Regularly varying functions

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Contents

1 Motivation: maximum of iid random variables

This part is mainly from Feller [5, Chapter VIII.8].

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $X, X_1, X_2, \ldots$ are iid random variables on it such that $F(x) = \mathbf{P}(X \leq x) < 1$ for any $x \in \mathbb{R}$. Let $M_n = \max\{X_1, \ldots, X_n\}$ the partial maximum. In probability theory we are often interested in the following type of question:

What are the necessary and sufficient conditions for the existence of a sequence a_n such that M_n/a_n converges in distribution to a nondegenerate limit?

Recall that converges in distribution to Y, $Y_n \xrightarrow{\mathcal{D}} Y$, if $\lim_{n\to\infty} \mathbf{P}(Y_n \leq y) = \mathbf{P}(Y \leq y) = G(y)$ for any $y \in C_G$, where C_G stands for the continuity points of G.

For maximum we can easily calculate the distribution function. Indeed,

$$\mathbf{P}(M_n/a_n \leq x) = \mathbf{P}(M_n \leq a_n x) = F(a_n x)^n.$$

Thus we need that

$$\lim_{n\to\infty} F(a_n x)^n = G(x) \qquad \text{for all } x \in C_G.$$

Taking logarithms, and using that $\log(1+x) \sim x$ as $x \to 0$,

$$\lim_{n\to\infty} n\overline{F}(a_n x) = -\log G(x), \tag{1}$$

where $\overline{F}(x) = 1 - F(x)$. It turns out that the simple limit relation in (1) forces the structure of both the limit function and F. We need the following lemma.

Lemma 1. *Let b_n be a sequence for which $b_{n+1}/b_n \to 1$, $a_n \to \infty$, and U is a monotone function (increasing or decreasing). Assume that*

$$\lim_{n\to\infty} b_n U(a_n x) = h(x) \leq \infty$$

for all $x \in D$, where D is a dense subset, and the limit h is finite and strictly positive on an interval. Then $h(x) = cx^\rho$, for some $c \in \mathbb{R}$, $\rho \in \mathbb{R}$.

Proof. Easy. $\square$

Definition 1. A function $U : [0,\infty) \to [0,\infty)$ is regularly varying with index ρ, $U \in \mathcal{RV}_\rho$, if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{U(\lambda x)}{U(x)} = \lambda^\rho.$$

For $\rho = 0$, i.e. when for $\lambda > 0$

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1,$$

ℓ is slowly varying, $\ell \in \mathcal{SV}$.

This is regular variation at infinity. Regular variation at zero can be defined similarly, by changing $x \to \infty$ to $x \downarrow 0$.

From the definition we see that if $U \in \mathcal{RV}_\rho$ then $U(x) = x^\rho \ell(x)$, where ℓ is slowly varying.

Example 1. The constant function is trivially slowly varying. Moreover, any function with a strictly positive finite limit is slowly varying. More interesting examples are:

- $\log x$;
- $(\log x)^\alpha$, $\alpha \in \mathbb{R}$;
- $\log\log x$;
- $\exp\{(\log x)^\alpha\}$, $\alpha \in (0,1)$.

Going back to our maximum process, we see form Lemma 1 and from (1) that the limiting distribution function has to be of the form $G(x) = e^{-cx^\rho}$, for $x > 0$, and 0 otherwise, for some $\rho < 0$. In fact we have the following.

Theorem 1. *Assume the $F(x) < 1$ for all $x \in \mathbb{R}$. There exist a_n such that*

$$\frac{M_n}{a_n} \xrightarrow{\mathcal{D}} Z,$$

with a nondegenerate limit Z, if and only if $\overline{F}$ is regularly varying with index $\rho < 0$. In this case $\mathbf{P}(Z \le x) = G(x) = e^{-cx^\rho}$ for $x > 0$, and 0 otherwise.

The limiting distribution is the so-called *Fréchet* distribution. There are three type of extreme value distribution; see Exercises 4, 5.

Proof. Choose a_n such that $n\overline{F}(a_n) \to 1$. □

1.1 Exercises

1. Show that $\ell_1(x) = e^{(\log x)^\alpha}$ is slowly varying for $\alpha \in (0,1)$, and not slowly varying for $\alpha \geq 1$.

2. Show that $f(x) = 2 + \sin x$ is not slowly varying.

3. Show that $\ell_2(x) = \exp\left\{(\log x)^{1/3} \cos\left((\log x)^{1/3}\right)\right\}$ is slowly varying, and $\liminf_{x\to\infty} \ell_2(x) = 0$, $\limsup_{x\to\infty} \ell_2(x) = \infty$.

4. Let $X, X_1, X_2, \ldots$ be iid Exponential(1) random variables, and let $M_n = \max\{X_1, \ldots, X_n\}$ denote the partial maxima. Find a sequence a_n such that $M_n - a_n$ converges in distribution to a nondegenerate limit. The limiting distribution is the Gumbel distribution.

5. Let $X, X_1, X_2, \ldots$ be iid Uniform$(0,1)$ random variables, and let $M_n = \max\{X_1, \ldots, X_n\}$ denote the partial maxima. Find a sequence a_n, b_n such that $a_n(M_n - b_n)$ converges in distribution to a nondegenerate limit. Determine the limit distribution.

2 Steinhaus theory and the Cauchy functional equation

Main theory on regular variation follows Bingham et al.[1].

Theorem 2. *Let $A \subset \mathbb{R}$ be a measurable set with positive Lebesgue measure. Then $A - A$ contains in interval.*

Theorem 3. *Let $A, B \subset \mathbb{R}$ be measurable sets with positive Lebesgue measure. Then $A - B$ contains in interval.*

Corollary 1. *Let $A \subset \mathbb{R}$ be a measurable set with positive Lebesgue measure. Then $A + A$ contains in interval.*

Corollary 2. *(i) If $S \subset \mathbb{R}$ is and additive subgroup, and S contains a set of positive measure, then $S = \mathbb{R}$. (ii) If $S \subset (0,\infty)$ is and additive semigroup, and S contains a set of positive measure, then there exists $b > 0$ such that $S \supset (b, \infty)$.*

Definition 2. A function $k : \mathbb{R} \to \mathbb{R}$ is additive if $k(x + y) = k(x) + k(y)$ for all x, y.

Lemma 2. *If k is additive and bounded above on a set A with positive measure, then k is bounded in the neighborhood of the origin.*

Theorem 4. *Let k be additive and bounded above on a set A with positive measure. Then $k(x) = cx$ for some $c \in \mathbb{R}$.*

Corollary 3. *If k is additive and measurable then $k(x) = cx$.*

There are pathological solutions to the Cauchy functional equations. Consider $\mathbb{R}$ as a vector space above $\mathbb{Q}$, and let B be a Hamel base. This exist by the Zorn lemma, and the cardinality of B is continuum. For $b_0 \in B$ fixed let $k(b_0) = b_0$, and $k(b) = 0$ for $b \in B$, $b \neq b_0$. Define

$$k(x) = \sum_{i=1}^{n} r_i k(b_i), \qquad \text{if } x = \sum_{i=1}^{n} r_i b_i.$$

Then k is additive, but not of the form $k(x) = cx$.

2.1 Exercises

6. (i) If $S \subset \mathbb{R}$ is and additive subgroup, and S contains a set of positive measure, then $S = \mathbb{R}$. (ii) If $S \subset (0, \infty)$ is and additive semigroup, and S contains a set of positive measure, then there exists $b > 0$ such that $S \supset (b, \infty)$.

3 Slowly varying functions

Definition 3. A nonnegative measurable function $\ell : [a, \infty) \to [0, \infty)$, $a \geq 0$, is slowly varying, if

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \qquad \text{for each} \quad \lambda > 0. \tag{2}$$

For simplicity, we assume that $a = 0$.

Theorem 5. *Uniform convergence theorem. Let ℓ be a slowly varying function. Then (2) holds uniformly on each compact set of $(0, \infty)$; that is for each $\varepsilon > 0$, $K < \infty$*

$$\sup_{\lambda \in [\varepsilon, K]} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 \right| = 0.$$

Proof. I. Direct proof

II: Indirect proof by Erdős and Csiszár. □

Theorem 6. *Representation theorem. Let ℓ be a nonnegative measurable function. It is slowly varying if and only if*

$$\ell(x) = c(x) \exp\left\{\int_a^x \frac{\varepsilon(u)}{u} \mathrm{d}u\right\}, \quad x > a,$$

where $a \geq 0$, $\lim_{x\to\infty} c(x) = c \in (0, \infty)$, $\lim_{x\to\infty} \varepsilon(x) = 0$.

Changing to the additive notation $h(x) = \log \ell(e^x)$, we have

$$h(x) = c_1(e^x) + \int_{\log a}^x \varepsilon(e^x)\mathrm{d}x =: d(x) + \int_b^x e(x)\mathrm{d}x. \tag{3}$$

Proof. Sufficiency is clear.

For the necessity, write

$$h(x) = \int_x^{x+1} [h(x) - h(t)]\,\mathrm{d}t + \int_{x_0}^x [h(t+1) - h(t)]\,\mathrm{d}t + \int_{x_0}^{x_0+1} h(t)\mathrm{d}t.$$

The last term is constant. In the second term is integrand $e(t) = h(t+1) - h(t) \to 0$ as $t \to \infty$. While the first

$$\int_x^{x+1} [h(x) - h(t)]\,\mathrm{d}t = \int_0^1 [h(x) - h(x+u)]\mathrm{d}u,$$

and here the integrand tends to 0 uniformly by the UCT. □

We use the following lemma without explicitly mentioning.

Lemma 3. *If $\ell \in \mathcal{SV}$ then ℓ is locally bounded far enough to the right; i.e. there exists $a > 0$ such that $\sup_{x\in[a,a+n]} \ell(x) < \infty$ for each n.*

Proposition 1. *Let ℓ, ℓ_1, ℓ_2 be slowly varying functions. Then*

1. $(\log \ell(x))/\log x \to 0$;
2. $(\ell(x))^\alpha$ *is slowly varying for each* $\alpha \in \mathbb{R}$;
3. $\ell_1\ell_2$, $\ell_1 + \ell_2$ *are slowly varying;*
4. *for each* $\varepsilon > 0$ $\lim_{x\to\infty} x^\varepsilon \ell(x) = \infty$, $\lim_{x\to\infty} x^{-\varepsilon}\ell(x) = 0$.

3.1 Exercises

7. Show that the representation theorem implies the UCT.

8. Let ℓ, ℓ_1, ℓ_2 be slowly varying functions. Then

1. $(\log \ell(x))/\log x \to 0$;
2. $(\ell(x))^\alpha$ is slowly varying for each $\alpha \in \mathbb{R}$;
3. $\ell_1 \ell_2$, $\ell_1 + \ell_2$ are slowly varying;
4. for each $\varepsilon > 0$ $\lim_{x\to\infty} x^\varepsilon \ell(x) = \infty$, $\lim_{x\to\infty} x^{-\varepsilon}\ell(x) = 0$.

4 The limit function

Let $f : [0, \infty) \to (0, \infty)$ be a measurable function, and assume that

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = g(\lambda) \in (0, \infty), \qquad \lambda \in S, \tag{4}$$

for some set S. Then $\lambda, \mu \in S$ implies $\lambda\mu \in S$ and $g(\lambda\mu) = g(\lambda)g(\mu)$. Also $\lambda \in S$ implies $1/\lambda \in S$ and $g(1/\lambda) = 1/g(\lambda)$. Thus S is a multiplicative subgroup of $(0, \infty)$.

Changing to the additive notation $h(x) = \log f(e^x)$, $k(x) = \log g(e^x)$, we have that $k(u+v) = k(u)+k(v)$ for $u, v \in T$, where T is an additive subgroup of $\mathbb{R}$.

Theorem 7 (Characterization theorem)**.** *Assume that* (4) *holds and S has positive measure. Then*

(i) $\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)}$ *exists for all* $\lambda > 0$.

(ii) $g(\lambda) = \lambda^\rho$ *for some* $\rho \in \mathbb{R}$.

(iii) $f(x) = x^\rho \ell(x)$ *for some* $\ell \in \mathcal{SV}$.

Proof. This follows from Corollary 2. □

Definition 4. A positive measurable function f is regularly varying with index $\rho \in \mathbb{R}$ if

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho \qquad \text{for all } \lambda > 0.$$

Regular variation at 0 defined similarly, but $x \downarrow 0$ instead of $x \to \infty$. Simply $f(x)$ is regularly varying at 0 if and only if $f(1/x)$ is regularly varying at infinity.

There are more general characterization theorems.

Theorem 8. *Let f be positive measurable function and assume that for $g^*(\lambda) = \limsup_{x\to\infty} f(\lambda x)/f(x)$, we have $\limsup_{\lambda\downarrow 1} g^*(\lambda) \le 1$. Then the following are equivalent.*

(i) There is a $\rho \in \mathbb{R}$ such that $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\rho$ for all $\lambda > 0$.

(ii) $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists and finite on a set of positive measure.

(iii) $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists and finite on a dense subset of $(0,\infty)$.

(iv) $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists and finite for $\lambda = \lambda_1, \lambda_2$, where $\log \lambda_1/\log \lambda_2$ is irrational.

5 Regularly varying functions: first properties

An immediate consequence of Proposition 1 is the following.

Proposition 2. *For $f \in \mathcal{RV}_\rho$, as $x \to \infty$*

$$f(x) \to \begin{cases} \infty, & \rho > 0, \\ 0, & \rho < 0. \end{cases}$$

Theorem 9 (Uniform convergence theorem for regularly varying functions)**.** *Let $f \in \mathcal{RV}_\rho$ locally bounded on $[0,\infty)$. Then $f(\lambda x)/f(x) \to \lambda^\rho$ uniformly in λ*

- *on each $[a,b] \subset (0,\infty)$ for $\rho = 0$;*
- *on each $(0,b] \subset (0,\infty)$ for $\rho > 0$;*
- *on each $[a,\infty) \subset (0,\infty)$ for $\rho < 0$.*

Proof. The case $\rho = 0$ is the UCT for slowly varying functions. We only prove the statement for $\rho > 0$, the other case is similar.

By the UCT for slowly varying functions it is enough to prove on $(0,1]$. By the representation theorem

$$f(x) = x^\rho \ell(x) = x^\rho c(x) \exp\left\{ \int_0^x \varepsilon(u)/u \mathrm{d}u \right\}.$$

There exists $x_0 > 0$ such that for $x \geq x_0$ $c(x) \in (c/2, 2c)$ and $|\varepsilon(x)| < 1$. Thus, whenever $\lambda x \geq x_0$

$$\frac{f(\lambda x)}{f(x)} \leq \lambda^\rho \frac{2c}{c/2} e^{\log \lambda} = 4\lambda^{\rho+1}.$$

Let $\varepsilon > 0$ be fix. If $\lambda \leq \varepsilon^{1/(\rho+1)}$ then for $\lambda x \geq x_0$

$$\frac{f(\lambda x)}{f(x)} \leq 4\varepsilon.$$

Therefore, if $\lambda \leq \varepsilon^{1/(\rho+1)}$ and $\lambda x \geq x_0$

$$\left| \frac{f(\lambda x)}{f(x)} - \lambda^\rho \right| \leq 4\varepsilon + \epsilon^{\rho/(\rho+1)}.$$

On the other hand, if $\lambda x \leq x_0$ then

$$\left| \frac{f(\lambda x)}{f(x)} - \lambda^\rho \right| \leq \frac{\sup_{y \in (0, x_0]} f(y)}{f(x)} + \left(\frac{x_0}{x} \right)^\rho.$$

The latter bound goes to 0 as $x \to \infty$ (uniformly in λ, since it does not contain any λ).

Finally, for $\lambda \in [\varepsilon^{1/(\rho+1)}, 1]$ the UCT works. $\square$

As a consequence we obtain that a regularly varying function with index $\rho \neq 0$ is asymptotically equivalent to a monotone function.

Theorem 10. *Let $f \in \mathcal{RV}_\rho$ locally bounded on $[a, \infty)$. If $\rho > 0$ then*

(i) $\overline{f}(x) = \sup\{f(t) : 0 \leq t \leq x\} \sim f(x)$*;*

(ii) $\underline{f}(x) = \inf\{f(t) : t \geq x\} \sim f(x)$*.*

If $\rho < 0$ then $\sup\{f(t) : t \geq x\} \sim f(x)$ *and* $\inf\{f(t) : a \leq t \leq x\} \sim f(x)$*.*

Theorem 11 (Potter bounds)**.** *(i) Let ℓ be a slowly varying function. Then for each $A > 1$, $\delta > 0$ there exists x_0 such that for each $x, y \geq x_0$*

$$\frac{\ell(x)}{\ell(y)} \leq A \max\left\{ \left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta \right\}.$$

(ii) If ℓ is bounded away from 0 and ∞ on every compact subset of $[0,\infty)$ then for each $\delta > 0$ there exists and $A = A(\delta)$ such that for each x, y

$$\frac{\ell(x)}{\ell(y)} \leq A \max\left\{ \left(\frac{x}{y}\right)^{\delta}, \left(\frac{y}{x}\right)^{\delta} \right\}.$$

(iii) If $f \in \mathcal{RV}_\rho$ then for each $A > 1$, $\delta > 0$ there exist $x_0 > 0$ such that for $x, y \geq x_0$

$$\frac{f(x)}{f(y)} \leq A \max\left\{ \left(\frac{x}{y}\right)^{\rho+\delta}, \left(\frac{x}{y}\right)^{\rho-\delta} \right\}.$$

Proof. (i) follows from the representation theorem. (iii) is immediate from (i). (ii) follows from the local boundedness and strict positivity. □

Proposition 3. *(i) If $f \in \mathcal{RV}_\rho$ then $f^\alpha \in \mathcal{RV}_{\rho\alpha}$.*
(ii) If $f_i \in \mathcal{RV}_{\rho_i}$, $i = 1, 2$, and $f_2(x) \to \infty$, then $f_1(f_2(x)) \in \mathcal{RV}_{\rho_1\rho_2}$.
(iii) If $f_i \in \mathcal{RV}_{\rho_i}$, $i = 1, 2$, then $f_1 + f_2 \in \mathcal{RV}_{\max\{\rho_1,\rho_2\}}$.

5.1 Exercises

9. Prove Proposition 3.

6 Karamata's theorem

Proposition 4. *Let $\ell \in \mathcal{SV}$ be locally bounded on $[a, \infty)$, $\alpha > -1$. Then*

$$\int_a^x t^\alpha \ell(t) \mathrm{d}t \sim x^{\alpha+1} \ell(x) \frac{1}{\alpha + 1}.$$

Proof. We have

$$\begin{aligned} \frac{\int_{a'}^x t^\alpha \ell(t) \mathrm{d}t}{x^{\alpha+1}\ell(x)} &= \int_{a'/x}^1 u^\alpha \frac{\ell(ux)}{\ell(x)} \mathrm{d}u \\ &= \int_0^1 u^\alpha \frac{\ell(ux)}{\ell(x)} I_{[a'/x,1]}(u) \mathrm{d}u. \end{aligned}$$

The integrand converges pointwise to u^α. Choose a' so that the Potter bound can be applied to the ratio with $A = 2$ and $\delta < \alpha + 1$. The statement follows from Lebesgue's dominated convergence theorem. □

We need $\alpha > -1$ for the integrability of the integrand. However, the result hold true in the following sense.

Proposition 5. *Let $\ell \in \mathcal{SV}$ be locally bounded on $[a, \infty)$. Then*

$$\widetilde{\ell}(x) = \int_a^x t^{-1}\ell(t)\mathrm{d}t$$

is slowly varying, and $\widetilde{\ell}(x)/\ell(x) \to \infty$.

Proof. Let $c \in (0, 1)$. For $x > a/c$, by the uniform convergence theorem

$$\begin{aligned}\widetilde{\ell}(x) &= \int_a^x \frac{\ell(t)}{t}\mathrm{d}t \geq \int_{x/c}^x \frac{\ell(t)}{t}\mathrm{d}t \\ &= \int_{1/c}^1 \frac{\ell(xu)}{u}\mathrm{d}u \sim \ell(x)\int_{1/c}^1 \frac{1}{u}\mathrm{d}u \\ &= \ell(x)\log c^{-1}.\end{aligned}$$

Thus

$$\liminf_{x\to\infty} \frac{\widetilde{\ell}(x)}{\ell(x)} \geq \log c^{-1} \to \infty \quad \text{as } c \to 0.$$

To show that $\widetilde{\ell}$ is slowly varying let

$$\varepsilon(x) = \frac{\ell(x)}{\widetilde{\ell}(x)}.$$

We have already shown that $\varepsilon(x) \to 0$ as $x \to \infty$. By the definition of $\widetilde{\ell}$, Lebesgue almost everywhere

$$\widetilde{\ell}'(x) = \frac{\ell(x)}{x} = \frac{\varepsilon(x)\widetilde{\ell}(x)}{x}.$$

Since $\widetilde{\ell}$ is absolutely continuous, so is $\log \widetilde{\ell}$, and

$$\frac{\mathrm{d}}{\mathrm{d}x}\log \widetilde{(}x) = \frac{\varepsilon(x)}{x} \quad \text{a.e.}$$

Integrating out, the representation theorem implies the statement. $\square$

The following versions can be proved similarly.

Proposition 6. *If* $\int_x^\infty \ell(t)/t\,\mathrm{d}t < \infty$ *then*

$$\widetilde{\ell}(x) = \int_x^\infty \frac{\ell(t)}{t}\mathrm{d}t$$

is slowly varying and $\widetilde{\ell}(x)/\ell(x) \to \infty$.

Proposition 7. *Let* $\ell \in \mathcal{SV}$, $\alpha < -1$. *Then*

$$\int_x^\infty t^\alpha \ell(t)\mathrm{d}t < \infty$$

and

$$\frac{x^{\alpha+1}\ell(x)}{\int_x^\infty t^\alpha \ell(t)\mathrm{d}t} \to -\alpha - 1.$$

Summarizing, we proved the following.

Theorem 12 (Karamata's theorem, direct part)**.** *Let* $f \in \mathcal{RV}_\rho$ *be locally bounded on* $[a,\infty)$. *Then*

(i) *for* $\sigma \geq -(\rho+1)$

$$\frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)\mathrm{d}t} \to \sigma + \rho + 1;$$

(ii) *for* $\sigma < -(\rho+1)$

$$\frac{x^{\sigma+1}f(x)}{\int_x^\infty t^\sigma f(t)\mathrm{d}t} \to -(\sigma + \rho + 1).$$

(The latter also holds for $\sigma = -(\rho+1)$ *if the integral is finite.)*

It turns out that this behavior also characterizes regular variation.

Theorem 13 (Karamata's theorem, converse part)**.** *Let* f *be a positive, measurable, locally integrable function on* $[a,\infty)$.

(i) *If for some* $\sigma > -(\rho+1)$

$$\frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)\mathrm{d}t} \to \sigma + \rho + 1,$$

then $f \in \mathcal{RV}_\rho$*;*

(ii) If for $\sigma < -(\rho+1)$

$$\frac{x^{\sigma+1}f(x)}{\int_x^\infty t^\sigma f(t)\mathrm{d}t} \to -(\sigma+\rho+1),$$

then $f \in \mathcal{RV}_\rho$.

Proof. We only prove (i), the other is similar. Put

$$g(x) = \frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)\mathrm{d}t}.$$

Then $g(x) \to \sigma+\rho+1$, and for some $b > a$ fix

$$\int_b^x \frac{g(t)}{t}\mathrm{d}t = \log\left(\int_a^x t^\sigma f(t)\mathrm{d}t/C\right),$$

with $C = \int_a^b t^\sigma f(t)\mathrm{d}t$. This follows by differentiating both sides. Then

$$f(x) = Cb^{-(\rho+\sigma+1)}g(x)x^\rho \exp\left\{\int_b^\sigma \varepsilon(t)/t\mathrm{d}t\right\},$$

and the result follows from the representation theorem. □

6.1 Exercises

10. Let ℓ be a slowly varying function which is locally bounded on $[0,\infty)$. Assume further that $\int_1^\infty \ell(t)/t\,\mathrm{d}t < \infty$. Show that $\widetilde{\ell}(x) = \int_x^\infty \ell(t)/t\,\mathrm{d}t$ is slowly varying and $\widetilde{\ell}(x)/\ell(x) \to \infty$ as $x \to \infty$.

11. Let $\ell_0(x) \equiv 1$, and let $\ell_{i+1}(x) = \int_1^x \ell_i(t)/t\,\mathrm{d}t$, $i = 0, 1, 2, \ldots$. Find ℓ_i.

12. Let ℓ be slowly varying, locally bounded, and $\alpha < -1$. Show that $\int_x^\infty t^\alpha \ell(t)\mathrm{d}t < \infty$, and

$$\lim_{x\to\infty} \frac{x^{\alpha+1}\ell(x)}{\int_x^\infty t^\alpha \ell(t)\mathrm{d}t} = -\alpha - 1.$$

7 Monotone density theorem

Karamata's theorems show how to integrate regularly varying function. Next we turn to the question of differentiating absolutely continuous regularly varying functions. Assume that

$$U(x) = \int_0^x u(t)\mathrm{d}t,$$

for some nonnegative measurable u. Assume that U is regularly varying. Under some additional assumption it follows that u is regularly varying too. A function is *ultimately monotone* if it is monotone (increasing or decreasing) for x large enough.

Theorem 14. *Let $U(x) = \int_0^x u(t)\mathrm{d}t \sim cx^\rho \ell(x)$ as $x \to \infty$ for $c \geq 0$, $\rho \geq 0$, ℓ slowly varying, and assume that u is ultimately monotone. Then*

$$u(x) \sim c\rho x^{\rho-1}\ell(x).$$

Proof. Assume that u is eventually nondecreasing. Then for $a < b$

$$U(bx) - U(ax) = \int_{ax}^{bx} u(t)\mathrm{d}t \leq (b-a)xu(bx).$$

Dividing both sides by $x^\rho \ell(x)$ we obtain

$$\limsup \frac{u(ax)}{x^{\rho-1}\ell(x)} \leq c\frac{b^\rho - a^\rho}{b-a}.$$

Choosing $a = 1$ and letting $b \downarrow 1$ we obtain

$$\limsup \frac{u(x)}{x^{\rho-1}\ell(x)} \leq c\rho.$$

The lim inf result can be shown similarly, and the statement follows. □

Versions of this theorem remain true.

Theorem 15. *Let $U(x) = \int_0^x u(t)\mathrm{d}t \sim cx^\rho \ell(x)$ as $x \downarrow 0$ for $c \geq 0$, $\rho \geq 0$, ℓ slowly varying at 0, and assume that u is ultimately monotone. Then as $x \downarrow 0$*

$$u(x) \sim c\rho x^{\rho-1}\ell(x).$$

8 Inversion

Let f be positive locally bounded function on $[a, \infty)$ tending to ∞. Put

$$f^{\leftarrow}(x) = \inf\{y \geq a \,:\, f(y) > x\}.$$

Clearly $f^{\leftarrow}$ is monotone increasing.

Theorem 16. *For $f \in \mathcal{RV}_\alpha$, $\alpha > 0$, there exists $g \in \mathcal{RV}_{1/\alpha}$ such that*

$$f(g(x)) \sim g(f(x)) \sim x \qquad \text{as } x \to \infty.$$

Furthermore, g is uniquely determined up to asymptotic equivalence, and a version of g is $f^{\leftarrow}$.

Proof. We prove that $f(f^{\leftarrow}(x)) \sim x$. Let $A > 1$, $\lambda > 1$, $\delta > 0$. By Potter's bound there is an x_0 such that for $u \geq x_0$

$$\frac{1}{A\lambda^{\alpha+\delta}} \leq \frac{f(u)}{f(v)} \leq A\lambda^{\alpha+\delta} \qquad \text{for } v \in [u/\lambda, u\lambda].$$

Choose x so large that $f^{\leftarrow}(x) \geq x_0$. There exists $y \in [f^{\leftarrow}(x), \lambda f^{\leftarrow}(x)]$ such that $f(y) > x$, and there exists $y' \in [\lambda^{-1} f^{\leftarrow}(x), f^{\leftarrow}(x)]$ such that $f(y') \leq x$. Choosing $u = f^{\leftarrow}(x)$ we obtain

$$\frac{1}{A\lambda^{\alpha+\delta}} \leq \liminf_{x\to\infty} \frac{f(f^{\leftarrow}(x))}{x} \leq \limsup_{x\to\infty} \frac{f(f^{\leftarrow}(x))}{x} \leq A\lambda^{\alpha+\delta}.$$

Letting $A \downarrow 1$, $\lambda \downarrow 1$, the statement follows.

Next we show that $f^{\leftarrow}$ is regularly varying with index $1/\alpha$. Fix $\lambda > 1$. We have

$$\frac{f(\lambda^{1/\alpha} f^{\leftarrow}(x))}{f(f^{\leftarrow}(\lambda x))} = \frac{\lambda x}{f(f^{\leftarrow}(\lambda x))} \frac{f(f^{\leftarrow}(x))}{x} \frac{f(\lambda^{1/\alpha} f^{\leftarrow}(x))}{\lambda f(f^{\leftarrow}(x))},$$

where each factor in the product tends to 1. The first two by the fact that $f(f^{\leftarrow}(x)) \sim x$, the third by the regular variation of f. Therefore

$$\frac{f(\lambda^{1/\alpha} f^{\leftarrow}(x))}{f(f^{\leftarrow}(\lambda x))} \to 1.$$

The regular variation of f implies that

$$f^{\leftarrow}(\lambda x) \sim \lambda^{1/\alpha} f^{\leftarrow}(x),$$

i.e. $f^{\leftarrow}$ is regularly varying with index $1/\alpha$.

Next we show that $f^{\leftarrow}(f(x)) \sim x$. Since $f(f^{\leftarrow}(x)) \sim x$ we have

$$f(f^{\leftarrow}(f(x))) \sim f(x),$$

which, by the regular variation of f implies $f^{\leftarrow}(f(x)) \sim x$.

Finally, $g(f(x)) \sim x$ implies $g(f(f^{\leftarrow}(x))) \sim f^{\leftarrow}(x)$, thus $g(x) \sim f^{\leftarrow}(x)$ as claimed. $\square$

As a simple consequence we obtain the following.

Theorem 17 (de Bruijn conjugate)**.** *For any $\ell \in \mathcal{SV}$ there exists $\ell^{\sharp} \in \mathcal{SV}$ unique up to asymptotic equivalence such that*

$$\ell(x)\ell^{\sharp}(x\ell(x)) \to 1 \quad \textit{and} \quad \ell^{\sharp}(x)\ell(x\ell^{\sharp}(x)) \to 1.$$

Moreover, $(\ell^{\sharp})^{\sharp} \sim \ell$.

8.1 Exercises

13. Find an asymptotic inverse of the following functions and prove that it is indeed an asymptotic inverse.

(a) $f_1(x) = x \log x$;

(b) $f_2(x) = x^2 \log\log x$;

(c) $f_3(x) = x^2 (\log x)^3$.

14. Let $f \in \mathcal{RV}_\alpha$, and g is a positive measurable function such that

$$\lim_{x\to\infty} \frac{f(g(x)\lambda^{1/\alpha})}{f(g(\lambda x))} = 1.$$

Show that $g \in \mathcal{RV}_{1/\alpha}$.

9 Laplace–Stieltjes transforms

In the following U is a nondecreasing right-continuous function on $\mathbb{R}$ such that $U(x) = 0$ for $x < 0$. Its Laplace–Stieltjes transform is

$$\widehat{U}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}U(x).$$

Theorem 18. *Let U be as above, $c \geq 0$, $\rho \geq 0$, $\ell \in \mathcal{SV}$. The following are equivalent:*

(i) $U(x) \sim cx^\rho \ell(x) \frac{1}{\Gamma(1+\rho)}$ *as* $x \to \infty$*;*

(ii) $\widehat{U}(s) \sim cs^{-\rho} \ell(1/s)$ *as* $s \downarrow 0$*.*

The following version can be proved in the same way.

Theorem 19. *Let U be as above, $c \geq 0$, $\rho \geq 0$, $\ell \in \mathcal{SV}$. The following are equivalent:*

(i) $U(x) \sim cx^\rho \ell(x) \frac{1}{\Gamma(1+\rho)}$ *as* $x \downarrow 0$*;*

(ii) $\widehat{U}(s) \sim cs^{-\rho} \ell(1/s)$ *as* $s \to \infty$*.*

9.1 Exercices

15. Show that $\sum_{n=1}^\infty e^{-2^n} 2^{\rho n} < \infty$ for any ρ.

10 Tails of nonnegative random variables

In the following let X be a nonnegative random variable, and $F(x) = \mathbf{P}(X \leq x)$ its distribution function. The tail of the distribution function is $\overline{F}(x) = 1 - F(x)$. The Laplace transform of F, or X is

$$\widehat{F}(s) = \mathbf{E} e^{-sX} = \int_{[0,\infty)} e^{-sx} \mathrm{d}F(x), \quad s \geq 0.$$

Further, let μ_n denote the moments of F, i.e.

$$\mu_n = \mathbf{E} X^n = \int_{[0,\infty)} x^n \mathrm{d}F(x).$$

We are interested in the relation of $\overline{F}$ at infinity and $\widehat{F}$ at zero. By the Taylor formula, whenever $\mathbf{E}X^n = \mu_n < \infty$

$$\widehat{F}(s) = \sum_{k=0}^n \mu_k \frac{(-s)^k}{k!} + o(s^n) \quad \text{as} \quad s \downarrow 0.$$

Introduce the notation for $n \geq 0$

$$\begin{aligned} f_n(s) &= (-1)^{n+1} \left(\widehat{F}(s) - \sum_{k=0}^{n} \mu_k \frac{(-s)^k}{k!} \right) \\ g_n(s) &= \frac{\mathrm{d}^n}{\mathrm{d}s^n} f_n(s) = \mu_n + (-1)^{n+1} \widehat{F}^{(n)}(s). \end{aligned} \tag{5}$$

In particular, $f_0(s) = g_0(s) = 1 - \widehat{F}(s)$.

The following theorem is due to Bingham and Doney (1974), see Theorem 8.1.6 in [1].

Theorem 20. *Let $\ell \in \mathcal{SV}$, $\mu_n < \infty$, $\alpha = n + \beta$ for $\beta \in [0, 1]$. The following are equivalent:*

(i) $f_n(s) \sim s^\alpha \ell(1/s)$ *as* $s \downarrow 0$;

(ii) $g_n(s) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} s^\beta \ell(1/s)$ *as* $s \downarrow 0$;

(iii) as $x \to \infty$

$$\begin{aligned} \int_{(x,\infty)} t^n \mathrm{d}F(t) &\sim n! \ell(x) \quad \textit{if} \ \ \beta = 0 \\ \overline{F}(x) &\sim \frac{(-1)^n}{\Gamma(1-\alpha)} x^{-\alpha} \ell(x) \quad \textit{if} \ \ \beta \in (0,1) \\ \int_{[0,x]} t^{n+1} \mathrm{d}F(t) &\sim (n+1)! \ell(x) \quad \textit{if} \ \ \beta = 1. \end{aligned}$$

For $\beta > 0$ these are further equivalent to

(iv) $(-1)^{n+1} \widehat{F}^{(n+1)}(s) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta)} s^{\beta-1} \ell(1/s)$ *as* $s \downarrow 0$.

Proof. The equivalence of (i) and (ii) follows from the monotone density theorem. By the same reason, for $\beta > 0$ these are equivalent to (iv).

For $\beta = 1$ the function $(-1)^{n+1} \widehat{F}^{(n+1)}(s)$ is the Laplace–Stieltjes transform of $\int_{[0,x]} t^{n+1} \mathrm{d}F(t)$, thus the equivalence of (iii) and (iv) are follows from the Tauberian theorem for the Laplace transform. Thus in the following we may assume that $\beta < 1$.

Next we show the equivalence of (ii) and (iii). Put

$$U(x) = \int_0^x \int_{(t,\infty)} y^n \mathrm{d}F(y) \, \mathrm{d}t.$$

Then integrations by parts shows

$$\widehat{U}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}U(x) = s^{-1} \left[\mu_n + (-1)^{n+1} \widehat{F}^{(n)}(s) \right] = \frac{g_n(s)}{s}.$$

Thus by the Tauberian theorem

$$\text{(ii)} \quad \Longleftrightarrow \quad U(x) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(2-\beta)} x^{1-\beta} \ell(x). \tag{6}$$

By the monotone density theorem again, the right-hand side of (6) is further equivalent to

$$\begin{aligned} T_n(x) := \int_{(x,\infty)} y^n \mathrm{d}F(y) &\sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(2-\beta)} (1-\beta) x^{-\beta} \ell(x) \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(1-\beta)} x^{-\beta} \ell(x). \end{aligned} \tag{7}$$

Thus the statement is proved for $\beta = 0$. Assume now $\beta \in (0,1)$. Then integration by parts gives

$$T_n(x) = x^n \overline{F}(x) + n \int_x^\infty y^{n-1} \overline{F}(y) \mathrm{d}y.$$

If (iii) holds then by Karamata's theorem (7), and thus (ii) follows. For the converse, assume that (ii), thus (7) holds. Then, after some integration by parts formulas, we obtain

$$\frac{x^n \overline{F}(x)}{T_n(x)} = 1 - \frac{n x^n}{T_n(x)} \int_x^\infty y^{-n-1} T_n(y) \mathrm{d}y.$$

Thus the theorem follows again by an application of Karamata's theorem. $\square$

The most important special case is when $n = 0$.

Corollary 4. *Let $\ell \in \mathcal{SV}$, $\alpha \in [0,1]$. Then the following are equivalent:*

(i) $1 - \widehat{F}(s) \sim s^\alpha \ell(1/s)$ *as* $s \downarrow 0$*;*

(ii) *as* $x \to \infty$

$$\begin{aligned} \overline{F}(x) &\sim \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} \ell(x) \quad \textit{if} \ \ \alpha \in [0,1) \\ \int_{[0,x]} t \mathrm{d}F(t) &\sim \ell(x) \quad \textit{if} \ \ \alpha = 1 \\ \int_0^x \overline{F}(t) \mathrm{d}t &\sim \ell(x) \quad \textit{if} \ \ \alpha = 1. \end{aligned}$$

The importance of the tail behavior of random variables is explained by the following classical result.

Theorem 21 (Domains of attraction; Doeblin, Gnedenko)**.** *Let $X, X_1, X_2, \ldots$ be iid random variables with distribution function F, and let $S_n = X_1 + \ldots + X_n$ denote their partial sum. Then there exist centering and norming sequences a_n and c_n such that $(S_n - c_n)/a_n$ converges in distribution to a nondegenerate random variable Z if and only if one of the following two conditions holds:*

(i) Z a normal, and the truncated second moment

$$V(x) = \int_{[-x,x]} y^2 \mathrm{d}F(y)$$

is slowly varying;

(ii) for some $\alpha \in (0,2)$ and a slowly varying function ℓ

$$F(-x) + 1 - F(x) = \frac{\ell(x)}{x^\alpha}$$

and $\lim_{x\to\infty} F(-x)/(1-F(x))$ exists (0 or ∞ allowed).

Thus the possible limits can be characterized by the parameter $\alpha \in (0,2]$. The case $\alpha = 2$ corresponds to Gaussian limit, while for $\alpha \in (0,2)$ corresponds to non-Gaussian stable limit. If the assumption of the theorem holds then X or its distribution F is in the domain of attraction of the stable law with parameter α, written $F \in D(\alpha)$. Note that if X is nonnegative then $F \in D(\alpha)$ simply means that $\overline{F}(x) = 1 - F(x)$ is regularly varying with parameter $-\alpha \in (-2,0)$.

Example 2. Let X be a nonnegative random variable with distribution function $F(x) = 1 - x^{-\alpha}$, $x \geq 1$. This is the Pareto distribution with parameter $\alpha > 0$. By Theorem 20

$$1 - \mathbf{E}e^{-sX} \sim \Gamma(1-\alpha)s^\alpha \quad \text{as } s \downarrow 0.$$

Therefore, for the partial sum $S_n = X_1 + \ldots + X_n$ with the sequence $a_n = n^{1/\alpha}$

$$\mathbf{E}e^{s\frac{S_n}{a_n}} = \exp\left\{n \log \mathbf{E}e^{-sX/a_n}\right\} \sim e^{-\Gamma(1-\alpha)s^\alpha},$$

which implies that $S_n/n^{1/\alpha}$ converges in distribution.

10.1 Exercises

16. Determine the Laplace transform of the following distributions.

(a) $X \sim \text{Bernoulli}(p)$;

(b) $X \sim \text{Binomial}(n, p)$;

(c) $X \sim \text{Poisson}(\lambda)$;

(d) $X \sim \text{Uniform}(a, b)$;

(e) $X \sim \text{Exp}(\lambda)$.

17. Let $X \geq 0$, $\alpha > 0$. Show that $\mathbf{E}X^\alpha < \infty$ implies $\lim_{x\to\infty} x^\alpha[1 - F(x)] = 0$. Give a counterexample to show that the converse is not true. (It is almost true, see the next exercise.)

18. Let $X \geq 0$, $\alpha > 0$. Show that $\lim_{x\to\infty} x^\alpha[1-F(x)] = 0$ implies $\mathbf{E}X^\beta < \infty$ for any $\beta < \alpha$.

19. Let X be a nonnegative random variable, F its distribution function, and $\widehat{F}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}F(x)$ its Laplace transform. Assume that $\mu_n = \mathbf{E}X^n < \infty$. Define

$$f_n(s) = (-1)^{n+1} \left(\widehat{F}(s) - \sum_{k=0}^{n} \mu_k (-s)^k / k! \right)$$

$$g_n(s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n} f_n(s).$$

Let ℓ be a slowly varying function, $\alpha = n + \beta$ with $\beta \in [0, 1]$. Show that $f_n(s) \sim s^\alpha \ell(1/s)$ if and only if $g_n(s) \sim \Gamma(\alpha + 1)/\Gamma(\beta + 1)\, s^\beta \ell(1/s)$.

20. Show that the Laplace transform of the standard normal distribution is $e^{s^2/2}$.

11 Sum and maxima of iid random variables

In the following $X, X_1, X_2, \ldots$ are nonnegative iid random variables with distribution function $\mathbf{P}(X \leq x) = F(x)$. Let $M_n = \max\{X_1, \ldots, X_n\}$ and $S_n = X_1 + \ldots + X_n$ denote the partial maxima and partial sum. We are interested in the behavior of the ration M_n/S_n.

Darling [4] proved that if $\overline{F}(x) = 1 - F(x)$ is slowly varying then the maximum term dominates the whole sum.

Theorem 22. *If $\overline{F}$ is slowly varying then $S_n/M_n \to 1$ in probability (and in L^1).*

Before the proof we need the conditional distribution of S_n given M_n.

Lemma 4. *Assume that F is continuous with density function f. Then*

$$\mathcal{L}(S_n | M_n = m) = \mathcal{L}(S_{n-1}^{(m)} + m),$$

where $S_k^{(m)} = Y_1^{(m)} + \ldots + Y_k^{(m)}$, with $Y^{(m)}, Y_1^{(m)}, \ldots$ being iid random variables with distribution function $\mathbf{P}(Y^{(m)} \le y) = \mathbf{P}(X \le y | X \le x)$.

Proof. It is a long but straightforward calculation. $\square$

Next we prove the theorem.

Proof of Theorem 22. Assume that F is continuous. This can be dropped by adding iid uniform$(0,1)$ random variables.

Note that S_n/M_n for n fix is a bounded nonnegative random variable which is ≥ 1. Therefore its mean can be calculated as the derivative of its Laplace transform at 0. Since $S_n/M_n \ge 1$, it is enough to show that $\mathbf{E} S_n/M_n \to 1$ as $n \to \infty$.

Let $\lambda \ge 0$. Using Lemma 4 (and the notation there) we have

$$\begin{aligned}\phi_n(\lambda) := \mathbf{E} e^{-\lambda \frac{S_n}{M_n}} &= \int_{[0,\infty)} \mathbf{E}^{-\lambda \frac{S_{n-1}^{(x)} + x}{x}} \mathrm{d}\mathbf{P}(M_n \le x) \\ &= \int_{[0,\infty)} n e^{-\lambda} \left(\int_{[0,x]} e^{-\lambda y/x} \mathrm{d}F(y) \right)^{n-1} \mathrm{d}F(x).\end{aligned} \tag{8}$$

Differentiating and substituting $\lambda = 0$

$$\mathbf{E}\frac{S_n}{M_n} = -\phi'(0) = 1 + \int_{[0,\infty)} n(n-1)F(x)^{n-2} \int_{[0,x]} \frac{y}{x} \mathrm{d}F(y)\mathrm{d}F(x). \tag{9}$$

Integration by parts gives

$$\int_{[0,x]} y \mathrm{d}F(y) = x \int_0^1 \left[\overline{F}(ux) - \overline{F}(x) \right] \mathrm{d}u.$$

Substituting back into (9)

$$\mathbf{E}\frac{S_n}{M_n} = 1 + \int_{[0,\infty)} n(n-1)F(x)^{n-2}\overline{F}(x)A(x)\mathrm{d}F(x), \tag{10}$$

where

$$A(x) = \int_0^1 \left(\frac{\overline{F}(ux)}{\overline{F}(x)} - 1 \right) \mathrm{d}u.$$

The integrand in $A(x)$ converges pointwise to 0 by the slow variation of $\overline{F}$, and Potter's bound provides an integrable majorant ($u^{-1/2}$ say). Therefore, by Lebesgue's dominated convergence theorem $\lim_{x\to\infty} A(x) = 0$. Let $\varepsilon > 0$ be fixed. Then there exists x_0 such that $A(x) \le \varepsilon$ for all $x \ge x_0$. Further, there exists n_0 such that $n(n-1)F(x_0)^{n-2} \sup_{y\in[0,x]} A(y) \le \varepsilon$ for $n \ge n_0$. Thus

$$\int_{[0,x_0]} n(n-1)F(x)^{n-2}\overline{F}(x)A(x)\mathrm{d}F(x) \le \varepsilon \int_{[0,\infty)} \mathrm{d}F(x) = \varepsilon.$$

On the other hand

$$\begin{aligned}
&\int_{(x_0,\infty)} n(n-1)F(x)^{n-2}\overline{F}(x)A(x)\mathrm{d}F(x) \\
&\le \varepsilon \int_{(x_0,\infty)} n(n-1)F(x)^{n-2}\overline{F}(x)\mathrm{d}F(x) \\
&\le \varepsilon \int_0^1 n(n-1)u^{n-2}(1-u)\mathrm{d}u = \varepsilon,
\end{aligned}$$

proving the statement. □

In fact the slow variation of $\overline{F}$ is necessary to the domination of the maxima.

Theorem 23 (Maller & Resnick, 1984)**.** *The following are equivalent:*

(i) $M_n/S_n \xrightarrow{\mathbf{P}} 1$*;*

(ii) $\overline{F}$ *is slowly varying.*

The other extremal situation is when the maxima is asymptotically negligible compared to the sum.

Theorem 24 (O'Brien, 1980)**.** *The following are equivalent:*

(i) $M_n/S_n \xrightarrow{\mathbf{P}} 0$*;*

(ii) $\int_{[0,x]} y\mathrm{d}F(y)$ *is slowly varying.*

Next we turn to the intermediate case.

Theorem 25 (Darling, 1952)**.** *If $\overline{F}$ is regularly varying with parameter $-\alpha \in (-1,0)$ then*

$$\frac{S_n}{M_n} \xrightarrow{\mathcal{D}} W, \qquad \text{where } \mathbf{E}e^{-\lambda W} = \frac{e^{-\lambda}}{1-\alpha\int_0^1 (e^{-\lambda u}-1)u^{-\alpha-1}\mathrm{d}u}.$$

Proof. Assume that F is continuous. Recall from (8) that

$$\phi_n(\lambda) = \int_{[0,\infty)} ne^{-\lambda}\left(\int_{[0,x]} e^{-\lambda y/x}\mathrm{d}F(y)\right)^{n-1}\mathrm{d}F(x).$$

Integration by parts gives

$$\int_{[0,x]} e^{-\lambda y/x}\mathrm{d}F(y) = 1-\overline{F}(x)-\overline{F}(x)\int_0^1\left(\frac{\overline{F}(ux)}{\overline{F}(x)}-1\right)\lambda e^{-\lambda u}\mathrm{d}u \qquad (11)$$

As $x\to\infty$, by the regular variation combined with Potter bounds and Lebesgue's dominated convergence we have

$$\int_0^1\left(\frac{\overline{F}(ux)}{\overline{F}(x)}-1\right)\lambda e^{-\lambda u}\mathrm{d}u \to \int_0^1 (u^{-\alpha}-1)\lambda e^{-\lambda u}\mathrm{d}u.$$

Since the integrand is exponentially small on any finite interval, we obtain for any K large

$$\begin{aligned}\phi_n(\lambda) &\sim e^{-\lambda}\int_K^\infty n\left[1-\overline{F}(x)\left(1+\int_0^1 (u^{-\alpha}-1)\lambda e^{-\lambda u}\mathrm{d}u\right)\right]\mathrm{d}F(x)\\ &\sim e^{-\lambda}\mathbf{E}\left[n(1-Uc_\lambda)^{n-1}I(U<\delta)\right],\end{aligned}$$

where $U\sim$Uniform$(0,1)$, $\delta=\overline{F}(K)$ and

$$c_\lambda = 1+\int_0^1 (u^{-\alpha}-1)\lambda e^{-\lambda u}\mathrm{d}u.$$

Now, simple analysis shows that

$$\lim_{n\to\infty}\mathbf{E}\left[n(1-Uc_\lambda)^{n-1}I(U<\delta)\right] = c_\lambda^{-1},$$

and the theorem follows.

The continuity assumption can be dropped by adding iid uniform$(0,1)$ random variables. □

The converse result is due to Breiman [2].

Theorem 26 (Breiman, 1965)**.** *If S_n/M_n converges in distribution to a non-degenerate limit then $\overline{F}$ is regularly varying with parameter $-\alpha \in (-1, 0)$.*

Proof. Again, assume that F is continuous.

The distributional convergence of S_n/M_n implies that

$$\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda) \tag{12}$$

exists for all $\lambda \geq 0$. Put

$$U(\lambda, x) = \int_{[0,x]} e^{-\lambda y} \mathrm{d}F(y). \tag{13}$$

We have seen in (8) that

$$\phi_n(\lambda) = e^{-\lambda} \int_{[0,\infty)} nU(\lambda/x, x)^{n-1} F(\mathrm{d}x).$$

The monotonicity of U and (12) implies that n can be exchanged to the continuous parameter t, i.e.

$$\lim_{t\to\infty} e^{-\lambda} \int_{[0,\infty)} tU(\lambda/x, x)^t F(\mathrm{d}x) = \phi(\lambda). \tag{14}$$

We have seen in (11) that

$$U(\lambda/x, x) = 1 - \overline{F}(x) \left(1 + \int_0^1 \left(\frac{\overline{F}(ux)}{\overline{F}(x)} - 1\right) \lambda e^{-\lambda u} \mathrm{d}u\right). \tag{15}$$

Note that $U(\lambda/x, x)$ is increasing in x, and it is strictly increasing for x large. Moreover, $\lim_{x\to 0} U(\lambda/x, x) = 0$, and $\lim_{x\to\infty} U(\lambda/x, x) = 1$. For $\lambda \geq 0$ fixed, put

$$V(x) = -\log U(\lambda/x, x), \tag{16}$$

and let $G(t) = \mu_F(\{y : V(y) \leq t\})$. By the transformation theorem

$$\int_{[0,\infty)} U(\lambda/x, x)^t F(\mathrm{d}x) = \int_{[0,\infty)} e^{-tV(x)} F(\mathrm{d}x) = \int_{[0,\infty)} e^{-ty} G(\mathrm{d}y).$$

Thus, by Karamata's Tauberian theorem (14) is equivalent to

$$G(y) \sim y\phi(\lambda)e^{\lambda} \quad \text{as } y \downarrow 0. \tag{17}$$

By the continuity of F

$$G(V(x)) = \mu_F(\{u : V(u) \le V(x)\}) = \mu_F(\{u :\ u \ge x\}) = \overline{F}(x-) = \overline{F}(x),$$

which, combined with (17) and (15)

$$\overline{F}(x) \sim e^{\lambda}\phi(\lambda)\left(\overline{F}(x)e^{-\lambda} + \int_0^1 \overline{F}(ux)e^{-\lambda u}\mathrm{d}u\right).$$

Therefore, we obtain that

$$\lim_{x\to\infty}\int_0^1 \frac{\overline{F}(ux)}{\overline{F}(x)}e^{-\lambda u}\mathrm{d}u \tag{18}$$

exists for all λ. We need the following lemma.

Lemma 5. *Let $J_n(u)$ be a sequence of nonincreasing functions such that for all $\lambda \ge 0$*

$$\lim_{n\to\infty}\int_0^1 e^{-\lambda u}J_n(u)\mathrm{d}u = h(\lambda)$$

for some $h(\lambda)$. Then there exists $J(u)$ nonincreasing such that $J_n(x) \to J(x)$ for all $x \in C_J$, and

$$h(\lambda) = \int_0^1 e^{-\lambda u}J(u)\mathrm{d}u.$$

Proof. The existence of a subsequential limit follows from Helly's selection theorem. The uniqueness of the limit follows from the continuity theorem. □

The lemma and (18) implies that the limit $\overline{F}(ux)/\overline{F}(x)$ exists for each u, which implies that $\overline{F}$ is regularly varying. □

12 Breiman's conjecture

Breiman's motivation in his 1965 paper was the following. Let $S_1, S_2, \ldots$ be a simple symmetric random walk, and let $Y, Y_1, Y_2, \ldots$ be the interarrival times between the consecutive zeros of $S_1, S_2, \ldots$. Independently of S, let $X, X_1, X_2, \ldots$ be iid 0/1 random variables such that $\mathbf{P}(X = 0) = \frac{1}{2} = \mathbf{P}(X = 1)$. Then

$$R_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

is the proportion of the time that the random walk spends in $[0, \infty)$.

In this case the well-known arcsine law holds.

Theorem 27 (Arcsine law). *Let the X's and Y's be as above. Then*

$$\lim_{n \to \infty} \mathbf{P}(R_n \leq x) = \frac{2}{\pi} \arcsin \sqrt{x}.$$

Moreover, in this case $\overline{G}(y) = \mathbf{P}(Y > y) \sim c y^{-1/2}$, in particular it is regularly varying with parameter $1/2$.

In general, let $Y, Y_1, Y_2, \ldots$ be nonnegative iid random variables with distribution function G, and independently let $X, X_1, X_2, \ldots$ be iid random variables with distribution function F, and assume that $\mathbf{E}|X| < \infty$. What is the necessary and sufficient condition on G such that

$$R_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

has a nondegenerate limit as $n \to \infty$?

Remark 1. If $\mathbf{E}Y < \infty$, then

$$\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i} = \frac{\frac{\sum_{i=1}^n X_i Y_i}{n}}{\frac{\sum_{i=1}^n Y_i}{n}} \xrightarrow{\text{a.s.}} \mathbf{E}X,$$

so the limit exists, and it is degenerate. Therefore, the interesting situation is when $\mathbf{E}Y = \infty$.

Breiman proved the following.

Theorem 28 (Breiman, 1965). *If $Y \in D(\alpha)$ then for every F with finite expectation*

$$R_n \xrightarrow{\mathcal{D}} V,$$

where

$$\mathbf{P}(V \le x) = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\frac{\varphi_1(x) - \varphi_2(x)}{\varphi_1(x) + \varphi_2(x)} \tan \frac{\alpha\pi}{2} \right),$$

with

$$\varphi_1(x) = \int_{y \le x} |y - x|^\alpha \mathrm{d}G(y),$$
$$\varphi_2(x) = \int_{y \ge x} |y - x|^\alpha \mathrm{d}G(y).$$

Conversely, if R_n converges in distribution for every *F, and the limit is non-degenerate for at least one F, then $Y \in D(\alpha)$, for some $\alpha \in [0, 1)$, i.e. $\overline{G}$ is regularly varying with parameter $-\alpha \in (-1, 0]$.*

The necessity part is the difficult one. We do not prove this result, only sketch the idea; for the details see Breiman [2]. The idea of his proof is to prove that the existence of the limit for *all* X implies the existence of the distributional limit of

$$\frac{\max\{Y_1, \ldots, Y_n\}}{Y_1 + \ldots + Y_n}$$

which, by Theorem 26 implies the regular variation.

The sufficiency follows from the following important observation, which was proved in a slightly weaker form by Breiman [2]; see also [3, Lemma B.5.1].

Breiman's Lemma. *If X and Y are independent, nonnegative random variables, $Y \in D(\alpha)$ and $\mathbf{E}X^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$ then $XY \in D(\alpha)$.*

Proof. Conditioning on X, for any $m > 0$

$$\begin{aligned} \frac{\mathbf{P}(XY > z)}{\mathbf{P}(Y > z)} &= \int_{[0,\infty)} \frac{\mathbf{P}(Y > z/x)}{\mathbf{P}(Y > z)} \mathrm{d}F(x) \\ &= \int_{[0,m]} \frac{\mathbf{P}(Y > z/x)}{\mathbf{P}(Y > z)} \mathrm{d}F(x) + \int_{(m,\infty)} \frac{\mathbf{P}(Y > z/x)}{\mathbf{P}(Y > z)} \mathrm{d}F(x). \end{aligned} \tag{19}$$

By the uniform convergence theorem the integrand in the first term goes to x^α, thus as $m \to \infty$

$$\lim_{m\to\infty} \lim_{z\to\infty} \int_{[0,m]} \frac{\mathbf{P}(Y > z/x)}{\mathbf{P}(Y > z)} \mathrm{d}F(x) = \int_{[0,\infty)} x^\alpha \mathrm{d}F(x) = \mathbf{E}X^\alpha.$$

To finish the proof we have to show that the second term in (19) is negligible as $m \to \infty$, i.e.

$$\lim_{m\to\infty} \limsup_{z\to\infty} \int_{(m,\infty)} \frac{\mathbf{P}(Y > z/x)}{\mathbf{P}(Y > z)} \mathrm{d}F(x).$$

This follows by the Potter bounds combined with Lebesgue's dominated convergence theorem. $\square$

The existence of the limit for *all* X is an essential assumption in the proof, though Breiman conjectured it is not necessary. This is the Breiman conjecture, which is still open.

Breiman conjecture. *If T_n has a non-degenerate limit for some F, then $Y \in D(\alpha)$ for some $\alpha \in [0,1)$.*

A partial solution was obtained by Mason and Zinn [9].

Theorem 29 (Mason & Zinn, 2005)**.** *If $\mathbf{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$, and T_n converges in distribution to a nondegenerate limit then $Y \in D(\alpha)$ for some $\alpha \in (0,1)$.*

Proof. We may and do assume that $\mathbf{E}X = 0$. By the Jensen inequality

$$\mathbf{E}R_n^{2+\delta} \le \mathbf{E}\frac{\sum_{i=1}^n |X_i|^{2+\delta} Y_i}{\sum_{i=1}^n Y_i} = \mathbf{E}|X|^{2+\delta},$$

implying that $\mathbf{E}R_n^2$ is uniformly integrable. Therefore $R_n \xrightarrow{\mathcal{D}} V$ implies

$$\lim_{n\to\infty} \mathbf{E}R_n^2 = \mathbf{E}R^2.$$

By the independence of X and Y, and the fact that $\mathbf{E}X = 0$ we have

$$\mathbf{E}R_n^2 = \mathbf{E}\frac{\left(\sum_{i=1}^n X_i Y_i\right)^2}{\left(\sum_{i=1}^n Y_i\right)^2} = \mathbf{E}X^2 \, n\mathbf{E}\frac{Y_1^2}{\left(\sum_{i=1}^n Y_i\right)^2}.$$

Thus we have that for some $\alpha \in [0,1)$

$$\lim_{n\to\infty} n\mathbf{E}\frac{Y_1^2}{\left(\sum_{i=1}^n Y_i\right)^2} = 1 - \alpha. \tag{20}$$

The following proposition, which was proved independently by Fuch, Joffe, and Teugels [6] and Mason and Zinn [9], implies the theorem. $\square$

Proposition 8. *The limit relation* (20) *holds if and only if* $Y \in D(\alpha)$.

Proof. We have that

$$\begin{aligned}
\mathbf{E}\frac{\sum_{i=1}^n Y_i^2}{\left(\sum_{i=1}^n Y_i\right)^2} &= n\mathbf{E}\frac{Y_1^2}{\left(\sum_{i=1}^n Y_i\right)^2} \\
&= n\mathbf{E}\int_0^\infty Y_1^2 e^{-t\sum_{i=1}^n Y_i}\, t\, \mathrm{d}t \\
&= n\int_0^\infty t\mathbf{E}\left(e^{-tY_1}Y_1^2\right)\left(\mathbf{E}e^{-tY_1}\right)^{n-1}\mathrm{d}t \\
&= n\int_0^\infty t\phi''(t)\phi(t)^{n-1}\mathrm{d}t,
\end{aligned}$$

where $\phi(t) = \mathbf{E}e^{-tY}$. As in the proof of Theorem 26 we obtain that n can be changed to the continuous parameter t, i.e.

$$s\int_0^\infty t\phi''(t)e^{s\log\phi(t)}\mathrm{d}t \to (1-\alpha), \quad s\to\infty, \tag{21}$$

where $0 < \gamma \leq 1$. For $y \geq 0$, let $q(y)$ denote the inverse of $-\log\varphi(t)$. Changing the variables to $y = -\log\phi(t)$ and $t = q(y)$, we get from (21) that

$$s\int_0^\infty (q(y))\,\phi''(q(y))\exp(-sy)\,\mathrm{d}q(y) \to (1-\alpha), \qquad \text{as } s\to\infty.$$

By Karamata's Tauberian theorem we conclude that

$$v^{-1}\int_0^v q(x)\phi''(q(x))\mathrm{d}q(x) \to (1-\alpha), \qquad \text{as } v\downarrow 0,$$

which, in turn, by the change of variable $y = q(x)$ gives

$$\frac{\int_0^t y\phi''(y)\mathrm{d}y}{-\log\phi(t)} \to (1-\alpha), \qquad \text{as } t\downarrow 0.$$

Now using that $-\log\phi(t) \sim 1-\phi(t)$ as $t\downarrow 0$, we end up with

$$\lim_{t\downarrow 0}\frac{\int_0^t y\phi''(y)\mathrm{d}y}{1-\phi(t)} = 1-\alpha. \tag{22}$$

Since $\phi_\alpha(y) = \int_0^\infty e^{-uy} u^\alpha G(\mathrm{d}u)$, by Fubini's theorem

$$\begin{aligned}
\int_0^t y^{\alpha-1}\phi_\alpha(y)\mathrm{d}y &= \int_0^\infty u^\alpha G(\mathrm{d}u) \int_0^t y^{\alpha-1}e^{-uy}\mathrm{d}y \\
&= \int_0^\infty G(\mathrm{d}u) \int_0^{ut} z^{\alpha-1}e^{-z}\mathrm{d}z \\
&= \int_0^\infty \overline{G}(z/t) z^{\alpha-1}e^{-z}\mathrm{d}z \\
&= t^\alpha \int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}\mathrm{d}u.
\end{aligned}$$

A partial integration gives

$$1 - \phi_0(t) = t\int_0^\infty \overline{G}(u)e^{-ut}\mathrm{d}u.$$

So

$$t^{\alpha-1}\frac{\int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}\mathrm{d}u}{\int_0^\infty \overline{G}(u)e^{-ut}\mathrm{d}u} \to \gamma\Gamma(\alpha), \text{ as } t \searrow 0, \tag{23}$$

with $0 < \gamma \leq 1$. Let us define the function for $t > 0$

$$f(t) = \int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}\mathrm{d}u. \tag{24}$$

Clearly, f is monotone decreasing and since $\mathbf{E}Y = \infty$, $\lim_{t\to 0} f(t) = \infty$. We shall show that f is regularly varying at 0, which by Lemma 3 of Pitman [10], implies that $\overline{G}$ is regularly varying at infinity. We use the identity

$$u^{1-\alpha}e^{-ut} = \frac{1}{\Gamma(\alpha-1)}\int_0^\infty y^{\alpha-2}e^{-(y+t)u}\mathrm{d}y,$$

which holds for $u > 0$ and $\alpha \in (1, 2]$. (This is the *Weyl-transform*, or *Weyl-fractional integral* of the function e^{-ut}.) This identity combined with Fubini's theorem (everything is nonnegative) gives

$$\begin{aligned}
\frac{1}{\Gamma(\alpha-1)}\int_0^\infty y^{\alpha-2}f(y+t)\mathrm{d}y &= \int_0^\infty \overline{G}(u)u^{\alpha-1}\mathrm{d}u \frac{1}{\Gamma(\alpha-1)}\int_0^\infty y^{\alpha-2}e^{-(y+t)u}\mathrm{d}y \\
&= \int_0^\infty \overline{G}(u)e^{-ut}\mathrm{d}u.
\end{aligned}$$

So we can rewrite (23) as

$$\lim_{t \searrow 0} \frac{t^{\alpha-1} f(t)}{\int_0^\infty y^{\alpha-2} f(t+y) \mathrm{d}y} = \frac{\gamma \Gamma(\alpha)}{\Gamma(\alpha-1)} = \gamma(\alpha-1). \tag{25}$$

A change of variable gives

$$\int_0^\infty y^{\alpha-2} f(t+y) \mathrm{d}y = t^{\alpha-1} \int_1^\infty (u-1)^{\alpha-2} f(ut) \mathrm{d}u,$$

and so we have

$$\lim_{t \searrow 0} \int_1^\infty (u-1)^{\alpha-2} \frac{f(ut)}{f(t)} \mathrm{d}u = [\gamma(\alpha-1)]^{-1}. \tag{26}$$

We can rewrite f as

$$f(t) = \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} \mathrm{d}u = t^{-\alpha} \int_0^\infty \overline{G}(u/t) u^{\alpha-1} e^{-u} \mathrm{d}u,$$

from which we see that the function

$$g(t) = \int_0^\infty \overline{G}(u/t) u^{\alpha-1} e^{-u} \mathrm{d}u = t^\alpha f(t)$$

is bounded and nondecreasing. Substituting g into (26) we obtain

$$\lim_{t \to 0+} \int_1^\infty (u-1)^{\alpha-2} u^{-\alpha} \frac{g(ut)}{g(t)} \mathrm{d}u = [\gamma(\alpha-1)]^{-1}. \tag{27}$$

□

The next proposition is extension of the previous result.

Proposition 9 (Kevei & Mason [8]). *If*

$$\mathbb{E} S_n(\alpha) \to \gamma, \ \text{as } n \to \infty \tag{28}$$

holds with some $\gamma \in (0,1]$ then $Y \in D(\beta)$, for some $\beta \in [0,1)$, where $-\beta \in (-1,0]$ is the unique solution of

$$\mathrm{Beta}(\alpha-1, 1-\beta) = \frac{\Gamma(\alpha-1)\Gamma(1-\beta)}{\Gamma(\alpha-\beta)} = \frac{1}{\gamma(\alpha-1)}.$$

In particular, $Y \in D(0)$ for $\gamma = 1$.

Conversely, if $Y \in D(\beta)$, $0 \le \beta < 1$, then (28) holds with

$$\gamma = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1-\beta)} = \frac{1}{(\alpha-1)\mathrm{Beta}(\alpha-1, 1-\beta)}.$$

The best result on Breiman's conjecture is the following.

Theorem 30 (Kevei & Mason [8]). *Assume that for some* $X \in \mathcal{X}_0$, $1 < \alpha \leq 2$, *positive slowly varying function* L *at zero and* $c > 0$,

$$\frac{-\log(\Re\phi_X(t))}{|t|^\alpha L(|t|)} \to c, \ \text{as } t \to 0. \tag{29}$$

Whenever

$$T_n \to_d T, \ \text{as } n \to \infty, \ \text{with } T \text{ nondegenerate,} \tag{30}$$

holds then $Y \in D(\beta)$ *for some* $\beta \in [0, 1)$.

12.1 Exercises

21. Show that in the proof of Breiman's lemma

$$\lim_{m\to\infty} \limsup_{z\to\infty} \int_{(m,\infty)} \frac{\mathbf{P}(Y > z/x)}{\mathbf{P}(Y > z)} \mathrm{d}F(x).$$

22. Version of Breiman's lemma. Assume that X and Y are independent nonnegative random variables such that $\mathbf{P}(Y > y) \sim cy^{-\alpha}$ for some $c > 0$, and $\mathbf{E}X^\alpha < \infty$. Show that

$$\mathbf{P}(XY > z) \sim \mathbf{E}X^\alpha \mathbf{P}(Y > z) \quad \text{as } \ z \to \infty.$$

13 Renewal theory

Let $X, X_1, \ldots$ be iid nonnegative random variables with distribution function $F(x) = \mathbf{P}(X \leq x)$. Put $S_n = X_1 + \ldots + X_n$, $S_0 = 0$. Let

$$U(x) = \sum_{n=0}^{\infty} F^{*n}(x), \quad x \geq 0,$$

be the renewal function.

The elementary renewal theorem states that

$$\lim_{x\to\infty} \frac{U(x)}{x} = \mu^{-1}, \tag{31}$$

where $\mu = \mathbf{E}X \in (0, \infty]$. If $\mu = \infty$, then the limit above is 0.

A random variable is *arithmetic* (or centered lattice) if $\mathbf{P}(X \in \delta\mathbb{Z}) = 1$ for some $\delta > 0$. The following renewal theorem is a more precise version of (31).

Theorem 31 (Blackwell's theorem)**.** *Assume that X is nonarithmetic. Then for each $h > 0$*

$$\lim_{x\to\infty}[U(x+h) - U(x)] = \frac{h}{\mu}.$$

Lemma 6. *Let $z(x)$ be a bounded measurable function, $z(x) = 0$ for $x < 0$. The equation*

$$Z(x) = z(x) + \int_{[0,x]} Z(x-y)\mathrm{d}F(y), \quad x \geq 0,$$
$$Z(x) = z(x) = 0, \quad x \leq 0,$$

has a unique solution

$$Z(x) = \int_{[0,x]} z(x-y)\mathrm{d}U(y),$$

which is bounded on finite intervals.

Since $U(y)$ behaves as $\mu^{-1}\cdot y$, one expects that $\lim_{x\to\infty}\int_0^x z(x-y)\mathrm{d}U(y) = \mu^{-1}\int_0^\infty z(y)\mathrm{d}y$. However, an extra condition is needed for z.

A function $z \geq 0$ is *directly Riemann integrable* (dRi) if $\lim_{h\downarrow 0}[\overline{U}(h) - \underline{U}(h)] = 0$, where

$$\overline{U}(h) = h\sum_{k=0}^{\infty}\sup_{u\in[kh,(k+1)h]} z(u)$$
$$\underline{U}(h) = h\sum_{k=0}^{\infty}\inf_{u\in[kh,(k+1)h]} z(u).$$

Theorem 32 (Key Renewal Theorem)**.** *If z is dRi, F is nonarithmetic, then*

$$\lim_{x\to\infty} Z(x) = \lim_{x\to\infty}\int_0^x z(x-y)\mathrm{d}U(y) = \frac{1}{\mu}\int_0^\infty z(y)\mathrm{d}y.$$

Infinite mean case...

13.1 Exercises

23. Let $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$. Determine the corresponding renewal function $U(x) = \sum_{n=0}^\infty F^{*n}(x)$.

Hint: Using the convolution formula for continuous random variables we can determine the density function of F^{*n}, which can be summed.

24. Let X be a random variables with distribution function F. Using the renewal theorem show that if F is nonarithmetic, $\mathbf{E}(X) = \mu$, $\mathbf{Var}(X) = \sigma^2$, and $U = \sum_{n=0}^{\infty} F^{*n}$ is the corresponding renewal function, then

$$0 \leq U(t) - \frac{t}{\mu} \to \frac{\sigma^2 + \mu^2}{2\mu^2}.$$

25. Let X be a random variables with distribution function F. Show that if $F_0(t) = \mu^{-1} \int_0^t [1 - F(y)] \mathrm{d}y$ then $V(t) = t/\mu$ is the solution of the renewal equation

$$V = F_0 + F * V.$$

That is, a delayed renewal process with delay distribution F_0 has constant renewal rate.

26. Let $X, X_1, X_2, \ldots$ be iid nonnegative random variables with distribution function F, and finite mean $\mathbf{E}X = \mu$. Let N_t be the corresponding renewal process, i.e. $S_{N_t} \leq t < S_{N_t+1}$.

(a) Determine the limit distribution of the waiting time $t - S_{N_t}$ as $t \to \infty$.

(b) Determine the limit distribution of the interarrival time $S_{N_t+1} - S_{N_t}$ as $t \to \infty$.

14 Implicit renewal theory

The most investigated stochastic fixed point equation is the *perpetuity equation*, which is

$$X \stackrel{\mathcal{D}}{=} AX + B, \tag{32}$$

where the random vector (A, B) and X on the right-hand side are independent. Other examples are

$$X \stackrel{\mathcal{D}}{=} AX \vee B, \tag{33}$$

and

$$X \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} A_i + B, \tag{34}$$

where X is independent of $B, A_1, \ldots$, and A's are iid conditionally on the environment ϵ.

The latter corresponds to the stationary distribution of a Galton–Watson branching process with immigration in random environment.

In general a random fixed point equation is a distributional equation of the form

$$X \stackrel{\mathcal{D}}{=} \Psi(X), \tag{35}$$

where $\Psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a random operator, independent of X on the right-hand side.

In the 3 examples above, we see that for large X the operator Ψ is close to a multiplication. Indeed, for (32) and (33) it is a multiplication by A, and in case (34), by the law of large numbers it is a multiplication by the conditional expectation $m(\epsilon)$. Goldie's implicit renewal theory [7] is tailor-made to these situations.

Assumptions: Assume that A is a nonnegative random variable, $\log A$ is nonarithmetic conditioned on being nonzero, and $\mathbf{E}A^\kappa = 1$, $\mathbf{E}A^\kappa \log A < \infty$ for some $\kappa > 0$.

Theorem 33 (Goldie 1991, [7])**.** *Let X be the solution of* (35)*. If the assumptions above hold and $\mathbf{E}|\Psi(X)^\kappa - (AX)^\kappa| < \infty$ then*

$$\mathbf{P}(X > x) \sim cx^{-\kappa} \quad \text{as } x \to \infty,$$

where

$$c = \frac{1}{\kappa \mathbf{E}A^\kappa \log A} \mathbf{E}[\Psi(X)^\kappa - (AX)^\kappa] \geq 0.$$

Proof. Using renewal theorem. See [7] or [3]. □

The problem is that the constant c can be 0, and in general it is a difficult task to show that it is strictly positive.

The first result of this kind was proved for the perpetuity equation.

Theorem 34 (Kesten (1973), Grincevičius (1975))**.** *If* $\mathbf{E}A^\kappa = 1, \mathbf{E}A^\kappa \log_+ A < \infty$*,* $\log A$ *is nonarithmetic,* $\mathbf{E}B^\kappa < \infty$ *then for the solution to the equation* $X \stackrel{\mathcal{D}}{=} AX + B$ *we have*

$$\mathbf{P}\{X > x\} \sim cx^{-\kappa},$$

with $c > 0$*.*

In this case the regular variation is caused by the multiplicative factor A alone. It is also possible that the additive term B dominates, and causes regular variation.

Theorem 35 (Grincevičius (1975), Grey (1994))**.** *If* $A \geq 0$*,* $\mathbf{E}A^\kappa < 1$*,* $\mathbf{E}A^{\kappa+\epsilon} < \infty$ *then the tail of X is regularly varying with parameter $-\kappa$ if and only if the tail of B is.*

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