Strong Renewal Theorem and Local Limit Theorem in the Abscence of Regular Variation

Péter Kevei

University of Szeged

Joint work with Dalia Terhesiu (Leiden).

Outline

Outline

Setup

$X, X_1, X_2, \ldots$ nonnegative, iid random variables,
$F(x) = \mathbf{P}(X \le x)$, $\overline{F}(x) = 1 - F(x)$
$S_n = X_1 + \ldots + X_n$.
Renewal function: $U(x) = \sum_{n=0}^{\infty} F^{*n}(x) = \sum_{n=0}^{\infty} \mathbf{P}(S_n \le x)$.

Terminology

If $X \in a + h\mathbb{Z}$, then X is lattice.
If $a = 0$ then X is arithmetic (centered lattice).

Renewal theorems

Elementary renewal theorem:

$$\lim_{x\to\infty} \frac{U(x)}{x} = \frac{1}{\mathbf{E}X}.$$

Renewal theorems

Elementary renewal theorem:

$$\lim_{x \to \infty} \frac{U(x)}{x} = \frac{1}{\mathbf{E}X}.$$

Blackwell theorem/ Strong renewal theorem:

$$\lim_{x \to \infty} U(x+h) - U(x) = \frac{h}{\mathbf{E}X},$$

for any $h > 0$ if X is nonarithmetic, for $h = \delta$, if X is arithmetic with span δ.

Outline

Infinite mean

$$\lim_{x \to \infty} \frac{U(x)}{x} = \frac{1}{\mathbf{E}X} = 0.$$

Better?

Infinite mean

$$\lim_{x\to\infty} \frac{U(x)}{x} = \frac{1}{\mathbf{E}X} = 0.$$

Better?

$$\widehat{U}(s) = \int_0^\infty e^{-sx} U(\mathrm{d}x) = \sum_{n=0}^{\infty} \int_0^\infty e^{-sx} F^{*n}(\mathrm{d}x) = \frac{1}{1-\widehat{F}(s)}.$$

If $\overline{F}(x) = \ell(x) x^{-\alpha} \Gamma(1-\alpha)^{-1}$, $\alpha \in (0,1)$, then

$$\overline{F}(x) \cdot U(x) \longrightarrow \frac{\sin \pi\alpha}{\pi\alpha}.$$

Regular variation

$\ell : (0, \infty) \to (0, \infty)$ is slowly varying if for every $\lambda > 0$

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

Regular variation

$\ell : (0, \infty) \to (0, \infty)$ is slowly varying if for every $\lambda > 0$

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

f is regularly varying with parameter $-\alpha$, $f \in \mathcal{RV}_{-\alpha}$ if $f(x) = \ell(x) x^{-\alpha}$.

Dynkin–Lamperti problem

$$\lim_{x\to\infty} \overline{F}(x) \cdot U(x) = c \quad \Rightarrow \quad \overline{F} \in \mathcal{RV}?? \text{ OPEN}$$

Infinite mean SRT

Assume $\overline{F}(x) \in \mathcal{RV}(-\alpha)$,

$$m(x) = \int_0^x \overline{F}(y)\mathrm{d}y.$$

Infinite mean SRT

Assume $\overline{F}(x) \in \mathcal{RV}(-\alpha)$,

$$m(x) = \int_0^x \overline{F}(y) \mathrm{d}y.$$

Infinite mean analogue of SRT

$$\lim_{x \to \infty} m(x)[U(x+h) - U(x)] = hC_\alpha, \quad \forall h > 0.$$

Infinite mean SRT

Assume $\overline{F}(x) \in \mathcal{RV}(-\alpha)$,

$$m(x) = \int_0^x \overline{F}(y)\mathrm{d}y.$$

Infinite mean analogue of SRT

$$\lim_{x\to\infty} m(x)[U(x+h) - U(x)] = hC_\alpha, \quad \forall h > 0.$$

- Garsia & Lamperti (1963), arithmetic case, $\alpha \in (1/2, 1]$
- Erickson (1970), nonarithmetic, $\alpha \in (1/2, 1]$.
- $\alpha \le 1/2$ sufficient conditions by Doney (1997), Vatutin, Topchii (2014), Chi (2015).

NASC

NASC for nonnegative random variables was given independently by Caravenna (2015+) and Doney (2015+) (Caravenna–Doney 2019, EJP):

$$\lim_{\delta\to 0} \limsup_{x\to\infty} x\overline{F}(x) \int_1^{\delta x} \frac{1}{y\overline{F}(y)^2} F(x - \mathrm{d}y) = 0.$$

Aim

Without regular variation?

Outline

- Renewal theory
  - Finite mean
  - Infinite mean
- Semistable laws
  - Definition and properties
  - Possible limits
- Results
  - Renewal theorems
  - Local limit theorems

Stable laws, domain of attraction

V is *stable*, if there exist $X, X_1, X_2, \ldots$ iid, $a_n > 0, c_n \in \mathbb{R}$, such that

$$\frac{1}{a_n}\left(\sum_{i=1}^{n} X_i - c_n\right) \xrightarrow{\mathcal{D}} V.$$

$F \in D(\alpha)$ iff $1 - F(x) = \ell(x)x^{-\alpha}$.

Semistable laws

V is *stable*, if there exist $X, X_1, X_2, \dots$ iid, $a_n > 0$, c_n such that

$$\frac{1}{a_n}\left(\sum_{i=1}^{n} X_i - c_n\right) \xrightarrow{\mathcal{D}} V.$$

W is *semistable*, if there exist $X, X_1, X_2, \dots$ iid, $a_n > 0$, c_n, n_k geometrically increasing ($= c^k$), such that

$$\frac{1}{a_{n_k}}\left(\sum_{i=1}^{n_k} X_i - c_{n_k}\right) \xrightarrow{\mathcal{D}} W.$$

Semistable laws

Paul Lévy 1935 (István Berkes: Some forgotten results of Paul Lévy)
Kruglov, Mejzler, Pillai, Shimizu, Grinevich, Khokhlov
Martin-Löf, Sándor Csörgő, Dodunekova, Berkes, Csáki, Megyesi, Györfi, K
Meerschaert, Scheffler, Kern, Wedrich
Sato, Watanabe, Yamamuro

Characteristic function

Characteristic function of a nonnegative semistable random variable V:

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{\mathrm{i}ta + \int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x)\right\},$$

where $a \geq 0$
$M : (0, \infty) \to (0, \infty)$ logarithmically periodic $M(c^{1/\alpha}x) = M(x)$
$-R(x) := M(x)/x^\alpha$ is nonincreasing for $x > 0$, $\alpha \in (0, 1)$.

Domain of geometric partial attraction

Grinevich, Khokhlov (1995); Megyesi (2000)
$X, X_1, X_2, \ldots$ iid $F(x) = \mathbf{P}(X \le x)$. $V = V(R)$ semistable

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{ \int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x) \right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}.$$

$X \in D_g(V)$ if $\exists k_n$, A_n

$$\frac{\sum_{i=1}^{k_n} X_i}{A_{k_n}} \xrightarrow{\mathcal{D}} V.$$

Domain of geometric partial attraction

Grinevich, Khokhlov (1995); Megyesi (2000)
$X, X_1, X_2, \ldots$ iid $F(x) = \mathbf{P}(X \le x)$. $V = V(R)$ semistable

$$\mathbf{E} e^{\mathrm{i}tV} = \exp\left\{ \int_0^\infty (e^{\mathrm{i}tx} - 1) \mathrm{d}R(x) \right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}.$$

$X \in D_g(V)$ if $\exists k_n$, A_n

$$\frac{\sum_{i=1}^{k_n} X_i}{A_{k_n}} \xrightarrow{\mathcal{D}} V.$$

$F \in D_g(V)$ iff $1 - F(x) = \ell(x) M(x) x^{-\alpha}$.

St. Petersburg distribution

Nicolaus Bernoulli (1713): $\mathbf{P}(X = 2^k) = 2^{-k}$, $k = 1, 2, \ldots$

$$\mathbf{E}X = \sum_{k=1}^{\infty} 2^k 2^{-k} = \infty.$$

St. Petersburg paradox

St. Petersburg distribution

Nicolaus Bernoulli (1713): $\mathbf{P}(X = 2^k) = 2^{-k}$, $k = 1, 2, \ldots$

$$\mathbf{E}X = \sum_{k=1}^{\infty} 2^k 2^{-k} = \infty.$$

St. Petersburg paradox

$$1 - F(x) = \mathbf{P}(X > x) = \frac{2^{\{\log_2 x\}}}{x}.$$

X is not in the domain of attraction of any stable law

St. Petersburg distribution

Martin-Löf (1985): Resolution of the St. Petersburg paradox

$$\frac{S_{2^n}}{2^n} - n \xrightarrow{\mathcal{D}} V$$

In fact

$$\frac{S_n}{n} - \log_2 n$$

has infinitely many different limits along subsequences.

Regularly log-periodic functions

$$x^{\beta}\ell(x)p(x),$$

where for some $r > 0$, $p(rx) = p(x)$, for all $x > 0$.
Appear naturally in

- semistable laws
- fixed points of smoothing transforms
- supercritical branching processes

Regularly log-periodic functions

$$x^{\beta}\ell(x)p(x),$$

where for some $r > 0$, $p(rx) = p(x)$, for all $x > 0$.
Appear naturally in

- semistable laws
- fixed points of smoothing transforms
- supercritical branching processes

Buldygin, Pavlenkov (2013): Karamata theorem
Buldygin, Indlekofer, Klesov, Steinebach: Pseudo Regularly Varying Functions (2018)
K (2020): Tauberian and Karamata theorems, and applications

Outline

Circular convergence

For $x > 0$ (large) we define the position parameter as

$$\gamma_x = \gamma(x) = \frac{x}{c^n}, \quad \text{where } c^{n-1} < x \leq c^n.$$

$$c^{-1} = \liminf_{x\to\infty} \gamma_x < \limsup_{x\to\infty} \gamma_x = 1.$$

Limits on subsequences

$$\mathbf{E} e^{\mathrm{i}tV} = \exp\left\{ \int_0^\infty (e^{\mathrm{i}tx} - 1) \mathrm{d}R(x) \right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}$$

Limits on subsequences

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{\int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x)\right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}$$

Theorem (Csörgő & Megyesi (2002))

$$\frac{\sum_{i=1}^{n_r} X_i}{n_r^{1/\alpha}\ell_1(n_r)} \xrightarrow{\mathcal{D}} V_\lambda \quad \text{as } r \to \infty,$$

whenever $\gamma_{n_r} \stackrel{cir}{\to} \lambda$. *Here*

$$\mathbf{E}e^{\mathrm{i}tV_\lambda} = \exp\left\{\int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R_\lambda(x)\right\}, \quad R_\lambda(x) = -\frac{M(\lambda^{1/\alpha}x)}{x^\alpha}.$$

Merging

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{\int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x)\right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}$$

$$\mathbf{E}e^{\mathrm{i}tV_\lambda} = \exp\left\{\int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R_\lambda(x)\right\}, \quad R_\lambda(x) = -\frac{M(\lambda^{1/\alpha}x)}{x^\alpha}.$$

$$\gamma_x = \gamma(x) = \frac{x}{c^n}, \quad \text{where } c^{n-1} < x \le c^n.$$

Theorem (Csörgő & Megyesi (2002))

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbf{P}\left(\frac{S_n}{n^{1/\alpha}\ell_1(n)} \le x\right) - \mathbf{P}(V_{\gamma_n} \le x) \right| = 0.$$

Outline

$$\widehat{U}(s) = \frac{1}{1 - \widehat{F}(s)}.$$

If

$$\overline{F}(x) = \ell(x) x^{-\alpha} p_0(x), \quad \text{with } p_0 \in \mathcal{P}_r,$$

then

$$\lim_{n \to \infty} \frac{U(r^n z) \ell(r^n)}{(r^n z)^\alpha} = p_1(z),$$

where p_1 can be determined explicitly.

G_λ df of the possible limits
$\gamma(x)$ positional parameter
ℓ slowly varying from the domain of attraction condition

Theorem (K-Terhesiu (2021))

X is nonnegative from the domain of attraction of a semistable with $\alpha \in (0,1)$. Set $B(x) = x^\alpha \ell(x)^{-1}$. Then

$$\lim_{y\to\infty} \left| y^{-\alpha}\ell(y)U(y) - \alpha \int_0^\infty G_{\gamma(B(y)x^{-\alpha})}(x) x^{-\alpha-1} \, \mathrm{d}x \right| = 0.$$

Arithmetic setup

Assume that X is integer valued,

$$u_n = \sum_{k=0}^{\infty} \mathbf{P}(S_k = n).$$

Theorem (K-Terhesiu (2021))

Assume that X is a nonnegative integer valued from the domain of attraction of a semistable with $\alpha \in (1/2, 1)$. Set $B(x) = x^{\alpha} \ell(x)^{-1}$. Then

$$\lim_{n\to\infty} \left| n^{1-\alpha} \ell(n) u_n - \alpha \int_0^{\infty} g_{\gamma(B(n)x^{-\alpha})}(x)\, x^{-\alpha}\, \mathrm{d}x \right| = 0.$$

Nonarithmetic setup

Theorem (K-Terhesiu (2021))

Assume that X is a nonnegative from the domain of attraction of a semistable with $\alpha \in (1/2, 1)$. Set $B(x) = x^\alpha \ell(x)^{-1}$. Then for any $h > 0$,

$$\lim_{y \to \infty} \left| \frac{y^{1-\alpha} \ell(y)}{2h} \left(U(y+h) - U(y-h) \right) - \alpha \int_0^\infty g_{\gamma(B(y)x^{-\alpha})}(x)\, x^{-\alpha}\, \mathrm{d}x \right| = 0.$$

$\alpha = 1$

Uchiyama (2020: A renewal theorem for relatively stable variables): If

$$m(x) = \int_0^x \overline{F}(y) \mathrm{d}y$$

is slowly varying then

$$m(x)(U(x+h) - U(x)) \to h.$$

Berger (2019): Cauchy domain of attraction.

Outline

Lattice

Extension of Gnedenko's local limit theorem:

Theorem (K-Terhesiu (2021))

Let $X, X_1, \ldots$ be integer valued iid random variables with span 1 from the domain of attraction of a semistable. Then

$$\lim_{n\to\infty} \sup_k |A_n \mathbf{P}(S_n = k) - g_{\gamma_n}((k - C_n)/A_n)| = 0.$$

Fourier analytic proof, inversion formula

$$\mathbf{P}(S_n = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}tk} \varphi(t)^n \, \mathrm{d}t,$$

merging, asymptotics of the characteristic function.

Nonlattice

Extension of Stone's local limit theorem.

Theorem (K-Terhesiu (2021))

Let $X, X_1, \ldots$ be iid nonlattice random variables from the domain of attraction of a semistable. Then for any $h > 0$

$$\lim_{n\to\infty} \sup_x \left| \frac{A_n}{2h} \mathbf{P}(S_n \in (x-h, x+h]) - g_{\gamma_n}((x - C_n)/A_n) \right| = 0.$$