{dist}(y_s, \mathcal{O}) \to 0$ as $s \to \infty$, we obtain that

$$\mathrm{dist}_{C^1}(x_t, \mathcal{O}) \to 0 \quad \text{as } t \to -\infty, \quad \mathrm{dist}_{C^1}(y_s, \mathcal{O}) \to 0 \quad \text{as } s \to \infty.$$

Then we find $t_1 \leq 0$ and $s_1 \geq 0$ so that

$$x_t \in N_{c,\epsilon} + y_0 \qquad \text{for all } t \leq t_1,$$

$$y_s \in N_{d,\epsilon} + x_0 \qquad \text{for all } s \geq s_1.$$

Let $t \leq t_1$ and $s \geq s_1$. Then $y_{s-t} - x_0 \in N_{d,\epsilon}$ and $x_{t-s} - y_0 \in N_{c,\epsilon}$. Consequently,

$$2N = V(x_0 - y_{s-t}) \leq V(x_t - y_s) \leq V(x_{t-s} - y_0) = 2N.$$

2. Set $t_2 = t_1$ and $s_2 = s_1 + 3$. Then, for all $t \leq t_2$ and $s \geq s_2$, we have

$$V(x_t - y_s) = V(x_{t-3} - y_{s-3}) = 2N.$$

It follows from Lemma 2.3(iii) that $x_t - y_s \in R$. A corollary of this fact is that the curves

$$(-\infty, t_2] \ni t \mapsto \Pi_2 x_t \in \mathbb{R}^2$$

and

$$[s_2, \infty) \ni s \mapsto \Pi_2 y_s \in \mathbb{R}^2$$

do not intersect.

3. From $\operatorname{dist}(x_t, \mathcal{O}) \to 0$ as $t \to -\infty$, by using Eq. (1.1), we get that

$$\operatorname{dist}_{C^1}(x_t, \mathcal{O}) \to 0 \qquad \text{as } t \to -\infty.$$

Differentiating Eq. (1.1) it also follows that $x_t \in C^2$, $t \leq 0$, $\mathcal{O} \subset C^2$, moreover

$$\operatorname{dist}_{C^2}(x_t, \mathcal{O}) \to 0 \qquad \text{as } t \to -\infty.$$

Let t^0 denote the maximal zero of x on $(-\infty, 0]$ with $\dot{x}(t^0) > 0$. If t^n is defined for some $n \in -\mathbb{N}$, then let t^{n-1} be the greatest zero of x on $(-\infty, t^n)$. Then, applying $x_t \in R$ for all $t \leq 0$, for the sequence $(t^n)_{-\infty}^0$ in $(-\infty, 0]$ we obtain

$$x(t^n) = 0 \qquad \text{for all } n \in -\mathbb{N},$$

$$\dot{x}(t^{2n}) > 0,\ \dot{x}(t^{2n-1}) < 0 \qquad \text{for all } n \in -\mathbb{N},$$

$$t^n \to -\infty \qquad \text{as } n \to -\infty.$$

Recall from Section 2 that $p_t \in R$ and $\dot{p}_t \in R$ for all $t \in \mathbb{R}$, and thus all zeros of p and $\dot{p}$ are simple. These results, the fact $\operatorname{dist}_{C^2}(x_t, \mathcal{O}) \to 0 \quad$ as $t \to -\infty$, and Proposition 2.9 combined imply that $x_{t^{2n}} \to p_0$ as $n \to -\infty$, and that there exists $n_0 \in -\mathbb{N}$ such that

$$\dot{x} \text{ has exactly one zero in } (t^{n-1}, t^n) \text{ for all } n_0 \geq n \in -\mathbb{N}.$$

Then, for every integer $n \leq n_0$, the curve

$$[t^{2n-2}, t^{2n}) \ni t \mapsto \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \mathbb{R}^2$$

is injective. Set

$$X : (-\infty, 0] \ni t \mapsto \Pi_2 x_t \in \mathbb{R}^2.$$

As $x(t)$ and $x(t-1)$ uniquely determine $\dot{x}(t)$, it follows that, for each integer $n \leq n_0$, the restriction $X|_{[t^{2n-2}, t^{2n})}$ is also injective. Observe that $x(t^{2n} - 1) > 0$ and $x(t^{2n-1} - 1) < 0$ for all $n \in -\mathbb{N}$ since $x_t \in R$, $t \leq 0$. Then we have

$$X|_{[t^{2n-2}, t^{2n})} \cap \{(0, v)^{tr} \in \mathbb{R}^2 : v \in \mathbb{R}\} = \{(0, x(t^{2n-2} - 1))^{tr}, (0, x(t^{2n-1} - 1))^{tr}\}.$$

4. Assume that $\{\Pi_2 x_t : t \le 0\}$ and $\{\Pi_2 y_s : s \ge 0\}$ belong to $\mathrm{ext}(\Pi_2\mathcal{O})$. The unbounded set

$$U = \{(0,v)^{tr} \in \mathbb{R}^2 : v > p(-1)\}$$

is a subset of $\mathrm{ext}(\Pi_2\mathcal{O})$. By Proposition 2.9,

$$V = \{(0,v)^{tr} \in \mathbb{R}^2 : 0 \le v < p(-1)\} \subset \mathrm{int}(\Pi_2\mathcal{O}).$$

As $V(y_s) = 2N \ge 2$ and $y_s \in R$ for all $s \ge 0$, there exists $s^* \ge s_2$ such that $y(s^*) = 0$ and $\dot{y}(s^*) > 0$. Then $y(s^* - 1) > 0$. Using also $\Pi_2 y_{s^*} \in \mathrm{ext}(\Pi_2\mathcal{O})$ we find a real $v_1 > p(-1)$ so that

$$\Pi_2 y_{s^*} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \in U.$$

From $x_{t^{2n}} \to p_0$ as $n \to -\infty$ and $x(t^{2n}) = 0$, $x(t^{2n} - 1) > 0$, $n \in -\mathbb{N}$, we obtain an integer $k \le n_0$, reals v_2, v_3 such that $t^{2k} < t_2$, $p(-1) < v_2 < v_3 < v_1$ and

$$X(t^{2k-2}) = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \qquad X(t^{2k}) = \begin{pmatrix} 0 \\ v_3 \end{pmatrix}.$$

The line segment connecting $(0, v_2)^{tr}$, $(0, v_3)^{tr}$, and the injective curve $X|_{[t^{2k-2}, t^{2k})}$ form a simple closed curve γ. It is easy to see that the sets $\{(0,v)^{tr} \in \mathbb{R}^2 : 0 \le v < v_2\}$ and $\{(0,v)^{tr} \in \mathbb{R}^2 : v_3 < v\}$ belong to different connected components of $\mathbb{R}^2 \setminus |\gamma|$. As $\{(0,v)^{tr} \in \mathbb{R}^2 : v_3 < v\}$ is unbounded, we conclude

$$0 \in \mathrm{int}(\gamma), \qquad (0, v_1)^{tr} \in \mathrm{ext}(\gamma).$$

Using $|\gamma| \subset \mathrm{ext}(\Pi_2\mathcal{O})$, we obtain

$$\Pi_2\mathcal{O} \subset \mathrm{int}(\gamma).$$

We claim that

$$\Pi_2 y_s \in \mathrm{ext}(\gamma) \qquad \text{for all } s \ge s^*.$$

If this is not true, then there exists $s^{**} > s^*$ so that $\Pi_2 y_s \in \mathrm{ext}(\gamma)$ for $s^* \le s < s^{**}$ and $\Pi_2 y_{s^{**}} \in |\gamma|$. By the result of part 2, $\Pi_2 y_{s^{**}} \notin X|_{[t^{2k-2}, t^{2k}]}$. So, there is $v_4 \in (v_2, v_3)$ so that

$$\Pi_2 y_{s^{**}} = \begin{pmatrix} y(s^{**}) \\ y(s^{**} - 1) \end{pmatrix} = \begin{pmatrix} 0 \\ v_4 \end{pmatrix}$$

We can find a $\delta > 0$ such that

$$\{(u,v)^{tr} \in \mathbb{R}^2 : -\delta < u < 0,\ |v - v_4| < \delta\} \subset \mathrm{int}(\gamma)$$

and

$$\{(u,v)^{tr} \in \mathbb{R}^2 : 0 < u < \delta,\ |v - v_4| < \delta\} \subset \mathrm{ext}(\gamma).$$

On the other hand, the equation for y yields $\dot{y}(s^{**}) > 0$. This implies $\Pi_2 y_s \in \mathrm{int}(\gamma)$ for some $s < s^{**}$ sufficiently close to s^{**}. This is a contradiction.

The above claim, the facts $\Pi_2\mathcal{O} \subset \mathrm{int}\,\gamma$ and $\mathrm{dist}(y_s, \mathcal{O}) \to 0$ as $s \to \infty$ combined give a contradiction.

The assumption $\{\Pi_2 x_t : t \le 0\} \subset \mathrm{int}(\Pi_2\mathcal{O})$, $\{\Pi_2 y_s : s \ge 0\} \subset \mathrm{int}(\Pi_2\mathcal{O})$ analogously leads to a contradiction. □

Now we prove that in case $W^u(\mathcal{O}) \ne W^u_{str}(\mathcal{O})$ there is a globally defined solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with some of the properties assumed in Proposition 4.1.

Proposition 4.2 *Assume that hypotheses (H0) and (H1) hold. Let* $N \in \mathbb{N} \setminus \{0\}$ *be given so that* $\operatorname{Re} \lambda_N > 0$. *Let* $\mathcal{O} = \mathcal{O}_N$ *be the periodic orbit of Eq. (1.1) in* $V^{-1}(2N)$ *given by Proposition 2.8. If* $W^u(\mathcal{O}) \neq W^u_{str}(\mathcal{O})$ *then there exists a bounded solution* $x : \mathbb{R} \to \mathbb{R}$ *of Eq. (1.1) so that*

$$\alpha(x) = \mathcal{O},$$

$$V(x_t) = V(x_t - \psi) = 2N \qquad \text{for all } t \leq 0,\ \psi \in \mathcal{O},$$

$$x_t \in R,\ x_t - \psi \in R \qquad \text{for all } t \leq 0,\ \psi \in \mathcal{O}.$$

Proof Let $p : \mathbb{R} \to \mathbb{R}$ be the periodic solution of Eq. (1.1) such that $p(0) = 0$, $p(-1) > 0$ and $\mathcal{O} = \{p_t : t \in \mathbb{R}\}$. Assume $W^u(\mathcal{O}) \neq W^u_{str}(\mathcal{O})$. Then there exist $\phi \in W^u(\mathcal{O}) \setminus W^u_{str}(\mathcal{O})$ and a bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) so that $x_0 = \phi$, $\alpha(x) = \mathcal{O}$ and $x_t \notin W^u(p_0, F(\omega, \cdot), N^u)$ for all $t \in \mathbb{R}$. Observe that the set $\{x_t : t \in \mathbb{R}\} \cup \mathcal{O}$ is a subset of the global attractor.

1. First we claim that

$$V(x_t - \psi) \leq 2N \qquad \text{for all } t \in \mathbb{R},\ \psi \in \mathcal{O}.$$

Let $t \in \mathbb{R}$ and $\psi = p_\tau$ be fixed. From $\alpha(x) = \mathcal{O}$ it follows that there exist a sequence $(t_n)^0_{-\infty}$ and reals $r, s \in [0, \omega)$ such that $t_n \to -\infty$ and

$$x_{t+t_n} \to p_r, \quad p_{\tau+t_n} \to p_s \qquad \text{as } n \to -\infty.$$

The above sequences converge in the C^1-topology as well. If $r \neq s$, then $p_r - p_s \in R$ and $V(p_r - p_s) = 2N$ by Proposition 2.10. Then Lemma 2.3(ii) yields

$$V(x_{t+t_n} - p_{\tau+t_n}) = 2N \qquad \text{for all sufficiently large negative } n.$$

Hence the monotonicity of V gives $V(x_t - p_\tau) \leq 2N$. If $r = s$ then $p_{r+\epsilon} \neq p_r$ for all $\epsilon \in (0, \epsilon_0)$ with $\epsilon_0 = \omega - r$. Then the above proof shows also $V(x_{t+\epsilon} - p_\tau) \leq 2N$. By the lower semicontinuity of V,

$$V(x_t - p_\tau) \leq \liminf_{\epsilon \to 0+} V(x_{t+\epsilon} - p_\tau) \leq 2N.$$

2. Assume that $s \leq 0$ has the property that the sequence $(\phi_n)^0_{-\infty}$, defined by $\phi_n = x^\phi_{s+n\omega}$ for all $n \in -\mathbb{N}$, has a subsequence converging to p_0 as $n \to -\infty$. We claim that $V(\phi_n - p_0) = 2N$ for all sufficiently large negative integers n.

Recall from Section 2 that $\lambda \in (0, 1)$ is fixed so that

$$\lambda > \max\left\{ \max_{\zeta \in \sigma, |\zeta| > 1} \frac{1}{|\zeta|}, \max_{\zeta \in \sigma, |\zeta| < 1} |\zeta| \right\},$$

where σ denotes the spectrum of the monodromy operator $M = D_2F(\omega, p_0)$.

We first show that $(\lambda^n(\phi_n - p_0))^0_{-\infty}$ does not converge to 0 as $n \to -\infty$. Assume $\lambda^n(\phi_n - p_0) \to 0$ as $n \to -\infty$. For every negative integer k, $\eta_n = \phi_{k+n}$ satisfies

$$\lambda^n(\eta_n - p_0) = \lambda^n(\phi_{k+n} - p_0) = \lambda^{-k}\lambda^{k+n}(\phi_{k+n} - p_0).$$

Therefore, if k is a sufficiently large negative integer, then $\lambda^n(\eta_n - p_0) \in N^u$ for all $n \in -\mathbb{N}$, and $\lim_{n \to -\infty} \lambda^n(\eta_n - p_0) = 0$. Thus $\eta_0 = \phi_k = x^\phi_{s+k\omega} \in W^u(p_0, F(\omega, \cdot), N^u)$, which is a contradiction to our assumption.

Choose $\lambda^* \in (0, \lambda)$ such that

$$\lambda^* > \max_{\zeta \in \sigma, |\zeta| > 1} \frac{1}{|\zeta|}.$$

Let $\nu \in \left(1/\lambda^*, \min_{\zeta\in\sigma, |\zeta|>1} |\zeta|\right)$. By Theorem I.1 in [**16**], there exists an equivalent norm $|\cdot|_e$ on C such that

$$|M\psi|_e \geq \nu|\psi|_e \qquad \text{for all } \psi \in C_{1<}.$$

Choose $\delta > 0$ so that $\{\psi \in C : |\psi|_e < \delta\} \subset N^u$.

We construct a subsequence $(\phi_{n_k})_{-\infty}^0$ of $(\phi_n)_{-\infty}^0$ such that $\phi_{n_k} \to p_0$ as $k \to -\infty$ and

$$\frac{|\lambda^{*n_k-1}(\phi_{n_k-1} - p_0)|_e}{|\lambda^{*n_k}(\phi_{n_k} - p_0)|_e} \geq 1 \tag{4.1}$$

holds for all $k \in -\mathbb{N}$. In order to define n_0, choose an integer $m \in -\mathbb{N}$ such that $|\phi_m - p_0|_e < \delta = \frac{\delta}{2^0}$. If (4.1) holds with $n_k = m$, then let $n_0 = m$. Otherwise, we note that if (4.1) does not hold with $n_k = m, m-1, \ldots, m-j$ for some $j \in \mathbb{N}$, then

$$\begin{aligned} &|\phi_{m-j-1} - p_0|_e < \lambda^*|\phi_{m-j} - p_0|_e \\ &< {\lambda^*}^2|\phi_{m-j+1} - p_0|_e < \ldots < {\lambda^*}^{j+1}|\phi_m - p_0|_e. \end{aligned} \tag{4.2}$$

If this is the case for all $j \in \mathbb{N}$, then with $\eta_n = \phi_{m+n}$, $n \in -\mathbb{N}$, it follows that

$$|\lambda^n(\eta_n - p_0)|_e = |\lambda^n(\phi_{m+n} - p_0)|_e \leq \left(\frac{\lambda}{\lambda^*}\right)^n |\phi_m - p_0|_e < \delta$$

for all $n \in -\mathbb{N}$, and $\lim_{n\to-\infty} \lambda^n(\eta_n - p_0) = 0$. Hence

$$\phi_m = x^\phi_{s+m\omega} \in W^u(p_0, F(\omega, \cdot), N^u)$$

follows, a contradiction. Therefore, there is a maximal $j \in \mathbb{N}$ such that (4.2) holds. Now define $n_0 = m - j - 1$. Then (4.1) holds with $k = 0$. Assume that $n_0, n_{-1}, \ldots, n_l$ are defined for some $l \in -\mathbb{N}$. In order to define n_{l-1}, we choose $m \in -\mathbb{N}$ such that $m < n_l$ and

$$|\phi_m - p_0|_e < \frac{\delta}{2^{-l+1}}.$$

If (4.1) holds with $n_k = m$, then let $n_{l-1} = m$. If (4.1) does not hold with $n_k = m$ then the same argument as above shows that there exists a maximal $j \in \mathbb{N}$ such that (4.2) holds. In this case define $n_{l-1} = m - j - 1$. Then $(\phi_{n_k})_{-\infty}^0$ is defined by induction and has the desired properties.

For every $k \in -\mathbb{N}$ the function

$$z^k = \frac{1}{|\phi_{n_k} - p_0|_e}\left(x^{\phi_{n_k}} - p\right)$$

is a solution of the equation

$$\dot{z}(t) = -\mu z(t) + \int_0^1 f'\left(ux^{\phi_{n_k}}(t-1) + (1-u)p(t-1)\right) du\, z(t-1)$$

with $|z_0^k|_e = 1$, and $V(z_t^k) \leq 2N$ for all $t \in \mathbb{R}$ by part 1. As $\phi_{n_k} \to p_0$ and F_A is a continuous flow on A,

$$b^k(t) = \int_0^1 f'\left(ux^{\phi_{n_k}}(t-1) + (1-u)p(t-1)\right) du \to f'(p(t-1)) \qquad \text{as } k \to -\infty$$

uniformly on compact subsets of $\mathbb{R}$. As x is bounded, we can find positive constants b_0 and b_1 such that $b_0 \leq b^k(t) \leq b_1$ for all $t \leq 0$ and $k \in -\mathbb{N}$. Then, by Lemma 2.4, there is $c > 0$ with $||z_t^k|| \leq ce^{c|t|}$ for all $t \leq 0$ and $k \in -\mathbb{N}$. Using the differential equations for z^k we can apply the Arzela–Ascoli theorem to get a subsequence $(z^{k_i})_{-\infty}^0$ of $(z^k)_{-\infty}^0$ and a C^1-function $z : (-\infty, 0] \to \mathbb{R}$ such that $z^{k_i}|_{(-\infty,0]} \to z$

and $\dot{z}^{k_i}|_{(-\infty,0]} \to \dot{z}$ as $i \to -\infty$ uniformly on compact subsets of $(-\infty, 0]$, and z satisfies

$$\dot{z}(t) = -\mu z(t) + f'(p(t-1))z(t-1) \tag{4.3}$$

for all $t \leq 0$. It also follows that $|z_0|_e = 1$. From

$$|z^k_{-\omega}|_e = \frac{|x^{\phi_{n_k}}_{-\omega} - p_{-\omega}|_e}{|\phi_{n_k} - p_0|_e} = \frac{|\lambda^{*n_k-1}(\phi_{n_k-1} - p_0)|_e}{|\lambda^{*n_k}(\phi_{n_k} - p_0)|_e}\lambda^*$$

and property (4.1) of $(\phi_{n_k})^0_{-\infty}$, we infer $|z^k_{-\omega}|_e \geq \lambda^*$, $k \in -\mathbb{N}$. Hence

$$|z_{-\omega}|_e \geq \lambda^*.$$

Suppose the statement

$$V(\phi_n - p_0) = 2N \qquad \text{for all sufficiently large negative integers } n$$

is false. Then, from $V(\phi_n - p_0) \leq 2N$ for all $n \in -\mathbb{N}$ and from the monotonicity of V, we get $V(\phi_n - p_0) \leq 2N-2$ for all $n \in -\mathbb{N}$. Extending z to a solution $v : \mathbb{R} \to \mathbb{R}$ of (4.3) and using the monotonicity of V, we find

$$V(v_t) \leq 2N-2 \qquad \text{for all } t \in \mathbb{R}.$$

Then, Proposition 2.11(iii) implies $v_{-\omega} = z_{-\omega} \in C_{1<}$. Then

$$1 = |z_0|_e = |Mz_{-\omega}|_e \geq \nu|z_{-\omega}|_e \geq \nu\lambda^* > 1,$$

a contradiction. So, we have $V(\phi_n - p_0) = 2N$ for all sufficiently large negative integers n.

3. We prove that there exists $T_0 < 0$ with $V(x_t - p_0) = 2N$ for all $t \leq T_0$.

Assume that there exists a sequence $(t_n)^0_{-\infty}$ in $(-\infty, 0)$ with $t_n \to -\infty$ as $n \to -\infty$ and

$$V(x_{t_n} - p_0) \leq 2N-2 \qquad \text{for all } n \in -\mathbb{N}.$$

We claim that $x_{t_n} \to p_0$ as $n \to -\infty$. If not, then there exist a subsequence $(t_{n_k})^0_{-\infty}$ of $(t_n)^0_{-\infty}$ and $\tau \in (0,\omega)$ such that $x_{t_{n_k}} \to p_\tau$ as $k \to -\infty$ since $\alpha(x) = \mathcal{O}$. As $\tau \in (0,\omega)$, we have $V(p_\tau - p_0) = 2N$ by Proposition 2.10, and hence $V(x_{t_{n_k}} - p_0) = 2N$ for all sufficiently large negative integers k, a contradiction.

Each t_n can be uniquely written as $t_n = m_n\omega + \tau_n$ for some $m_n \in -\mathbb{N}$ and $\tau_n \in [0,\omega)$. We may assume (replacing $(t_n)^0_{-\infty}$ with a subsequence if necessary)

$$\tau_n \to \tau^* \in [0,\omega] \qquad \text{as } n \to -\infty.$$

We claim that

$$V(x_{\tau^*+u} - p_u) \leq 2N-2 \qquad \text{for all } u \in \mathbb{R}. \tag{4.4}$$

If (4.4) is not satisfied then there exists $\hat{u} < 0$ so that

$$V(x_{\tau^*+\hat{u}} - p_{\hat{u}}) = 2N.$$

By continuity, there is $\epsilon \in (0,1)$ so that

$$V(x_{\tau+\hat{u}} - p_{\hat{u}}) = 2N \qquad \text{for all } \tau \in \mathbb{R} \quad \text{with } |\tau - \tau^*| < \epsilon.$$

Choose $n \in -\mathbb{N}$ such that $|\tau_n - \tau^*| < \epsilon$ and $t_n = m_n\omega + \tau_n < \hat{u}$. Using

$$x_{\tau_n+\hat{u}} = F(\hat{u} - m_n\omega, x_{\tau_n+m_n\omega}), \quad p_{\hat{u}} = F(\hat{u} - m_n\omega, p_0)$$

and the monotonicity of V, we get

$$2N = V(x_{\tau_n+\hat{u}} - p_{\hat{u}}) \leq V(x_{\tau_n+m_n\omega} - p_0) = V(x_{t_n} - p_0) \leq 2N-2,$$

a contradiction. Therefore, (4.4) holds.

We also notice that

$$x_{\tau^*+m_n\omega} \to p_0 \quad \text{as} \quad n \to -\infty,$$

since $x_{\tau^*+m_n\omega} = F_A(\tau^* - \tau_n, F_A(m_n\omega + \tau_n, \phi))$ and $\tau_n \to \tau^*$, $F_A(m_n\omega + \tau_n, \phi) = x_{t_n} \to p_0$ as $n \to -\infty$. Then the result of part 2 with $s = \tau^* - \omega$ implies that $V(x_{\tau^*+n\omega} - p_0) = 2N$ for all sufficiently large negative integers n. This contradicts (4.4) and proves the existence of T_0.

4. We show that $V(x_t - \psi) = 2N$ for all $\psi \in \mathcal{O}$ and for all $t \le T_0 - \omega$.

Let $u \in [-\omega, 0]$. Then, from part 3 and the monotonicity of V it follows that

$$2N = V(x_t - p_0) \le V(x_{t+u} - p_u) \qquad \text{for all } t \le T_0.$$

On the other hand, $V(x_{t+u} - p_u) \le 2N$ by the result of part 1. This proves the assertion.

5. The facts $\alpha(x) = \mathcal{O}$, $\mathcal{O} \subset R \cap V^{-1}(2N)$ and that the C and C^1 topologies on A are equivalent give $T_1 \in \mathbb{R}$ such that $V(x_t) = 2N$ for all $t \le T_1$. Then the results of part 4, Lemma 2.3(iii) and a time shift easily show the existence of a globally defined solution with the required properties. □

Now we can prove the main result of this section.

Theorem 4.3 *Assume that hypotheses (H0), (H1) and (H2) hold, $N > 0$ is an integer so that*

$$f'(0) > \frac{\mu}{\cos\theta_N}$$

where $\theta_N \in (2N\pi - \pi/2, 2N\pi)$ is the unique solution of $\theta = -\mu\tan\theta$. Then Eq. (1.1) has a unique periodic orbit $\mathcal{O}_N$ in $V^{-1}(2N)$, and

$$W^u(\mathcal{O}_N) = W^u_{str}(\mathcal{O}_N).$$

Proof Propositions 2.2, 2.7 and 2.8 give that there is a unique periodic orbit $\mathcal{O}_N$ in $V^{-1}(2N)$. By Proposition 2.8 there exists a solution $z : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) so that

$$z_t \in R,\ z_t - \psi \in R,\ V(z_t) = V(z_t - \psi) = 2N \qquad \text{for all } t \in \mathbb{R} \text{ and } \psi \in \mathcal{O}_N$$

and

$$\alpha(z) = \{0\}, \quad \omega(z_0) = \mathcal{O}_N.$$

Then the curves $\mathbb{R} \ni t \mapsto \Pi_2 z_2 \in \mathbb{R}^2$ and $\Pi_2\mathcal{O}_N$ do not intersect. By Proposition 2.9 we have $0 \in \text{int}(\Pi_2\mathcal{O}_N)$. From $\alpha(z) = \{0\}$ it follows that $\Pi_2 z_t \in \text{int}(\Pi_2\mathcal{O}_N)$ for all sufficiently large negative t. Consequently,

$$\Pi_2 z_t \in \text{int}(\Pi_2\mathcal{O}_N) \qquad \text{for all } t \in \mathbb{R}.$$

Using $\lim_{\xi\to 0} \frac{\xi f'(\xi)}{f(\xi)} = 1$, (H2) yields $\frac{\xi f'(\xi)}{f(\xi)} < 1$ for all $\xi > 0$. Hence it follows that $(0,\infty) \ni \xi \mapsto \frac{f(\xi)}{\xi} \in \mathbb{R}$ is strictly decreasing. By assumption we have $f'(0) > \mu$. These facts and the oddness of f combined give that $0, \xi_-, \xi_+$ are the only stationary points of F.

Theorem 3.2 shows the existence of a solution $y : [-1,\infty) \to \mathbb{R}$ of Eq. (1.1) such that

$$y_t \in R,\ y_t - \psi \in R,\ V(y_t) = V(y_t - \psi) = 2N \qquad \text{for all } t \ge 0,\ \psi \in \mathcal{O}_N$$

and

$$\Pi_2 y_t \in \text{ext}(\Pi_2\mathcal{O}_N) \qquad \text{for all } t \ge 0.$$

Then $0 \notin \omega(y_0)$ since $0 \in \mathrm{int}(\Pi_2 \mathcal{O}_N)$ by Proposition 2.9. By Proposition 2.13(i), we have $\omega(y_0) \subset V^{-1}(2N) \cup \{0\}$. Thus, $\omega(y_0) \cap \{0, \xi_-, \xi_+\} = \emptyset$ follows. As $0, \xi_-, \xi_+$ are the only stationary points of F, and $\mathcal{O}_N$ is the only periodic orbit in $V^{-1}(2N)$, Proposition 2.6 implies $\omega(y_0) = \mathcal{O}_N$.

Assume $W^u(\mathcal{O}_N) \neq W^u_{str}(\mathcal{O}_N)$. Then Proposition 4.2 guarantees the existence of a solution $x : \mathbb{R} \to \mathbb{R}$ so that

$$x_t \in R,\ x_t - \psi \in R,\ V(x_t) = V(x_t - \psi) = 2N \qquad \text{for all } t \leq 0,\ \psi \in \mathcal{O}_N,$$

and $\alpha(x) = \mathcal{O}_N$. Then $\{\Pi_2 x_t : t \leq 0\} \cap \Pi_2 \mathcal{O}_N = \emptyset$.

In case $\{\Pi_2 x_t : t \leq 0\} \subset \mathrm{ext}(\Pi_2 \mathcal{O}_N)$, by Proposition 4.1 we have a contradiction. In case $\{\Pi_2 x_t : t \leq 0\} \subset \mathrm{int}(\Pi_2 \mathcal{O}_N)$, Proposition 4.1 with $y = z$ leads again to a contradiction. □

5 The structure of the global attractor

The equality $W^u(\mathcal{O}) = W^u_{str}(\mathcal{O})$ of Theorem 4.3 implies a result on the structure of the global attractor A. This is formulated in the next theorem.

Theorem 5.1 *Assume that hypotheses (H0), (H1) and (H2) hold, and let* $N > 0$ *be an integer such that*

$$\frac{\mu}{\cos\theta_N} < f'(0) < \frac{\mu}{\cos_{N+1}} \tag{5.1}$$

is satisfied where θ_N, θ_{N+1} *denote the unique solution of* $\theta = -\mu \tan\theta$ *in* $(2N\pi - \pi/2, 2N\pi)$, $(2(N+1)\pi - \pi/2, 2(N+1)\pi)$, *respectively. Then the semiflow* F *has exactly* 3 *stationary points* $0, \xi_-, \xi_+$ *and* N *periodic orbits* $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$, *and, for the global attractor* A *of* F, *we have*

$$A = \{\xi_-, \xi_+\} \cup W^u_{str}(0) \cup \left(\bigcup_{k=1}^{N} W^u_{str}(\mathcal{O}_k) \right). \tag{5.2}$$

Proof Proposition 2.1 and the remarks following it show that the semiflow has a global attractor A. As in the proof of Theorem 4.3 we obtain that $0, \xi_-, \xi_+$ are the only stationary points of F. We also saw that $\frac{\xi f'(\xi)}{f(\xi)} < 1$ for all $\xi > 0$. Hence $1 > \frac{\xi^+ f'(\xi^+)}{f(\xi^+)} = \frac{f'(\xi^+)}{\mu}$, that is $f'(\xi^+) < \mu$. From the oddness of f it also follows that $f'(\xi^-) < \mu$. Therefore, ξ_- and ξ_+ are locally asymptotically stable stationary points. By Proposition 2.2 and assumption (5.1), 0 is a hyperbolic and unstable stationary point. In particular, $W^u(0) = W^u_{str}(0)$.

Propositions 2.2, 2.7 and 2.8 imply that F has exactly N periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$, and $\mathcal{O}_k \subset V^{-1}(2k)$, $k \in \{1, 2, \ldots, N\}$. Theorem 4.3 shows $W^u(\mathcal{O}_k) = W^u_{str}(\mathcal{O}_k)$, $k \in \{1, 2, \ldots, N\}$.

Let $\phi \in A \setminus \{\xi_-, \xi_+\}$. By the invariance of A, there exists a solution $x : \mathbb{R} \to \mathbb{R}$ so that $x_0 = \phi$ and $x_t \in A$ for all $t \in \mathbb{R}$. Proposition 2.6 and the above facts give that either $\alpha(x) = \mathcal{O}_k$ for some $k \in \{1, 2, \ldots, N\}$ or, for every solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 \in \alpha(x)$, the sets $\alpha(y)$ and $\omega(y_0)$ consist of stationary points of F. In order to show (5.2) it suffices to verify that in case $\alpha(x)$ is not a periodic orbit we have $\alpha(x) = \{0\}$. Suppose

$$\alpha(x) \neq \mathcal{O}_k \qquad \text{for all } k \in \{1, 2, \ldots, N\}.$$

Then $\alpha(x) \cap \{0, \xi_-, \xi_+\} \neq \emptyset$. As ξ_- and ξ_+ are locally asymptotically stable stationary points, we conclude $\alpha(x) \cap \{\xi_-, \xi_+\} = \emptyset$. So, 0 is the only stationary point in $\alpha(x)$. Assume $\alpha(x) \neq \{0\}$. Then there exist $\psi \in \alpha(x) \setminus \{0\}$ and a solution

$y : \mathbb{R} \to \mathbb{R}$ with $y_0 = \psi$ and $\alpha(y) \cup \omega(y_0) \subset \alpha(x) \cap \{0, \xi_-, \xi_+\} = \{0\}$. Thus, $\alpha(y) = \omega(y_0) = \{0\}$. Proposition 2.12 gives the contradiction

$$2N + 2 \le V(\psi) \le 2N.$$

Consequently, $\alpha(x) = \{0\}$ and (5.2) holds. □

Remarks. 1. We emphasize that no hyperbolicity condition on the periodic orbits is assumed in Theorem 5.1. The stationary points $0, \xi_-, \xi_+$ are supposed to be hyperbolic, which can be checked by Proposition 2.2. We believe that Theorem 5.1 remains true if (5.1) is replaced by

$$\frac{\mu}{\cos\theta_N} < f'(0) \le \frac{\mu}{\cos\theta_{N+1}},$$

that is the hyperbolicity of the stationary point 0 can be omitted.

2. As the maps $F(t,\cdot)$ and $D_2F(t,\cdot)$ are injective for all $t \ge 0$, Theorem 6.1.9 in Henry [**12**] can be used to show that the strong unstable sets

$$W^u_{str}(\mathcal{O}_1), \dots, W^u_{str}(\mathcal{O}_N)$$

in formula (5.2) are C^1 immersed submanifolds of C. In a subsequent paper we show that these strong unstable sets are also C^1-submanifolds of C.

3. We mentioned in Section 1 that Theorem 5.1 implies a Morse decomposition of the global attractor A with Morse sets

$$S_0 = \{\xi_-, \xi_+\},\ S_{2k} = \mathcal{O}_k \quad \text{for all } k \in \{1, 2, \dots, N\},\ S_{2N+1} = \{0\}.$$

Introducing the connecting sets

$$\begin{aligned} C^k_l = \{\phi \in A :& \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1.1)} \\ & \text{with } x_0 = \phi,\ \alpha(x) \in S_k,\ \omega(\phi) \in S_l\} \end{aligned}$$

for integers $k > l$ in $\{0, 2, \dots, 2N, 2N+1\}$, one has

$$A = \Big(\bigcup_{k \in \{0,2,\dots,2N,2N+1\}} S_k \Big) \cup \Big(\bigcup_{k>l,\, k,l \in \{0,2,\dots,2N,2N+1\}} C^k_l \Big).$$

Clearly,

$$W^u_{str}(0) \setminus \{0\} = \bigcup_{k \in \{0,2,\dots,2N\}} C^{2N+1}_k$$

and

$$W^u_{str}(\mathcal{O}_k) \setminus \mathcal{O}_k = \bigcup_{l \in \{0,2,\dots,2k-2\}} C^k_l \qquad \text{for } k \in \{1, 2, \dots, N\}.$$

A description of the connecting sets C^k_l would give a finer structure of the global attractor A than formula (5.2). We refer to Fiedler and Mallet-Paret [**9**], McCord and Mischaikow [**22**], Krisztin, Walther and Wu [**16**], Krisztin and Wu [**17**] for some results on connecting sets.

4. In the particular case

$$f(\xi) = \alpha \tanh(\beta\xi)$$

with parameters $\alpha > 0$ and $\beta > 0$, which is used in neural network theory, the conditions of Theorem 5.1 are satisfied if

$$\alpha\beta > \mu$$

and

$$2N\pi - \arccos \frac{\mu}{\alpha\beta} < \sqrt{\alpha^2\beta^2 - \mu^2} < 2(N+1)\pi - \arccos \frac{\mu}{\alpha\beta};$$

or equivalently,

$$\frac{\alpha\beta}{\mu} \in \left(\frac{1}{\cos\theta_N}, \frac{1}{\cos\theta_{N+1}} \right)$$

with θ_N, θ_{N+1} defined in Theorem 5.1.

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