where we use Lemma 4. The lower bound is a divergent series in n, therefore the event A_n occur infinitely often. On the other hand by (7) (for $-W_t$)

$$-W_{\theta^{n+1}} \leq 2h(\theta^{n+1}) \leq 4\theta^{1/2}h(\theta^n)$$

for all $n \geq N(\omega)$. Therefore whenever A_n occur

$$\frac{W_{\theta^n}(\omega)}{h(\theta^n)} \geq \sqrt{1-\theta} - 4\sqrt{\theta}.$$

Letting $n \to \infty$ we have

$$\limsup_{t \downarrow 0} \frac{W_t}{h(t)} \geq \sqrt{1-\theta} - 4\sqrt{\theta},$$

and the result follows by letting $\theta \downarrow 0$. □

Exercise 25. Show that if W is SBM then for any λ

$$X_t = \exp\left\{\lambda W_t - \frac{\lambda^2}{2}t\right\}$$

is a martingale.

4 Stochastic integral

Here we define the integration with respect to the Brownian motion. Note that SBM is not of bounded variation, therefore we cannot define the integral pathwise. This is the major difficulty in the theory.

4.1 Integration of simple processes

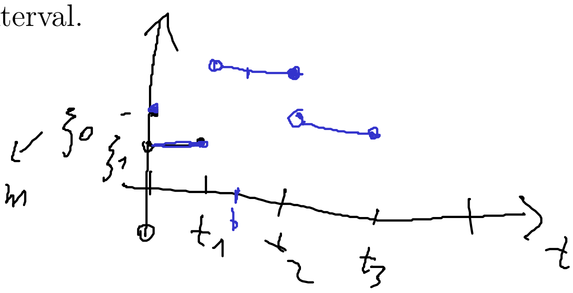
In what follows we work on $[0,T]$, for $T < \infty$. Let $(W_t, \mathcal{F}_t)$ be SBM.

The process (X_t) is a *simple process*, if

$$X_t(\omega) = \xi_0(\omega)\mathbf{I}_{\{0\}}(t) + \sum_{i=1}^{n-1} \xi_i(\omega)\mathbf{I}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ is a partition of $[0,T]$, and ξ_i is $\mathcal{F}_{t_i}$-measurable.

That is $(X_t(\omega))$ is a step function for each $\omega \in \Omega$, where the step sizes are random. Note that ξ_i is measurable with respect to the σ-algebra corresponding to the left end point of the interval.



(W_t) $\int dW_t$. ← random

A_t increasing / non-decreasing random

$\int_0^T Y_s \, dA_s$ ← pointwise

F nondecreasing function $F: \mathbb{R} \to \mathbb{R}$

↓ induced measure

$\mu_F((a,b]) = F(b) - F(a)$

↳ extension (Carathéodory)

μ_F: measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\int f \, d\mu_F$ can be defined.

"dF" (prob. theory: integral with respect to distribution function)

–

G is of bounded var. $\Rightarrow$ $G = G^+ - G^-$

G^+ and G^- are nondec.

$\int dG = \int dG^+ - \int dG^-$.

←

G bounded var. $\Rightarrow$ we can define $\int \cdot \, dG$

If it is random, we can define the integral pathwise (for each ω).

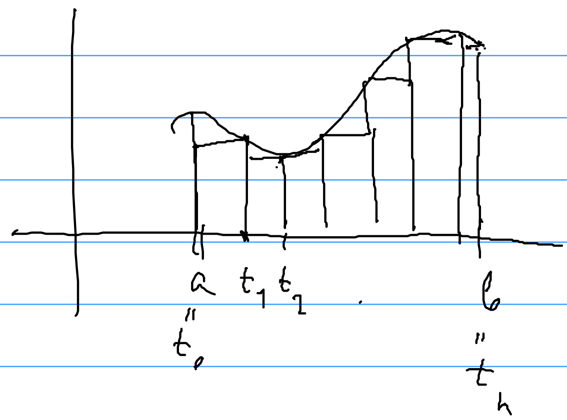
(W_t) SBM is not of bounded variation.

Need something new.

- first define for simple functions

<u>Riemann integral</u>

take the limit

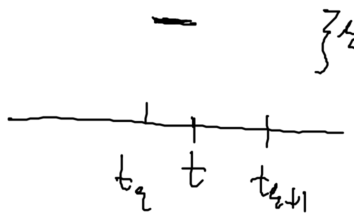


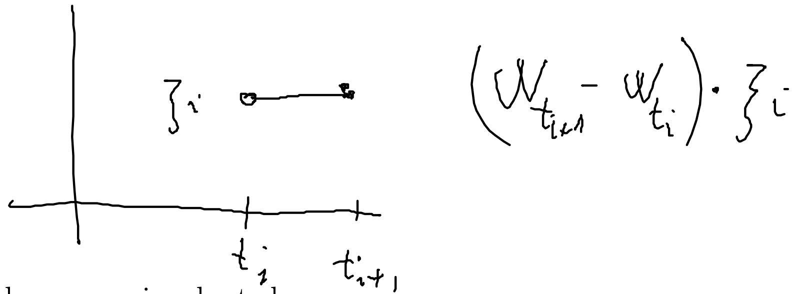
<u>Lebesgue</u>: simple function

$$\int \sum_i \alpha_i \cdot I_{A_i} \, d\mu \quad (A_i)_i \text{ are disjoint}$$

$$= \sum_i \alpha_i \, \mu(A_i))$$

+ limiting





Exercise 26. Show that a simple process is adapted.

The definition of the integral of simple processes is straightforward. Let k be such that $t \in (t_k, t_{k+1}]$. Then

$$I_t(X) = \int_0^t X_s \mathrm{d}W_s = \sum_{i=0}^{k-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_k (W_t - W_{t_k}), \quad t \in [0, T].$$

Note that we defined the process for each $t \in [0, T]$.

{thm:stint-prop}

Theorem 25. *Let X, Y be simple processes with square integrable coefficients.*

(i) $I_t(X)$ is a continuous martingale, $I_0(X) = 0$ a.s.

(ii) For $t > s$

$$\mathbf{E}\left[\left(\int_s^t X_u \mathrm{d}W_u \right)^2 \middle| \mathcal{F}_s \right] = \mathbf{E}\left[\int_s^t X_u^2 \mathrm{d}u \middle| \mathcal{F}_s \right];$$

in particular $\mathbf{E} I_t(X)^2 = \mathbf{E} \int_0^t X_u^2 \mathrm{d}u$.

(iii) The integral is linear, that is

$$I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y), \quad \alpha, \beta \in \mathbb{R}.$$

(iv) $\mathbf{E} \sup_{0 \le t \le T} \left(\int_0^t X_u \mathrm{d}W_u \right)^2 \le 4 \mathbf{E} \int_0^T X_u^2 \mathrm{d}u$.

Proof. (iii) is clear. (iv) follows from Doob's maximal inequality.

(i) The continuity is obvious and $I_0(X) = 0$. We prove that (I_t) is martingale. Let $s < t$ and $s \in (t_k, t_{k+1}], t \in (t_m, t_{m+1}]$. Then

$$\int_0^t X_u \mathrm{d}W_u = \sum_{i=0}^{k-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_k (W_s - W_{t_k})$$
$$+ \xi_k (W_{t_{k+1}} - W_s) + \sum_{i=k+1}^{m-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_m (W_t - W_{t_m}).$$

By the tower rule

$$\begin{aligned} \mathbf{E}[\xi_i (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_s] &= \mathbf{E}\left[\mathbf{E}[\xi_i (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}] | \mathcal{F}_s \right] \\ &= \mathbf{E}\left[\xi_i \mathbf{E}[W_{t_{i+1}} - W_{t_i} | \mathcal{F}_{t_i}] | \mathcal{F}_s \right] \\ &= \mathbf{E}[\xi_i \cdot 0 | \mathcal{F}_s] = 0. \end{aligned}$$

$$I_t(X) = \sum_{i=0}^{m-1} \xi_i \cdot (W_{t_{i+1}} - W_{t_i}) + \xi_m \cdot (W_t - W_{t_m})$$

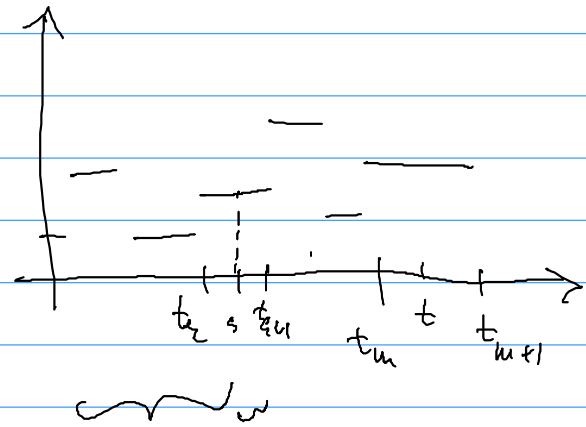
where $t_m < t \leq t_{m+1}$

I_t mtg. $(I_t, \mathcal{F}_t)$ mtg. | and cont.

Need to prove:

$s < t$. $E[I_t | \mathcal{F}_s] = I_s$ a.s.

$t_l < s \leq t_{l+1}$



$$I_t = I_s + \xi_l \cdot (W_{t_{l+1}} - W_s) + \sum_{i=l+1}^{m-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_m \cdot (W_t - W_{t_m})$$

$\mathcal{F}_s$-mean.

$$E[I_t | \mathcal{F}_s] = E\Big[I_s + \xi_l (W_{t_{l+1}} - W_s) + \sum_{i=l+1}^{m-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_m (W_t - W_{t_m}) \,\Big|\, \mathcal{F}_s\Big]$$

$$= I_s + 0$$

tower rule

$$E[\xi_i (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_s] = \qquad t_i > s$$

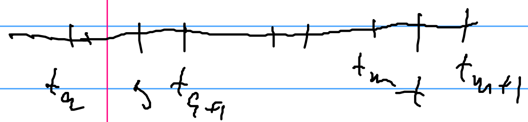
$$= E\big[E[\xi_i (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}] \,|\, \mathcal{F}_s\big]$$

$$= E[\,0 \mid \mathcal{F}_s] = 0.$$

ii) $$E\left[\left(\int_s^t X_u\, dW_u\right)^2 \Big| \mathcal{F}_s\right] =$$

$t_q < s \le t_{q+1},\ t_m < t \le t_{m+1}$

$$= E\left[\left(\xi_q (W_{t_{q+1}} - W_s) + \sum_{i=q+1}^{m-1} (W_{t_{i+1}} - W_{t_i})\,\xi_i + \xi_m (W_t - W_{t_m})\right)^2 \Big| \mathcal{F}_s\right]$$



$i \neq j$

mixed terms: $\xi_i \xi_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})$

squares: $\xi_i^2 (W_{t_{i+1}} - W_{t_i})^2$

$i<j$: $$E[\xi_i \xi_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_s] =$$

$$= E\left[E[\xi_i \xi_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_j}] \mid \mathcal{F}_s\right] = 0.$$

($\mathcal{F}_{t_j}$-meas.)

squares: $E\left[\xi_i^2 (W_{t_{i+1}} - W_{t_i})^2 \mid \mathcal{F}_s\right]$

$$= E\left[E\left[\xi_i^2 (W_{t_{i+1}} - W_{t_i})^2 \mid \mathcal{F}_{t_i}\right] \mid \mathcal{F}_s\right]$$

indep. of $\mathcal{F}_{t_i}$

$$= E\left[\xi_i^2 (t_{i+1} - t_i) \mid \mathcal{F}_s\right]$$

$$= E\left[\int_{t_i}^{t_{i+1}} X_u^2 \, du \mid \mathcal{F}_s\right]$$

summing ✓

$$E\left[\left(\int_s^t X_u \, dW_u\right)^2 \mid \mathcal{F}_s\right] = E\left[\int_s^t X_u^2 \, du \mid \mathcal{F}_s\right]$$ ✓

The first and last term can be handled similarly.

(ii) We showed that

$$\int_s^t X_u \mathrm{d}W_u = \xi_k (W_{t_{k+1}} - W_s) + \sum_{i=k+1}^{m-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_m (W_t - W_{t_m}).$$

Taking square and conditional expectaion we end up with sum of terms

$$\mathbf{E}[\xi_i (W_{t_{i+1}} - W_{t_i}) \xi_j (W_{t_{j+1}} - W_{t_j}) | \mathcal{F}_s]$$

We show that this equals 0, whenever $i \neq j$. Indeed,

$$\begin{aligned} &\mathbf{E}[\xi_i (W_{t_{i+1}} - W_{t_i}) \xi_j (W_{t_{j+1}} - W_{t_j}) | \mathcal{F}_s] \\ &= \mathbf{E}\left[\mathbf{E}[\xi_i (W_{t_{i+1}} - W_{t_i}) \xi_j (W_{t_{j+1}} - W_{t_j}) | \mathcal{F}_{t_j}] | \mathcal{F}_s \right] = 0. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbf{E}\left[\left(\int_s^t X_u \mathrm{d}W_u \right)^2 | \mathcal{F}_s \right] \\ &= \mathbf{E}\left[\xi_k^2 (W_{t_{k+1}} - W_s)^2 + \sum_{i=k+1}^{m-1} \xi_i^2 (W_{t_{i+1}} - W_{t_i})^2 + \xi_m^2 (W_t - W_{t_m})^2 | \mathcal{F}_s \right]. \end{aligned}$$

By the tower rule again

$$\begin{aligned} \mathbf{E}[\xi_i^2 (W_{t_{i+1}} - W_{t_i})^2 | \mathcal{F}_s] &= \mathbf{E}\left[\mathbf{E}[\xi_i^2 (W_{t_{i+1}} - W_{t_i})^2 | \mathcal{F}_{t_i}] \mathcal{F}_s \right] \\ &= \mathbf{E}[\xi_i^2 (t_{i+1} - t_i) | \mathcal{F}_s] \\ &= \mathbf{E}\left[\int_{t_i}^{t_{i+1}} X_u^2 \mathrm{d}u | \mathcal{F}_s \right]. \end{aligned}$$

Summing we obtain the result. □

4.2 Extending the definition

The idea is the following. We defined the integral for simple processes. Adapted processes can be approximated by simple processes, so we can define the integral of adapted process as a limit and hope for the best. This was the method at the definition of both Riemann and Lebesgue integral.

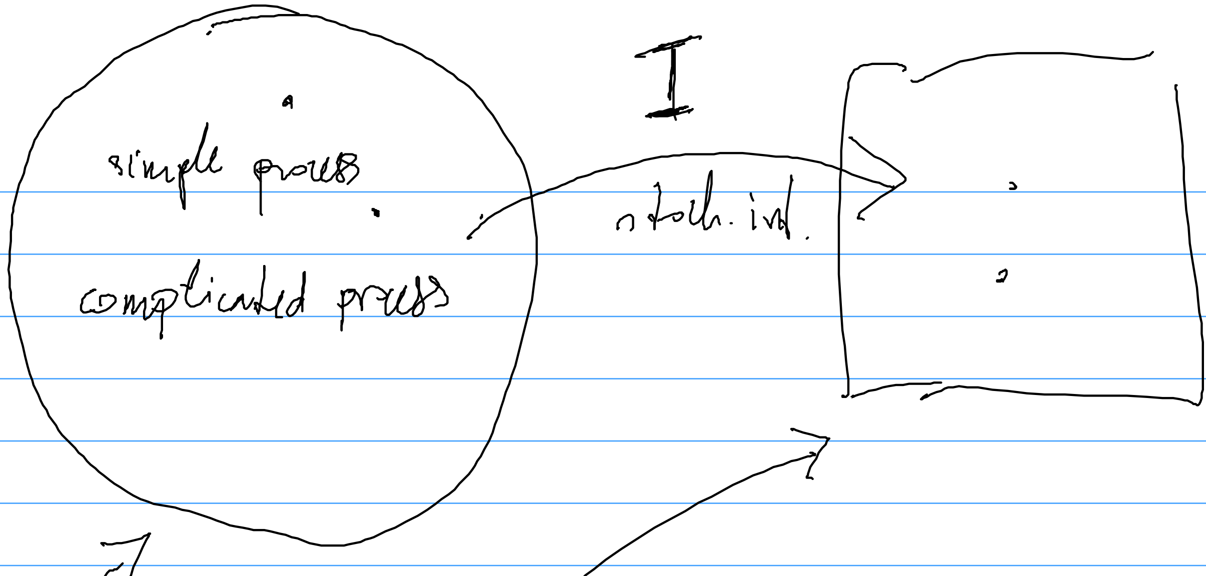
Let

$$\mathcal{H} = \left\{ (X_t) \, : \mathcal{F}_t\text{-adapted and } \mathbf{E} \int_0^T X_u^2 \mathrm{d}u < \infty \right\}.$$

We extend the definition to the class $\mathcal{H}$.

random

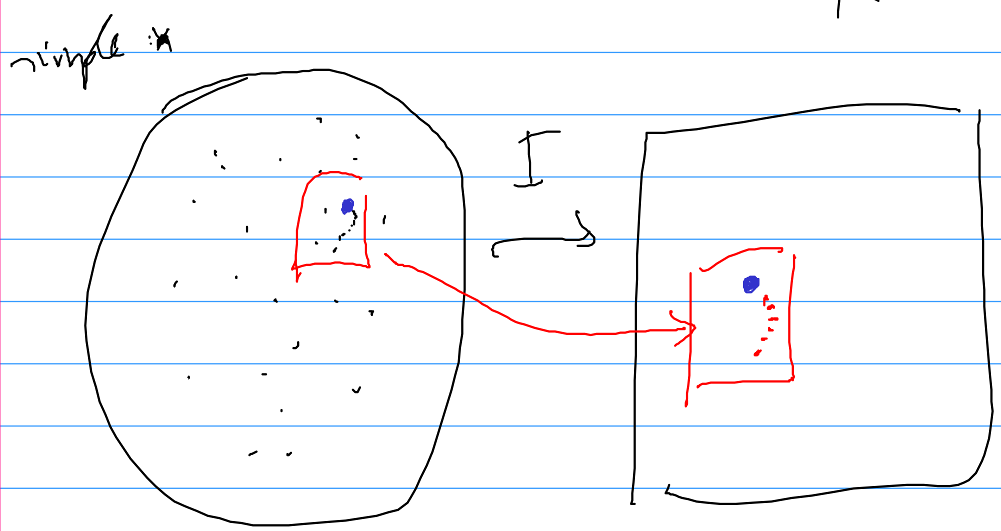
Idea



$$E\left(I_t(X)^2\right) = E\left(\int_0^t X_u^2\,du\right)$$

metric L_2-norm

operator I (stoch integral) preserves the norm | isometry |



Lemma 5. *Let $(X_t) \in \mathcal{H}$. There exists a sequence of simple processes $\{(X_t^n)\}_n$ such that*

$$\lim_{n\to\infty} \mathbf{E} \int_0^T (X_s - X_s^n)^2 \, \mathrm{d}s = 0.$$

Proof. We only prove in the special case when X is bounded and continuous. Let

$$X_t^n(\omega) = X_0(\omega)\mathbf{I}_{\{0\}}(t) + \sum_{k=0}^{2^n-1} X_{\frac{kT}{2^n}}(\omega)\mathbf{I}_{(\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]}(t).$$

These are simple processes. Since contiuous function is uniformly continous on compacts, almost surely

$$\int_0^T |X_u^n - X_u|^2 \mathrm{d}t \to 0.$$

Lebesgue's dominated convergence gives the proof. □

Let $X \in \mathcal{H}$ and $\{X^n\}_n$ given in the lemma. By Theorem 25 (iv)

$$\mathbf{E} \sup_{t\in[0,T]} \left(\int_0^t (X_u^n - X_u^m) \mathrm{d}W_u \right)^2 \leq 4\mathbf{E} \int_0^T (X_u^n - X_u^m)^2 \mathrm{d}u$$

The right-hand side tends to 0 by the lemma above, therefore the left-hand side too. Thus there exists a sequence $\{n_k\}$ such that

$$\mathbf{E} \sup_{t\in[0,T]} \left(\int_0^t (X_u^{n_{k+1}} - X_u^{n_k}) \mathrm{d}W_u \right)^2 \leq 2^{-k}. \qquad (8)$$ {eq:unif-conv-ineq

The first Borel–Cantelli lemma implies

$$I(X^{n_k}) \to I(X), \text{ uniformly on } [0,T]\text{-n a.s.}$$

As $I(X^{n_k})$ is continuous, so is $I(X)$. We have to show that $I(X)$ does not depend on the subsequence. In (8) letting $m \to \infty$

$$\mathbf{E} \sup_{t\in[0,T]} (I_t(X) - I_t(X^n))^2 \leq 4\mathbf{E} \int_0^T (X_u - X_u^n)^2 \mathrm{d}u,$$

so $I(X)$ does not depend on the subsequence.

$$E\left[\sup_{t\leq T}\left(\int_0^t \left(X_u^{n_{k+1}} - X_u^{n_k}\right) dW_u\right)^2\right] \leq 2^{-k}$$

$$I_t(X^{n_k}) = \int_0^t X_u^{n_k}\, dW_u$$

$$P\left(\sup_{t\leq T}\left|I_t(X^{n_{k+1}}) - I_t(X^{n_k})\right| > \varepsilon\right)$$

$$\overset{\text{chebyshev}}{\leq} \frac{E\left[\left(\cdots\right)^2\right]}{\varepsilon^2} \leq 2^{-k}\cdot\frac{1}{\varepsilon^2} = \frac{k^4}{2^k}$$

$\varepsilon_k = k^{-2} = \frac{1}{k^2}$ — summable in k

Borel-Cantelli

$$\Rightarrow \sup\left|I_t(X^{n_{k+1}}) - I_t(X^{n_k})\right| \leq k^{-2}$$

for k large enough. a.s.

$$\Rightarrow \sum\left[I_t(X^{n_{k+1}}) - I_t(X^{n_k})\right] = \lim_{k\to\infty} I_t(X^{n_k})$$

$\Rightarrow$ the limit exists and continuous. $I(X)$

Next we show that $I(X)$ is martingale, i.e. for any $s < t$

$$\mathbf{E}[I_t(X)|\mathcal{F}_s] = I_s(X).$$

For any n

$$\|\mathbf{E}[I_t(X)|\mathcal{F}_s] - I_s(X)\|_{L^2} \leq \|\mathbf{E}[I_t(X) - I_t(X^n)|\mathcal{F}_s]\|_{L^2} + \|\mathbf{E}[I_t(X^n) - I_s(X^n)|\mathcal{F}_s]\|_{L^2} + \|I_s(X^n) - I_s(X)\|_{L^2},$$

where $\|X\|_{L^2} = \sqrt{\mathbf{E}X^2}$. The second term on the RHS equals 0, since $I(X^n)$ is martingale, while the first and third term can be arbitrarily small. So $I(X)$ is indeed a martingale.

Summarizing, for $X \in \mathcal{H}$ we defined the stochastic integral

$$I_t(X) = \int_0^t X_u \mathrm{d}W_u$$

and showed that it satisfies the properties of Theorem 25.

We note that the definition of the integral can be further extended from $\mathcal{H}$ to the larger class

$$\mathcal{H}' = \{(X_t) : \mathcal{F}_t\text{-adapted and } \int_0^T X_u^2 \mathrm{d}u < \infty \text{ a.s.}\}$$

such that Theorem 25 remains true.

{example:W-appr}

Example 9 (Approximation of $\int_0^t W_s \mathrm{d}W_s$)**.** Fix $\varepsilon \in [0,1]$ and consider

$$S_\varepsilon(\Pi) = \sum_{i=0}^{n-1} \left(\varepsilon W_{t_{i+1}} + (1-\varepsilon) W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right).$$

We prove that

$$\lim_{\|\Pi\|\to 0} S_\varepsilon(\Pi) \stackrel{L^2}{=} \frac{1}{2} W_t^2 + \left(\varepsilon - \frac{1}{2}\right) t. \qquad (9)$$

{eq:W-int}

We know that $(W_t^2 - t)$ is martingale, thus the limit above is martingale iff $\varepsilon = 0$, which corresponds to the definition of Itô stochastic integral. There are other stochastic integrals: $\varepsilon = 1/2$ corresponds to the *Fisk–Stratonovich integral*, and $\varepsilon = 1$ corresponds to the *backward Itô integral*.

By (9)

$$\int_0^t W_s \mathrm{d}W_s = \frac{W_t^2 - t}{2}.$$

Next we prove (9). Since

$$\varepsilon W_{t_{i+1}} + (1-\varepsilon)W_{t_i} = \frac{W_{t_{i+1}} + W_{t_i}}{2} + \left(\varepsilon - \frac{1}{2}\right)\left(W_{t_{i+1}} - W_{t_i}\right),$$

we have to determine the limits

$$\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2, \quad \sum_{i=0}^{n-1} (W_{t_{i+1}}^2 - W_{t_i}^2).$$

The first is exactly the quadratic variation of SBM, therefore converges to t in L^2, while the second is a telescopic sum, giving W_t^2.

{example:exp}

Example 10. Let X be simple process and W SBM. Let

$$\zeta_t^s(X) = \int_s^t X_u \mathrm{d}W_u - \frac{1}{2}\int_s^t X_u^2 \mathrm{d}u, \quad \zeta_t = \zeta_t^0.$$

We show that $(Y_t = e^{\zeta_t})$ is martingale.

Since X is simple, we have

$$X_t = \xi_0 \mathbf{I}_{\{0\}}(t) + \sum_{i=0}^{n-1} \xi_i \mathbf{I}_{(t_i, t_{i+1}]}(t),$$

where ξ_i is $\mathcal{F}_{t_i}$-measurable. Thus if $s \in (t_k, t_{k+1}]$, $t \in (t_m, t_{m+1}]$, then

$$\zeta_t^s = \xi_k (W_{t_{k+1}} - W_s) - \frac{\xi_k^2}{2}(t_{k+1} - s) + \sum_{i=k+1}^{m-1}\left[\xi_i (W_{t_{i+1}} - W_{t_i}) - \frac{\xi_i^2}{2}(t_{i+1} - t_i)\right] + \xi_m (W_t - W_{t_m}) - \frac{\xi_m^2}{2}(t - t_m). \tag{10}$$

{eq:zeta-felbontas

Since ζ_s is $\mathcal{F}_s$-measurable we obtain

$$\mathbf{E}[e^{\zeta_t} | \mathcal{F}_s] = e^{\zeta_s} \mathbf{E}[e^{\zeta_t^s} | \mathcal{F}_s].$$

We only have to show that

$$\mathbf{E}[e^{\zeta_t^s} | \mathcal{F}_s] = 1.$$