Financial mathematics

Péter Kevei

December 6, 2021

Contents

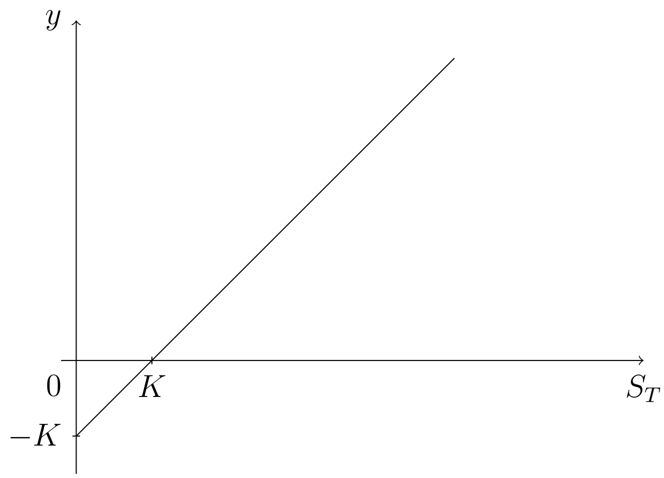

Figure 1: Payoff of a forward

1 Introduction

These notes are based on the Hungarian lecture notes by Gáll and Pap [2], on Shiryaev's monograph [3], and on Elliott and Kopp [1].

There are two type of financial instruments: the basic financial units and their derivatives.

Underlying:

- bond: risk-free asset, basically money. Its price is deterministic B_t;
- stock: risky asset. Its price is a random, modeled by a stochastic process $S_t = (S_t^1, \ldots, S_t^d)$.

Derivatives are bets on the underlying. They are used to share or reduce risk. Here we consider forward contracts and options.

1.1 Forward

A forward contract is an agreement to buy or sell an asset (stock) for a price previously agreed K in the future time T.

From the buyers point of view, at time T his wealth is $S_T - K$, that is the payoff function is $f(s) = s - K$.

We want to determine the fair price of this contract, and to understand the meaning of 'fair'. Assume $B_0 = 1$.

Seller's point of view: At time 0, we can buy a stock for S_0. Then at time T selling a stock for K and paying back the loan $S_0 \cdot B_T$, we have $K - S_0 B_T$. Therefore,

$$K \geq S_0 B_T.$$

Buyer's point of view: At time 0, we sell a stock for S_0. At time T we pay K for a stock, and the our wealth is $S_0 B_T - K$. Thus,

$$K \leq S_0 B_T.$$

We see that the fair price has to be $K = S_0 B_T$. Otherwise, either the seller or the buyer would have a strategy providing riskless profit (arbitrage).

Example 1. Let $S_0 = 40$, $B_t = e^{rt}$, $r = 0.1$ being the annual interest, $T = 1$ year. What is the fair price of this forward, and what is the value of the contract after half a year if $S_{0.5} = 45$?

The forward price at time 0 is

$$K = S_0 B_1 = 40 \cdot e^{0.1} = 44.2.$$

At time $t = 0.5$ the forward price

$$K_2 = S_{0.5} B_{0.5} = 45 \cdot e^{\frac{1}{2}0.1} = 47.3.$$

Thus the current value of the contract

$$e^{-\frac{1}{2}r}(47.3 - 44.2) = 2.9.$$

1.2 Options

An option is *right* to do something but not an obligation. European option can be executed only at the expiration date, while American options can be executed at any time.

The writer of a European call option agrees to sell a stock for a previously agreed price K. Clearly, the buyer of this option will not use his right if $S_T < K$. The payoff function for the buyer is $f(s) = (s - K)_+$

In case of a put option the writer agrees to buy a stock for K. The payoff function of the buyer

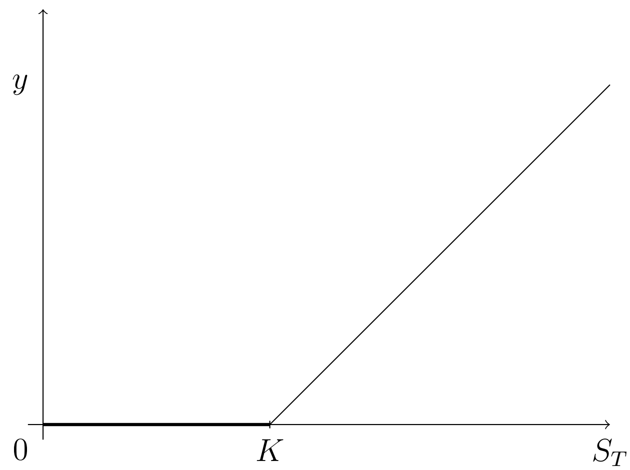

Figure 2: Payoff a call option

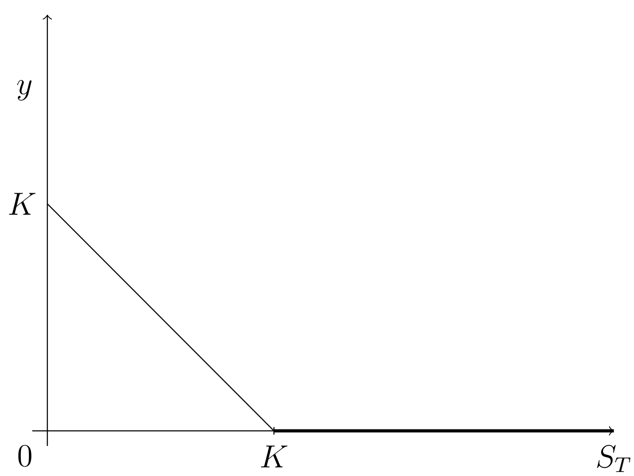

Figure 3: Payoff of a put option

1.3 Put–call parity

The aim of the course is to determine the fair price of an option, and understand the fairness. However, there is a simple relation between call and put prices regardless of the underlying market model.

Let C_K be the fair price of the call, and P_K be the fair price of the put, both with strike price K. Then, from the payoff functions it is easy to see that having put, a stock, and -1 call results at the expiration date (regardless of the stock price) a wealth K. That is, after discounting

$$\frac{K}{B_T} = P + S_0 - C.$$

This is the put-call parity.

2 Portfolio, claim, and hedging in discrete time

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. In the discrete time case we always assume (if not stated otherwise) that Ω is finite, and $\mathbf{P}(\{\omega\}) > 0$ for each $\omega \in \Omega$. We assume that transactions are made only at the time instants $0, 1, \ldots, N$. Let $(\mathcal{F}_n)_{n=0,1,\ldots,N}$ be a filtration, an increasing sequence of σ-algebras, such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$. Assume that there are d risky assets and a bond. The price of the risky asset i at time is S_n^i, an $\mathcal{F}_n$-measurable random variable, and the bond price at time n is B_n.

2.1 Portfolio

An investment portfolio (strategy) is $\pi_n = (\beta_n, \gamma_n)$, where $\beta_n \in \mathbb{R}$ represents the amount of bonds in the portfolio at time n, while $\gamma_n = (\gamma_n^1, \ldots, \gamma_n^d) \in \mathbb{R}^d$, where γ_n^i represents the amount of type-i stock at time n. The random variables (β_n, γ_n) are $\mathcal{F}_{n-1}$-measurable, which means the investor has to decide at time $n-1$ how to invest on time n. That is the sequence (β_n, γ_n) is *predictable*. For simplicity

$$\gamma_n S_n = \sum_{i=1}^{d} \gamma_n^i S_n^i.$$

The wealth of the investor at time n under the strategy π is

$$X_n^\pi = \beta_n B_n + \gamma_n S_n.$$

This is the *value process* of the investment portfolio.

A strategy is *self-financing (SF)* if the investor does not take out money from, and does not invest money to the portfolio after time 0. That is π is self-financing if

$$X_{n-1}^\pi = \beta_n B_{n-1} + \gamma_n S_{n-1} \quad \text{for all } n.$$

For a sequence a_n put $\Delta a_n = a_n - a_{n-1}$.

{lemma:SF}

Lemma 1. *The following are equivalent:*

(i) π *is SF;*

(ii) $\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n \Delta S_n$*;*

(iii) $B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = 0$*.*

Proof. We have

$$\begin{aligned}
&\Delta X_n = X_n - X_{n-1}\\
&= \beta_n B_n - \beta_{n-1}B_{n-1} + \gamma_n S_n - \gamma_{n-1}S_{n-1}\\
&= \beta_n(B_n - B_{n-1}) + (\beta_n - \beta_{n-1})B_{n-1} + \gamma_n(S_n - S_{n-1}) + (\gamma_n - \gamma_{n-1})S_{n-1}\\
&= \beta_n \Delta B_n + \Delta\beta_n B_{n-1} + \gamma_n \Delta S_n + \Delta\gamma_n S_{n-1},
\end{aligned}$$

and the equivalence follows. $\square$

In what follows, unless otherwise stated all the strategies are meant to be SF.

We can decompose the value process as

$$\begin{aligned}
X_n^\pi &= X_{n-1}^\pi + \Delta X_n^\pi = \dots\\
&= X_0^\pi + \sum_{i=1}^n (\beta_i \Delta B_i + \gamma_i \Delta S_i)\\
&=: X_0^\pi + G_n^\pi,
\end{aligned}$$

where G_n^π is the *gain process*. So the value of the strategy is the initial investment plus the gain.

2.2 Strategies in a more general market

Previously, we assumed that there are no transaction cost (market is frictionless), shares pay no dividend, and apart from time 0, there is neither investment, nor consumption. Here we see how to handle this.

2.2.1 Dividend

Assume that stock-i pays a dividend $\delta_n^i = D_n^i - D_{n-1}^i \geq 0$ at n, where δ_n^i, and D_n^i are adapted processes. Then the change in the value process is

$$\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \delta_n),$$

and the value of the portfolio

$$X_n^\pi = \beta_n B_n + \gamma_n(S_n + \delta_n).$$

Then, π is self-financing portfolio if

$$B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = \delta_{n-1}\gamma_{n-1}.$$

Indeed, the dividend obtained in time $n-1$ is reinvested in the portfolio.

2.2.2 Consumption and investment

The consumption and investment can be included as well. Let (C_n), (I_n) be adapted nondecreasing random sequences with $C_0 = I_0 = 0$. Then, if an investor takes out ΔC_n and invests ΔI_n then

$$\begin{aligned} \Delta X_n^\pi &= \beta_n \Delta B_n + \gamma_n \Delta S_n + \Delta I_n - \Delta C_n \\ X_n^\pi &= \beta_n B_n + \gamma_n S_n. \end{aligned}$$

2.2.3 Transaction costs

If $\Delta\gamma_n > 0$ then we buy share, and pay an extra cost λ, that is we pay $(1+\lambda)S_{n-1}\Delta\gamma_n$. While if $\Delta\gamma_n < 0$ we sell, and paying transaction cost means receiving less money, say $-(1-\mu)S_{n-1}\Delta\gamma_n$. Then an SF strategy satisfies (see (iii) in Lemma 1)

$$B_{n-1}\Delta\beta_n + (1+\lambda)S_{n-1}\Delta\gamma_n I(\Delta\gamma_n > 0) + (1-\mu)S_{n-1}\Delta\gamma_n I(\Delta\gamma_n < 0) = 0.$$

2.3 Claim and hedging

Let f_N be a nonnegative random variable, which is the *payoff function*, or *obligation*, or *contingent claim*. A strategy π is an *upper* (x, f_N)*-hedge*, if $\mathbf{P}$-almost surely

$$X_0^\pi = x, \quad X_N^\pi \geq f_N.$$

It is a *lower* (x, f_N)*-hedge*, if a.s.

$$X_0^\pi = x, \quad X_N^\pi \leq f_N.$$

The hedge is perfect if $=$ holds a.s.

Put

$$C^*(f_N) = \inf\{x : \exists \text{ upper } (x, f_N)\text{-hedge }\},$$

and similarly

$$C_*(f_N) = \sup\{x : \exists \text{ lower } (x, f_N)\text{-hedge }\}.$$

For the class of upper (x, f_N)-hedge strategies put $H^*(x, f_N, \mathbf{P})$, and for the lower $H_*(x, f_N, \mathbf{P})$.

{lemma:hedge}

Lemma 2. *For any payoff function f_N there exists an x such that there is an upper (x, f_N)-hedge.*

Proof. Put

$$x = \frac{B_0}{B_N} \max_{\omega \in \Omega} |f_N(\omega)|.$$

Then the (trivial) strategy $\pi_n \equiv (\frac{x}{B_0}, 0)$ (start with enough money and don't do anything) is an upper hedge. □

2.4 Binomial market

{ss:bin}

2.4.1 One-step market

Consider a one-step binomial market with $d = 1$ stock. That is $\Omega = \{0, 1\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = 2^\Omega$. Assume that $\mathbf{P}(\{0\}) \in (0, 1)$. The bond price $B_1 = (1 + r)B_0$, that is $r > -1$ is the interest rate, and for some $a < b$, $S_1 = (1 + \rho)S_0$, $\rho \in \{a, b\}$. Say, $\rho(1) = b$, $\rho(0) = a$. Let f be a payoff, that is $f(0) = f_0$, $f(1) = f_1$. We construct a perfect hedge.

Using the strategy $\pi_1 = (\beta_1, \gamma_1)$ we want that

$$X_1^\pi = \beta_1 B_1 + \gamma_1 S_1 = f \quad \text{a.s.}$$

Since there are only two possibilities, a.s. means

$$\beta_1 B_0(1+r) + \gamma_1 S_0(1+a) = f_0$$
$$\beta_1 B_0(1+r) + \gamma_1 S_0(1+b) = f_1.$$

Solving the linear system

$$\gamma_1 = \frac{1}{S_0}\frac{f_1 - f_0}{b-a}, \quad \beta_1 = \frac{f_1 - (1+b)\frac{f_1-f_0}{b-a}}{B_0(1+r)}.$$

This is deterministic, so $\mathcal{F}_0$-measurable, as it should be. The initial cost of this strategy is

$$X_0^\pi = B_0\beta_1 + S_0\gamma_1 = \frac{1}{1+r}\left(\frac{r-a}{b-a}f_1 + \frac{b-r}{b-a}f_0.\right)$$

If $a < r < b$ this can be written as

$$X_0^\pi = \frac{1}{1+r}\mathbf{E}_\mathbf{Q} f,$$

with the probability measure $\mathbf{Q}(\{0\}) = (b-r)/(b-a)$, $\mathbf{Q}(\{1\}) = (r-a)/(b-a)$.

This shows that the 'fair' price of the payoff is $\mathbf{E}_\mathbf{Q} f/(1+r)$. Note that this does not depend on the probability measure $\mathbf{P}$.

2.4.2 N-step market

Assume we have only one stock, $d = 1$. For the bond $B_n = (1+r_n)B_{n-1}$, and for the share $S_n = (1+\rho_n)S_{n-1}$, where $\rho_n \in \{a_n, b_n\}$.

Exercise 1. Give a concrete construction of the probability space and the filtration!

Solution 1. Let

$$\Omega = \{0,1\}^N = \{\omega = (\omega_1, \dots, \omega_N) : \omega_i \in \{0,1\}\}.$$

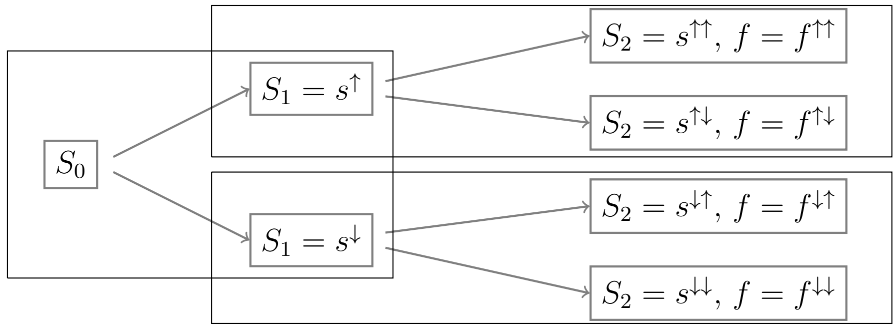

Figure 4: 2-step binary market as 3 1-step binary market

Define the random variables $\rho_n : \Omega \to \{a_n, b_n\}$ as

$$\rho_n(\omega) = \begin{cases} a_n, & \text{if } \omega_n = 0, \\ b_n, & \text{if } \omega_n = 1. \end{cases}$$

For the filtration let $\mathcal{F}_n = \sigma(\rho_1, \ldots, \rho_n)$, i.e. the natural filtration generated by the variables $\rho_1, \ldots, \rho_n$.

Consider any payoff function f_N. A perfect hedge can be constructed recursively, using the simple one-step market. Indeed, a two-step model can be seen as 3 one-step markets.

3 Arbitrage and pricing in discrete time

3.1 Arbitrage

A SF strategy π is an *arbitrage strategy* if

- $X_0^\pi = 0$;
- $X_n^\pi \geq 0$ for all $n = 0, 1, \ldots, N$;
- $\mathbf{P}(X_N^\pi > 0) > 0$.

That is, using the strategy π with 0 money we have riskless profit.

If the second assumption only holds for $n = N$ then π is a *weak arbitrage strategy*. According to the following if weak arbitrage strategy exists, then also arbitrage strategy exists.

{lemma:arbitrage}

Lemma 3. *Assume that π is a weak arbitrage strategy. Then there exists an arbitrage strategy π'.*

Proof. If $X_n^\pi \geq 0$ a.s. for all n, then we are ready. Otherwise, there exists $m < N$ such that $\mathbf{P}(X_m^\pi < 0) > 0$, and $X_n^\pi \geq 0$ for any $n \geq m+1$. Let

$$A_m = \{X_m < 0\} \in \mathcal{F}_m.$$

Consider the strategy

$$\beta_n' = I_{A_m} I_{n>m} \left(\beta_n - \frac{X_m}{B_m} \right), \quad \gamma_n' = I_{A_m} I_{n>m} \gamma_n.$$

It is easy to check that this strategy is predictable, SF, and arbitrage strategy. Indeed,

(i) predictable: for $n \leq m$ this is clear, since $\beta_n' = 0$ and $\gamma_n' = 0$, while for $n > m$ A_m is $\mathcal{F}_m$-measurable and thus $\mathcal{F}_{n-1}$-measurable as well, and β_n, γ_n are $\mathcal{F}_{n-1}$-measurable by the assumption.

(ii) SF: for $n \leq m$ this is again clear. For $n = m+1$

$$B_m \Delta \beta_{m+1}' + S_m \Delta \gamma_{m+1}' = I_{A_m} \left(B_m \beta_{m+1}(\omega) - X_m^\pi(\omega) + S_m \gamma_{m+1}(\omega) \right) = 0,$$

since π is SF. For $n > m+1$ we have $\Delta \beta_n' = I_{A_m} \Delta \beta_n$, and $\Delta \gamma_n' = I_{A_m} \Delta \gamma_n$, and the result follows, using again that π is SF.

(iii) arbitrage: we have

$$X_n^{\pi'} = I_{A_m} I_{n>m} \left(\beta_n B_n + \gamma_n S_n - \frac{X_m^\pi B_n}{B_m} \right),$$

where the sum of the first two terms in the bracket is nonnegative by the definition of m and the last is strictly negative on A_m, which proves the statement.

□

Exercise 2. Assume that $a < b < r$ in the one-step binomial model. Give an arbitrage strategy.

Assume that $a_n < b_n < r_n$ for some n in the N-step binomial model. Give an arbitrage strategy.

3.2 Martingale measures

A probability measure $\mathbf{Q}$ is called *equivalent martingale measure* (EMM) if $\mathbf{P} \sim \mathbf{Q}$ and $(S_n^i/B_n, \mathcal{F}_n)$ is a $\mathbf{Q}$-martingale for each $i = 1, 2, \ldots, d$.

3.2.1 EMM in binomial markets

In a one-step binomial market the martingale property is easy to check. Indeed, $(S_i/B_i)_{i=0,1}$ is a martingale iff

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{S_1}{B_1} \middle| \mathcal{F}_0 \right] = \frac{S_0}{B_0}.$$

We have

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}} \left[\frac{S_1}{B_1} \middle| \mathcal{F}_0 \right] &= \mathbf{E}_{\mathbf{Q}} \frac{S_1}{B_1} \\
&= \mathbf{Q}(\rho = a) \frac{(1+a)S_0}{(1+r)B_0} + (1 - \mathbf{Q}(\rho = a)) \frac{(1+b)S_0}{(1+r)B_0} \\
&= \frac{S_0}{B_0}.
\end{aligned}$$

Solving the equation we obtain that

$$\mathbf{Q}(\rho = a) = \frac{b-r}{b-a}, \quad \text{and} \quad \mathbf{Q}(\rho = b) = \frac{r-a}{b-a}.$$

That is $\mathbf{Q}(\{0\}) = (b-r)/(b-a)$, $\mathbf{Q}(\{1\}) = (r-a)/(b-a)$. This is the probability measure $\mathbf{Q}$ we obtained at pricing.

Let us see the general N-step model. Then

$$S_n = \prod_{i=1}^{n} (1 + \rho_i) \, S_0,$$

thus the martingale property reads as

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{S_n}{B_n} \middle| \mathcal{F}_{n-1} \right] = \frac{S_{n-1}}{B_{n-1}} \quad n = 0, 1, \ldots N.$$

Using the properties of conditional expectation we have

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{S_n}{B_n} \middle| \mathcal{F}_{n-1} \right] = \frac{S_{n-1}}{B_{n-1}} \frac{1}{1+r_n} \mathbf{E}_{\mathbf{Q}}[1 + \rho_n | \mathcal{F}_{n-1}].$$

Therefore S_n/B_n is a $\mathbf{Q}$-martingale iff

$$\mathbf{E}_{\mathbf{Q}}[\rho_n|\mathcal{F}_{n-1}] = r_n.$$

This condition exactly tells that under the new measure $\mathbf{Q}$ the risky asset behaves as the bond on average. Using that $\rho_n \in \{a_n, b_n\}$, we obtain as above

$$\mathbf{Q}(\rho_n = a_n|\mathcal{F}_{n-1}) = \frac{b_n - r_n}{b_n - a_n}, \quad \text{and } \mathbf{Q}(\rho_n = b_n|\mathcal{F}_{n-1}) = \frac{r_n - a_n}{b_n - a_n}.$$

Note the conditioning on $\mathcal{F}_{n-1}$ gives a constant, meaning that ρ_n is independent of $\mathcal{F}_{n-1}$ under the measure $\mathbf{Q}$.

We obtained the following.

{thm:binom-EMM}

Theorem 1. *In the binomial market if $a_n < r_n < b_n$ for each n then there exists a unique EMM $\mathbf{Q}$ given by the formulas above. Moreover, under $\mathbf{Q}$ the random variables $\rho_1, \ldots, \rho_N$ are independent.*

In the proof we used the following simple result.

Exercise 3. Assume that $Y \in \{a, b\}$ and

$$\mathbf{P}(Y = a|\mathcal{F}) = p \text{ a.s.}$$

Show that Y is independent of $\mathcal{F}$.

Note that the original measure $\mathbf{P}$ is irrelevant.

In the special case of the homogeneous binomial market we get that

$$\mathbf{Q}(S_N = S_0(1+b)^k(1+a)^{N-k}) = \binom{N}{k} q^k (1-q)^{N-k}, \quad k = 0, 1, \ldots, N.$$

3.2.2 Pricing with EMM

{prop:Xbar-mtg}

Proposition 1. *If $\mathbf{Q}$ is an EMM then $(\overline{X}_n^\pi = X_n^\pi / B_n)_n$ is a $\mathbf{Q}$-martingale for any SF strategy π.*

Proof. Easily follows from the SF property. Indeed, using that β_n, γ_n are $\mathcal{F}_{n-1}$-measurable

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}}\left[\frac{X_n^\pi}{B_n}\middle|\mathcal{F}_{n-1}\right] &= \mathbf{E}_{\mathbf{Q}}\left[\beta_n + \gamma_n \frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right] \\ &= \beta_n + \gamma_n \mathbf{E}_{\mathbf{Q}}\left[\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right] \\ &= \beta_n + \gamma_n \frac{S_{n-1}}{B_{n-1}} \\ &= \frac{\beta_n B_{n-1} + \gamma_n S_{n-1}}{B_{n-1}} \\ &= \frac{X_{n-1}^\pi}{B_{n-1}},\end{aligned}$$

where the last equality follow from the self-financing property. □

The following main result is the *first fundamental theorem of asset pricing.*

{thm:emm-arb}

Theorem 2. *There exists an EMM if and only if the market is arbitrage-free.*

Proof. Let $\mathbf{Q}$ be an EMM and π be any strategy with $X_0^\pi = 0$. Then, by the previous statement

$$\mathbf{E}_{\mathbf{Q}} \frac{X_N^\pi}{B_N} = \mathbf{E}_{\mathbf{Q}} \frac{X_0^\pi}{B_0} = 0.$$

Thus $X_N \geq 0$ $\mathbf{P}$-a.s., then also $\mathbf{Q}$-a.s., which implies $X_N \equiv 0$ $\mathbf{Q}$-a.s., thus $\mathbf{P}$-a.s.

We prove the converse later. □

Assume that f_N is a replicable payoff, i.e. there is a prefect hedge π. This means that

$$X_N^\pi = f_N \quad \text{a.s.}$$

Then the fair price for f_N is the initial cost of the portfolio, $X_0^\pi = x$. By the martingale property

$$\mathbf{E}_{\mathbf{Q}} \frac{f_N}{B_N} = \mathbf{E}_{\mathbf{Q}} \frac{X_N^\pi}{B_N} \overset{\text{mtg}}{=} \mathbf{E}_{\mathbf{Q}} \frac{X_0^\pi}{B_0} = \frac{x}{B_0}.$$

That is, the fair price x for a replicable payoff f_N is

$$x = \frac{B_0}{B_N} \mathbf{E}_{\mathbf{Q}} f.$$

In particular, it also follows that for a replicable f, the value $\mathbf{E}_{\mathbf{Q}} f$ is the same for any EMM $\mathbf{Q}$.

Summarizing, we proved the following:

{thm:pricing}

Theorem 3. *Consider an arbitrage-free market and let f be a replicable payoff. Then the fair price of f is*

$$C(f) = C_* = C^* = \frac{B_0}{B_N} \mathbf{E}_{\mathbf{Q}} f,$$

where $\mathbf{Q}$ is any EMM.

3.3 General one-step market

Assume that $B_1 = B_0(1 + r)$ with a deterministic interest rate $r > -1$ and

$$S_1 = S_0(1 + \rho),$$

where $\rho > -1$ is a random variable, the unique source of randomness in the model. Let

$$F(x) = \mathbf{P}(\rho \leq x), \quad x \in \mathbb{R},$$

be the distribution function of ρ. Then F induces a probability measure (denoted by $\mathbf{P}$) on the Borel sets of $(-1, \infty)$ (or $\mathbb{R}$). If F is concentrated on $\{a, b\}$ then we get back the previous one-step binomial model.

Assume without loss of generality that $B_0 = 1$. Consider a payoff function $f : \mathbb{R} \to \mathbb{R}$ as a function of the stock price S_1. A strategy π is an upper hedge if

$$\beta(1 + r) + \gamma S_0(1 + \rho) \geq f(S_0(1 + \rho)) \quad \text{a.s.} \tag{1}$$ {eq:1step-hedge}

A probability measure on $(\mathbb{R}, \mathcal{B})$ is $\mathbf{Q}$ is EMM if $\mathbf{P} \sim \mathbf{Q}$ (meaning that $\mathbf{P}$ is absolutely continuous to $\mathbf{Q}$ ($\mathbf{P}(A) = 0$ whenever $\mathbf{Q}(A) = 0$) and conversely) if and only if S_n/B_n is $\mathbf{Q}$-martingale, that is

$$\mathbf{E}_{\mathbf{Q}} \frac{S_1}{B_1} = \frac{S_0}{B_0}.$$

This means

$$\mathbf{E}_{\mathbf{Q}} \rho = r.$$

That is a probability measure $\mathbf{Q}$ which is equivalent to $\mathbf{P}$ is EMM iff

$$\int_{\mathbb{R}} \rho \mathrm{d}\mathbf{Q}(\rho) = r.$$

Taking expectation with respect to the EMM $\mathbf{Q}$

$$\beta(1 + r) + S_0 \gamma(1 + r) \geq \mathbf{E}_{\mathbf{Q}} f(S_0(1 + \rho)).$$

For the initial cost $\beta + \gamma S_0$ we have

$$\beta + \gamma S_0 \geq \mathbf{E}_{\mathbf{Q}} \frac{f(S_0(1 + \rho))}{1 + r}.$$

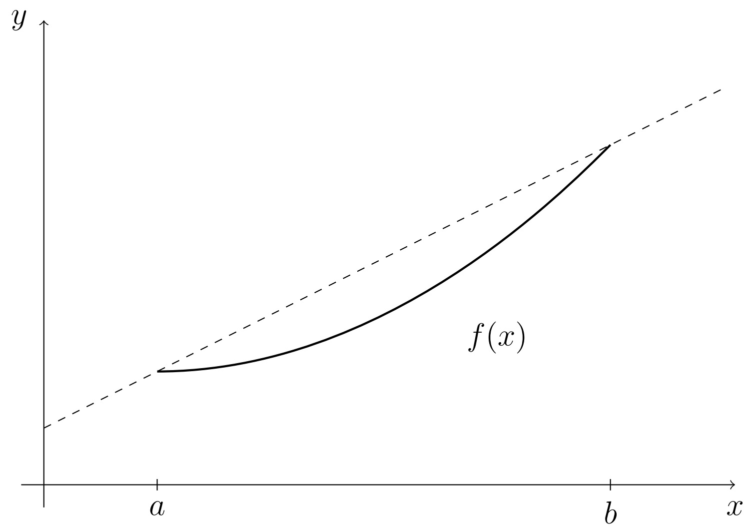

Figure 5: Bounding the upper price C^*

For the class of EMM's put

$$\mathcal{P}(\mathbf{P}) = \{\mathbf{Q} : \mathbf{Q} \text{ probability measure }, \mathbf{Q} \sim \mathbf{P}, (S_n/B_n)_n \text{ is } \mathbf{Q}\text{-martingale}\}.$$

Then

$$\begin{aligned} C^*(f) &= \inf\{\beta + \gamma S_0 \ : (\beta, \gamma) \text{ is an upper hedge}\} \\ &\geq \sup_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(S_0(1+\rho))}{1+r}. \end{aligned} \tag{2}$$

{eq:onestep-C*lowe

Similarly, for the lower price

$$\begin{aligned} C_*(f) &= \sup\{\beta + \gamma S_0 \ : (\beta, \gamma) \text{ is a lower hedge}\} \\ &\leq \inf_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(S_0(1+\rho))}{1+r}. \end{aligned} \tag{3}$$

{eq:onestep-C_*uppe

Assume now that $\rho \in [a, b]$ for some $-1 \leq a < b < \infty$. To ease notation put

$$f(x) = f(S_0(1+x)), \quad x \in [a, b], \tag{4}$$

{eq:onestep-f}

and assume that f is convex and continuous on $[a, b]$. By convexity,

$$\begin{aligned} f(x) &\leq \frac{f(b) - f(a)}{b - a}(1 + x) + \frac{(1+b)f(a) - (1+a)f(b)}{b-a} \\ &=: \mu S_0(1 + x) + \nu. \end{aligned} \tag{5}$$

{eq:conv-ineq}

Indeed, the left-hand side is a linear function equals to $f(a)$ at a, and $f(b)$ at b. Introduce the strategy,

$$\pi^* = (\beta^*, \gamma^*) := \left(\frac{\nu}{1+r}, \mu\right).$$

Then, by (5)

$$X_1^{\pi^*} = \nu + \mu S_0(1+\rho) \geq f(\rho),$$

that is, π^* is an upper hedge. Therefore,

$$C^*(f) = \inf_{\pi \text{upper hedge}} X_0^\pi \leq \beta^* + S_0\gamma^* = \frac{\nu}{1+r} + \mu S_0. \tag{6}$$ {eq:C*lower}

Assumption (weak limit): In the set $\mathcal{P}(\mathbf{P})$ there exists a sequence P_n, such that P_n converges weakly to a measure $\mathbf{Q}^*$ supported on $\{a, b\}$. Since

$$\mathbf{E}_{P_n}\rho = r,$$

the equality holds for the limit

$$\mathbf{E}_{\mathbf{Q}^*}\rho = r.$$

Since $\mathbf{Q}^*$ is supported on $\{a, b\}$

$$\mathbf{E}_{\mathbf{Q}^*}\rho = \mathbf{Q}^*(\{a\})a + (1 - \mathbf{Q}^*(\{a\}))b = r,$$

implying (as in the binomial market setup) that

$$\mathbf{Q}^*(\{a\}) = \frac{b-r}{b-a}, \quad \mathbf{Q}^*(\{b\}) = \frac{r-a}{b-a}.$$

Note that $\mathbf{Q}^*$ is, in general, not equivalent to $\mathbf{P}$. In fact, it is only equivalent in the binomial market setup.

By the convergence of P_n (here we use the continuity of f)

$$\begin{aligned}
\sup_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} &\geq \lim_{n\to\infty} \mathbf{E}_{P_n} \frac{f(\rho)}{1+r} \\
&= \mathbf{E}_{\mathbf{Q}^*} \frac{f(\rho)}{1+r} \\
&= \mathbf{Q}^*(\{a\})\frac{f(a)}{1+r} + (1 - \mathbf{Q}^*(\{a\}))\frac{f(b)}{1+r} \\
&= \beta^* + \gamma^* S_0 \geq C^*(f).
\end{aligned}$$

Combining with (6) we obtained the following.

Theorem 4. *Assume that the payoff function is convex and continuous on $[a, b]$, and that the weak limit assumption holds. Then*

$$C^*(f) = \sup_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} = \frac{b-r}{b-a}\frac{f(a)}{1+r} + \frac{r-a}{b-a}\frac{f(b)}{1+r},$$

and the supremum is attained on the measure $\mathbf{Q}^*$.

Exercise 4. Let ρ be uniform random variable on $[a, b]$. Show that the weak limit property holds. Construct P_n explicitly!

Try to weaken the condition on the distribution of ρ.

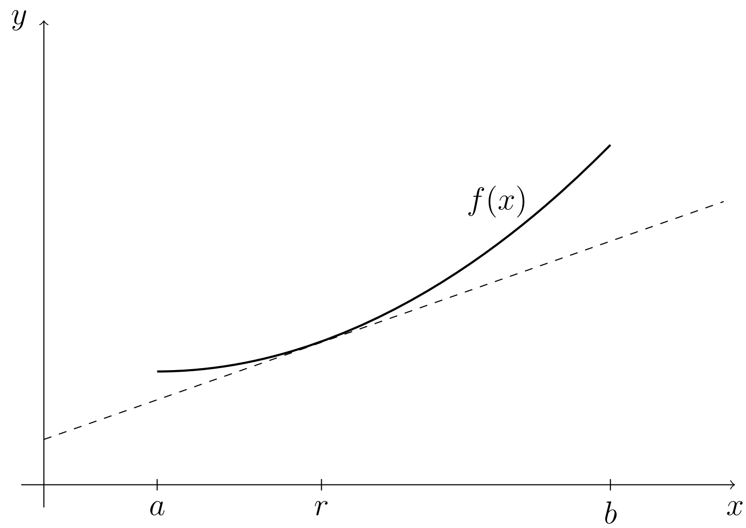

Figure 6: Bounding the lower price C_*

Let's see the lower price $C_*(f)$. Assume again that f in (4) is continuous and convex. Then

$$f(\rho) \geq f(r) + (\rho - r)\lambda(r), \tag{7}$$ {eq:onestep-convex-

for some $\lambda(r)$. Here $\lambda(r) = f'(r)$ if f is smooth, but this is not assumed.

If $\mathbf{Q} \in \mathcal{P}(\mathbf{P})$ then taking expectation in (7) and noting that $\mathbf{E_Q}\rho = r$ we have

$$\inf_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E_Q} f(\rho) \geq f(r).$$

Consider the strategy

$$\beta_* = \frac{f(r)}{1+r} - \lambda(r), \quad \gamma_* = \frac{\lambda(r)}{S_0}.$$

Then, by (7), the value at 1

$$X_1^{\pi_*} = \beta_*(1+r) + \gamma_* S_0(1+\rho) = f(r) + (\rho - r)\lambda(r) \leq f(\rho),$$

that is (β_*, γ_*) is a lower hedge.

Assumption (weak limit-2): In the set $\mathcal{P}(\mathbf{P})$ there exists a sequence P_n, such that P_n converges weakly to a measure $\mathbf{Q}_*$ concentrated at r.

Again note that $\mathbf{Q}_*$ does not belong to $\mathcal{P}(\mathbf{P})$, as it is not equivalent to any nondegenerate measure.

Then by the continuity

$$\begin{aligned}
\inf_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E_Q} \frac{f(\rho)}{1+r} &\leq \lim_{n\to\infty} \mathbf{E}_{P_n} \frac{f(\rho)}{1+r} \\
&= \frac{f(r)}{1+r} = \beta_* + S_0\gamma_* \\
&\leq \sup\{\beta + \gamma S_0 : (\beta, \gamma) \text{ lower hedge }\} = C_*.
\end{aligned}$$

Combined with (2) we obtain the following.

Theorem 5. *Let f be a convex continuous function on $[a,b]$, and assume that the weak limit-2 assumption holds. Then*

$$C_*(f) = \inf_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} = \frac{f(r)}{1+r},$$

and the infimum is attained at the measure $\mathbf{Q}_*$.

3.4 Complete markets

We proved that if EMM exists then we have the fair price for any replicable payoff. A market is *complete* if any payoff is replicable.

We have seen in Theorem 3 that on a complete arbitrage-free market any payoff f has a unique well-defined fair price $B_0\mathbf{E}_{\mathbf{Q}}f/B_N$.

In section 2.4 we showed that a binomial market is complete.

The second fundamental theorem of asset pricing is the following.

{thm:complete-mark

Theorem 6. *Consider an arbitrage-free market with EMM $\mathbf{Q}$. Then the following are equivalent:*

(i) the market is complete;

(ii) $\mathbf{Q}$ is the unique EMM;

(iii) for any $\mathbf{Q}$-martingale (M_n) there exists a predictable sequence γ_n such that M_n can be represented as

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \left(\frac{S_k}{B_k} - \frac{S_{k-1}}{B_{k-1}} \right) = M_0 + \sum_{k=1}^{n}\sum_{i=1}^{d} \gamma_k^i \left(\frac{S_k^i}{B_k} - \frac{S_{k-1}^i}{B_{k-1}} \right).$$

Proof. We prove again the easy parts (i) $\Rightarrow$ (ii), and (iii) $\Leftrightarrow$ (i), and postpone the difficult (ii) $\Rightarrow$ (i) implication later.

(i) $\Rightarrow$ (ii): Assume that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are EMM's. Consider any $A \in \mathcal{F}$. We show that $\mathbf{Q}_1(A) = \mathbf{Q}_2(A)$ implying the uniqueness. Let π be a perfect hedge to $f = I_A$. Then X_n^π/B_n is both $\mathbf{Q}_1$ and $\mathbf{Q}_2$ martingale, so

$$\mathbf{Q}_1(A) = \mathbf{E}_{\mathbf{Q}_1} f = \mathbf{E}_{\mathbf{Q}_1} X_N^\pi = B_N \mathbf{E}_{\mathbf{Q}_1} \frac{X_N^\pi}{B_N} = B_N \frac{X_0^\pi}{B_0} = \ldots = \mathbf{Q}_2(A).$$

(i) $\Rightarrow$ (iii): Consider a $\mathbf{Q}$-martingale M_n. There exists a strategy π_n such that a.s.

$$X_N^\pi = B_N M_N.$$

Using that both M_n and X_n^π/B_n are martingales

$$M_n = \mathbf{E_Q}[M_N|\mathcal{F}_n] = \mathbf{E_Q}\left[\frac{X_N^\pi}{B_N}|\mathcal{F}_n\right] = \frac{X_n^\pi}{B_n} = \beta_n + \gamma_n \frac{S_n}{B_n}.$$

Thus, using that π is SF

$$\begin{aligned}
M_n - M_{n-1} &= \Delta\beta_n + \gamma_n \frac{S_n}{B_n} - \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}} \\
&= \gamma_n \left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right) + \frac{1}{B_{n-1}}\left(B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n\right) \\
&= \gamma_n \left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right),
\end{aligned}$$

as claimed.

(iii) $\Rightarrow$ (i): Consider a payoff f. We are looking for a strategy π such that $X_N^\pi = f$ $\mathbf{Q}$-a.s. We know that $(X_n^\pi/B_n)_n$ is a martingale, so this should be (M_n). Now the following choice is clear: let

$$M_n = \mathbf{E_Q}\left[\frac{f}{B_N}|\mathcal{F}_n\right].$$

Then M_n is a martingale, therefore by the assumption

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \Delta \frac{S_k}{B_k}.$$

Let

$$\beta_n = M_n - \gamma_n \frac{S_n}{B_n},$$

and consider the strategy $\pi_n = (\beta_n, \gamma_n)$. To see that this is indeed a strategy we have to show that it is predictable and SF. The sequence γ_n is predictable by the assumption (iii), and β_n is predictable because all the terms in M_n are $\mathcal{F}_{n-1}$-measurable except $\gamma_n S_n/B_n$, which is subtracted. To see that it is SF note that

$$\begin{aligned}
&B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n \\
&= B_{n-1}\left(M_n - M_{n-1} - \gamma_n \frac{S_n}{B_n} + \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}}\right) + S_{n-1}\Delta\gamma_n \\
&= B_{n-1}\left(\gamma_n \Delta \frac{S_n}{B_n} - \gamma_n \frac{S_n}{B_n} + \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}}\right) + S_{n-1}\Delta\gamma_n = 0,
\end{aligned}$$

showing that π is SF. It is clearly a perfect hedge since

$$X_N^\pi = \beta_N B_N + \gamma_N S_N = B_N M_N = f,$$

as claimed. $\square$

3.5 Proof of the difficult part of Theorem 2

Here we use strongly that Ω is finite, and let $|\Omega| = k$.

Assume that there is no arbitrage strategy. Let

$$\mathcal{V}_0 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | \exists \pi : X_0^\pi = 0 \text{ and } X_N^\pi = X\},$$

and

$$\mathcal{V}_1 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | X \geq 0, \mathbf{E}X \geq 1\}.$$

We identify a random variable $X : \Omega \to \mathbb{R}$ with a vector in $\mathbb{R}^k$, as $X \leftrightarrow (X(\omega_1), \ldots, X(\omega_k))$. Clearly, $\mathcal{V}_0$ is a linear subspace and $\mathcal{V}_1$ is convex set in $\mathbb{R}^k$.

Since there is no arbitrage strategy, $\mathcal{V}_0 \cap \mathcal{V}_1 = \emptyset$. Therefore, by the Kreps–Yan theorem, there exists a linear functional $\ell : \mathbb{R}^k \to \mathbb{R}$ such that $\ell|_{\mathcal{V}_0} \equiv 0$ and $\ell(v_1) > 0$ for all $v_1 \in \mathcal{V}_1$. A linear function in $\mathbb{R}^k$ (in any Hilbert space) is a inner product, thus there exists $q \in \mathbb{R}^k$ such that

$$\ell(v) = \langle v, q \rangle.$$

Define the random variables

$$X_i(\omega_j) = \delta_{i,j} \frac{1}{\mathbf{P}(\{\omega_i\})}.$$

Then $X_i \geq 0$ and $\mathbf{E}X_i = 1$, so $X_i \in \mathcal{V}_1$. Furthermore

$$\ell(X_i) = \frac{q_i}{\mathbf{P}(\{\omega_i\})} > 0,$$

implying $q_i > 0$ for any i. Define the probability measure $\mathbf{Q}$ as

$$\mathbf{Q}(\{\omega_i\}) = \frac{q_i}{\sum_{i=1}^k q_i}.$$

It is clear that $\mathbf{Q} \sim \mathbf{P}$. We have to check that (S_n/B_n) is a $\mathbf{Q}$-martingale. First we need a lemma.

Lemma 4. *Let $(X_n)_{n=1}^N$ be an adapted process. If for any stopping time $\tau : \Omega \to \{0, \ldots, N\}$*

$$\mathbf{E}X_\tau = \mathbf{E}X_0,$$

then (X_n) is martingale.

Proof. We show that $X_n = \mathbf{E}[X_N|\mathcal{F}_n]$, which implies that X is martingale.

Let $A \in \mathcal{F}_n$ and consider the stopping time

$$\tau_A(\omega) = \begin{cases} n, & \omega \in A, \\ N, & \text{otherwise.} \end{cases}$$

This is indeed a stopping time, since $\{\tau_A \leq k\} = \emptyset$ for $k < n$, and A for $k \geq n$, which is $\mathcal{F}_k$-measurable. Then, by the assumption

$$\mathbf{E}X_0 = \mathbf{E}X_{\tau_A} = \mathbf{E}X_n I(A) + \mathbf{E}X_N I(A^c).$$

With $A = \emptyset$ we see that $\mathbf{E}X_0 = \mathbf{E}X_N$, implying

$$\mathbf{E}X_n I(A) = \mathbf{E}X_N I(A).$$

This exactly means that

$$X_n = \mathbf{E}[X_N|\mathcal{F}_n],$$

as claimed. □

We show that (S_n/B_n) satisfies the condition of the lemma above. Let τ be a stopping time and define the strategy

$$\beta_n = \frac{S_\tau}{B_\tau} I(\tau \leq n-1) - \frac{S_0}{B_0}, \quad \gamma_n = I(\tau > n-1).$$

Since $\{\tau < n\} = \{\tau \leq n-1\} \in \mathcal{F}_{n-1}$, the sequence (β_n, γ_n) is predictable. Furthermore,

$$B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = \frac{S_\tau}{B_\tau} B_{n-1} I(\tau = n-1) - S_{n-1} I(\tau = n-1) = 0,$$

so it is SF. Finally,

$$X_0^\pi = -\frac{S_0}{B_0} B_0 + S_0 = 0,$$

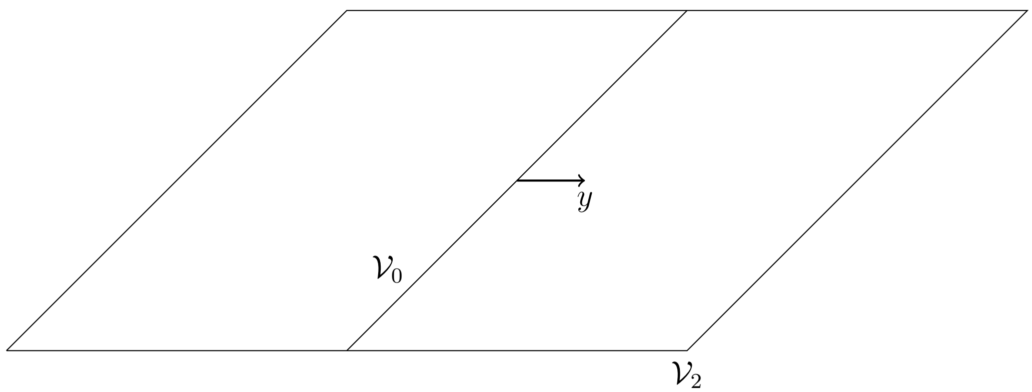

Figure 7: Choice of y

so $X_N^\pi \in \mathcal{V}_0$. Therefore

$$\begin{aligned} 0 &= \mathbf{E_Q} X_N^\pi = \mathbf{E_Q} \beta_N B_N + \gamma_N S_N \\ &\mathbf{E_Q}\left(\left(\frac{S_\tau}{B_\tau} I(\tau \le N-1) - \frac{S_0}{B_0}\right) B_N + \frac{S_\tau}{B_\tau} I(\tau = N) B_N\right) \\ &= B_N \mathbf{E_Q}\left(\frac{S_\tau}{B_\tau} - \frac{S_0}{B_0}\right). \end{aligned}$$

That is (S_n/B_n) is indeed a $\mathbf{Q}$-martingale.

3.6 Proof of the difficult part of Theorem 6

Here we prove the implication (ii) $\Rightarrow$ (i).

We use the notation of the previous proof. Let

$$\mathcal{V}_2 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | \mathbf{E_Q} X = 0\}.$$

Then $\mathcal{V}_2$ is a linear subspace in $\mathbb{R}^k$ and we have seen in the previous proof that $\mathcal{V}_0 \subset \mathcal{V}_2$. We claim that equality holds.

Assume first that this is indeed true. Then for any claim X the centered version $X - \mathbf{E_Q} X \in \mathcal{V}_2 = \mathcal{V}_0$, meaning that there is a perfect hedge. Thus the market is complete. So we only have to show that $\mathcal{V}_0 = \mathcal{V}_2$.

Assume on the contrary that $\mathcal{V}_0 \neq \mathcal{V}_2$. Then there is an $y \in \mathcal{V}_2$, which is orthogonal to $\mathcal{V}_0$. Since $q_i > 0$ (see the previous proof) for all $i = 1, \ldots, k$, we may choose $\varepsilon > 0$ small enough such that

$$q_i' = q_i - \varepsilon y_i > 0 \quad \text{for all } i.$$

As both q and y are orthogonal to $\mathcal{V}_0$, q' is also orthogonal. Define the measure

$$\mathbf{Q}'(\{\omega_i\}) = \frac{q_i'}{\sum_{i=1}^k q_i'}.$$

Exactly as in the previous proof we can show that $\mathbf{Q}'$ is EMM. The uniqueness of the EMM implies

$$\frac{q_i'}{\sum_{i=1}^k q_i'} = \frac{q_i}{\sum_{i=1}^k q_i},$$

that is, using also the definition of q',

$$q = \alpha q' = \alpha q - \alpha \varepsilon y,$$

with $\alpha = \sum q_i / \sum q_i'$. Thus

$$(1 - \alpha) q = -\alpha \varepsilon y.$$

But y and q are orthogonal, which is a contradiction. The proof is complete.

4 Girsanov's theorem in discrete time

4.1 Second proof of the difficult part of Theorem 2

Assume that $d = 1$ and first consider the one-step model with $B_0 = B_1 = 1$. The stock price S_0 is known, and the only randomness here is S_1.

Exercise 5. The no arbitrage assumption (in this simple market) is equivalent to

$$\mathbf{P}(\Delta S_1 > 0)\mathbf{P}(\Delta S_1 < 0) > 0.$$

Furthermore, (S_n) is $\mathbf{Q}$-martingale if

$$\mathbf{E}_\mathbf{Q} S_1 = S_0.$$

Therefore we have to construct a measure $\mathbf{Q}$ such that $\mathbf{E}_\mathbf{Q} \Delta S_1 = 0$. This is done in the following lemma.

Lemma 5. *Let X be a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbf{P})$ such that $\mathbf{P}(X > 0)\mathbf{P}(X < 0) > 0$. Then there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that $\mathbf{E}_\mathbf{Q} X = 0$. Furthermore, for any $a \in \mathbb{R}$*

$$\mathbf{E}_\mathbf{Q} e^{aX} < \infty.$$

Proof. Define the probability measure

$$P_1(\mathrm{d}x) = ce^{-x^2} F(\mathrm{d}x),$$

where $F(x) = \mathbf{P}(X \le x)$ and $c^{-1} = \int_{\mathbb{R}} e^{-x^2} F(\mathrm{d}x)$. That is

$$P_1(A) = \int_A ce^{-x^2} F(\mathrm{d}x).$$

Then P_1 is equivalent to F. (Recall that μ is absolute continuous with respect to ν, $\mu \ll \nu$ if $\mu(A) = 0$ whenever $\nu(A) = 0$. And μ and ν are equivalent, $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.) Let

$$\varphi(a) = \mathbf{E}_{P_1} e^{aX} = \int_{\mathbb{R}} e^{ax} P_1(\mathrm{d}x) = c \int_{\mathbb{R}} e^{ax-x^2} F(\mathrm{d}x).$$

Clearly, $\varphi(a) < \infty$ for any a as the function e^{ax-x^2} is bounded on $\mathbb{R}$. Note that φ is convex, because $\varphi'' > 0$. Put

$$Z_a(x) = \frac{e^{ax}}{\varphi(a)}.$$

Then

$$Q_a(\mathrm{d}x) = Z_a(x) P_1(\mathrm{d}x)$$

is a probability measure for any a, and $Q_a \sim P_1 \sim F$. Again, this means

$$Q_a(A) = \int_A Z_a(x) P_1(\mathrm{d}x) = \frac{c}{\varphi(a)} \int_A e^{ax-x^2} F(\mathrm{d}x).$$

Let

$$\varphi_* = \inf_{a \in \mathbb{R}} \varphi(a).$$

Since $P_1(X > 0) > 0$ and $P_1(X < 0) > 0$ we obtain that

$$\lim_{a \to \pm\infty} \varphi(a) = \infty.$$

Therefore, the infimum is attained, i.e. there is a_* such that $\varphi(a_*) = \varphi_*$. Then $\varphi'(a_*) = 0$, thus

$$0 = \varphi'(a_*) = \mathbf{E}_{P_1} X e^{a_* X} = \varphi(a_*) \mathbf{E}_{P_1} X \frac{e^{a_* X}}{\varphi(a_*)} = \varphi(a_*) \mathbf{E}_{Q_{a_*}} X.$$

Thus the measure Q_{a_*} works. □

Exercise 6. Prove rigorously that

$$\lim_{a \to \pm\infty} \varphi(a) = \infty.$$

Exercise 7. Let $X \sim N(\mu, \sigma^2)$. Determine the measure constructed above explicitly.

Next we extend the previous lemma for a general N-step market.

Exercise 8. The no arbitrage assumption implies that for any n a.s.

$$\mathbf{P}(\Delta S_n > 0 | \mathcal{F}_{n-1}) \mathbf{P}(\Delta S_n < 0 | \mathcal{F}_{n-1}) > 0.$$

As a preliminary result we have to understand how to compute conditional expectation under different measures.

{lemma:condexp-mea

Lemma 6. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,\ldots,N}, \mathbf{P})$ a filtered probability space, and Z a nonnegative random variable $\mathbf{E}_\mathbf{P} Z = 1$. Define the new probability measure $\mathbf{Q}$ as*

$$\mathrm{d}\mathbf{Q} = Z \mathrm{d}\mathbf{P},$$

that is

$$\mathbf{Q}(A) = \int_A Z \mathrm{d}\mathbf{P}.$$

Put $Z_n = \mathbf{E}_\mathbf{P}[Z|\mathcal{F}_n]$. For any adapted process (X_n)

$$Z_{n-1}\mathbf{E}_\mathbf{Q}[X_n|\mathcal{F}_{n-1}] = \mathbf{E}_\mathbf{P}[X_n Z_n|\mathcal{F}_{n-1}].$$

Proof. Both sides are $\mathcal{F}_{n-1}$-measurable. We have to prove that for any $A \in \mathcal{F}_{n-1}$

$$\int_A Z_{n-1}\mathbf{E}_\mathbf{Q}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} = \int_A X_n Z_n \mathrm{d}\mathbf{P}. \tag{8}$$ {eq:cemlemma-0}

First note that

$$\mathbf{E}_\mathbf{P}[Z X_n|\mathcal{F}_n] = X_n \mathbf{E}_\mathbf{P}[Z|\mathcal{F}_n] = X_n Z_n. \tag{9}$$ {eq:cemlemma-1}

Therefore, for an $\mathcal{F}_{n-1}$-measurable Y

$$\mathbf{E}_\mathbf{P}[Z_{n-1}Y|\mathcal{F}_{n-1}] = Y\mathbf{E}_\mathbf{P}[Z|\mathcal{F}_{n-1}],$$

implying for any $A \in \mathcal{F}_{n-1}$ that

$$\begin{aligned}\int_A Z_{n-1} Y \mathrm{d}\mathbf{P} &= \int_A Y\mathbf{E}_\mathbf{P}[Z|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} \\ &= \int_A \mathbf{E}_\mathbf{P}[ZY|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} = \int_A YZ\mathrm{d}\mathbf{P}.\end{aligned}$$

Choosing $Y = \mathbf{E}_\mathbf{Q}[X_n|\mathcal{F}_{n-1}]$ we obtain

$$\begin{aligned}\int_A Z_{n-1}\mathbf{E}_\mathbf{Q}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} &= \int_A \mathbf{E}_\mathbf{Q}[X_n|\mathcal{F}_{n-1}]Z\mathrm{d}\mathbf{P} \\ &= \int_A \mathbf{E}_\mathbf{Q}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{Q} && \text{definition of } \mathbf{Q} \\ &= \int_A X_n \mathrm{d}\mathbf{Q} && \text{conditional exp.} \\ &= \int_A X_n Z\mathrm{d}\mathbf{P} && \text{definition of } \mathbf{Q} \\ &= \int_A X_n Z_n \mathrm{d}\mathbf{P}, && \text{by (9)}\end{aligned}$$

which is (8). □

As a simple but useful corollary we obtain the following.

{cor:p-q-mtg}

Corollary 1. *The adapted process (X_n) is $\mathbf{Q}$-martingale if and only if $(X_n Z_n)$ is $\mathbf{P}$-martingale.*

{lemma:existence-e

Lemma 7. *Let $(X_n)_{n=1}^N$ be an adapted process, and assume that*

$$\mathbf{P}(X_n > 0 | \mathcal{F}_{n-1}) \mathbf{P}(X_n < 0 | \mathcal{F}_{n-1}) > 0.$$

Then there exists a probabilty measure $\mathbf{Q} \sim \mathbf{P}$ such that (X_n) is a $\mathbf{Q}$-martingale difference.

Proof. First let

$$P_1(\mathrm{d}\omega) = c \exp\left\{ -\sum_{i=0}^N X_i^2(\omega) \right\} \mathbf{P}(\mathrm{d}\omega),$$

where c is the normalizing factor, i.e.

$$c^{-1} = \int_\Omega \exp\left\{ -\sum_{i=0}^N X_i^2 \right\} \mathrm{d}\mathbf{P} = \mathbf{E} \exp\left\{ -\sum_{i=0}^N X_i^2 \right\}.$$

This means that for $A \in \mathcal{F}$

$$P_1(A) = c \int_A \exp\left\{ -\sum_{i=0}^N X_i^2 \right\} \mathrm{d}\mathbf{P}.$$

Let

$$\varphi_n(a) = \mathbf{E}_{P_1}[e^{aX_n} | \mathcal{F}_{n-1}].$$

Note that this is an $\mathcal{F}_{n-1}$-measurable random variable. As in the proof of the previous lemma there is a unique finite a_n (random!) such that the infimum of φ_n is attained at a_n. Since φ_n is $\mathcal{F}_{n-1}$-measurable so is a_n.

Let $Z_0 = 1$, and recursively

$$Z_n = Z_{n-1} \frac{e^{a_n X_n}}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]}.$$

Then (Z_n) is a P_1-martingale, since

$$\mathbf{E}_{P_1}[Z_n | \mathcal{F}_{n-1}] = Z_{n-1}.$$

Then the probability measure

$$\mathbf{Q}(\mathrm{d}\omega) = Z_N(\omega) P_1(\mathrm{d}\omega)$$

works. Indeed,

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}[X_n | \mathcal{F}_{n-1}] &= \frac{1}{Z_{n-1}} \mathbf{E}_{P_1}[Z_n X_n | \mathcal{F}_{n-1}] && \text{by Lemma 6} \\
&= \frac{1}{Z_{n-1}} \frac{Z_{n-1}}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]} \mathbf{E}_{P_1}[X_n e^{a_n X_n} | \mathcal{F}_{n-1}] && \text{definition} \\
&= \frac{1}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]} \cdot 0 = 0. && \text{choice of } a_n
\end{aligned}$$

□

Exercise 9. Show that a_n is $\mathcal{F}_{n-1}$-measurable.

Now we can return to the proof of Theorem 2. The existence of the martingale measure follows from the previous lemma applied to $X_n = \Delta S_n$.

4.2 ARCH processes

Autoregressive conditional heteroscedasticity (ARCH) models were introduced by Robert Engle in 1982 to model log-returns. In 2003 he obtained Nobel prize in economics for this model. The novelty in these models is the stochastic volatility term.

Let

$$R_n = \log \frac{S_n}{S_{n-1}}$$

denote the log-return of the stock, and assume that

$$R_n = \mu_n + \sqrt{\beta + \lambda R_{n-1}^2}\xi_n,$$

where ξ_n's are iid $N(0,1)$ random variables. Then (R_n) is an ARCH(1) process. That is conditionally on $\mathcal{F}_{n-1}$ the log-return R_n is Gaussian with mean μ_n, and variance $\beta + \lambda R_{n-1}^2$. Write $\sigma_n = \beta + \lambda R_{n-1}^2$. Then for S_n we obtain

$$\begin{aligned} S_n = S_{n-1}e^{R_n} &= S_0 \exp\left\{ \sum_{k=1}^n \left(\mu_k + \sqrt{\beta + \lambda R_{k-1}^2}\xi_k \right) \right\} \\ &= S_0 \exp\left\{ \sum_{k=1}^n (\mu_k + \sigma_k \xi_k) \right\}. \end{aligned}$$

In what follows we only assume that μ_n and σ_n are $\mathcal{F}_{n-1}$-measurable, i.e. the sequence $(\mu_n, \sigma_n)_n$ is predictable, and (ξ_n) is adapted, ξ_n is independent of $\mathcal{F}_{n-1}$, and $N(0,1)$ distributed. Put $h_n = \mu_n + \sigma_n \xi_n$. For simplicity we assume that $B_n \equiv 1$.

We construct a measure $\mathbf{Q}$ such that (S_n) is a $\mathbf{Q}$-martingale. Let

$$Z_N = \prod_{n=1}^N z_n := \prod_{n=1}^N \frac{e^{a_n h_n}}{\mathbf{E}_\mathbf{P}[e^{a_n h_n} | \mathcal{F}_{n-1}]},$$

where

$$a_n = -\frac{\mu_n}{\sigma_n^2} - \frac{1}{2}. \tag{10}$$ {eq:disc-girs-0}

Introduce the new measure $\mathbf{Q}$ as

$$\mathrm{d}\mathbf{Q} = Z_N \mathrm{d}\mathbf{P},$$

and let $Z_n = \mathbf{E}_\mathbf{P}[Z_N | \mathcal{F}_n] = \prod_{i=1}^n z_i$.

By Corollary 1, to show that S_n is $\mathbf{Q}$-martingale we have to show that $S_n Z_n$ is a $\mathbf{P}$-martingale. We have

$$\mathbf{E}_\mathbf{P}[S_n Z_n | \mathcal{F}_{n-1}] = S_{n-1} Z_{n-1} \frac{\mathbf{E}_\mathbf{P}[e^{h_n(1+a_n)} | \mathcal{F}_{n-1}]}{\mathbf{E}_\mathbf{P}[e^{a_n h_n} | \mathcal{F}_{n-1}]}.$$

Therefore we have to check that

$$\mathbf{E}_\mathbf{P}[e^{h_n(1+a_n)} | \mathcal{F}_{n-1}] = \mathbf{E}_\mathbf{P}[e^{a_n h_n} | \mathcal{F}_{n-1}]. \tag{11}$$ {eq:disc-girs-1}

Recall that for a standard normal ξ

$$\mathbf{E}e^{t\xi} = e^{\frac{t^2}{2}},$$

thus

$$\mathbf{E}e^{\mu + \sigma\xi} = e^{\mu + \frac{\sigma^2}{2}}.$$

Since a_n in (11) is $\mathcal{F}_{n-1}$-measurable and given $\mathcal{F}_{n-1}$ the variable h_n is Gaussian $\mathrm{N}(\mu_n, \sigma_n^2)$, we obtain

$$\mathbf{E}_\mathbf{P}[e^{h_n(1+a_n)} | \mathcal{F}_{n-1}] = e^{\mu_n(1+a_n) + \frac{\sigma_n^2(1+a_n)^2}{2}},$$

and

$$\mathbf{E}_{\mathbf{P}}[e^{h_n a_n}|\mathcal{F}_{n-1}] = e^{\mu_n a_n + \frac{\sigma_n^2 a_n^2}{2}},$$

By the choice of a_n in (10)

$$\mu_n(1+a_n) + \frac{\sigma_n^2(1+a_n)^2}{2} = \mu_n a_n + \frac{\sigma_n^2 a_n^2}{2}.$$

Indeed, by (10)

$$\mu_n + \sigma_n^2\left(\frac{1}{2} + a_n\right) = 0.$$

That is, (11) holds.

We proved the following.

Theorem 7 (Discrete Girsanov's theorem)**.** *Let $(\mu_n, \sigma_n)_n$ be a predictable sequence and assume that the stock prices are given by*

$$S_n = e^{\sum_{k=1}^n (\mu_k + \sigma_k \xi_k)},$$

where $(\xi_n)_n$ is a adapted sequence of $N(0,1)$ random variables, ξ_n is independent of $\mathcal{F}_{n-1}$. Further, let $B_n \equiv 1$. Then, under the new measure

$$\mathrm{d}\mathbf{Q} = Z_N \mathrm{d}\mathbf{P}$$

(S_n) is a martingale.

5 Pricing and hedging European options

In this section we summarize our findings on pricing and hedging, and consider some special cases in detail.

5.1 Complete markets

Consider an arbitrage-free complete market. The *fair price* of the contingent claim f_N is

$$C(f_N) = \inf\{x : \exists \pi, X_0^\pi = x, X_N^\pi = f_N\}.$$

Then, by Theorems 2 and 6 there exists a unique EMM $\mathbf{Q}$. Since (X_n^π/B_n) is $\mathbf{Q}$-martingale

$$\mathbf{E}_{\mathbf{Q}}\frac{f_N}{B_N} = \mathbf{E}_{\mathbf{Q}}\frac{X_N^\pi}{B_N} = \mathbf{E}_{\mathbf{Q}}\frac{x}{B_0} = \frac{x}{B_0},$$

therefore

$$C(f_N) = x = \frac{B_0}{B_N}\mathbf{E}_{\mathbf{Q}} f_N.$$

Note that x is independent of the hedge π itself, that is for different hedges the initial value is the same.

For a hedge we need to know not only the fair price C, but also the strategy π itself. For the given claim f_N consider the martingale

$$M_n = \mathbf{E}_{\mathbf{Q}} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_n \right].$$

By Theorem 6 there exists a representation

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \Delta \frac{S_k}{B_k},$$

with a predictable sequence (γ_n). Let

$$\beta_n = M_n - \frac{\gamma_n S_n}{B_n}.$$

We proved that $\pi = (\beta_n, \gamma_n)_n$ is an SF strategy and is a perfect hedge for f_N.

Summarizing, we obtained the following.

Theorem 8. *In an arbitrary arbitrage-free complete market the price of the contingent claim* f_N *is*

$$C(f_N) = B_0 \mathbf{E}_{\mathbf{Q}} \frac{f_N}{B_N}.$$

Moreover, there exists a strategy π *which is a perfect hedge of* f_N*, i.e.*

$$X_N^\pi = f_N,$$

where (β_n, γ_n) *are given above. The value process is determined by*

$$X_n^\pi = B_n \mathbf{E}_{\mathbf{Q}} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_n \right].$$

5.2 Homogeneous binomial market – CRR formula

Consider a homogeneous binomial N-step market with $a < r < b$. That is

$$B_n = (1+r)^n, \quad S_n = S_0 \prod_{k=1}^{n} (1 + \rho_k),$$

where $\rho_k \in \{a, b\}$. We proved that this market is arbitrage-free and complete, and the unique EMM is given by

$$\mathbf{Q}(\rho_i = a) = \frac{b-r}{b-a},$$

and ρ_i's are independent. If the claim f_N only depends on the final price S_N, and not on the whole trajectory, i.e.

$$f_N(\omega) = f_N(S_N(\omega)),$$

then the pricing formula simplifies, and we obtain the Cox–Ross-Rubinstein formula:

$$C(f_N) = \frac{1}{(1+r)^N} \sum_{k=0}^{N} f_N(S_0(1+b)^k(1+a)^{N-k}) \binom{N}{k} q^k(1-q)^{N-k},$$

where $q = \frac{r-a}{b-a}$.

5.3 Incomplete markets

We assume that the market is arbitrage-free, but there are various EMM's. Let $\mathcal{P}(\mathbf{P})$ be the set of EMM's.

In incomplete markets there are contingent claims which are not replicable, that is, there is no perfect hedge. The upper price of a claim f_N is

$$C^*(f_N) = \inf\{x : \pi,\, X_0^\pi = x,\, X_N^\pi \geq f_N\}.$$

We proved the following result in a one-step market. Without a proof we state the general version.

Theorem 9. *The upper price of the claim f_N in an arbitrage-free incomplete market is given by*

$$C^*(f_N) = \sup_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} B_0 \mathbf{E}_{\mathbf{Q}} \frac{f_N}{B_N}.$$

6 American options

While European options can be exercised only at the terminal date N, American options can be exercised at any time. Formally, instead of a fixed random payoff function f_N, a sequence of payoffs $(f_n)_{n=0,1,\dots,N}$ is given, where f_n is $\mathcal{F}_n$-measurable, i.e. $(f_n)_n$ is adapted to $(\mathcal{F}_n)_n$. So f_n is the random payoff if the option is exercised at time n. Clearly, the exercise time has to be a stopping time.

6.1 Optimal stopping problems

Consider a probability space with a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,\dots,N}, \mathbf{P})$, and let

$$\mathcal{M}_n^N = \{\tau : \tau \text{ is a stopping time}, \tau \in \{n, \dots, N\}\}.$$

To ease notation we suppress N in the upper index. Consider a sequence of nonnegative adapted random variables $(X_n)_n$, and define by backward induction its Snell-envelope $(Z_n)_n$ as follows. We are interested in the value

$$Z_N = X_N, \quad Z_n = \max\{X_n, \mathbf{E}[Z_{n+1}|\mathcal{F}_n]\}, \ n < N.$$

For a stopping time τ the stopped process is denoted by Z^τ, i.e.

$$Z_n^\tau = Z_{\tau \wedge n},$$

where $a \wedge b = \min\{a, b\}$.

Proposition 2. *Let (Z_n) be the Snell-envelope of (X_n) with $X_n \geq 0$ a.s.*

(i) Z is the smallest supermartingale dominating X.

(ii) The random variable $\tau^* = \min\{n : Z_n = X_n\}$ is a stopping time and the stopped process $Z_{n\wedge\tau^*} = Z_n^{\tau^*}$ is martingale.

Proof. From the definition it is clear that Z is supermartingale and dominates X. Let Y be another supermartingale dominating X. Then $Y_N \geq X_N = Z_N$. Assuming that $Y_n \geq Z_n$ we have

$$Y_{n-1} \geq \max\{\mathbf{E}[Y_n|\mathcal{F}_{n-1}], X_{n-1}\} \geq \max\{\mathbf{E}[Z_n|\mathcal{F}_{n-1}], X_{n-1}\} = Z_{n-1}.$$

Thus the minimality follows.

To see that τ^* is stopping time note that

$$\{\tau^* = n\} = \cap_{k=0}^{n-1}\{Z_k > X_k\} \cap \{Z_n = X_n\}.$$

For the last assertion note that

$$Z_n^{\tau^*} - Z_{n-1}^{\tau^*} = I(\tau^* \geq n)(Z_n - Z_{n-1}).$$

On the event $\{\tau^* \geq n\}$ we have $Z_{n-1} = \mathbf{E}[Z_n|\mathcal{F}_{n-1}]$ therefore

$$\mathbf{E}[I(\tau^* \geq n)(Z_n - Z_{n-1})|\mathcal{F}_{n-1}] = 0.$$

□

A stopping time σ is optimal if

$$\mathbf{E}X_\sigma = \sup_{\tau \in \mathcal{M}_0} \mathbf{E}X_\tau.$$

Proposition 3. *The stopping time τ^* is optimal for X, and*

$$Z_0 = \mathbf{E}X_{\tau^*} = \sup_{\tau \in \mathcal{M}_0} \mathbf{E}X_\tau.$$

Proof. Since Z^{τ^*} is martingale

$$Z_0 = Z_0^{\tau^*} = \mathbf{E}Z_N^{\tau^*} = \mathbf{E}Z_{\tau^*} = \mathbf{E}X_{\tau^*}.$$

On the other hand for any stopping time τ the process Z^τ is supermartingale (by Doob's optional sampling), thus

$$Z_0 = \mathbf{E}Z_0^\tau \geq \mathbf{E}Z_\tau \geq \mathbf{E}X_\tau.$$

□

Proposition 4. *The stopping time σ is optimal iff the following two conditions hold.*

(i) $Z_\sigma = X_\sigma$;

(ii) Z^σ *is martingale.*

Proof. If (i) and (ii) hold than σ is optimal. This follows exactly as the optimality of τ^*.

Conversely, assume that σ is optimal. We have seen that $\sup_\tau \mathbf{E}X_\tau = Z_0$ thus

$$Z_0 = \mathbf{E}X_\sigma \leq \mathbf{E}Z_\sigma,$$

by the dominance of Z. By Doob's optional stopping theorem Z^σ is supermartingale, therefore $\mathbf{E}Z_\sigma \leq Z_0$, implying that

$$\mathbf{E}X_\sigma = \mathbf{E}Z_\sigma.$$

Since $Z_n \geq X_n$ this implies $X_\sigma = Z_\sigma$ a.s., proving (i).

By the optimality $\mathbf{E}Z_\sigma = Z_0$, while the supermartingale property implies

$$Z_0 \geq \mathbf{E}Z_{\sigma\wedge n} \geq \mathbf{E}Z_\sigma.$$

Thus

$$EZ_{\sigma\wedge n} = \mathbf{E}Z_\sigma = \mathbf{E}\mathbf{E}[Z_\sigma|\mathcal{F}_n].$$

Furthermore, by Doob's optional stopping

$$Z_{\sigma\wedge n} \geq \mathbf{E}[Z_\sigma|\mathcal{F}_n],$$

implying $Z_{\sigma\wedge n} = \mathbf{E}[Z_\sigma|\mathcal{F}_n]$. Thus (Z_n^σ) is indeed a martingale. $\square$

6.2 Pricing American options

Let us return to our pricing problem. Assume that we have an arbitrage-free complete market, that is the EMM $\mathbf{Q}$ is unique. Let $(f_n)_{n=0,\ldots,N}$ be the payoff of an American option. A hedging strategy now has to fulfill the conditions

$$X_n^\pi \geq f_n, \quad n = 0, 1, \ldots, N,$$

as the option can be exercised at any time. A hedge is *minimal*, if for a stopping time τ^* we have $X_{\tau^*}^\pi = f_{\tau^*}$.

By Doob's optional stopping $(X_0^\pi/B_0, X_\tau^\pi/B_\tau)$ is martingale for any stopping time τ, i.e.

$$\frac{x}{B_0} = \mathbf{E}_\mathbf{Q}\frac{X_0^\pi}{B_0} = \mathbf{E}_\mathbf{Q}\frac{X_\tau^\pi}{B_\tau} \geq \mathbf{E}_\mathbf{Q}\frac{f_\tau}{B_\tau}.$$

Therefore the initial cost of the hedge is at least

$$x \geq B_0 \sup_{\tau\in\mathcal{M}_0^N} \mathbf{E}_\mathbf{Q}\frac{f_\tau}{B_\tau}.$$

At time N we need

$$X_N^\pi \geq f_N.$$

At time $N-1$ the holder either exercise the option or continues to time N, (in that case we discount the price), therefore

$$X_{N-1}^\pi \geq \max\left\{f_{N-1}, \frac{B_{N-1}}{B_N}\mathbf{E}_\mathbf{Q}[f_N|\mathcal{F}_{N-1}]\right\}.$$

Dividing by B_{N-1}

$$\frac{X_{N-1}^{\pi}}{B_{N-1}} \geq \max \left\{ \frac{f_{N-1}}{B_{N-1}}, \mathbf{E}_{\mathbf{Q}} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_{N-1} \right] \right\}.$$

Thus, we see the connection with the Snell-envelope.

For a hedging strategy π we have that

(i) $(X_n^{\pi}/B_n)_n$ is a $\mathbf{Q}$-martingale (since $\mathbf{Q}$ is EMM and π is SF), and

(ii) (X_n^{π}/B_n) dominates (f_n/B_n) (since π is a hedge).

Therefore, the value process of a hedge is larger than the Snell-envelope of (f_n/B_n), i.e.

$$\frac{X_n^{\pi}}{B_n} \geq Z_n, \quad n = 0, 1, \ldots, N, \tag{12}$$ {eq:di-american-1}

where (Z_n) is the Snell-envelope of (f_n/B_n). The Snell-envelope (Z_n) is a supermartingale, therefore by the Doob-decomposition (that's stated for submartingale, but multiply by -1) we have

$$Z_n = M_n - A_n, \quad n = 0, 1, \ldots, N, \tag{13}$$ {eq:di-american-2}

where M_n is a $\mathbf{Q}$-martingale, and (A_n) is an increasing predictable sequence, $A_0 = 0$. Comparing (12) and (13) we see that for $n \leq \tau^*$

$$\frac{X_n^{\pi}}{B_n} \geq M_n.$$

On the other hand, the market is complete, which implies (see the easy parts of the proof of Theorem 6) that there exists a strategy π such that

$$\frac{X_n^{\pi}}{B_n} = M_n, \quad n = 0, 1, \ldots, N.$$

This is a minimal hedging strategy with initial cost

$$\frac{x}{B_0} = \frac{X_0^{\pi}}{B_0} = M_0 = Z_0.$$

{thm:price-di-amer

Theorem 10. *Consider an aribtrage-free complete market with unique EMM* $\mathbf{Q}$. *Let* (f_n) *be the nonnegative payoff sequence of an American option. Let* (Z_n) *be the Snell-envelope of the discounted payoff sequence* (f_n/B_n). *The fair price for this option is*

$$C = B_0 Z_0 = B_0 \sup_{\tau \in \mathcal{M}_0^N} \mathbf{E}_{\mathbf{Q}} \frac{f_\tau}{B_\tau} = B_0 \mathbf{E}_{\mathbf{Q}} \frac{f_{\tau^*}}{B_{\tau^*}},$$

where τ^* is an (not unique in general) optimal exercise time given by

$$\tau^* = \min \left\{ n : \frac{f_n}{B_n} = Z_n \right\}.$$

Furthermore, there exists a SF strategy π which is an optimal hedge with initial cost C and

$$X^{\pi}_{\tau^*} = \frac{f_{\tau^*}}{B_{\tau^*}}.$$

6.3 American vs. European options

Clearly, an American option with payoff sequence $(f_n)_{n=0,1,\dots,N}$ worth at least as a European option with payoff f_N. However, in some cases the fair prices are equal.

Consider an American call option with strike price K, that is

$$f_n = f(S_n) = (S_n - K)_+.$$

Assume that the deterministic sequence (B_n) is nondecreasing (i.e. the interest rate is nonnegative). Let (Z_n) denote the Snell envelope of (f_n/B_n), that is

$$Z_N = \frac{f_N}{B_N}, \quad Z_n = \max \left\{ \frac{f_n}{B_n}, \mathbf{E}\left[Z_{n+1} | \mathcal{F}_n\right] \right\}, \quad n = 0, 1, \dots, N-1.$$

Using that (S_n/B_n) is a $\mathbf{Q}$-martingale, by Jensen's inequality

$$\begin{aligned}
\frac{f_{N-1}}{B_{N-1}} &= \frac{(S_{N-1} - K)_+}{B_{N-1}} \\
&= \left(\frac{S_{N-1}}{B_{N-1}} - \frac{K}{B_{N-1}} \right)_+ \\
&\leq \mathbf{E_Q} \left[\left(\frac{S_N}{B_N} - \frac{K}{B_{N-1}} \right)_+ \middle| \mathcal{F}_{N-1} \right] && \text{Jensen's inequality} \\
&\leq \mathbf{E_Q} \left[\left(\frac{S_N}{B_N} - \frac{K}{B_N} \right)_+ \middle| \mathcal{F}_{N-1} \right] && \text{by } B_N \geq B_{N-1} \\
&= \mathbf{E_Q} \left[\frac{(S_N - K)_+}{B_N} \middle| \mathcal{F}_{N-1} \right] \\
&= \mathbf{E_Q}[Z_N | \mathcal{F}_{N-1}].
\end{aligned}$$

This means that at time $N-1$ it is always good to hold the option and continue to step N.

An induction argument shows that at any time it is better to hold the option. Indeed, assume for some n

$$\frac{f_n}{B_n} \leq \mathbf{E}_{\mathbf{Q}}[Z_{n+1}|\mathcal{F}_n].$$

We just proved this for $n = N-1$. The same way as above we have

$$\begin{aligned}
\frac{f_{n-1}}{B_{n-1}} &= \frac{(S_{n-1}-K)_+}{B_{n-1}} \\
&= \left(\frac{S_{n-1}}{B_{n-1}} - \frac{K}{B_{n-1}}\right)_+ \\
&\leq \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_n}{B_n} - \frac{K}{B_{n-1}}\right)_+ \middle| \mathcal{F}_{n-1}\right] && \text{Jensen's inequality} \\
&\leq \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_n}{B_n} - \frac{K}{B_n}\right)_+ \middle| \mathcal{F}_{n-1}\right] && \text{by } B_n \geq B_{n-1} \\
&= \mathbf{E}_{\mathbf{Q}}\left[\frac{(S_n-K)_+}{B_n} \middle| \mathcal{F}_{n-1}\right] \\
&= \mathbf{E}_{\mathbf{Q}}\left[\frac{f_n}{B_n} \middle| \mathcal{F}_{n-1}\right] \\
&\leq \mathbf{E}_{\mathbf{Q}}\left[\mathbf{E}_{\mathbf{Q}}[Z_{n+1}|\mathcal{F}_n] \middle| \mathcal{F}_{n-1}\right] && \text{induction} \\
&\leq \mathbf{E}_{\mathbf{Q}}[Z_n|\mathcal{F}_{n-1}] && Z \text{ supermartingale}
\end{aligned}$$

Thus $\tau^* \equiv N$ is an optimal stopping time, which means that no matter what happens, we wait until the end. Then the American option behaves as the European, so the prices are equal.

Theorem 11. *Assume that the market is arbitrage free and complete, and the interest rate is nonnegative. Then the price of a European call option equals to the price of the American call option.*

7 Stochastic integration

7.1 Lévy characterization

One can define stochastic integral with respect to more general processes. The process (X_t) is a continuous *semimartingale* if

$$X_t = M_t + A_t,$$

where M_t is a continuous martingale and A_t is of bounded variation, and both are adapted.

We can define stochastic integral with respect to semimartingales. Indeed, integral with respect to A_t can be defined pathwise, since A is of bounded variation, and integration with respect to continuous M_t can be defined similarly as for SBM.

We recall Itô's formula.

Theorem 12 (Itô formula for semimartingales)**.** *Let $X_t = M_t + A_t$ be a continuous semimartingale, and let $f \in C^2$. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle M\rangle_s.$$

We have seen that if W_t is SBM, then it W_t is a continuous martingale, and $W_t^2 - t$ is a martingale. It turns out that this characterizes SBM.

Theorem 13 (Lévy's characterization of SBM)**.** *Let M_t be a continuous martingale, such that $M_0 = 0$, and $M_t^2 - t$ is martingale. Then M_t is SBM.*

Proof. We determine the conditional characteristic function of M_t with respect to $\mathcal{F}_s$, $t > s$. Apply Itô with $f(x) = e^{\mathrm{i}ux}$, where $u \in \mathbb{R}$ is arbitrary but fixed. Since $f'(x) = \mathrm{i}ue^{\mathrm{i}ux}$, $f''(x) = -u^2e^{\mathrm{i}ux}$, and by assumption $\langle M\rangle_t = t$, therefore

$$e^{\mathrm{i}uM_t} - e^{\mathrm{i}uM_s} = \int_s^t \mathrm{i}ue^{\mathrm{i}uM_v}\mathrm{d}M_v + \frac{1}{2}\int_s^t (-u^2)e^{\mathrm{i}uM_v}\mathrm{d}v.$$

Let $A \in \mathcal{F}_s$ arbitrary. Multiplying by $e^{-\mathrm{i}uM_s}$, and integrating on A we get

$$\mathbf{E}\left[e^{\mathrm{i}u(M_t-M_s)}I_A\right] = \mathbf{P}(A) - \frac{u^2}{2}\int_s^t \mathbf{E}\left[e^{\mathrm{i}u(M_v-M_s)}I_A\right]\mathrm{d}v.$$

With A and s fixed, define

$$g_{A,s}(t) = g(t) = \mathbf{E}\left[e^{\mathrm{i}u(M_t - M_s)} I_A\right].$$

With this notation

$$g(t) = \mathbf{P}(A) - \frac{u^2}{2} \int_s^t g(v)\mathrm{d}v.$$

Differentiating we obtain

$$g'(t) = -\frac{u^2}{2} g(t), \quad g(s) = \mathbf{P}(A).$$

Therefore, the solution

$$g(t) = \mathbf{P}(A) \cdot e^{-\frac{u^2}{2}(t-s)}.$$

This holds for any $A \in \mathcal{F}_s$, which means that

$$\mathbf{E}\left[e^{\mathrm{i}u(M_t - M_s)} | \mathcal{F}_s\right] = e^{-\frac{u^2}{2}(t-s)}$$

for $u \in \mathbb{R}$. That is the increment $M_t - M_s$ is independent of $\mathcal{F}_s$, and it is Gaussian with mean 0 and variance $(t - s)$. Since it is continuous, it is SBM. $\square$

Note that the continuity assumption is important. Indeed, if N_t is a Poisson process with intensity 1, then both $(N_t - t)$ and $(N_t - t)^2 - t$ are martingales.

7.2 Girsanov's theorem

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $(\mathcal{F}_t)$ a filtration. Let $\mathbf{Q}$ be another probability measure on $(\Omega, \mathcal{A})$, which is absolute continuous with respect to $\mathbf{P}$, i.e. $\mathbf{Q} \ll \mathbf{P}$. Let M_∞ denote the Radon–Nikodym-derivative,

$$M_\infty = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}},$$

that is

$$\mathbf{Q}(A) = \int_A M_\infty \mathrm{d}\mathbf{P}.$$

In what follows, we have more (usually 2) probability measures, therefore we put in the lower index of $\mathbf{E}$ the corresponding measure. That is $\mathbf{E}_\mathbf{P} X = \int_\Omega X \mathrm{d}\mathbf{P}$, and $\mathbf{E}_\mathbf{Q} X = \int_\Omega X \mathrm{d}\mathbf{Q}$. Note that the notion of martingale does depend on the underlying measure. Therefore, we have $\mathbf{P}$-martingale, and $\mathbf{Q}$-martingale.

Define the $\mathbf{P}$-martingale

$$M_t = \mathbf{E}_\mathbf{P}[M_\infty | \mathcal{F}_t].$$

{lemma:p-q-mtg}

Lemma 8. *The adapted process (X_t) is $\mathbf{Q}$-martingale if and only if $(M_t X_t)$ is $\mathbf{P}$-martingale.*

Proof. Since

$$\mathbf{E}_\mathbf{P}[M_\infty X_t | \mathcal{F}_t] = X_t M_t,$$

for each $A \in \mathcal{F}_t$

$$\int_A X_t M_\infty \mathrm{d}\mathbf{P} = \int_A X_t M_t \mathrm{d}\mathbf{P}.$$

Therefore, if $A \in \mathcal{F}_s \subset \mathcal{F}_t$, then

$$\begin{aligned} \int_A X_t \mathrm{d}\mathbf{Q} &= \int_A X_t M_\infty \mathrm{d}\mathbf{P} = \int_A X_t M_t \mathrm{d}\mathbf{P} \\ \int_A X_s \mathrm{d}\mathbf{Q} &= \int_A X_s M_\infty \mathrm{d}\mathbf{P} = \int_A X_s M_s \mathrm{d}\mathbf{P}. \end{aligned}$$

Then (X_t) is $\mathbf{Q}$-martingale if the left-hand sides are equal for each $A \in \mathcal{F}_s$, $s < t$, which is obviously equivalent to the equality of the right-hand sides, which means that $(M_t X_t)$ is $\mathbf{P}$-martingale. $\square$

Let

$$\zeta_t^s = \int_s^t \theta_u \mathrm{d}W_u - \frac{1}{2} \int_s^t \theta_u^2 \mathrm{d}u, \quad \zeta_t = \zeta_t^0,$$

where θ_t is adapted. Then $Z_t = e^{\zeta_t}$ satisfies the SDE

$$Z_t = 1 + \int_0^t Z_s X_s \mathrm{d}W_s. \tag{14}$$

{eq:Gir-sde}

We use this formula in the proof of Girsanov's theorem. We can write the SDE above as

$$\mathrm{d}Z_t = Z_t X_t \mathrm{d}W_t, \quad Z_0 = 1.$$

Indeed, rewriting ζ as an Itô process

$$\zeta_t = \int_0^t -\frac{1}{2}\theta_u^2 \mathrm{d}u + \int_0^t \theta_u \mathrm{d}W_u.$$

Using the Itô formula with $f(x) = e^x$, we obtain

$$\begin{aligned} Z_t = e^{\zeta_t} &= 1 + \int_0^t e^{\zeta_s}\mathrm{d}\zeta_s + \frac{1}{2}\int_0^t e^{\zeta_s}\theta_s^2 \mathrm{d}s \\ &= 1 + \int_0^t e^{\zeta_s}\left(-\frac{1}{2}\theta_s^2\mathrm{d}s + X_s \mathrm{d}W_s\right) + \frac{1}{2}\int_0^t e^{\zeta_s}\theta_s^2\mathrm{d}s \\ &= 1 + \int_0^t e^{\zeta_s}\theta_s \mathrm{d}W_s \\ &= 1 + \int_0^t Z_s\theta_s\mathrm{d}W_s, \end{aligned}$$

as claimed. We also see that Z_t is a martingale.

Exercise 10. Let ζ_t as above. Prove that $Y_t = e^{-\zeta_t}$ satisfies the SDE

$$\mathrm{d}Y_t = Y_t\theta_t^2\mathrm{d}t - \theta_t Y_t \mathrm{d}W_t, \quad Y_0 = 1.$$

{thm:Girsanov}

Theorem 14 (Girsanov's theorem)**.** *Let (θ_t) be an adapted process, such that $\int_0^T \theta_s^2 \mathrm{d}s < \infty$ a.s., and assume that*

$$\Lambda_t = \exp\left\{-\int_0^t \theta_s\mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2\mathrm{d}s\right\} \tag{15}$$

{eq:Lambda}

is $\mathbf{P}$-martingale, where (W_t) is $\mathbf{P}$-SBM. Define $\mathbf{Q}_\theta = \mathbf{Q}$

$$\left.\frac{\mathrm{d}\mathbf{Q}_\theta}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = \Lambda_T.$$

Then $\widetilde{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$ is $\mathbf{Q}$-SBM.

Remark 1. We have seen above that Λ_t is martingale. In fact, in general it is only local martingale, and we need integrability conditions. These technical assumptions are omitted.

Proof. First we show that $\mathbf{Q}$ is indeed a probability measure. By (14)

$$\Lambda_t = 1 - \int_0^t \Lambda_s \theta_s \mathrm{d}W_s,$$

which is martingale, so

$$\mathbf{E_P}\Lambda_T = \mathbf{E_P}\Lambda_0 = 1.$$

Since $\Lambda_T > 0$ we see that $\mathbf{Q}$ is probability measure.

Next we show that $\widetilde{W}$ satisfies the conditions of the Lévy characterization.

The continuity is clear, since W is SBM and $\mathbf{Q} \ll \mathbf{P}$. By Lemma 8 $(\widetilde{W}_t)$ is $\mathbf{Q}$-martingale iff $(\widetilde{W}_t\Lambda_t)$ is $\mathbf{P}$-martingale. We apply the Itô formula with $f(x, y) = xy$ and the Itô process

$$\begin{aligned} \widetilde{W}_t &= \int_0^t \theta_s \mathrm{d}s + \int_0^t 1 \mathrm{d}W_s \\ \Lambda_t &= 1 - \int_0^t \Lambda_s \theta_s \mathrm{d}W_s. \end{aligned}$$

Then

$$\begin{aligned} \Lambda_t \widetilde{W}_t &= \int_0^t \widetilde{W}_s \mathrm{d}\Lambda_s + \int_0^t \Lambda_s \mathrm{d}\widetilde{W}_s + \int_0^t -\Lambda_s \theta_s \mathrm{d}s \\ &= -\int_0^t \widetilde{W}_s \Lambda_s \theta_s \mathrm{d}W_s + \int_0^t \Lambda_s \left(\theta_s \mathrm{d}s + \mathrm{d}W_s\right) - \int_0^t \Lambda_s \theta_s \mathrm{d}s \\ &= \int_0^t \Lambda_s (1 - \theta_s \widetilde{W}_s) \mathrm{d}W_s, \end{aligned}$$

which is $\mathbf{P}$-martingale. Thus $(\widetilde{W}_t)$ is $\mathbf{Q}$-martingale.

Next we show that $(\widetilde{W}_t^2 - t)$ is $\mathbf{Q}$-martingale. Again, by Itô's formula with $f(x) = x^2$

$$\widetilde{W}_t^2 = 2\int_0^t \widetilde{W}_s \mathrm{d}\widetilde{W}_s + \frac{1}{2}\int_0^t 2\mathrm{d}t,$$

from which

$$\widetilde{W}_t^2 - t = 2\int_0^t \widetilde{W}_s \left(\theta_s \mathrm{d}s + \mathrm{d}W_s\right).$$

Therefore

$$\begin{aligned} \Lambda_t(\widetilde{W}_t^2 - t) &= \int_0^t \Lambda_s 2\widetilde{W}_s \left(\theta_s \mathrm{d}s + \mathrm{d}W_s\right) + \int_0^t (\widetilde{W}_s^2 - s)\mathrm{d}\Lambda_s - \int_0^t \Lambda_s \theta_s 2\widetilde{W}_s \mathrm{d}s \\ &= \int_0^t \left[2\Lambda_s \widetilde{W}_s - (\widetilde{W}_s^2 - s)\Lambda_s \theta_s\right] \mathrm{d}W_s, \end{aligned}$$

which is $\mathbf{P}$-martingale. Thus $(\widetilde{W}_t^2 - t)$ is $\mathbf{Q}$-martingale, and the proof is complete. □

Finally, we state without proof (and precise statement) the martingale representation theorem.

{thm:martingale-re

Theorem 15 (Martingale representation theorem)**.** *Let (W_t) SBM on $(\Omega, \mathcal{A}, \mathbf{P})$, and let $(\mathcal{F}_t)$ the generated filtration, together with the $\mathbf{P}$-zero sets. If (M_t) is continuous square integrable martingale with $M_0 = 0$ a.s., then there exists an adapted (Y_t) such that*

$$M_t = \int_0^t Y_s \mathrm{d}W_s.$$

8 Continuous time markets in general

The general notations are the same as in the discrete time setup.

In what follows, we work on the finite time horizon $[0, T]$, $T < \infty$. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $(\mathcal{F}_t)$ a filtration. There are two financial instruments on the market, the bond, which is the riskless asset, and the stock, which is the risky asset. The price process of the bond is given by the deterministic process $(B_t = e^{rt})$, $r \in \mathbb{R}$ being the continuous interest rate, while the price process of the stock is (S_t), which is nonnegative, adapted to $(\mathcal{F}_t)$. Furthermore, we assume that (S_t) is an Itô process.

A *strategy / portfolio* is a process $(\pi_t = (\beta_t, \gamma_t))$, where the components are adapted and

$$\int_0^T |\beta_t| \mathrm{d}t < \infty, \quad \int_0^T \gamma_t^2 \mathrm{d}t < \infty, \text{ a.s.}$$

The process β_t represents the amount of bonds at time t, while γ_t is the amount of stock. Both processes are real valued (short selling is possible).

The value of the portfolio (π) at t is

$$X_t^\pi = \beta_t B_t + \gamma_t S_t. \tag{16}$$ {eq:ertekfoly}

Recall that in discrete time an equivalent formulation of self-financing portfolio is

$$X_{n+1} - X_n = \beta_{n+1}(B_{n+1} - B_n) + \gamma_{n+1}(S_{n+1} - S_n).$$

The continuous time analogue of the above is the SDE

$$\mathrm{d}X_t^\pi = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}S_t.$$

The strategy $(\pi_t = (\beta_t, \gamma_t))$ is *self-financing* (SF) if it satisfies the SDE

$$\mathrm{d}X_t^\pi = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}S_t. \tag{17}$$ {eq:onfin}

In what follows all strategies are SF unless otherwise stated.

The discounted processes are defined as $(\overline{S}_t = S_t B_0 / B_t)$ and $(\overline{X}_t^\pi = X_t^\pi B_0 / B_t)$.

{all:onfin-ekv}

Proposition 5. *A strategy $(\pi_t = (\beta_t, \gamma_t))$ is SF iff*

$$\overline{X}_t^\pi = X_0^\pi + \int_0^t \gamma_s \mathrm{d}\overline{S}_s, \quad t \in [0, T].$$

Proof. Assume that π is SF. Then, by Itô's formula

$$\begin{aligned}
\mathrm{d}\overline{X}_t^\pi &= \mathrm{d}\left(e^{-rt} X_t^\pi\right) = -re^{-rt} X_t^\pi \mathrm{d}t + e^{-rt} \mathrm{d}X_t \\
&= -re^{-rt}(\beta_t e^{rt} + \gamma_t S_t)\mathrm{d}t + e^{-rt}\left(\beta_t \mathrm{d}e^{rt} + \gamma_t \mathrm{d}S_t\right) \\
&= -re^{-rt}\gamma_t S_t \mathrm{d}t + e^{-rt}\gamma_t \mathrm{d}S_t \\
&= \gamma_t \mathrm{d}\left(e^{-rt} S_t\right),
\end{aligned}$$

as claimed.

For the reverse direction, we have

$$\mathrm{d}\overline{X}_t^\pi = \gamma_t \mathrm{d}\overline{S}_t.$$

Since $X_t^\pi = \beta_t e^{rt} + \gamma_t S_t$, so

$$\mathrm{d}\overline{X}_t^\pi = -re^{-rt} X_t^\pi \mathrm{d}t + e^{-rt}\mathrm{d}X_t^\pi = -e^{-rt}\beta_t \mathrm{d}B_t - re^{-rt}\gamma_t S_t \mathrm{d}t + e^{-rt}\mathrm{d}X_t^\pi.$$

The right-hand side

$$\gamma_t \mathrm{d}\overline{S}_t = -re^{-rt}\gamma_t S_t \mathrm{d}t + \gamma_t e^{-rt}\mathrm{d}S_t.$$

The equality of the sides gives

$$\mathrm{d}X_t^\pi = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}S_t,$$

which is the definition of SF. $\square$

An SF strategy π is *arbitrage*, if $X_0^\pi = 0$ a.s., $X_T \geq 0$ a.s., and $\mathbf{P}(X_T^\pi > 0) > 0$. The market is *arbitrage free* if there exists no arbitrage strategy.

A probability measure $\mathbf{Q}$ is *equivalent martingale measure* (EMM) if $\mathbf{P} \sim \mathbf{Q}$ (that is $\mathbf{P} \ll \mathbf{Q}$ and $\mathbf{Q} \ll \mathbf{P}$), and $(\overline{S}_t)$ is $\mathbf{Q}$-martingale.

We have seen in the discrete time setup that the existence of EMM is equivalent to the arbitrage free property. One of the implications is rather simple in the continuous time setup. Assume that $\mathbf{Q}$ is EMM, and let π be an (SF) strategy. By Proposition 5 the discounted value process has the representation

$$\overline{X}_t^\pi = X_0^\pi + \int_0^t \gamma_s \mathrm{d}\overline{S}_s.$$

Since $(\overline{S}_t)$ is $\mathbf{Q}$-martingale, and $\overline{X}_t^\pi$ is a stochastic integral with respect to $\overline{S}$, we see that $(\overline{X}_t^\pi)$ is $\mathbf{Q}$-martingale. (Recall the discrete time analogue of this statement.) Therefore

$$\mathbf{E_Q}\overline{X}_T^\pi = \mathbf{E_Q} X_0^\pi.$$

Since $\mathbf{P} \sim \mathbf{Q}$, $X_0^\pi = 0$, $X_T^\pi \geq 0$ $\mathbf{P}$-a.s., implies $\mathbf{Q}$-a.s. Then $\mathbf{E_Q}\overline{X}_T^\pi = \mathbf{E_Q} X_0^\pi = 0$, from which $X_T^\pi \equiv 0$ $\mathbf{Q}$-a.s., and so $\mathbf{P}$-a.s.

We proved the following.

Theorem 16. *Assume that on the market* $(\Omega, \mathcal{A}, \mathbf{P}, (S_t), (B_t = e^{rt}), (\mathcal{F}_t))$ *there exists EMM. Then the market is arbitrage free.*

9 Black–Scholes model

In a special model we explicitly construct the EMM via Girsanov's theorem, and compute the fair price of a payoff. In particular, we prove the Nobel-prize winner Black–Scholes pricing formula, which gives the fair price of a European call option.

9.1 The model

Fix $r > 0, \mu \in \mathbb{R}$ and $\sigma > 0$. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, (W_t) SBM on $[0, T]$, $T < \infty$, and $\mathcal{F}_t$ be the generated filtration. The bond and stock price in the *Black–Scholes-model* is given by

$$\begin{aligned} \mathrm{d}B_t &= rB_t\,\mathrm{d}t, \quad B_0 = 1, \\ \mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad S_0 = S_0., \end{aligned} \tag{18}$$ {eq:black-shcholes

From the form of S_t we immediately see that S_t is a martingale if and only if $\mu = 0$.

The bond price is simply $B_t = e^{rt}$.

Writing S_t as an Itô process

$$S_t = S_0 + \int_0^t \mu S_s \mathrm{d}s + \int_0^t \sigma S_s \mathrm{d}W_s.$$

Applying Itô with $f(x) = \log x$

$$\begin{aligned}\log S_t &= \log S_0 + \int_0^t \frac{1}{S_s}\left(\mu S_s \mathrm{d}s + \sigma S_s \mathrm{d}W_s\right) + \frac{1}{2}\int_0^t -\frac{1}{S_s^2}\sigma^2 S_s^2 \mathrm{d}s \\ &= \log S_0 + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t.\end{aligned}$$

From which

$$S_t = S_0 \cdot e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t}. \qquad (19) \quad \{eq:exp-BM\}$$

This is the *exponential Brownian motion*.

Note that the proof is not complete, because the logarithm is not smooth at 0. The argument above only helps to find out the solution. (A more constructive approach is to apply Itô with a general f, and then choose f to obtain a solvable equation.)

Exercise 11. Prove that (19) is indeed a solution.

9.2 Equivalent martingale measure and the fair price

As an application of Girsanov's theorem, we construct a new measure, such that $\overline{S}_t$ is a martingale under this measure.

By (18)

$$\mathrm{d}\overline{S}_t = \overline{S}_t\left((\mu - r)\mathrm{d}t + \sigma \mathrm{d}W_t\right) = \overline{S}_t \sigma \mathrm{d}\widetilde{W}_t^\mu, \qquad (20) \quad \{eq:tildeS\}$$

where

$$\widetilde{W}_t^\mu = W_t + \frac{\mu - r}{\sigma}t. \qquad (21) \quad \{eq:tildeW\}$$

Therefore, we need a measure $\mathbf{Q}$ such that the process $\widetilde{W}_t^\mu$ is $\mathbf{Q}$-SBM. Then, by (20) $(\overline{S}_t)$ is $\mathbf{Q}$-martingale.

Let $\theta_t \equiv \theta = \frac{\mu - r}{\sigma}$, and

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = \Lambda_T = \exp\left\{ -\int_0^T \theta \mathrm{d}W_s - \frac{1}{2}\int_0^T \theta^2 \mathrm{d}s \right\} = e^{-\theta W_T - \frac{\theta^2 T}{2}}.$$

By Girsanov's theorem (Theorem 14) the shifted process $(\widetilde{W}_t^\mu)$ is $\mathbf{Q}$-SBM, thus $(\overline{S}_t)$ is $\mathbf{Q}$-martingale. Since $\Lambda_T > 0$ a.s., $\mathbf{P} \sim \mathbf{Q}$, therefore $\mathbf{Q}$ is EMM. By (20)

$$\overline{S}_t = S_0 \cdot e^{\sigma \widetilde{W}_t^\mu - \frac{\sigma^2}{2} t}. \tag{22}$$ {eq:S-mu}

Next, we determine the fair price of a claim f_T, for which $\mathbf{E} f_T^2 < \infty$. Let

$$N_t = \mathbf{E}_\mathbf{Q}\left[e^{-rT} f_T | \mathcal{F}_t \right], \ 0 \le t \le T.$$

By the martingale representation theorem (Theorem 15) there exists an adapted process Y_t, such that

$$N_t = N_0 + \int_0^t Y_s \mathrm{d}\widetilde{W}_s^\mu, \tag{23}$$ {eq:N-def}

where $N_0 = \mathbf{E}_\mathbf{Q} e^{-rT} f_T$. Define the strategy $\pi_t = (\beta_t, \gamma_t)$ as

$$\beta_t = N_t - \frac{Y_t}{\sigma}, \quad \gamma_t = \frac{Y_t e^{rt}}{\sigma S_t}.$$

Lemma 9. *The strategy $(\pi_t = (\beta_t, \gamma_t))$ is self-financing and $\overline{X}_t^\pi = N_t$.*

Proof. By the definition

$$X_t^\pi = \beta_t B_t + \gamma_t S_t = \left(N_t - \frac{Y_t}{\sigma} \right) e^{rt} + \frac{Y_t}{\sigma} e^{rt} = e^{rt} N_t,$$

i.e. $\overline{X}_t^\pi = N_t$.

In order to show that π is SF, by Proposition 5 we need that $\mathrm{d}\overline{X}_t^\pi = \gamma_t \mathrm{d}\overline{S}_t$. By (23)

$$\mathrm{d}\overline{X}_t^\pi = \mathrm{d}N_t = Y_t \mathrm{d}\widetilde{W}_t^\mu,$$

while (20) gives

$$\gamma_t \mathrm{d}\overline{S}_t = \gamma_t \overline{S}_t \sigma \mathrm{d}\widetilde{W}_t^\mu = Y_t \mathrm{d}\widetilde{W}_t^\mu.$$

$\square$

Since
$$X_T^\pi = e^{rT} N_T = e^{rT} \mathbf{E}_{\mathbf{P}_\mu} \left[e^{-rT} f_T | \mathcal{F}_T \right] = f_T,$$
π is a perfect hedge for f_T, and $X_0^\pi = N_0 = \mathbf{E}_{\mathbf{Q}} e^{-rT} f_T$. Therefore, we proved the following.

{thm:bs-price}

Theorem 17. *In the Black–Scholes model the fair price of the contingent claim f_T is*
$$C_T(f_T) = \mathbf{E}_{\mathbf{Q}} e^{-rT} f_T.$$
Furthermore, $\pi_t = (\beta_t, \gamma_t)$,
$$\beta_t = N_t - \frac{Y_t}{\sigma}, \quad \gamma_t = \frac{Y_t e^{rt}}{\sigma S_t},$$
is a perfect hedge, where $N_t = \mathbf{E}_{\mathbf{Q}}[e^{-rT} f_T | \mathcal{F}_t]$, and $N_t = N_0 + \int_0^t Y_s \mathrm{d}\widetilde{W}_s^\mu$.

9.3 Black–Scholes formula

The famous Black–Scholes formula gives the fair price of a European call option. In this case the payoff function is $f_T = (S_T - K)_+$, where K is the strike price. By Theorem 17, the fair price is
$$C_T(K) = \mathbf{E}_{\mathbf{Q}} \left(e^{-rT} (S_T - K)_+ \right).$$
By (22)
$$S_T = S_0 e^{rT} e^{\sigma \widetilde{W}_T^\mu - \frac{\sigma^2}{2} T},$$
where $\widetilde{W}_T^\mu \sim \mathrm{N}(0, T)$ under $\mathbf{Q}$. Therefore, writing Z for a standard normal
$$\begin{aligned} C_T(K) &= \mathbf{E}_{\mathbf{Q}} \left(e^{-rT} (S_T - K)_+ \right) \\ &= \mathbf{E}_{\mathbf{Q}} \left(S_0 e^{\sigma \widetilde{W}_T^\mu - \frac{\sigma^2}{2} T} - e^{-rT} K \right)_+ \\ &= \mathbf{E} \left(S_0 e^{\sigma \sqrt{T} Z - \frac{\sigma^2}{2} T} - e^{-rT} K \right)_+ \\ &= \frac{1}{\sqrt{2\pi}} \int_\gamma^\infty \left(S_0 e^{\sigma \sqrt{T} x - \frac{\sigma^2}{2} T} - e^{-rT} K \right) e^{-\frac{x^2}{2}} \, \mathrm{d}x \\ &= S_0 \frac{1}{\sqrt{2\pi}} \int_\gamma^\infty e^{-\frac{(x - \sigma\sqrt{T})^2}{2}} \, \mathrm{d}x - e^{-rT} K (1 - \Phi(\gamma)) \\ &= S_0 \left(1 - \Phi(\gamma - \sigma\sqrt{T}) \right) - e^{-rT} K (1 - \Phi(\gamma)), \end{aligned} \tag{24}$$
{eq:BS-calc}

where

$$\gamma = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{K}{S_0} + \left(\frac{\sigma^2}{2} - r\right)T\right].$$

The pricing formula

$$C_T(K) = S_0\left(1 - \Phi(\gamma - \sigma\sqrt{T})\right) - e^{-rT}K(1 - \Phi(\gamma))$$

is the *Black–Scholes formula*, which was published by Fischer Black and Myron Scholes in 1973. The underlying theory was generalized later by Merton. In 1997, Scholes and Merton received the Nobel prize for this (Black died in 1995).

9.4 From CRR to Black–Scholes

Here we derive the Black–Scholes formula as the limit of the discrete CRR pricing formula. This part is based on [1], section 2.6.

Consider the continuous model on $[0, T]$. Let $r > 0$ be the continuous interest rate and $\sigma > 0$ the volatility. In the approximating discrete model choose, for N fixed

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \quad \tau_i = \frac{i}{N}T$$

Put $h = T/N$. The parameters of the N-step homogeneous binomial market are r_N, a_N, and b_N. The price of the bond and stock is denoted by $B^N_{\tau_i}$ and $S^N_{\tau_i}$, respectively.

Choose

$$r_N = r\frac{T}{N} = rh, \quad \log\frac{1+b_N}{1+r_N} = \sigma\sqrt{h}, \quad \log\frac{1+a_N}{1+r_N} = -\sigma\sqrt{h}. \tag{25}$$ {eq:rab-choice}

It is easy to show that this implies

$$B^N_{\tau_{\frac{tN}{T}}} = (1+r_N)^{\lfloor\frac{tN}{T}\rfloor} \to e^{rt} = B_t,$$

which in fact suggests the choice of r_N. Similar, but more complicated calculations gives that with the choice above $\mathbf{Var}S^N_{\tau_N}$ converges.

In the homogeneous binomial model the EMM was given by the upwards step probability

$$p^*_N = \frac{r_N - a_N}{b_N - a_N}.$$

Under the EMM

$$S^N_{\tau_N} = S_0(1+b_N)^{Y_N}(1+a_N)^{N-Y_N} = S_0\left(\frac{1+b_N}{1+a_N}\right)^{Y_N}(1+a_N)^N, \qquad (26) \quad \texttt{\{eq:crrS\_N\}}$$

where $Y_N \sim \text{Binomial}(N, p^*_N)$.

The CRR pricing formula gives

$$C_N(K) = \mathbf{E}^*_N \frac{(S^N_{\tau_N} - K)_+}{B^N_{\tau_N}}. \qquad (27) \quad \texttt{\{eq:crr-ar\}}$$

By the central limit theorem (Lindeberg–Feller theorem)

$$\frac{Y_N - Np^*_N}{\sqrt{Np^*_N(1-p^*_N)}} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1), \;\; N \to \infty, \qquad (28) \quad \texttt{\{eq:Y\_N-conv\}}$$

whenever $0 < \liminf_{N\to\infty} p^*_N \le \limsup_{N\to\infty} p^*_N < 1$. Simple calculation gives that $\lim_{N\to\infty} p^*_N = 1/2$, so (28) holds. Rewriting (26)

$$\begin{aligned}
\left(\frac{1+b_N}{1+a_N}\right)^{Y_N}(1+a_N)^N &= \exp\left\{Y_N \log\frac{1+b_N}{1+a_N} + N\log(1+a_N)\right\} \\
&= \exp\left\{\frac{Y_N - Np^*_N}{\sqrt{Np^*_N(1-p^*_N)}}\sqrt{Np^*_N(1-p^*_N)}\log\frac{1+b_N}{1+a_N}\right. \\
&\qquad \left. + N\left(p^*_N \log\frac{1+b_N}{1+a_N} + \log(1+a_N)\right)\right\}.
\end{aligned}$$

By (28) we need to determine the limits

$$\lim_{N\to\infty}\sqrt{Np^*_N(1-p^*_N)}\log\frac{1+b_N}{1+a_N}, \text{ and } \lim_{N\to\infty} N\left(p^*_N\log\frac{1+b_N}{1+a_N} + \log(1+a_N)\right).$$

Taylor expansion and (25) gives

$$\begin{aligned}
1+b_N = e^{\sigma\sqrt{h}}(1+r_N) &= \left(1+\sigma\sqrt{h} + \frac{\sigma^2}{2}h + O(h^{3/2})\right)(1+rh) \\
&= 1+\sigma\sqrt{h} + \left(\frac{\sigma^2}{2}+r\right)h + O(h^{3/2}),
\end{aligned}$$

thus

$$b_N = \sigma\sqrt{h} + \left(\frac{\sigma^2}{2}+r\right)h + O(h^{3/2}).$$

Similarly,

$$a_N = -\sigma\sqrt{h} + \left(\frac{\sigma^2}{2} + r\right) h + O(h^{3/2}).$$

From this

$$\begin{aligned} p_N^* = \frac{r_N - a_N}{b_N - a_N} &= \frac{\sigma\sqrt{h} - \frac{\sigma^2}{2}h + O(h^{3/2})}{2\sigma\sqrt{h} + O(h^{3/2})} \\ &= \frac{1}{2 + O(h)} - \frac{\sigma\sqrt{h} + O(h)}{4 + O(h)} \\ &= \frac{1}{2} - \frac{\sigma}{4}\sqrt{h} + O(h). \end{aligned}$$

Substituting back, and using the second order expansion $\log(1+x) = x - x^2/2 + O(x^3)$, $x \to 0$, we obtain

$$\lim_{N\to\infty} \sqrt{N p_N^*(1-p_N^*)} \log \frac{1+b_N}{1+a_N} = \lim_{N\to\infty} \sqrt{p_N^*(1-p_N^*)} 2\sigma\sqrt{T} = \sigma\sqrt{T},$$

and

$$\begin{aligned} &\lim_{N\to\infty} N \left(p_N^* \log \frac{1+b_N}{1+a_N} + \log(1+a_N) \right) \\ &= \lim_{N\to\infty} N \left(\left[\frac{1}{2} - \frac{\sigma}{4}\sqrt{\frac{T}{N}} + O(N^{-1}) \right] 2\sigma\sqrt{\frac{T}{N}} - \sigma\sqrt{\frac{T}{N}} + r\frac{T}{N} + O(N^{-3/2}) \right) \\ &= \left(r - \frac{\sigma^2}{2} \right) T. \end{aligned}$$

Substituting back into (27)

$$\begin{aligned} \lim_{N\to\infty} C_N(K) &= e^{-rT} \mathbf{E}^* \left(S_0 e^{\sigma\sqrt{T} Z + T\left(r - \frac{\sigma^2}{2}\right)} - K \right)_+ \\ &= \mathbf{E}^* \left(S_0 e^{\sigma\sqrt{T} Z - \frac{\sigma^2}{2}T} - e^{-rT} K \right)_+, \end{aligned}$$

which is exactly the second line in (24).

In fact, we need the convergence of moments, that is, uniform integrability. That can be done with a little more work, but the details are skipped. It is important to note that not only the price convergence, but the whole process $(S_{\tau_i}^N)$ converges to the exponential Brownian motion. This follows from Donsker's theorem.

10 Interest rate models

10.1 The general setup

In what follows we are interested in options on bonds instead of stocks. Therefore, we assume that the stock price B_t is also random. The bond price is given by

$$B_t = \exp\left\{\int_0^t r_u \mathrm{d}u\right\}, \tag{29}$$ {eq:bond}

where r_t, *the interest rate* is an adapted stochastic process. The time interval is $[0, \mathcal{T}]$. The stock price is given by

$$S_t = S_0 + \int_0^t \mu(u) S_u \mathrm{d}u + \int_0^t \sigma_u S_u \mathrm{d}W_u, \tag{30}$$ {eq:stock}

with some adapted process μ and σ. Note that the bond price B_t is a stochastic process too, but it is much smoother than the stock price S_t, as it is the exponential of the Lebesgue integral of a stochastic process. In particular, the path of B_t are of bounded variation, while the path of S_t are not. (Recall that an Itô process is of bounded variation if and only if the stochastic integral part vanishes.)

We want to find an equivalent martingale measure. For the discounted stock price $\overline{S}_t = S_t/B_t$

$$\begin{aligned}
\mathrm{d}\frac{S_t}{B_t} &= \mathrm{d}\left(S_t e^{-\int_0^t r_u \mathrm{d}u}\right) \\
&= e^{-\int_0^t r_u} \mathrm{d}S_t + S_t(-r_t) e^{-\int_0^t r_u \mathrm{d}u} \mathrm{d}t \\
&= \overline{S}_t \left((\mu_t - r_t)\mathrm{d}t + \sigma_t \mathrm{d}W_t\right) \\
&= \overline{S}_t \sigma_t \mathrm{d}\widetilde{W}_t,
\end{aligned}$$

where

$$\widetilde{W}_t = \int_0^t \theta_s \mathrm{d}s + W_t,$$

with $\theta_s = \frac{\mu_s - r_s}{\sigma_s}$. Applying Girsanov's theorem $\widetilde{W}_t$ is SBM under the measure Q_θ, where

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}\mathbf{P}} = \exp\left\{-\int_0^T \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \theta_s^2 \mathrm{d}s\right\}.$$

Therefore, under Q_θ the discounted stock price $\overline{S}_t$ is a martingale, i.e. Q_θ is an equivalent martingale measure.

We are not interested in the specific form of the underlying risky asset (S_t) in (30), but we assume that there exists a unique equivalent martingale measure (that is (S_t/B_t) is martingale). This will be the only measure on the probability space, therefore it is denoted by $\mathbf{P}$ (instead of Q_θ).

Formally, let $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbf{P})$ be a filtered probability space, (r_u) an adapted stochastic process, and (B_t) is given by (29). We assume that the risky asset (S_t) is an adapted stochastic process, such that $(S_t/B_t)_t$ is a martingale under $\mathbf{P}$, and $\mathbf{P}$ is the unique such measure.

A *zero coupon bond (elemi kötvény)* maturing at time T is a claim that pays 1 at time T. Its value at time $t \in [0,T]$ is denoted by $P(t,T)$, $0 \le t \le T \le \mathcal{T}$.

From the pricing theorem we see that the fair price of the zero coupon bond at time 0 is

$$P(0,T) = \mathbf{E}\left[\frac{1}{B_T}\right],$$

thus at time $0 \le t \le T$

$$P(t,T) = B_t \mathbf{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] = \mathbf{E}\left[\exp\left\{-\int_t^T r_u \mathrm{d}u\right\}\Big|\mathcal{F}_t\right]. \qquad (31) \quad \{eq:P(tT)\}$$

A *term structure model (hozamgörbe modell)* is a mathematical model for the prices $P(t,T)$.

We are interested in pricing bond options. The fair price at time 0 of a *European call option* with strike price K at expiry date T_1 for a zero coupon bound with expiry date T_2, where $T_2 > T_1$, is given by

$$\mathbf{E}e^{-\int_0^{T_1} r_u \mathrm{d}u}\left(P(T_1,T_2) - K\right)_+. \qquad (32) \quad \{eq:bond-calleu\}$$

10.2 Short rate diffusion models

In short rate diffusion models the interest rate r_t is given as a solution of a stochastic differential equation.

10.2.1 Ornstein–Uhlenbeck process

Consider the *Langevin equation*

$$\mathrm{d}Y_t = -\mu Y_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t,$$

where $\mu > 0$, $\sigma > 0$, and Y_0 is independent of $\sigma(W_s : s \geq 0)$.

The solution of the homogeneous equation is $e^{-\mu t}$. Taking the derivative of $e^{\mu t} Y_t$ we obtain

$$\mathrm{d}\left(e^{\mu t} Y_t\right) = e^{\mu t}\,\mathrm{d}Y_t + \mu\, e^{\mu t} Y_t\,\mathrm{d}t = e^{\mu t}\,\sigma\,\mathrm{d}W_t,$$

which gives

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t e^{\mu s}\,\sigma\,\mathrm{d}W_s\right).$$

This is the *Ornstein–Uhlenbeck process*. The integral of a deterministic function with respect to SBM is Gaussian, thus

$$Y_t - e^{-\mu t} Y_0$$

is normal with mean and variance

$$\begin{aligned}
&\mathbf{E}Y_t = e^{-\mu t}\,\mathbf{E}Y_0,\\
&\mathbf{E}Y_t^2 = e^{-2\mu t}\,\mathbf{E}Y_0^2 + e^{-2\mu t}\int_0^t \sigma^2\, e^{2\mu s}\,\mathrm{d}s = e^{-2\mu t}\,\mathbf{E}Y_0^2 + \frac{\sigma^2}{2\mu}\,(1 - e^{-2\mu t}).
\end{aligned}$$

We see that as $t \to \infty$

$$Y_t \xrightarrow{\mathcal{D}} N(0, \sigma^2/(2\mu)).$$

Taking the limit for the initial distribution Y_0 we see that (Y_t) is Gaussian and

$$Y_t \sim \mathrm{N}\left(0, \frac{\sigma^2}{2\mu}\right).$$

Next we determine the covariance function of Y. Since

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t \sigma\, e^{\mu u}\,\mathrm{d}W_u\right)$$

we get

$$Y_t - e^{-\mu(t-s)} Y_s = e^{-\mu t}\int_s^t \sigma\, e^{\mu u}\,\mathrm{d}W_u,\ t > s, \tag{33}$$ {eq:ou-fgt}

which is independent of $\sigma(W_u : u \leq s)$ σ. Therefore,

$$\begin{aligned}
\mathbf{Cov}(Y_t, Y_s) = \mathbf{E}Y_t Y_s &= \mathbf{E}\left(Y_t - e^{-\mu(t-s)} Y_s + e^{-\mu(t-s)} Y_s\right) Y_s\\
&= e^{-\mu(t-s)}\,\mathbf{E}Y_s^2 = \frac{\sigma^2}{2\mu} e^{-\mu(t-s)},
\end{aligned}$$

which depends only on $t-s$. That is (Y_t) is stationary.

Using formula (33) for $A \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}
&\mathbf{P}(Y_t \in A | Y_u : u \leq s, Y_s = x) \\
&= \mathbf{P}(Y_t - e^{-\mu(t-s)} Y_s \in A - e^{-\mu(t-s)} x | Y_u : u \leq s, Y_s = x) \\
&= \mathbf{P}(Y_t - e^{-\mu(t-s)} Y_s \in A - e^{-\mu(t-s)} x).
\end{aligned}$$

The variable $Y_t - e^{-\mu(t-s)} Y_s$ is mean zero Gaussian with variance

$$\mathbf{E} \left(Y_t - e^{-\mu(t-s)} Y_s \right)^2 = e^{-2\mu t} \int_s^t \sigma^2 e^{2\mu u} \mathrm{d}u = \frac{\sigma^2}{2\mu} \left(1 - e^{-2\mu(t-s)} \right).$$

Substituting $s = 0$

$$p_t(\cdot|x) \sim \mathrm{N} \left(e^{-\mu t} x, \frac{\sigma^2}{2\mu} \left(1 - e^{-2\mu t} \right) \right),$$

that is, the transition density

$$\rho_t(y|x) = \sqrt{\frac{\mu}{\pi \sigma^2 (1 - e^{-2\mu t})}} \exp \left\{ - \frac{\mu (y - e^{-\mu t} x)^2}{\sigma^2 (1 - e^{-2\mu t})} \right\}.$$

We proved that (Y_t) is a continuous stationary Markov process. It can be shown that this characterizes the OU process.

Finally, we spell out the Kolmogorov equations. The backward is

$$\frac{\partial}{\partial t} \rho_t(y|x) = -\mu x \frac{\partial}{\partial x} \rho_t(y|x) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \rho_t(y|x),$$

which is called *Fokker–Planck equation*. The forward is

$$\frac{\partial}{\partial t} \rho_t(y|x) = - \frac{\partial}{\partial y} \left(-\mu y \rho_t(y|x) \right) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} \rho_t(y|x).$$

10.2.2 Vasicek model

For r_0, a, b, σ given positive numbers let r_t is given by the stochastic differential equation

$$\mathrm{d}r_t = a(b - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{34}$$ {eq:vasicek}

where W_t is a standard Brownian motion. Thus r_t is a translated Ornstein–Uhlenbeck process. Indeed, $X_t = r_t - b$ satisfies

$$\mathrm{d}X_t = \mathrm{d}r_t = -aX_t\mathrm{d}t + \sigma\mathrm{d}W_t,$$

thus

$$X_t = e^{-at}\left(X_0 + \int_0^t e^{as}\sigma\mathrm{d}W_s\right),$$

from which

$$r_t = b + e^{-at}\left(r_0 - b + \int_0^t e^{as}\sigma\mathrm{d}W_s\right).$$

Thus r_t is normally distributed for any fixed t with mean

$$\mathbf{E}r_t = b + e^{-at}(r_0 - b)$$

and variance

$$\mathbf{Var}(r_t) = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

This implies that r_t can take arbitrarily large negative values, which is not very realistic.

Now we determine the distribution of $P(t,T)$. By (31)

$$\begin{aligned} P(t,T) &= \mathbf{E}\left[\exp\left\{-\int_t^T r_u\mathrm{d}u\right\}\Big|\mathcal{F}_t\right] \\ &= e^{-b(T-t)}\mathbf{E}\left[\exp\left\{-\int_t^T X_u\mathrm{d}u\right\}\Big|\mathcal{F}_t\right], \end{aligned}$$

where $X_t = r_t - b$ as above. Since X_t is a Markov process, we have that

$$P(t,T) = e^{-b(T-t)}\mathbf{E}\exp\left\{-\int_0^{T-t}\widetilde{X}_u\mathrm{d}u\right\}, \tag{35}$$ {eq:vasicek-pt}

where $\widetilde{X}$ is the solution to the Langevin equation

$$\mathrm{d}\widetilde{X}_s = -a\widetilde{X}_s + \sigma\mathrm{d}W_s, \qquad \widetilde{X}_0 = x_0 = r_t - b. \tag{36}$$ {eq:vasicek-initia

Therefore, we need to determine the distribution of

$$\int_0^t \widetilde{X}_u\mathrm{d}u.$$

We have seen that (X_u) is a continuous Gaussian process, therefore its integral is Gaussian too. Since $\mathbf{E}\widetilde{X}_u = e^{-at}x_0$, we have

$$\mathbf{E}\int_0^t \widetilde{X}_u \mathrm{d}u = x_0 \int_0^t e^{-au}\mathrm{d}u = \frac{x_0}{a}(1-e^{-at}).$$

Furthermore, for $t \geq s$

$$\begin{aligned}
\mathbf{Cov}(\widetilde{X}_t, \widetilde{X}_s) &= \mathbf{E}e^{-at}\int_0^t \sigma e^{au}\mathrm{d}W_u e^{-as}\int_0^s \sigma e^{au}\mathrm{d}W_u \\
&= \sigma^2 e^{-a(t+s)}\mathbf{E}\left(\int_0^s e^{au}\mathrm{d}W_u\right)^2 \\
&= \sigma^2 e^{-a(t+s)}\int_0^s e^{2au}\mathrm{d}u \\
&= \frac{\sigma^2}{2a}e^{-a(t+s)}\left(e^{2as}-1\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{Var}\left(\int_0^t \widetilde{X}_u\mathrm{d}u\right) &= \mathbf{Cov}\left(\int_0^t \widetilde{X}_u\mathrm{d}u, \int_0^t \widetilde{X}_u\mathrm{d}u\right) \\
&= \mathbf{E}\int_0^t (\widetilde{X}_u - \mathbf{E}\widetilde{X}_u)\mathrm{d}v \int_0^t (\widetilde{X}_v - \mathbf{E}\widetilde{X}_v)\mathrm{d}v \\
&= \int_0^t\int_0^t \mathbf{E}(\widetilde{X}_u - \mathbf{E}\widetilde{X}_u)(\widetilde{X}_v - \mathbf{E}\widetilde{X}_v)\mathrm{d}u\mathrm{d}v \\
&= \int_0^t\int_0^t \mathbf{Cov}(\widetilde{X}_u, \widetilde{X}_v)\mathrm{d}u\mathrm{d}v \\
&= 2\int_0^t\int_0^v \mathbf{Cov}(\widetilde{X}_u, \widetilde{X}_v)\mathrm{d}u\mathrm{d}v \\
&= 2\int_0^t\int_0^v \frac{\sigma^2}{2a}e^{-a(u+v)}\left(e^{2au}-1\right)\mathrm{d}u\mathrm{d}v \\
&= \frac{\sigma^2}{2a^3}\left(at-3+4e^{-at}-e^{-2at}\right).
\end{aligned}$$

Thus we have the expectation and variance of the Gaussian random variable $\int_0^t \widetilde{X}_u\mathrm{d}u$. Since $\mathbf{E}e^{t(aZ+b)} = e^{a^2t^2/2+bt}$ for $Z \sim N(0,1)$, we have

$$\mathbf{E}\exp\left\{\int_0^t \widetilde{X}_u\mathrm{d}u\right\} = \exp\left\{-\frac{x_0}{a}(1-e^{-at}) + \frac{\sigma^2}{4a^3}\left(at-3+4e^{-at}-e^{-2at}\right)\right\}.$$

Substituting back into (35) and using the initial condition (36), we obtain

$$P(t,T) = \exp\Big\{ -b(T-t) - \frac{r_t - b}{a}(1 - e^{-a(T-t)}) + \frac{\sigma^2}{4a^3}\left(a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}\right)\Big\}.$$

The fair price of a European call option with strike price K at T_1 for a zero coupon bond with expiry $T_2 > T_1$ is

$$C(K;T_1,T_2) = \mathbf{E} e^{-\int_0^{T_1} r_t \mathrm{d}t} \left(P(T_1,T_2) - K\right)_+ . \tag{37}$$ {eq:vasicek-eucall

Since $P(T_1,T_2)$ is determined by r_{T_1}, to evaluate the latter integral we need the joint distribution of $\int_0^{T_1} r_t \mathrm{d}t$ and r_{T_1}. They are jointly Gaussian, and their covariance is

$$\begin{aligned}
\mathbf{Cov}\left(\int_0^t r_u \mathrm{d}u, r_t\right) &= \int_0^t \mathbf{Cov}(r_u, r_t)\mathrm{d}u \\
&= \int_0^t \frac{\sigma^2}{2a} e^{-a(t+u)}(e^{2au} - 1)\mathrm{d}u \\
&= \frac{\sigma^2}{2a^2}(1 - 2e^{-at} + e^{-2at}).
\end{aligned}$$

Therefore, the fair price in (37) is

$$\begin{aligned}
&C(K;,T_1,T_2) \\
&= \mathbf{E} e^{-U}\Bigg(\exp\Big\{ -b(T_2-T_1) - \frac{V-b}{a}(1 - e^{-a(T_2-T_1)}) \\
&+ \frac{\sigma^2}{4a^3}\left(a(T_2-T_1) - 3 + 4e^{-a(T_2-T_1)} - e^{-2a(T_2-T_1)}\right)\Big\} - K\Bigg)_+ ,
\end{aligned}$$

where (U,V) is a two dimensional normal random vector with covariance matrix

$$\begin{pmatrix} \frac{\sigma^2}{2a^3}\left(aT_1 - 3 + 4e^{-aT_1} - e^{-2aT_1}\right) & \frac{\sigma^2}{2a^2}(1 - 2e^{-aT_1} + e^{-2aT_1}) \\ \frac{\sigma^2}{2a^2}(1 - 2e^{-aT_1} + e^{-2aT_1}) & \frac{\sigma^2}{2a}(1 - e^{-2aT_1}). \end{pmatrix}$$

The main point here is that there exists an explicit formula, which can be computed numerically easily.

10.2.3 Hull–White model

This is a simple generalization of the Vasicek model, where we allow the parameters to be time dependent. Assume that for some deterministic functions a, b, and σ

$$\mathrm{d}r_t = (a(t) - b(t)r_t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t, \qquad r_0 = r_0 > 0. \tag{38}$$ {eq:HW}

Then the solution is

$$r(t) = e^{-\beta(t)}\left(r_0 + \int_0^t e^{\beta(u)}a(u)\mathrm{d}u + \int_0^t e^{\beta(u)}\sigma(u)\mathrm{d}W_u\right).$$

where $\beta(t) = \int_0^t b(u)\mathrm{d}u$.

10.2.4 Cox–Ingersoll–Ross model

In the Vasicek and the Hull–White model the distribution of r_t is normal for any t, therefore it can take any large negative number, which is not so realistic. In the following model r_t is nonnegative.

First consider n independent Ornstein–Uhlenbeck processes, that is

$$\mathrm{d}X_i(t) = -\frac{1}{2}\alpha X_i(t)\mathrm{d}t + \frac{\sigma}{2}\mathrm{d}W_i(t), \quad i = 1, 2, \ldots, n,$$

where $W_1, \ldots, W_n$ are independent standard Brownian motions. Then

$$X_i(t) = e^{-\frac{\alpha}{2}t}\left(X_i(0) + \frac{\sigma}{2}\int_0^t e^{\frac{\alpha}{2}s}\mathrm{d}W_i(s)\right).$$

Put

$$r_t = X_1^2(t) + \ldots + X_n^2(t).$$

Using the multivariate version of Itô's formula

$$\begin{aligned}
\mathrm{d}r_t &= \sum_{i=1}^n 2X_i(t)\mathrm{d}X_i(t) + \sum_{i=1}^n \frac{\sigma^2}{4}\mathrm{d}t \\
&= -\alpha r_t \mathrm{d}t + \sigma \sum_{i=1}^n X_i(t)\mathrm{d}W_i(t) + \frac{n\sigma^2}{4}\mathrm{d}t \\
&= \left(\frac{n\sigma^2}{4} - \alpha r_t\right)\mathrm{d}t + \sigma\sqrt{r_t}\sum_{i=1}^n \frac{X_i(t)}{\sqrt{r_t}}\mathrm{d}W_i(t).
\end{aligned}$$

The process

$$W_t = \sum_{i=1}^n \int_0^t \frac{X_i(u)}{\sqrt{r_u}} \mathrm{d}W_i(u)$$

is a continuous martingale, such that

$$\begin{aligned} W_t^2 &= 2\int_0^t W_u \mathrm{d}W_u + \sum_{i=1}^n \int_0^t \frac{X_i(u)^2}{r_u} \mathrm{d}u \\ &= 2\int_0^t W_u \mathrm{d}W_u + t, \end{aligned}$$

which means that $W_t^2 - t$ is a martingale too. Therefore, by Lévy's characterization of the Wiener process we obtain that W_t is a SBM. Substituting back we have

$$\mathrm{d}r_t = \left(\frac{n\sigma^2}{4} - \alpha r_t \right) \mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t$$

with W_t SBM. This is the definition of the Cox–Ingersoll–Ross (CIR) process.

The *CIR process* with parameters $a > 0$, $b > 0$, $\sigma > 0$ is the solution of the stochastic differential equation

$$\mathrm{d}r_t = (a - br_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t. \tag{39}$$

Note that existence and uniqueness result for SDE's does not apply here, because the function $\sqrt{x}$ is not Lipschitz at 0. However, it can be shown that a unique strictly positive solution exist for $a \geq \sigma^2/2$. We have seen this for $a = n\sigma^2/4$.

We have seen that at determining the fair price of a European call we need the joint distribution of $(r_t, \int_0^t r_u \mathrm{d}u)$. The joint Laplace transform of the vector can be determined explicitly. We state the following result without proof.

Theorem 18. *For any $u \geq 0$, $v \geq 0$*

$$\mathbf{E} \exp\left\{ -ur_t - v\int_0^t r_s \mathrm{d}s \right\} = e^{-a\phi_{u,v}(t) - r_0\psi_{u,v}(t)},$$

where

$$\begin{aligned} \phi_{u,v}(t) &= -\frac{2}{\sigma^2} \log\left(\frac{2\gamma e^{t(b+\gamma)/2}}{\sigma^2 u(e^{\gamma t} - 1) + \gamma - b + e^{\gamma t}(\gamma + b)} \right) \\ \psi_{u,v}(t) &= \frac{u(\gamma + b) + e^{\gamma t}(\gamma - b) + 2v(e^{\gamma t} - 1)}{\sigma^2 u(e^{\gamma t} - 1) + \gamma - b + e^{\gamma t}(\gamma + b)}, \end{aligned}$$

where $\gamma = \sqrt{b^2 + 2\sigma^2 v}$.

Therefore, using the result above and the Markov property the value of the zero coupon bound

$$\begin{aligned} P(t,T) &= \mathbf{E}\left[e^{-\int_t^T r_u \mathrm{d}u} \middle| \mathcal{F}_t \right] \\ &= \mathbf{E}\left[e^{-\int_0^{T-t} r_u \mathrm{d}u} \middle| r_0 = r_t \right] \\ &= \exp\left\{ -a\phi_{0,1}(T-t) - r_t \psi_{0,1}(T-t) \right\} \end{aligned}$$

The price of a European call option with strike price K at T_1 for a zero coupon bound with expiry $T_2 > T_1$

$$\begin{aligned} & C(K;T_1,T_2) \\ & = \mathbf{E}\left[e^{-\int_0^{T_1} r_u \mathrm{d}u} \left(\exp\left\{ -a\phi_{0,1}(T_2 - T_1) - r_{T_1}\psi_{0,1}(T_2 - T_1) \right\} - K \right)_+ \right]. \end{aligned}$$

This is not an explicit formula, but we now the joint Laplace transform of the vector $(\int_0^{T_1} r_u \mathrm{d}u, r_{T_1})$, therefore it is numerically computable.

10.3 The Heath–Jarrow–Morton model

10.3.1 Forward rate

Assume that at time t we buy one zero coupon bond with expiry T and short sell $P(t,T)/P(t,T+\varepsilon)$ unit zero coupon bond with expiry $T+\varepsilon$. The value of this portfolio at t

$$P(t,T) - \frac{P(t,T)}{P(t,T+\varepsilon)} P(t,T+\varepsilon) = 0,$$

so it costs nothing. What happens is that at time T we borrow 1 dollar, and we have to pay $P(t,T)/P(t,T+\varepsilon)$ at time $T+\varepsilon$. Therefore the interest rate we pay at time T is $R(t,T,T+\varepsilon)$

$$\frac{P(t,T)}{P(t,T+\varepsilon)} = e^{\varepsilon R(t,T,T+\varepsilon)},$$

that is

$$R(t,T,T+\varepsilon) = -\frac{1}{\varepsilon} \left(\log P(t,T+\varepsilon) - \log P(t,T) \right).$$

Thus the instantaneous forward interest rate at time T calculated at time t, called *forward rate* is

$$f(t,T) = \lim_{\varepsilon \downarrow 0} R(t,T,T+\varepsilon) = -\frac{\partial}{\partial T} \log P(t,T). \tag{40}$$

Intuitively, it is clear that at time t we predict the interest at time t to equal the short rate r_t, that is $r_t = f(t,t)$. In what follows we prove this statement.

{lemma:forward-sho

Lemma 10. *For any* $t \in [0, \mathcal{T}]$

$$f(t,t) = r_t.$$

Proof. As $\log P(t,t) = 0$

$$\log P(t,T) = \int_t^T \frac{\partial}{\partial T} \log P(t,u) \mathrm{d}u = -\int_t^T f(t,u) \mathrm{d}u,$$

we obtain

$$P(t,T) = e^{-\int_t^T f(t,u)\mathrm{d}u}. \tag{41}$$ {eq:P-f}

Differentiating

$$\frac{\partial}{\partial T} P(t,T) = -f(t,T)P(t,T),$$

which at $t = T$

$$\frac{\partial}{\partial T} P(t,T)\bigg|_{T=t} = -f(t,t),$$

On the other hand, differentiating

$$P(t,T) = \mathbf{E}\left[e^{-\int_t^T r_u \mathrm{d}u} \middle| \mathcal{F}_t \right]$$

we obtain

$$\frac{\partial}{\partial T} P(t,T) = \mathbf{E}\left[-r_T e^{-\int_t^T r_u \mathrm{d}u} \middle| \mathcal{F}_t \right]$$

which at $T = t$

$$\frac{\partial}{\partial T} P(t,T)\bigg|_{T=t} = -r_t,$$

and the statement follows. □

10.3.2 The Heath–Jarrow–Morton model

The *Heath–Jarrow–Morton (HJM) model* describes the dynamic of the forward rate $f(t,T)$ with the SDE

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t, \tag{42}$$ {eq:HJM}

which holds for every $0 \le t \le T \le \mathcal{T}$, where α and σ are adapted processes.

Note that the model has two time scales. In the followings we determine the necessary conditions on α and σ. We have

$$\begin{aligned}
\mathrm{d}\left(-\int_t^T f(t,u)\mathrm{d}u\right) &= f(t,t)\mathrm{d}t - \int_t^T (\mathrm{d}f(t,u))\mathrm{d}u \\
&= r_t\mathrm{d}t - \int_t^T (\alpha(t,u)\mathrm{d}t + \sigma(t,u)\mathrm{d}W_t)\mathrm{d}u \\
&= r_t\mathrm{d}t - \alpha^*(t,T)\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t,
\end{aligned}$$

where

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)\mathrm{d}u, \quad \sigma^*(t,T) = \int_t^T \sigma(t,u)\mathrm{d}u.$$

Here we use a stochastic version of Fubini's theorem, which we did not even formulate. Put

$$X_t = \log P(t,T) = -\int_t^T f(t,u)\mathrm{d}u.$$

Then the above calculation gives

$$\mathrm{d}X_t = (r_t - \alpha^*(t,T))\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t.$$

Thus

$$\begin{aligned}
\mathrm{d}P(t,T) &= e^{X_t}\left(r_t - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right)\mathrm{d}t - e^{X_t}\sigma^*(t,T)\mathrm{d}W_t \\
&= P(t,T)\left[\left(r_t - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right)\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t\right].
\end{aligned}$$

Under the equivalent martingale measure the discounted value process of a zero coupon bond

$$e^{-\int_0^t r_u \mathrm{d}u} P(t,T)$$

is a martingale. Since

$$\begin{aligned}
\mathrm{d}\left(e^{-\int_0^t r_u \mathrm{d}u} P(t,T)\right) &= e^{-\int_0^t r_u \mathrm{d}u} \mathrm{d}P(t,T) - r_t e^{-\int_0^t r_u \mathrm{d}u} P(t,T)\mathrm{d}t \\
&= e^{-\int_0^t r_u \mathrm{d}u} P(t,T) \left[\left(-\alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2 \right) \mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t \right],
\end{aligned}$$

which is martingale if and only if for any $0 \leq t \leq T \leq \mathcal{T}$

$$\alpha^*(t,T) = \frac{1}{2}\sigma^*(t,T)^2.$$

Substituting back the definition of α^* and σ^*, after differentiation we obtain that

$$\alpha(t,T) = \sigma(t,T) \int_0^T \sigma(t,u)\mathrm{d}u. \tag{43}$$

{eq:HJM-alpha-sigm

We proved the following.

Theorem 19. *If the HJM model is determined by the SDE* (42) *then necessarily* (43) *holds.*

References

[1] R. J. Elliott and P. E. Kopp. *Mathematics of financial markets.* Springer Finance. Springer-Verlag, New York, second edition, 2005.

[2] J. Gáll and G. Pap. *Bevezetés a pénzügyi matematikába.* Polygon, 2010.

[3] A. N. Shiryaev. *Essentials of stochastic finance*, volume 3 of *Advanced Series on Statistical Science & Applied Probability.* World Scientific Publishing Co., Inc., River Edge, NJ, 1999. Facts, models, theory, Translated from the Russian manuscript by N. Kruzhilin.