On the solution to the stochastic heat equation with Lévy noise

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Outline

Joint work with Carsten Chong (Columbia University).

Outline

Heat equation - deterministic case

$$\begin{aligned} \partial_t Y(t,x) &= \Delta Y(t,x) + h(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ Y(0,\cdot) &= f \end{aligned}$$

Δ Laplace operator, h external heating/cooling

Heat equation - deterministic case

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Δ Laplace operator, h external heating/cooling
Solution:

$$Y(t,x) = \int_{\mathbb{R}^d} g(t,x-y) f(y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y) h(s,y) \mathrm{d}y \mathrm{d}s,$$

where

$$g(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

Stochastic heat equation

$$\begin{aligned} \partial_t Y(t,x) &= \Delta Y(t,x) + \xi(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ Y(0,\cdot) &= f \end{aligned}$$

ξ is a space-time Lévy noise.

Consider the heat equation ($f \equiv 0$)

$$\partial_t Y(t,x) = \Delta Y(t,x) + \dot{\Lambda}(t,x),$$

where

$$\Lambda(\mathrm{d}t,\mathrm{d}x) = m\,\mathrm{d}t\mathrm{d}x + \int_{(1,\infty)} z\,\mu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{(0,1]} z\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z),$$

μ Poisson random measure on $(0,\infty) \times \mathbb{R}^d \times (0,\infty)$, intensity $\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) = \mathrm{d}t\,\mathrm{d}x\,\lambda(\mathrm{d}z)$. Solution:

$$Y(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) = \sum_{\tau_i \le t} g(t-\tau_i, x-\eta_i)\zeta_i.$$

$$g(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$$

Gaussian noise

ξ space-time white noise - $d = 1$
Khoshnevisan, Kim, Xiao, Conus, Foondun, Joseph, . . ., 2010-
Khoshnevisan, *Analyis of Stochastic Partial Differential Equations*, 2014, AMS

Outline

Moments

$$\begin{aligned}\mathbf{E}[Y(t,x)] &= \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t 1\,\mathrm{d}s = t, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d.\end{aligned}$$

Moments

$$\mathbf{E}[Y(t,x)] = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_0^t 1\,\mathrm{d}s = t, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d.$$

$$\operatorname{Var} Y(t,x) = \begin{cases} \frac{1}{4\sqrt{2\pi}}\sqrt{t}, & \text{for } d = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition

Let t_n be a sequence increasing to infinity. Then

$$\lim_{n\to\infty} \frac{Y_0(t_n)}{t_n} = 1 \quad a.s.$$

holds, if for some $\varepsilon > 0$

$$\begin{cases} \sum_{n=1}^{\infty} t_n^{\varepsilon-9/4} < \infty, & \text{for } d = 1, \\ \sum_{n=1}^{\infty} t_n^{\varepsilon-(1+2/d)} < \infty, & \text{for } d \geq 2. \end{cases}$$

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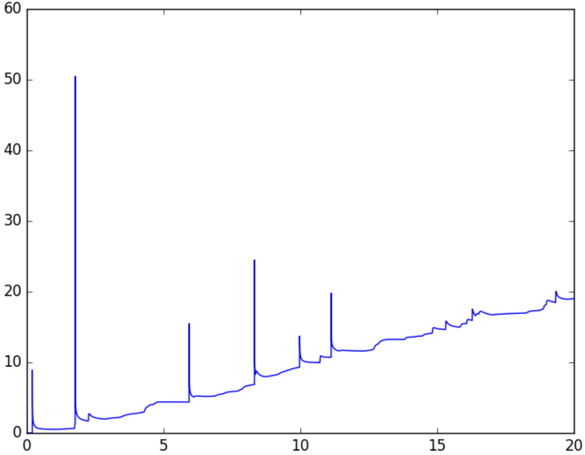
$$\begin{cases} \sum_{n=1}^{\infty} t_n^{\varepsilon - 9/4} < \infty, & \text{for } d = 1, \\ \sum_{n=1}^{\infty} t_n^{\varepsilon - (1+2/d)} < \infty, & \text{for } d \geq 2. \end{cases}$$

In any dimension $Y_0(n)/n \to 1$ a.s.

Motivation



It turns out that

$$\limsup_{t\to\infty} \frac{Y_0(t)}{t} = \infty \text{ a.s.}$$

Outline

$$Y(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y) = \sum_{\tau_i \le t} g(t-\tau_i, x-\eta_i)\zeta_i.$$

$$g(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$$

$$\eta(B) = \nu\Big(\{(s,y,z) :\ s \le t,\ g(s,y)z \in B\}\Big)$$

Theorem

$Y(t,x)$ exists iff

$$\int_{(1,\infty)} (\log z)^{d/2}\,\lambda(\mathrm{d}z) < \infty, \quad \begin{cases} \int_{(0,1]} z^2\,\lambda(\mathrm{d}z) < \infty & d = 1, \\ \int_{(0,1]} z^2 |\log z|\,\lambda(\mathrm{d}z) < \infty & d = 2, \\ \int_{(0,1]} z^{1+2/d}\,\lambda(\mathrm{d}z) < \infty & d \geq 3. \end{cases}$$

η is a Lévy measure and

$$\mathbf{E}[e^{\mathrm{i}\theta Y(t,x)}] = \exp\left\{ \mathrm{i}\theta A + \int_{(0,\infty)} \left(e^{\mathrm{i}\theta u} - 1 - \mathrm{i}\theta u \mathbb{1}(u \leq 1) \right) \eta(\mathrm{d}u) \right\}$$

Application of Rajput & Rosinski 1989

Outline

$$\eta(B) = \nu\Big(\{(s,y,z):\ s \le t,\ g(s,y)z \in B\}\Big),\ \nu = \mathrm{Leb} \times \lambda$$

Lemma

(i) If $m_{1+2/d}(\lambda) < \infty$

$$\overline{\eta}(r) \sim r^{-1-2/d} \frac{d^{d/2}}{2\pi(d+2)^{d/2+1}} m_{1+2/d}(\lambda), \qquad r \to \infty.$$

(ii) If $\overline{\lambda}(r) = \ell(r) r^{-\alpha}$ for $\alpha \in (0, 1+\frac{2}{d}]$,

$$\overline{\eta}(r) \sim \begin{cases} \ell(r) r^{-\alpha} \frac{D^{1+2/d-\alpha}}{2d\pi\alpha^{d/2}(1+\frac{2}{d}-\alpha)} & \text{if } \alpha < 1+\frac{2}{d}, \\ L(r) r^{-1-2/d} (2d\pi(1+\frac{2}{d})^{d/2})^{-1} & \text{if } \alpha = 1+\frac{2}{d}, \end{cases}$$

where $L(r) = \int_1^r \ell(u) u^{-1}\,\mathrm{d}u$, $D = (4\pi t)^{d/2}$.

Lemma

(iii) If $\overline{\lambda}(x) = \ell(x)$, then

$$\overline{\eta}(r) \sim L_0(r) \frac{D^{1+2/d}}{4\pi\Gamma(\frac{d}{2}+1)(1+\frac{2}{d})},$$

where

$$L_0(r) := \int_1^\infty \ell(ry) y^{-1} (\log y)^{d/2-1} \, \mathrm{d}y$$

is slowly varying and $L_0(r)/\ell(r) \to \infty$ as $r \to \infty$.

$\overline{Y}(t) = \sup_{\tau_i \le t} g(t - \tau_i, \eta_i)\zeta_i$

Theorem

(i) $\overline{\eta}$ *is subexponential.*
(ii) As $r \to \infty$,

$$\mathbf{P}(Y(t,x) > r) \sim \mathbf{P}(\overline{Y}(t) > r) \sim \overline{\eta}(r).$$

(iii) For $\alpha \in [0, 1 + \frac{2}{d})$, $\overline{\eta} \in \mathcal{RV}_{-\alpha}$ *iff* $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$.

Embrechts, Goldie, Veraverbeke ('79), Pakes ('04)

Outline

Introduction
- Setup
- Motivation

Fix t, x
- Existence
- Tail

Growth rate
- Continuous sampling
- Discrete sampling
- Spatial growth rate

Parabolic Anderson model
- Existence
- Growth

Simplification

Assume that $\lambda = \delta_1$, $\Lambda = N$ standard Poisson point process.

Theorem

Let $f : [0,\infty) \to [0,\infty)$ *be a nondecreasing function. If*

$$\int_1^\infty \frac{1}{f(t)} \mathrm{d}t = \infty,$$

then

$$\limsup_{t\to\infty} \frac{Y_0(t)}{f(t)} = \infty \quad a.s.$$

Conversely, if the integral above is finite then

$$\limsup_{t\to\infty} \frac{Y_0(t)}{f(t)} = 0 \quad a.s.$$

Furthermore, $\liminf_{t\to\infty} Y_0(t)/t = 1$ *a.s.*

Remarks

$$\limsup_{t\to\infty} \frac{Y_0(t)}{t} = \limsup_{t\to\infty} \frac{Y_0(t)}{t\log(t)} = \infty, \quad \text{a.s.}$$

but

$$\limsup_{t\to\infty} \frac{Y_0(t)}{t(\log(t))^{1.1}} = 0 \quad \text{a.s.}$$

Gaussian case

Solution is locally a fractional Brownian motion with Hurst index $1/4$. (Lei, Nualart 2009)

Theorem

Suppose that $\dot{\Lambda}$ is a Gaussian space–time white noise in one spatial dimension. Then, a.s.

$$\limsup_{t\to\infty} \frac{Y_0(t)}{(2t/\pi)^{1/4}\sqrt{\log\log t}} = -\liminf_{t\to\infty} \frac{Y_0(t)}{(2t/\pi)^{1/4}\sqrt{\log\log t}} = 1.$$

In particular, the SLLN holds: $\lim_{t\to\infty} \frac{Y_0(t)}{t} = 0$ *a.s.*

Follows from Watanabe (1970).

Theorem

Let $f : [0,\infty) \to [0,\infty)$ *be a nondecreasing function. If*

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Conversely, if the integral above is finite then

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Proof I

For $x \in \mathbb{R}^d$, $g(t,x)$ is increasing on $[0, |x|^2/(2d)]$, decreasing on $[|x|^2/(2d), \infty)$, and its maximum is

$$g(|x|^2/(2d), x) = \left(\frac{d}{2\pi e}\right)^{d/2} |x|^{-d}.$$

Jump at x causes a peak of size $|x|^{-d}$.

Proof I

Assume first $\int_1^\infty 1/f(t)\mathrm{d}t = \infty$. Let $K > 0$ be fix large.

$$A_n = \left\{ N\left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \; n \geq 0.$$

Proof I

Assume first $\int_1^\infty 1/f(t)\mathrm{d}t = \infty$. Let $K > 0$ be fix large.

$$A_n = \left\{ N\left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\},\ n \geq 0.$$

Since N is a homogeneous Poisson process, we have

$$\mathbf{P}(A_n) = 1 - e^{-v_d/[Kf(n+2)]} \sim \frac{v_d}{Kf(n+2)},$$

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$$\mathbf{P}(A_n) = 1 - e^{-v_d/[Kf(n+2)]} \sim \frac{v_d}{Kf(n+2)},$$

and thus $\sum_{n=1}^\infty \mathbf{P}(A_n) = \infty$. A_n's are independent, by Borel–Cantelli A_n occurs infinitely many times.

Proof I

$$A_n = \left\{ N\left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \ n \geq 0.$$

On A_n,

$$\sup_{t \in [n, n+2]} Y_0(t) \geq (2\pi e/d)^{-d/2} Kf(n+2),$$

Proof I

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On A_n,

$$\sup_{t \in [n,n+2]} Y_0(t) \geq (2\pi e/d)^{-d/2} K f(n+2),$$

that is

$$\sup_{t \in [n,n+2]} \frac{Y_0(t)}{f(t)} > (2\pi e/d)^{-1/2} K.$$

Since A_n occurs infinitely often, and K is arbitrary large, the result follows.

Proof II

Now assume $\int_1^\infty 1/f(t)\mathrm{d}t < \infty$. Let $K > 0$ be fix large.

Proof II

Now assume $\int_1^\infty 1/f(t)\mathrm{d}t < \infty$. Let $K > 0$ be fix large. Let us fix $t \in [n, n+1]$. Introduce the events (recent close jumps)

$$A_n = \left\{ N([n, n+1] \times B([K/f(n)]^{1/d}) \geq 1 \right\}$$

$$B_n = \left\{ N([n, n+1] \times B([K/f(n)]^{1/d}, [(K \log n)/n]^{1/d})) \geq 2 \right\}$$

$$C_n = \left\{ N([n, n+1] \times B([(K \log n)/n]^{1/d}, 1) \geq 6 \log n \right\}$$

Almost surely only finitely many of these events occur.

Outline

At which sequence we don’t see the superlinear part?

Theorem

Let $t_n \uparrow \infty$. Then

$$\lim_{n\to\infty} \frac{Y_0(t_n)}{t_n} = 1 \quad a.s.$$

holds, if

$$\sum_{n=1}^{\infty} \frac{t_n^{-2/d} \wedge \Delta t_n}{t_n} < \infty.$$

Special case $t_n = n^p$

If $t_n = n^p$ for some $p > d/(d+2)$, we have

$$\lim_{n\to\infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.},$$

while for $0 < p \leq d/(d+2)$, we have

$$\limsup_{n\to\infty} \frac{Y_0(t_n)}{t_n} = \infty, \quad \liminf_{n\to\infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.}$$

Outline

Fix t, and consider the behavior in x.

Theorem

Almost surely

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{f(x)} = \infty \qquad \textit{or} \qquad \limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{f(x)} = 0,$$

according to whether the following integral diverges or converges:

$$\int_1^\infty r^{d-1}\overline{\tau}(f(r))\,\mathrm{d}r,$$

where $\tau(B) = (\mathrm{Leb}\times\lambda)\left(\{(s,z):\ (4\pi s)^{-d/2}z \in B,\ s \le t\}\right)$.

$$\tau(B) = (\mathrm{Leb} \times \lambda)\left(\{(s,z) : (4\pi s)^{-d/2} z \in B,\ s \le t\}\right)$$

Lemma

(i) If $m_{2/d}(\lambda) < \infty$, then $\overline{\tau}(r) \sim (4\pi)^{-1} m_{2/d}(\lambda) r^{-2/d}$ as $r \to \infty$.
(ii) Assume that $\overline{\lambda}(r) = \ell(r) r^{-\alpha}$ for $\alpha \in [0, \frac{2}{d}]$, and further assume $\int_1^\infty \ell(u) u^{-1}\,\mathrm{d}u = \infty$ if $\alpha = \frac{2}{d}$. Then as $r \to \infty$

$$\overline{\tau}(r) \sim \begin{cases} \frac{2tD^{-\alpha}}{2-d\alpha} \ell(r) r^{-\alpha} & \text{if } \alpha < \frac{2}{d}, \\ \frac{1}{2\pi d} L(r) r^{-2/d} & \text{if } \alpha = \frac{2}{d}. \end{cases}$$

Special case

$$f(r) = r^p$$

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{x^p} = \infty \qquad \text{or} \;\; 0,$$

- if $m_{2/d}(\lambda) < \infty$: $p \le d^2/2$ or $p > d^2/2$
- if $\overline{\lambda}(r) = r^{-\alpha}$, $\alpha \in (0, 2/d)$: $p \le d/\alpha$ or $p > d/\alpha$.

Gaussian case (Khoshnevisan, Kim, Xiao 2017):

$$\limsup_{|x|\to\infty} \frac{Y(t,x)}{(\log|x|)^{1/2}} = \left(\frac{2t}{\pi}\right)^{1/4}$$

Outline

$$\partial_t Y(t,x) = \Delta Y(t,x) + Y(t,x)\dot{\Lambda}(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d,$$
$$Y(0,\cdot) \equiv 1,$$

Solution:

$$Y(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) Y(s,y) \Lambda(\mathrm{d}s, \mathrm{d}y)$$

Existence

$$Y(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) Y(s,y) \Lambda(\mathrm{d}s, \mathrm{d}y)$$

Berger & Chong & Lacoin 2022: Solution exists if

$$\int_{(1,\infty)} (\log z)^{d/2} \lambda(\mathrm{d}z) < \infty \quad \text{and}$$

$$\begin{cases} \int_{(0,1)} z^2 \lambda(\mathrm{d}z) < \infty, & d = 1, \ \text{(same as additive)} \\ \int_{(0,1)} z^2 |\log z| \lambda(\mathrm{d}z) < \infty, & d = 2, \ \text{(same as additive)} \\ \int_{(0,1)} z^{1+2/d} |\log z| \lambda(\mathrm{d}z) < \infty, & d \geq 3, \ \text{(log stronger than additive)} \end{cases}$$

Outline

Gaussian case

Conus & Khoshnevisan 2012, Khoshnevisan & Kim & Xiao 2017

$$\limsup_{|x|\to\infty} \frac{Y(t,x)}{(\log|x|)^{1/2}} = \left(\frac{4t}{\pi}\right)^{\frac{1}{4}} \quad \text{additive,}$$

$$\limsup_{|x|\to\infty} \frac{\log Y(t,x)}{(\log|x|)^{2/3}} = \left(\frac{9t}{32}\right)^{\frac{1}{3}} \quad \text{multiplicative}$$

Assume $\overline{\lambda}(x) = x^{-\alpha}$, $x > 1$, $\lambda((0,1)) = 0$.

Theorem

Suppose $\alpha > \frac{2}{d}$. Let f be nondecreasing. Then for both additive and multiplicative noise

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{f(x)} = \infty \text{ or } 0,$$

according to whether the integral

$$\int_1^\infty x^{d-1} f(x)^{-\frac{2}{d}} \, \mathrm{d}x$$

diverges or converges.

Theorem

Suppose $\alpha < \frac{2}{d}$.

- *additive noise:*

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{f(x)} = \infty \text{ or } 0,$$

depending on whether the following integral diverges or converges:

$$\int_1^\infty x^{d-1} f(x)^{-\alpha}\,\mathrm{d}x.$$

Theorem

- *multiplicative noise: there are* $0 < L_* \le L^* < \infty$ *such that for all* $L > L^*$,

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{x^{d/\alpha} e^{L(\log x)^{1/(1+\theta_\alpha)}}} = 0 \qquad a.s.,$$

while for all $L < L_*$,

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{x^{d/\alpha} e^{L(\log x)^{1/(1+\theta_\alpha)}}} = \infty \qquad a.s.$$

where $\theta_\alpha = 1 - \frac{d}{2}(\alpha - 1)$.

Theorem

- *If $\alpha > \frac{2}{d}$, in the additive case almost surely*

$$\limsup_{x\to\infty} \frac{\sup_{y\in\mathbb{Z}^d,|y|\le x} Y(t,y)}{f(x)} = \infty \quad \textit{or} \;\; 0,$$

according to whether

$$\int_1^\infty x^{d-1} f(x)^{-[(1+2/d)\wedge\alpha]}\,\mathrm{d}x.$$

diverges or converges.

Theorem

- *If* $\alpha \in (\frac{2}{d}, 1 + \frac{2}{d})$ *in the multiplicative case there are* $0 < M_* \leq M^* < \infty$ *such that for all* $M > M^*$,

$$\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{x^{d/\alpha} e^{M(\log x)^{1/(1+\theta_\alpha)}}} = 0 \quad a.s.,$$

while for $M < M_*$,

$$\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{x^{d/\alpha} e^{M(\log x)^{1/(1+\theta_\alpha)}}} = \infty \quad a.s.$$

Theorem

- *If $\alpha \geq 1 + \frac{2}{d}$ in the multiplicative case there are $0 < M_* \leq M^* < \infty$ such that for $M > M^*$,*

$$\limsup_{x\to\infty} \frac{\sup_{y\in\mathbb{Z}^d, |y|\leq x} Y(t,y)}{x^{d^2/(2+d)} e^{M(\log x)(\log\log\log x)/\log\log x}} = 0 \qquad a.s.$$

while for $M < M_*$,

$$\limsup_{x\to\infty} \frac{\sup_{y\in\mathbb{Z}^d, |y|\leq x} Y(t,y)}{x^{d^2/(2+d)} e^{M(\log x)(\log\log\log x)/\log\log x}} = \infty \qquad a.s.$$

Theorem

- *If $\alpha < \frac{2}{d}$, then in both cases* $\sup_{y \in \mathbb{Z}^d, |y| \le x} Y(t, y)$ *has the same asymptotic as* $\sup_{y \in \mathbb{R}^d, |y| \le x} Y(t, y)$

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