(i) $\{X_n, \mathcal{F}_n\}$ *szubmartingál, akkor* $\mathbf{E}[X_\tau|\mathcal{F}_1] \geq X_1$ *m.b., és persze* $\mathbf{E}X_\tau \geq \mathbf{E}X_1$*;*

(ii) $\{X_n, \mathcal{F}_n\}$ *martingál, akkor* $\mathbf{E}[X_\tau|\mathcal{F}_1] = X_1$ *m.b., és persze* $\mathbf{E}X_\tau = \mathbf{E}X_1$.

Fontos megjegyezni, hogy a tételben szereplő feltételek nem csupán technikai feltételek. Legyen S_n egy egyszerű szimmetrikus bolyongás az egyenesen. Ő martingál a az általa generált természetes filtrációra nézve. Tudjuk, hogy az egydimenziós bolyongás rekurrens, ezért majdnem biztosan eléri az 1-et. Legyen az elérés időpontja τ. Ekkor τ megállási idő, és persze $S_\tau \equiv 1 \neq S_0 = 0$. Csak a $\liminf_{n\to\infty} \int_{\{\tau>n\}} |X_n| \mathrm{d}\mathbf{P} = 0$ feltétellel lehet baj, és valóban, ez nem teljesül.

6.2 Optimal stopping problems

Consider a probability space with a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,\ldots,N}, \mathbf{P})$, and let

$$\mathcal{M}_n^N = \{\tau : \tau \text{ is a stopping time}, \tau \in \{n, \ldots, N\}\}.$$

To ease notation we suppress N in the upper index. Consider a sequence of nonnegative adapted random variables $(X_n)_n$, and define by backward induction its Snell-envelope $(Z_n)_n$ as follows. We are interested in the value

$$Z_N = X_N, \quad Z_n = \max\{X_n, \mathbf{E}[Z_{n+1}|\mathcal{F}_n]\}, \; n < N.$$

For a stopping time τ the stopped process is denoted by Z^τ, i.e.

$$Z_n^\tau = Z_{\tau\wedge n},$$

where $a \wedge b = \min\{a, b\}$.

Proposition 4. *Let* (Z_n) *be the Snell-envelope of* (X_n) *with* $X_n \geq 0$ *a.s.*

(i) Z *is the smallest supermartingale dominating* X*.*

(ii) The random variable $\tau^* = \min\{n : Z_n = X_n\}$ *is a stopping time and the stopped process* $Z_{n\wedge\tau^*} = Z_n^{\tau^*}$ *is martingale.*

Proof. From the definition it is clear that Z is supermatingale and dominates X. Let Y be another supermartingale dominating X. Then $Y_N \geq X_N = Z_N$. Assuming that $Y_n \geq Z_n$ we have

$$Y_{n-1} \geq \max\{\mathbf{E}[Y_n|\mathcal{F}_{n-1}], X_{n-1}\} \geq \max\{\mathbf{E}[Z_n|\mathcal{F}_{n-1}], X_{n-1}\} = Z_{n-1}.$$

Thus the minimality follows.

To see that τ^* is stopping time note that

$$\{\tau^* = n\} = \cap_{k=0}^{n-1}\{Z_k > X_k\} \cap \{Z_n = X_n\}.$$

For the last assertion note that

$$Z_n^{\tau^*} - Z_{n-1}^{\tau^*} = I(\tau^* \geq n)(Z_n - Z_{n-1}).$$

On the event $\{\tau^* \geq n\}$ we have $Z_{n-1} = \mathbf{E}[Z_n|\mathcal{F}_{n-1}]$ therefore

$$\mathbf{E}[I(\tau^* \geq n)(Z_n - Z_{n-1})|\mathcal{F}_{n-1}] = 0.$$

□

A stopping time σ is optimal if

$$\mathbf{E}X_\sigma = \sup_{\tau \in \mathcal{M}_0} \mathbf{E}X_\tau.$$

Proposition 5. *The stopping time τ^* is optimal for X, and*

$\tau^* = \min\{n: Z_n = X_n\}$.

$$Z_0 = \mathbf{E}X_{\tau^*} = \sup_{\tau \in \mathcal{M}_0} \mathbf{E}X_\tau.$$

$Z_N^{\tau^*} = Z_{\tau^* \wedge N} = Z_{\tau^*}$

Proof. Since Z^{τ^*} is martingale

$$Z_0 = Z_0^{\tau^*} = \mathbf{E}Z_N^{\tau^*} = \mathbf{E}Z_{\tau^*} = \mathbf{E}X_{\tau^*}.$$

On the other hand for any stopping time τ the process Z^τ is supermartingale (by Doob's optional sampling), thus

$$Z_0 = \mathbf{E}Z_0^\tau \geq \mathbf{E}Z_\tau \geq \mathbf{E}X_\tau.$$

□

Proposition 6. *The stopping time σ is optimal iff the following two conditions hold.*

(i) $Z_\sigma = X_\sigma$*;*

(ii) Z^σ *is martingale.*

Proof. If (i) and (ii) hold than σ is optimal. This follows exactly as the optimality of τ^*.

Conversely, assume that σ is optimal. We have seen that $\sup_\tau \mathbf{E}X_\tau = Z_0$ thus

$$Z_0 = \mathbf{E}X_\sigma \leq \mathbf{E}Z_\sigma,$$

by the dominance of Z. By Doob's otional stopping theorem Z^σ is supermartingale, therefore $\mathbf{E}Z_\sigma \leq Z_0$, implying that

$$\mathbf{E}X_\sigma = \mathbf{E}Z_\sigma.$$

Since $Z_n \geq X_n$ this implies $X_\sigma = Z_\sigma$ a.s., proving (i).

By the optimality $\mathbf{E}Z_\sigma = Z_0$, while the supermartingale property implies

$$Z_0 \geq \mathbf{E}Z_{\sigma\wedge n} \geq \mathbf{E}Z_\sigma.$$

Thus

$$EZ_{\sigma\wedge n} = \mathbf{E}Z_\sigma = \mathbf{E}\mathbf{E}[Z_\sigma|\mathcal{F}_n].$$

Furthermore, by Doob's optional stopping

$$Z_{\sigma\wedge n} \geq \mathbf{E}[Z_\sigma|\mathcal{F}_n],$$

implying $Z_{\sigma\wedge n} = \mathbf{E}[Z_\sigma|\mathcal{F}_n]$. Thus (Z_n^σ) is indeed a martingale. □

6.3 Pricing American options

Let us return to our pricing problem. Assume that we have an arbitrage-free complete market, that is the EMM $\mathbf{Q}$ is unique. Let $(f_n)_{n=0,\ldots,N}$ be the payoff of an American option. A hedging strategy now has to fulfil the conditions

$$X_n^\pi \geq f_n, \quad n = 0, 1, \ldots, N,$$

as the option can be exercised at any time. A hedge is *minimal*, if for a stopping time τ^* we have $X_{\tau^*}^\pi = f_{\tau^*}$.

By Doob's optional stopping $(X_0^\pi/B_0, X_\tau^\pi/B_\tau)$ is martingale for any stopping time τ, i.e.

$$\frac{x}{B_0} = \mathbf{E_Q}\frac{X_0^\pi}{B_0} = \mathbf{E_Q}\frac{X_\tau^\pi}{B_\tau} \geq \mathbf{E_Q}\frac{f_\tau}{B_\tau}.$$

Therefore the initial cost of the hedge is at least

$$x \geq B_0 \sup_{\tau\in\mathcal{M}_0^N} \mathbf{E_Q}\frac{f_\tau}{B_\tau}.$$

At time N we need

$$X_N^\pi \geq f_N.$$

At time $N-1$ the holder either exercise the option or continues to time N, (in that case we discount the price), therefore

$$X_{N-1}^\pi \geq \max \left\{ f_{N-1}, \frac{B_{N-1}}{B_N} \mathbf{E_Q}[f_N | \mathcal{F}_{N-1}] \right\}.$$

Dividing by B_{N-1}

$$\frac{X_{N-1}^\pi}{B_{N-1}} \geq \max \left\{ \frac{f_{N-1}}{B_{N-1}}, \mathbf{E_Q} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_{N-1} \right] \right\}.$$

Thus, we see the connection with the Snell-envelope.

For a hedging strategy π we have that

(i) $(X_n^\pi / B_n)_n$ is a $\mathbf{Q}$-martingale (since $\mathbf{Q}$ is EMM and π is SF), and

(ii) (X_n^π / B_n) dominates (f_n/B_n) (since π is a hedge).

Therefore, the value process of a hedge is larger than the Snell-envelope of (f_n/B_n), i.e.

$$\frac{X_n^\pi}{B_n} \geq Z_n, \quad n = 0, 1, \ldots, N, \tag{12}$$ {eq:di-american-1}

where (Z_n) is the Snell-envelope of (f_n/B_n). The Snell-envelope (Z_n) is a supermartingale, therefore by the Doob-decomposition (that's stated for submartingale, but multiply with -1) we have

$$Z_n = M_n - A_n, \quad n = 0, 1, \ldots, N, \tag{13}$$ {eq:di-american-2}

where M_n is a $\mathbf{Q}$-martingale, and (A_n) is an increasing predictable sequence, $A_0 = 0$. Comparing (12) and (13) we see that

$$\frac{X_n^\pi}{B_n} \geq M_n.$$

On the other hand, the market is complete, which implies (see the easy parts of the proof of Theorem 7) that there exists a strategy π such that

$$\frac{X_n^\pi}{B_n} = M_n, \quad n = 0, 1, \ldots, N.$$

This is a minimal hedging strategy with initial cost

$$\frac{x}{B_0} = \frac{X_0^\pi}{B_0} = M_0 = Z_0.$$

{thm:price-di-amer

Theorem 12. *Consider an aribtrage-free complete market with unique EMM* **Q**. *Let* (f_n) *be the nonnegative payoff sequence of an American option. Let* (Z_n) *be the Snell-envelope of the discounted payoff sequence* (f_n/B_n). *The fair price for this option is*

$$C = B_0 Z_0 = B_0 \sup_{\tau \in \mathcal{M}_0^N} \mathbf{E}_\mathbf{Q} \frac{f_\tau}{B_\tau} = B_0 \mathbf{E}_\mathbf{Q} \frac{f_{\tau^*}}{B_{\tau^*}},$$

where τ^* *is an (not unique in general) optimal exercise time given by*

$$\tau^* = \min \left\{ n : \frac{f_n}{B_n} = Z_n \right\}.$$

Furthermore, there exists a SF strategy π *which is an optimal hedge with inital cost* C *and*

$$X^\pi_{\tau^*} = \frac{f_{\tau^*}}{B_{\tau^*}}.$$

6.4 American vs. European options

Clearly, an American option witht payoff sequence $(f_n)_{n=0,1,\ldots,N}$ worth at least as a European option with payoff f_N. However, in some cases the fair prices are equal.

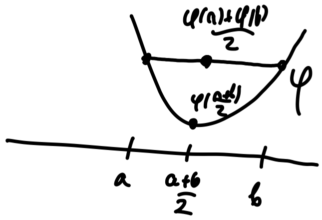
Consider an American call option with strike price K, that is

$$f_n = f(S_n) = (S_n - K)_+.$$

Assume that the deterministic sequence (B_n) is nondecreasing (i.e. the interest rate is nonnegative). Let (Z_n) denote the Snell envelope of (f_n/B_n), that is

$$Z_N = \frac{f_N}{B_N}, \quad Z_n = \max \left\{ \frac{f_n}{B_n}, \mathbf{E}\left[Z_{n+1} | \mathcal{F}_n \right] \right\}, \quad n = 0, 1, \ldots, N-1.$$

Using that (S_n/B_n) is a $\mathbf{Q}$-martingale, by Jensen's inequality

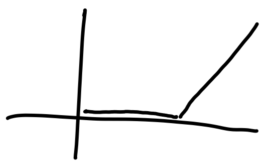


$$
\begin{aligned}
\frac{f_{N-1}}{B_{N-1}} &= \frac{(S_{N-1}-K)_+}{B_{N-1}} \\
&= \left(\frac{S_{N-1}}{B_{N-1}} - \frac{K}{B_{N-1}}\right)_+ \\
&\le \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_N}{B_N} - \frac{K}{B_{N-1}}\right)_+ \middle| \mathcal{F}_{N-1}\right] && \text{Jensen's inequality} \\
&\le \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_N}{B_N} - \frac{K}{B_N}\right)_+ \middle| \mathcal{F}_{N-1}\right] && \text{by } B_N \ge B_{N-1} \\
&= \mathbf{E}_{\mathbf{Q}}\left[\frac{(S_N-K)_+}{B_N} \middle| \mathcal{F}_{N-1}\right] \\
&= \mathbf{E}_{\mathbf{Q}}[Z_N|\mathcal{F}_{N-1}].
\end{aligned}
$$

This means that at time $N-1$ it is always good to hold the option and continue to step N.

An induction argument shows that at any time it is better to hold the option. Indeed, assume for some n

$$\frac{f_n}{B_n} \le \mathbf{E}_{\mathbf{Q}}[Z_{n+1}|\mathcal{F}_n].$$



We just proved this for $n = N - 1$. The same way as above we have

$$\begin{aligned}
\frac{f_{n-1}}{B_{n-1}} &= \frac{(S_{n-1} - K)_+}{B_{n-1}} \\
&= \left(\frac{S_{n-1}}{B_{n-1}} - \frac{K}{B_{n-1}}\right)_+ \\
&\leq \mathbf{E}_\mathbf{Q}\left[\left(\frac{S_n}{B_n} - \frac{K}{B_{n-1}}\right)_+ \middle| \mathcal{F}_{n-1}\right] && \text{Jensen's inequality} \\
&\leq \mathbf{E}_\mathbf{Q}\left[\left(\frac{S_n}{B_n} - \frac{K}{B_n}\right)_+ \middle| \mathcal{F}_{n-1}\right] && \text{by } B_n \geq B_{n-1} \\
&= \mathbf{E}_\mathbf{Q}\left[\frac{(S_n - K)_+}{B_n} \middle| \mathcal{F}_{n-1}\right] \\
&= \mathbf{E}_\mathbf{Q}\left[\frac{f_n}{B_n} \middle| \mathcal{F}_{n-1}\right] \\
&\leq \mathbf{E}_\mathbf{Q}\left[\mathbf{E}_\mathbf{Q}[Z_{n+1}|\mathcal{F}_n] \middle| \mathcal{F}_{n-1}\right] && \text{induction} \\
&\leq \mathbf{E}_\mathbf{Q}[Z_n|\mathcal{F}_{n-1}] && Z \text{ supermartingale}
\end{aligned}$$

Thus $\tau^* \equiv N$ is an optimal stopping time, which means that no matter what happens, we wait until the end. Then the American option behaves as the European, so the prices are equal.

Theorem 13. *Assume that the market is arbitrage free and complete, and the interest rate is nonnegative. Then the price of a European call option equals to the price of the American call option.*

7 Stochastic integration

7.1. Az Itô-formula

Ezek után belátjuk az Itô-formulát.

14. Theorem (Itô-formula (1944))**.** *Legyen* $X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$ *Itô-folyamat, és* $f \in C^2$ *kétszer folytonosan differenciálható függvény. Ekkor*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)H_s^2 \mathrm{d}s.$$