Nearly critical Galton–Watson processes

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Outline

Ongoing joint work with Kata Kubatovics (Szeged).

Outline

Galton – Watson process

ξ = number of offsprings

$$\mathbf{P}(\xi = k) = f[k], \quad k = 0, 1, 2, \ldots$$

$X_0 = 1$, and

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_{n,i},$$

where $\xi_{n,i}$ are iid ξ. The generating function:

$$f(s) = \sum_{k=0}^{\infty} f[k] s^k.$$

Extinction theorem

Theorem (Galton, Watson)

If $m = f'(1) \leq 1$ then $q = 1$, while for $m > 1$ q is the smallest root of $f(s) = s$.

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$m < 1$ subcritical, $m = 1$ critical, $m > 1$ supercritical case

Immigration

$$Y_n = \sum_{i=1}^{Y_{n-1}} \xi_{n,i} + \varepsilon_n$$

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Theorem (Heathcote (1965), Foster (1969))

(i) *If $m > 1$, or $m = 1$ and $f''(1) < \infty$, then* $\lim_{n\to\infty} \mathbf{P}(X_n = k) = 0$ *for* $k = 1, 2, \ldots$.

(ii) *If* $0 < h'(1) < \infty$, *and* $m < 1$, *then* $\lim_{n\to\infty} \mathbf{P}(X_n = k) = p_k$ *exists, and* $\{p_k\}$ *probability distribution.*

Outline

Varying environment

$X_0 = 1$, and

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j},$$

where $\{\xi_{n,j}\}_{n,j\in\mathbb{N}}$ are independent random variables, such that for each n, $\{\xi_n, \xi_{n,j}\}_{j\in\mathbb{N}}$ are identically distributed.

- 1970's: Church, Fearn, Jagers, Agresti
- 2017 Kersting, 2020 Kersting & Vatutin monograph (BPV/RE)
- 2020s: Bhattacharya & Perlman, Dolgopyat et al., Cardona-Tobón & Palau, González & Minuesa & del Puerto, ...

Varying environment- immigration

Inhomogeneous Galton–Watson process with immigration:
$Y_0 = 0$,

$$Y_n = \sum_{j=1}^{Y_{n-1}} \xi_{n,j} + \varepsilon_n$$

where $\{\xi_{n,j}, \varepsilon_n : n, j \in \mathbb{N}\}$ are independent nonnegative integer valued random variables, $\{\xi_{n,j} : j \in \mathbb{N}\}$ are iid.

Nearly critical process

$\bar{f}_n = f_n'(1) = \mathbf{E}\xi_n$. We assume the following conditions:

(C1) $\bar{f}_n < 1$, $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty}(1 - \bar{f}_n) = \infty$,

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\bar{f}_n} = \nu \in [0, \infty)$,

(C3) $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\bar{f}_n} = 0$, if $\nu > 0$.

C1

$\overline{f}_n < 1$, $\lim_{n\to\infty} \overline{f}_n = 1$, $\sum_{n=1}^{\infty}(1 - \overline{f}_n) = \infty$

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j},$$

$$\mathbf{E}X_n = \mathbf{E}\xi_1 \mathbf{E}\xi_2 \ldots \mathbf{E}\xi_n = \prod_{i=1}^{n} \overline{f}_i \to 0,$$

so (X_n) dies out a.s.

- conditioning on $X_n > 0$, Yaglom-type limit results
- adding immigration

INAR(1)

If the offspring distribution is Bernoulli(ρ_n): integer-valued autoregressive (INAR(1)) time series:

$$X_n = \rho_n \circ X_{n-1} + \varepsilon_n,$$

where $\rho \circ X$ is a Bernoulli thinning of X, $\circ$ is the *Steutel and van Harn operator*.

- introduced by Laci Györfi, Márton Ispány, Gyula Pap and Katalin Varga (2007)
- K (2011), weaking the Bernoulli offsrping assumption
- Györfi, Ispány, K, Pap (2014): multitype setup

Outline

Yaglom's theorem in the classical setup

Theorem (Yaglom)

If $m < 1$ *then* $\mathcal{L}(X_n | X_n > 0)$ *converges in distribution.*

Theorem (Yaglom)

If $m = 1$ *then* $\mathcal{L}(X_n / n | X_n > 0)$ *converges to the exponential distribution.*

Yaglom-type results

Theorem (K & Kubatovics (2022))

(C1) $\overline{f}_n \to 1$, $\overline{f}_n < 1$, $\sum_n (1 - \overline{f}_n) = \infty$

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\overline{f}_n} = \nu \in [0, \infty)$,

(C3) $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\overline{f}_n} = 0$, *if* $\nu > 0$.

Then

$$\mathcal{L}(X_n | X_n > 0) \xrightarrow{\mathcal{D}} \text{Geom}\left(\frac{2}{2+\nu}\right) \quad \text{as } n \to \infty,$$

Consequence: $\mathbf{P}(X_n > 0) \sim \frac{2}{2+\nu}\overline{f}_{0,n}$.

Outline

Notation

$f_n(s) = \mathbf{E} s^{\xi_n}$ g.f. in generation n.
For the composite g.f. $f_{n,n}(s) = s$, and for $j < n$

$$f_{j,n}(s) = f_{j+1} \circ \ldots \circ f_n(s),$$

and for the corresponding means $\overline{f}_{n,n} = 1$,

$$\overline{f}_{j,n} = \overline{f}_{j+1} \ldots \overline{f}_n, \quad j < n.$$

Then $\mathbf{E} s^{X_n} = f_{0,n}(s)$ and $\mathbf{E} X_n = \overline{f}_{0,n}$.

Shape function

For a g.f. f, with mean $\overline{f}$, define the *shape function* (Kersting 2017)

$$\varphi(s) = \frac{1}{1 - f(s)} - \frac{1}{\overline{f}(1-s)}, \ 0 \leq s < 1, \quad \varphi(1) = \frac{f''(1)}{2f'(1)^2}.$$

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$$\frac{1}{1 - f_{0,n}(s)} = \frac{1}{\bar{f}_1(1 - f_{1,n}(s))} + \varphi_1(f_{1,n}(s)),$$

therefore

$$\frac{1}{1 - f_{0,n}(s)} = \frac{1}{\bar{f}_{0,n}(1-s)} + \varphi_{0,n}(s),$$

where

$$\varphi_{0,n}(s) = \sum_{k=1}^{n} \frac{\varphi_k(f_{k,n}(s))}{\bar{f}_{0,k-1}}.$$

Example

Linear fractional g.f.:

$$f(s) = 1 - a\frac{1-s}{1-qs}, \quad f[k] = a(1-q)q^{k-1}, k > 0.$$

Then $\overline{f} = \frac{a}{1-q}$,

$$\frac{1}{1-f(s)} = \frac{1}{\overline{f} \cdot (1-s)} + \frac{q}{a}.$$

That is $\varphi(s) = \frac{q}{a}$.

Lemma (Kersting)

Assume $0 < \bar{f} < \infty$, $f''(1) < \infty$ and let $\varphi(s)$ be the shape function of f. Then, for $0 \leq s \leq 1$,

$$\frac{1}{2}\varphi(0) \leq \varphi(s) \leq 2\varphi(1).$$

Lemma

$$\lim_{n\to\infty} \frac{\bar{f}_{0,n}}{1 - f_{0,n}(s)} = \frac{1}{1-s} + \frac{\nu}{2}.$$

Consequence:

$$\mathbf{P}(X_n > 0) = 1 - f_{0,n}(0) \sim \bar{f}_{0,n} \frac{2}{2+\nu}$$

Proof

$$\frac{1}{1-f_{0,n}(s)} = \frac{1}{\overline{f}_{0,n}(1-s)} + \varphi_{0,n}(s), \quad \varphi_{0,n}(s) = \sum_{k=1}^{n} \frac{\varphi_k(f_{k,n}(s))}{\overline{f}_{0,k-1}}$$

We have to show that

$$\overline{f}_{0,n}\varphi_{0,n}(s) = \sum_{j=1}^{n} \overline{f}_{j-1,n}\varphi_j(f_{j,n}(s)) \to \frac{\nu}{2}.$$

Proof

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$$\sum_{j=1}^{n} \overline{f}_{j-1,n}\varphi_j(f_{j,n}(s)) \approx \sum_{j=1}^{n} \overline{f}_{j-1,n}\varphi_j(1)$$

$$= \sum_{j=1}^{n}(1-\overline{f}_j)\overline{f}_{j-1,n}\frac{f_j''(1)}{2f_j'(1)^2(1-\overline{f}_j)} \to \frac{\nu}{2} \quad \text{(Toeplitz-lemma)}$$

Proof of the Yaglom type theorem

Convergence of the conditional g.f.:

$$\mathbf{E}[s^{X_n} | X_n > 0] = \frac{f_{0,n}(s) - f_{0,n}(0)}{1 - f_{0,n}(0)}$$

$$= 1 - \frac{1 - f_{0,n}(s)}{1 - f_{0,n}(0)} \to \frac{2}{2+\nu} \frac{s}{1 - \frac{\nu}{\nu+2} s},$$

Outline

Bernoulli immigration

Theorem (Györfi, Ispány, Pap, Varga (2007))

Let $(Y_n)_{n\in\mathbb{N}}$ *be an inhomogeneous INAR(1) process, with* $\varepsilon_n \sim \mathrm{Bernoulli}(m_{n,1})$. *Assume that*

(i) $\overline{f}_n \to 1$, $\overline{f}_n < 1$, $\sum_n (1-\overline{f}_n) = \infty$,

(ii) $\lim_{n\to\infty} \frac{m_{n,1}}{1-\overline{f}_n} = \lambda$.

Then

$$Y_n \xrightarrow{\mathcal{D}} \mathrm{Poisson}(\lambda).$$

Negative binomial rv

X is negative binomial with parameters $r > 0$ and $p \in (0,1)$, NB(r,p), if $\mathbf{P}(X = k) = \binom{k+r-1}{r-1}(1-p)^r p^k$, $k = 0, 1, 2, \ldots$, where $\binom{k+r-1}{r-1} = \frac{(k+r-1)(k+r-2)\cdots r}{k!}$. The generating function is

$$\mathbf{E}s^X = \left(\frac{1-p}{1-ps} \right)^r .$$

Theorem (K 2011)

(Y_n) GW process with immigration, such that:

(i) $\bar{f}_n < 1, \bar{f}_n \to 1, \sum_{n=1}^{\infty}(1 - \bar{f}_n) = \infty$,

(ii) $\frac{f_n''(1)}{1-\bar{f}_n} \to \nu \in (0, \infty)$,

(iii) $\frac{f_n^{(s)}(1)}{1-\bar{f}_n} \to 0$, *for all* $s \geq 3$,

(iv) $\frac{m_{n,1}}{1-\bar{f}_n} \to \lambda$ *and* $\frac{m_{n,2}}{1-\bar{f}_n} \to 0$.

Then

$$Y_n \xrightarrow{\mathcal{D}} \mathrm{NB}(2\lambda/\nu, \nu/(2+\nu)).$$

Theorem (K - Kubatovics (2022))

Assume that (C1)–(C3) are satisfied and

(C4) $\lim_{n\to\infty} \frac{m_{n,k}}{k!(1-\bar{f}_n)} = \lambda_k, \quad k = 1, 2, \ldots, K$ *and* $\lambda_K = 0$.

$$Y_n \xrightarrow{\mathcal{D}} Y \quad \text{as } n \to \infty,$$

where Y is compound-Poisson with g.f.

$$\exp\left\{ -\sum_{k=1}^{K-1} \frac{2^k \lambda_k}{\nu^k} \left(\log\left(1 + \frac{\nu}{2}(1-s)\right) + \sum_{i=1}^{k-1} (-1)^i \frac{\nu^i}{i 2^i} (1-s)^i \right) \right\}.$$

Outline

Generating function

$$f_n(s) = \mathbf{E}s^{\xi_n},\ h_n(s) = \mathbf{E}s^{\varepsilon_n},\ g_n(s) = \mathbf{E}s^{Y_n}$$

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Using the branching property we obtain the recursion

$$\begin{aligned} g_n(s) &= \mathbf{E}\left[s^{\sum_{i=1}^{Y_{n-1}} \xi_{n,i}+\varepsilon_n}\right] = \mathbf{E}\left[\mathbf{E}\left(s^{\sum_{i=1}^{Y_{n-1}} \xi_{n,i}+\varepsilon_n}\middle| Y_{n-1}\right)\right] \\ &= \mathbf{E}\left[f_n(s)^{Y_{n-1}}\right] h_n(s) = g_{n-1}(f_n(x))h_n(s). \end{aligned}$$

Generating function

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$$g_n(s) = \mathbf{E}\left[s^{\sum_{i=1}^{Y_{n-1}} \xi_{n,i} + \varepsilon_n}\right] = \mathbf{E}\left[\mathbf{E}\left(s^{\sum_{i=1}^{Y_{n-1}} \xi_{n,i} + \varepsilon_n}\middle| Y_{n-1}\right)\right]$$
$$= \mathbf{E}\left[f_n(s)^{Y_{n-1}}\right] h_n(s) = g_{n-1}(f_n(x))h_n(s).$$

$$g_n(s) = \prod_{j=1}^{n} h_j(f_{j,n}(s)).$$

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Need to show that:

$$\lim_{n\to\infty} g_n(s) = f_Y(s), \quad s \in [0,1].$$

Introduce the accompanying law

$$\widehat{g}_n(s) = \prod_{j=1}^{n} e^{h_j(f_{j,n}(s))-1} = \exp \sum_{j=1}^{n} \left(h_j(f_{j,n}(s)) - 1 \right).$$

Proof

$$\widehat{g}_n(s) = \prod_{j=1}^{n} e^{h_j(f_{j,n}(s))-1} = \exp \sum_{j=1}^{n} \left(h_j(f_{j,n}(s)) - 1\right) .$$

- $g_n(s) - \widehat{g}_n(s) \to 0$;
- $\widehat{g}_n(s) \to f_Y(s)$
- g_n is a compound Poisson g.f., therefore f_Y too.

Work in progress

$$U_n(t) = X_{[nt]}$$

We showed that

$$\mathcal{L}(U_n(1) | U_n(1) > 0) \to \mathrm{Geom}\left(\frac{\nu}{2+\nu}\right).$$

Aim:

$$\mathcal{L}((U_n(t))_t | U_n(1) > 0) \to (U(t))_t.$$

Lemma

Assume

(i) $\bar{f}_n = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)$,

(ii) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\bar{f}_n} = \nu$,

(iii) $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\bar{f}_n} = 0$.

Then

$$\mathbf{E}\left[s^{X_{[nt]}} \middle| X_{[nu]} = 1\right] \to 1 - \left(\frac{u}{t}\right)^{\alpha} \left(\frac{1}{1-s} + \frac{\nu}{2}\left(1 - \left(\frac{u}{t}\right)^{\alpha}\right)\right)^{-1}$$

For $1 = t_0 < t_1 < \ldots < t_k$

$$\begin{aligned}&\mathbf{P}(X_{[nt_1]} = x_1, \ldots, X_{[nt_k]} = x_k | X_n = x_0) \\ &= \prod_{i=1}^{k} \mathbf{P}(X_{[nt_i]} = x_i | X_{[nt_{i-1}]} = x_{i-1}),\end{aligned}$$

thus the finite dimensional distributions converge.

The limit

$$\mathbf{E}\left[s^{U(t)} | U(u) = x_0\right] = \left(1 - \left(\frac{u}{t}\right)^{\alpha} \left(\frac{1}{1-s} + \frac{\nu}{2}\left(1 - \left(\frac{u}{t}\right)^{\alpha}\right)\right)^{-1}\right)^{x_0}.$$

Then $Z(t) = U(e^t)$ is time-homogeneous Markov process, a simple birth–death process, which dies out a.s.

Questions

- The limit

$$\lim_{n\to\infty} \mathbf{P}(X_{[nt_1]} = x_1, \ldots, X_{[nt_k]} = x_k | X_n = x_0)$$

  exists for $0 < t_1 < \ldots < t_k < 1$.
- tightness

- P. Kevei, K. Kubatovics, Nearly critical Galton–Watson processes. Arxiv.
- L. Györfi, M. Ispány, G. Pap and K. Varga, Poisson limit of an inhomogeneous nearly critical INAR(1) model. *Acta Sci. Math. (Szeged)* 73(3–4), (2007), 789–815.
- G. Kersting, A unifying approach to branching processes in a varying environment. *J. Appl. Probab.*, 57(1):196–220, 2020.
- G. Kersting and V. Vatutin, *Discrete Time Branching Processes in Random Environment*, 2017.