Exercise 39. Show that $(X_t, Y_t) = (\cos W_t, \sin W_t)$ is a solution to the SDE

$$\begin{cases} \mathrm{d}X_t = -\frac{1}{2}X_t \mathrm{d}t - Y_t \mathrm{d}W_t \\ \mathrm{d}Y_t = -\frac{1}{2}Y_t \mathrm{d}t + X_t \mathrm{d}W_t. \end{cases}$$

Show that $\sqrt{X_t^2 + Y_t^2}$ is a constant for any solution (X, Y)!

Exercise 40. Solve the SDE

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}t + \mathrm{d}W_t^{(1)} \\ \mathrm{d}Y_t = X_t \mathrm{d}W_t^{(2)}, \end{cases}$$

where $W^{(1)}$ and $W^{(2)}$ are independent SBMs.

Exercise 41. Solve the SDE

$$\begin{cases} \mathrm{d}X_t = Y_t \mathrm{d}t + \mathrm{d}W_t^{(1)} \\ \mathrm{d}Y_t = X_t \mathrm{d}t + \mathrm{d}W_t^{(2)}, \end{cases}$$

where $W^{(1)}$ and $W^{(2)}$ are independent SBMs.

6 General Markov processes

This part is from Breiman [1].

6.1 Transition probabilities and Chapman–Kolmogorov equations

The process (X_t) is a *Markov process*, if for each Borel set $B \in \mathcal{B}(\mathbb{R})$, and $t, \tau \mathbb{R}$

$\in$

$$\mathbf{P}(X_{t+\tau} \in B | \overbrace{X_s, s \le t}^{\mathcal{F}_t}) = \mathbf{P}(X_{t+\tau} \in B | \overbrace{X_t}^{X_t}).$$

Choosing natural filtration $\mathcal{F}_t = \sigma(X_s, s \le t)$, the definition is the same as in Subsection 3.4

Since regular conditional distributions exist, we may choose the probabilities

$$p_{t_2,t_1}(B|x) = \mathbf{P}(X_{t_2} \in B | X_{t_1} = x), \quad t_2 > t_1, B \in \mathcal{B},$$

such that

random

$P(X_{t_2} \in B | X_{t_1})(\omega)$

$$(*)\quad \mathbf{E}\left[h(X_{t_2}) \mid X_{t_1}\right] = \int h(x) \underbrace{P(X_{t_2} \in dx \mid X_{t_1})}_{\text{conditional distribution}}$$

- for x fixed, $p_{t_2,t_1}(\cdot|x)$ is a probability measure;
- for $B \in \mathcal{B}$ fixed, $p_{t_2,t_1}(B|\cdot)$ is measurable.

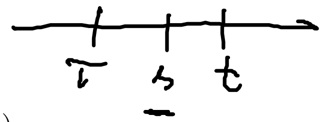
These probabilities are the *transition probabilities* of the Markov process (X_t).

Let $\tau < s < t$, $B \in \mathcal{B}$. By the tower rule, the Markov property, and the properties of regular conditional distribution

$$
\begin{aligned}
\mathbf{P}(X_t \in B|X_\tau) &= \mathbf{E}\left[\mathbf{P}(X_t \in B|\cancel{X_\tau}, X_s)|X_\tau\right] \leftarrow \text{tower rule} \\
&= \mathbf{E}\left[\mathbf{P}(X_t \in B|X_s)|X_\tau\right] \leftarrow \text{Markov property!} \\
&= \mathbf{E}\left[h(X_s)|X_\tau\right] \\
&= \int h(y)\mathbf{P}(X_s \in \mathrm{d}y|X_\tau) \leftarrow (*) \\
&= \int \mathbf{P}(X_t \in B|X_s = y)\mathbf{P}\left(X_s \in \mathrm{d}y|X_\tau\right) \\
&= \int_{\mathbb{R}} p_{t,s}(B|y)p_{s,\tau}(\mathrm{d}y|X_\tau). \quad (a.s.)
\end{aligned}
$$

$\mathbf{P}(X_t \in B|X_\tau) = E\left[I(X_t \in B)|X_\tau\right]$

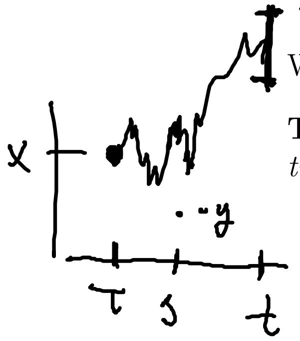
$E\left[I(X_t \in B)|X_\tau\right] = E\left[E\left[I(X_t \in B)|\underbrace{X_\tau, X_s}_{\sigma(X_\tau, X_s)}\right]\middle|X_\tau\right]$



That is

$$\boxed{p_{t,\tau}(B|x) = \int p_{t,s}(B|y)p_{s,\tau}(\mathrm{d}y|x).}\qquad x\text{-a.s.} \quad P(X_\tau \in dx)$$

We proved the following.



Theorem 33 (Chapman–Kolmogorov equations)**.** *The transition probabilities of a Markov process satisfies the equations*

$$p_{t,\tau}(B|x) = \int p_{t,s}(B|y)p_{s,\tau}(\mathrm{d}y|x), \quad \tau < s < t, B \in \mathcal{B}. \tag{23}$$

The expression $p_{t,\tau}(B|x)$ is the probability that starting from x in time τ we end up in B at time t. Consider any s between τ and t. The distribution of X_s given $X_\tau = x$ is $p_{s,\tau}(\cdot|x)$, that is the probability being in y is $p_{s,\tau}(\mathrm{d}y|x)$. Therefore, the Chapman–Kolmogorov equation is the law of total probability plus Markov property.

We are cheating again a bit. What we proved is that (23) holds for fixed $\tau < s < t$ *almost surely* with respect to the probability $\mathbf{P}(X_\tau \in \cdot)$. Indeed, in the proof we calculated conditional probabilities, where each equality is only an almost sure equality. In what follows *we assume that* (23) *holds for every* x.

time homogeneous

The Markov process (X_t) is *stationary* if the transition probabilities depend only on the time increment, i.e. $p_{t,\tau}(B|x) = p_{t-\tau}(B|x)$. Then $p_t(B|x) = p_{t,0}(B|x)$, and the Chapman–Kolmogorov equations simplify to

$$p_{t+s}(B|x) = \int p_t(B|y) p_s(\mathrm{d}y|x). \tag{24}$$

Assume that (X_t) is stochastically continuous at 0, that is

$$X_t \xrightarrow{\mathbf{P}} X_0, \quad t \to 0.$$

If (X_t) starts at x then its distribution is denoted by $\mathbf{P}_x$, and the corresponding expectation is $\mathbf{E}_x$, that is

$$\mathbf{P}_x(X_t \in B) = \mathbf{P}(X_t \in B | X_0 = x), \quad \mathbf{E}_x f(X_t) = \mathbf{E}\left[f(X_t)|X_0 = x\right].$$

Example 17 (Poisson process)**.** Let N_t be a standard Poisson process. Then $N_t - N_s \sim \text{Poisson}(t-s)$, so

$$\mathbf{P}_x(N_t = x + k) = p_t(\{x+k\}|x) = \frac{t^k}{k!} e^{-t},$$

or, what is the same

$$p_t(B|x) = \sum_{k : x+k \in B} \frac{t^k}{k!} e^{-t}.$$

The Chapman–Kolmogorov equation (24) become

$$p_{t+s}(\{k\}|0) = \sum_{\ell=0}^{\infty} p_t(\{k\}|\ell) p_s(\{\ell\}|0),$$

which is just a reformulation of the fact that the sum of two independent Poisson random variables is Poisson, and the parameter is the sum of the parameters.

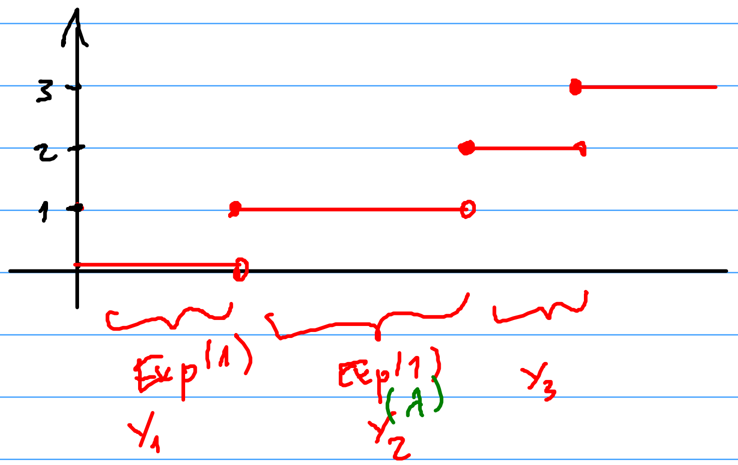
Example 18 (Wiener process)**.** Let W_t be SBM. Then

$$\begin{aligned} p_t(B|x) = \mathbf{P}_x(W_t \in B) &= \mathbf{P}_0(x + W_t \in B) = \mathbf{P}_0(W_t \in B - x) \\ &= \int_{B-x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \mathrm{d}y \\ &= \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \mathrm{d}y. \end{aligned}$$

Ex. 17. Poisson process $\qquad f$ measurable

$$E_x[f(N_t)] = E_0[f(x+N_t)] = \sum_{k=0}^{\infty} f(x+k) P_0(N_t = k) \overset{(*)}{=}$$

$(N_t) \sim$ Poisson process intensity 1 (λ)



$$\overset{(*)}{=} \sum_{k=0}^{\infty} f(x+k) \cdot \frac{t^k}{k!} e^{-t}$$

$Y_1, Y_2, \dots$ iid $Exp(1)$ (λ)

Markov prop.

$N_t \sim \text{Poisson}(t)$
(λt) $\quad \lambda = 1$

Chapman–Kolmogorov:

$$P_{t+s}(\{k\}|0) = \sum_{l=0}^{\infty} P_t(\{k\}|l)\, P_s(\{l\}|0)$$

$$= \sum_{l=0}^{\infty} P_t(\{k\}|l) \cdot \frac{s^l}{l!} \cdot e^{-s}$$

$$= \sum_{l=0}^{k} \frac{t^{k-l}}{(k-l)!} e^{-t} \frac{s^l}{l!} e^{-s}$$

$$P_t(\{k\}|l) = \begin{cases} \dfrac{t^{k-l}}{(k-l)!} e^{-t} & k \geq l \\ 0 & l > k \end{cases}$$

$$= e^{-(t+s)} \frac{1}{k!} \cdot \sum_{l=0}^{k} \binom{k}{l} \cdot t^{k-l} \cdot s^l$$

$$= e^{-(t+s)} \frac{1}{k!} (t+s)^k .$$

Wiener: (W_t) SBM

$$P_t(B|x) = P_x(W_t \in B) =$$

$$= P_0(x + W_t \in B) = P_0(W_t \in B - x)$$

$$= \int_{B-x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy$$

$$= \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

$$P_t(dy \mid x) = \underbrace{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2}}}_{\rho_t(y \mid x)} dy$$

Chapman-Kolmogorov:

$$P_{t+s}(B \mid x) = \int_{\mathbb{R}} P_s(B \mid y)\, P_t(dy \mid x)$$

$$\|$$

$$\int_B \rho_{t+s}(z \mid x)\, dz = \int_{\mathbb{R}} \int_B \rho_s(z \mid y)\, dz \cdot \rho_t(y \mid x)\, dy$$

$$\rho_{t+s}(z \mid x) = \int_{\mathbb{R}} \rho_s(z \mid y)\, \rho_t(y \mid x)\, dy$$

$$\rho_{t+s}(z - x) = \int \rho_s(z-y)\, \rho_t(y-x)\, dy$$

$x = 0$ $$\rho_{t+s}(z) = \int \rho_s(z-y)\, \rho_t(y)\, dy$$

convolution for densities

That is $p_t(B|x)$ is absolutely continuous with *transition density* $p_t(\mathrm{d}y|x) = \rho_t(y|x)\mathrm{d}y$

$$\rho_t(y|x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}.$$

The Chapman–Kolmogorov equation (24) become

$$p_{t+s}(B|x) = \int_{\mathbb{R}} p_t(B|y)\rho_s(y|x)\mathrm{d}y,$$

or for the densities

$$\rho_{t+s}(z|x) = \int_{\mathbb{R}} \rho_t(z|y)\rho_s(y|x)\mathrm{d}y.$$

This is a reformulation of the fact that the sum of independent normals is normal. Recall the convolution formula for densities.

Markov chain
$Y_0, Y_1, Y_2, \dots$
transition

6.2 Infinitesimal generator

The *infinitesimal generator* of X an operator defined by

$$f \mapsto Sf : Sf(x) = \lim_{t\to 0+} \frac{1}{t}\mathbf{E}_x\left[f(X_t) - f(x)\right], \tag{25}$$

whenever the limit exists. Its domain is denoted by $\mathcal{D}(S)$.

We determine the infinitesimal generator of the Poisson process and the Wiener process.

in cont.
Y_0, Y_t.

Example 19 (Poisson process). Let (N_t) be a Poisson process with intensity 1, and let f be a bounded measurable function. By definition $N_t - N_0 \sim \text{Poisson}(t)$, thus

$$\mathbf{E}_x f(N_t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} e^{-t} f(k+x).$$

$t \downarrow 0$

Since f is bounded the sum is finite, and as $t \downarrow 0$

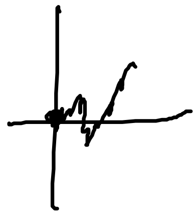
$$\mathbf{E}_x f(N_t) = f(x)e^{-t} + f(x+1)te^{-t} + O(t^2).$$

Thus

$$\begin{aligned} Sf(x) &= \lim_{t\to 0} \frac{1}{t}\mathbf{E}_x\left[f(N_t) - f(x)\right] \\ &= \lim_{t\to 0}\left(f(x)\frac{e^{-t}-1}{t} + f(x+1)e^{-t}\right) + \underbrace{\lim_{t\to 0}\frac{O(t^2)}{t}}_{=0} \\ &= f(x+1) - f(x). \end{aligned}$$

$Sf(x) = f(x+1) - f(x)$

$$\mathbf{E}_x f(W_t) = \mathbf{E}_0 f(x+W_t)$$



The limit exists for any bounded measurable function.

Example 20 (Wiener process)**.** Let (W_t) be SBM and $f \in C_c^2$ twice continuously differentiable function with compact support. Using Taylor expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + o(h^2), \qquad h \to 0$$

and since $\mathbf{E}_0 W_t = 0$, $\mathbf{E}_0 W_t^2 = t$, we have

$h = W_t$

$$\begin{aligned}
\mathbf{E}_x f(W_t) &= \mathbf{E}_0 f(x+W_t) \\
&= \mathbf{E}_0\left[f(x) + \underbrace{W_t f'(x)}_{\mathbf{E}=0} + \frac{W_t^2}{2}f''(x) + o(W_t^2)\right] \\
&= f(x) + \frac{t}{2}f''(x) + o(t).
\end{aligned}$$

local

Thus

$$Sf(x) = \lim_{t\to 0}\frac{1}{t}\mathbf{E}_x\left[f(W_t) - f(x)\right] = \frac{f''(x)}{2}.$$

We see that $C_c^2 \subset \mathcal{D}(S)$.

6.3 Kolmogorov equations

Backward. Let $t > 0$ fix, $B \in \mathcal{B}(\mathbb{R})$, $\tau > 0$ small. By the tower rule and the Markov property

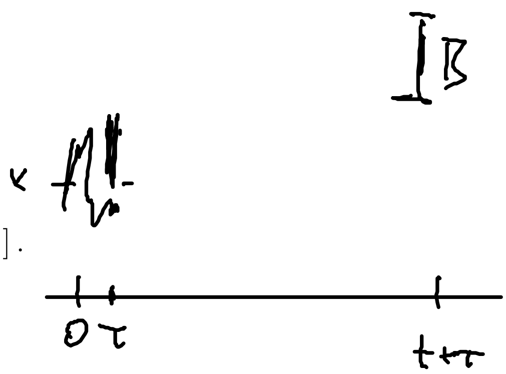
$$\varphi_{t+\tau}(x) = \mathbf{P}(X_{t+\tau} \in B | X_0 = x) = \mathbf{E}\left[\mathbf{P}(X_{t+\tau} \in B | X_\tau) | X_0 = x\right].$$

With the notation $\varphi_t(x) = p_t(B|x)$

$$\varphi_{t+\tau}(x) = \mathbf{E}_x \varphi_t(X_\tau),$$

which reads as

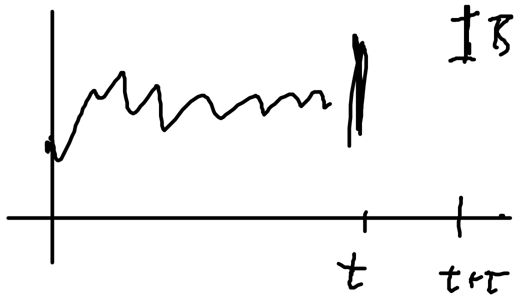
$$\frac{1}{\tau}\left[\varphi_{t+\tau}(x) - \varphi_t(x)\right] = \frac{1}{\tau}\mathbf{E}_x\left[\varphi_t(X_\tau) - \varphi_t(x)\right].$$



Letting τ tend to 0, we obtain

$\tau \downarrow 0$

$$\frac{\partial}{\partial t}\varphi_t(x) = (S\varphi_t)(x).$$



Substituting back the definition of φ, we obtain *Kolmogorov's backward equation*

$$\frac{\partial}{\partial t} p_t(B|x) = \left(S p_t(B|\cdot) \right)(x). \tag{26}$$

Forward. Let $t > 0$ fix, $f \in \mathcal{D}(S)$. By the tower rule and the Markov property

$$\mathbf{E}_x f(X_{t+\tau}) = \mathbf{E}_x \left[\mathbf{E}_x [f(X_{t+\tau}) | X_t] \right],$$

which can be rewritten as

$$\int f(y) p_{t+\tau}(\mathrm{d}y|x) = \int\!\!\int f(z) p_\tau(\mathrm{d}z|y) p_t(\mathrm{d}y|x) = \int \mathbf{E}_y f(X_\tau) p_t(\mathrm{d}y|x).$$

Subtracting

$$\mathbf{E}_x f(X_t) = \int f(y) p_t(\mathrm{d}y|x)$$

and dividing by τ

$$\int f(y) \frac{p_{t+\tau}(\mathrm{d}y|x) - p_t(\mathrm{d}y|x)}{\tau} = \int \frac{1}{\tau} \left[\mathbf{E}_y f(X_\tau) - f(y) \right] p_t(\mathrm{d}y|x).$$

Letting $\tau \downarrow 0$

$\downarrow \tau \downarrow 0$

$$\int f(y) \frac{\partial}{\partial t} p_t(\mathrm{d}y|x) = \int (Sf)(y) p_t(\mathrm{d}y|x). \tag{27}$$

The adjoint of the operator S is an operator S^* on the space of measures such that

$$\int (Sf)(y) \mu(\mathrm{d}y) = \int f(y) (S^* \mu)(\mathrm{d}y)$$

If this holds for sufficiently many f and μ, then it is unique.

$\mathcal{H}$ Hilbert

$S : \mathcal{H} \to \mathcal{H}$

$\langle Su, v \rangle = \langle u, S^* v \rangle$

$u, v \in \mathcal{H}$

Using the definition of adjoint in (27)

$$\int f(y) \frac{\partial}{\partial t} p_t(\mathrm{d}y|x) = \int f(y) \left(S^* p_t(\cdot|x) \right)(\mathrm{d}y),$$

from which we get *Kolmogorov's forward equation*

$$\frac{\partial}{\partial t} p_t(B|x) = \underbrace{\left(S^* p_t(\cdot|x) \right)}_{\text{unique}}(B). \tag{28}$$

. $\forall$ linear functional on $\mathcal{H}$

$\exists!\, v : Lu = \langle u, v \rangle$

Riesz repr.

Remark 2. The derivation of the forward equation is rather intuitive. What kind of space is the domain $\mathcal{D}(S)$, and how the adjoint operator defined? Furthermore, in (27)) we differentiated a family of measures with respect to t. If the measure are absolutely continuous, i.e.

$$p_t(\mathrm{d}y|x) = \rho_t(y|x)\mathrm{d}y,$$

then

$$\lim_{\tau \to 0} \frac{\rho_{t+\tau}(y|x) - \rho_t(y|x)}{\tau} = \frac{\partial}{\partial t}\rho_t(y|x).$$

In general, both for the backward and for the forward equations extra conditions are needed. As it can be guessed from the derivation, for the forward equation more restrictive conditions are needed.

The importance of the Kolmogorov equations (26) and (28) is that from infinitesimal conditions (from the generator S) one can determine the evolution of the whole process, that is the transition probabilities. In most of the cases the solution cannot be determined explicitly, only by simulation.

Example 21 (Poisson process)**.** Let (N_t) be a Poisson process with intensity 1. We proved that

$$(Sf)(x) = f(x+1) - f(x).$$

Therefore, the backward equation reads as

$$\frac{\partial}{\partial t}p_t(B|x) = p_t(B|x+1) - p_t(B|x). \quad = \big(S p_t(B|\cdot)\big)(x) \tag{29}$$

For the forward equation we determine the adjoint of S. We need an $S^*\mu$ such that

$$\int Sf(x)\mu(dx) \quad = \int [f(x+1) - f(x)]\mu(\mathrm{d}x) = \int f(x)(S^*\mu)(\mathrm{d}x).$$

From this form we can guess that

$$S^*\mu(A) = \mu(A-1) - \mu(A),$$

should work, where $A - 1 = \{a - 1 \,:\, a \in A\}$. This indeed holds, therefore the forward equation reads as

$$\frac{\partial}{\partial t}p_t(B|x) = p_t(B-1|x) - p_t(B|x).$$

The initial condition in both cases is

$$p_0(B|x) = \delta_x(B) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

In this special case we can solve the equation (29). Let $x = 0$ and $B = \{0\}$. Since the process have only upwards jumps $p_t(\{0\}|1) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\{0\}|0) = -p_t(\{0\}|0),$$

which together with the initial condition $p_0 = 1$ gives

$$p_t(\{0\}|0) = e^{-t}.$$

Now $B = \{1\}$ gives

$P_t(\{1\}|1) = P_t(\{0\}|0)$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\{1\}|0) = e^{-t} - p_t(\{1\}|0).$$

Multiplying by e^t

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^t p_t(\{1\}|0)\right) = 1,$$

$= e^t P_t + e^t(e^{-t} - P_t) = 1$

which with the initial condition $p_0(\{1\}|0) = 0$ gives

$$p_t(\{1\}|0) = te^{-t}.$$

$S \to (P_t)$

In general, induction gives that

$$p_t(\{k\}|0) = \frac{t^k}{k!} e^{-t}.$$

Example 22 (Wiener process)**.** Let (W_t) be SBM. Since $(Sf)(x) = f''(x)/2$, the backward equation is

$$\frac{\partial}{\partial t} p_t(B|x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(B|x).$$

$\sum \frac{\partial^2}{\partial x_i^2}$

For the density $p_t(\mathrm{d}y|x) = \rho_t(y|x)\mathrm{d}y$ we get

$$\frac{\partial}{\partial t} \rho_t(y|x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho_t(y|x).$$

$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u$

heat eq.

$$\int Sf \, d\mu = \int f \, dS^*\mu$$

This is the heat equation.

$g, f \in C_c^2$ $, 0$

For the forward equation we need again the adjoint of S. Let μ be absolutely continuous with respect to the Lebesgue measure, $\mu(\mathrm{d}y) = g(y)\mathrm{d}y$, and let $f \in C_c^2$. Integration by parts twice gives

$$\int f''(y)g(y)\mathrm{d}y = \int f(y)g''(y)\mathrm{d}y.$$

$$\int f''(x) g(x) dx = \left[f'(x) g(x) \right]_{-\infty}^{\infty} - \int f'(x) g'(x) dx = \int f(x) g''(x) dx$$

That is $(S^*\mu)(\mathrm{d}y) = \frac{1}{2}g''(y)\mathrm{d}y$. The forward equation is

$$\frac{\partial}{\partial t} p_t(y|x)\mathrm{d}y = \frac{1}{2}\frac{\partial^2}{\partial y^2} p_t(y|x)\mathrm{d}y,$$

which for the densities gives

$$\frac{\partial}{\partial t} \rho_t(y|x) = \frac{1}{2}\frac{\partial^2}{\partial y^2} \rho_t(y|x),$$

again the heat equation.

Recall that the *fundamental solution* to the heat equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x)$$

is

$$F(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},$$

$$\frac{1}{(2\pi t)^{d/2}} \cdot e^{-\frac{|x|^2}{2t}}$$

which is exactly the transition density of the SBM.

6.4 Diffusion processes

Diffusions can be handled as solution to SDEs. We showed that under general conditions unique strong solution to SDEs exists, implying the existence of diffusion processes. This is the probabilistic approach due to Lévy and Itô. Another more analytical approach to such processes was applied by Kolmogorov and Feller. They treated diffusions as general Markov processes and using tools from the theory of partial differential equations, they showed that under suitable conditions the Kolmogorov backward and forward equations have a unique solution. Then the existence of a desired Markov process follows from Kolmogorov's consistency theorem, and the continuity property