Trimmed Lévy processes

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Lévy processes and time series: in honour of Peter Brockwell and Ross Maller

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V_t subordinator with Lévy measure Λ and drift 0, i.e.

$$\mathbf{E} e^{-\lambda V_t} = \exp\left\{ -t \int_0^\infty \left(1 - e^{-\lambda v}\right) \Lambda(\mathrm{d}v) \right\},$$

where $\int_0^\infty \min\{1, x\} \Lambda(\mathrm{d}x) < \infty$

Assume $\overline{\Lambda}(0+) = \infty$, then there is an infinite number of jumps up to time t:

$$\Delta_t^{(1)} \geq \Delta_t^{(2)} \geq \ldots$$

the ordered jumps of V_s up to time t.

$$V_t^{(k)} = V_t - \sum_{j=1}^{k} \Delta_t^{(j)} \quad \text{trimmed subordinator}$$

$V_t^{(0)} = V_t$ is the subordinator, and $\Delta_t^{(1)}$ is the largest jump.

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Introduction

$$\Delta_t^{(1)} \geq \Delta_t^{(2)} \geq \ldots, \quad V_t^{(k)} = V_t - \sum_{j=1}^{k} \Delta_t^{(j)}$$

Problem: $V_t^{(k)}/\Delta_t^{(k+1)}$, and $\Delta_t^{(k+1)}/\Delta_t^{(k)}$ as $t \downarrow 0$ and $t \to \infty$.

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Ratio of the consecutive jumps

Theorem (K–Mason)

$\Delta_t^{(k+1)}/\Delta_t^{(k)}$ converges in distribution as $t \downarrow 0$ to Y_k iff one of the following holds:

(i) *$\overline{\Lambda}$ is regularly varying at 0 with parameter $-\alpha \in [-1, 0)$, in which case Y_k has beta$(k\alpha, 1)$ distribution, i.e.*

$$G_k(x) = \mathbf{P}\{Y_k \leq x\} = x^{k\alpha}, \quad x \in [0, 1];$$

(ii) *$\overline{\Lambda}$ is slowly varying at 0, in which case $Y_k = 0$ a.s.*

Poisson–Dirichlet laws

If S_t is a driftless α-stable subordinator, $\alpha \in (0,1)$, with jumps $J_1^{(1)} > J_1^{(2)} > \dots$. Then $(J_1^{(1)}/S_1, J_1^{(2)}/S_1, \dots)$ has Poisson–Dirichlet law with parameter α (PD_α).

PD laws: Poisson–Kingman partitions; fragmentation; sized biased reordering; ... Pitman, Yor;

Bertoin: Random fragmentation and coagulation processes.

The ratio of the $(k+1)^{\text{st}}$ and k^{th} element of a vector which has PD_α law has beta$(k\alpha, 1)$ distribution.

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Generalized Poisson–Dirichlet laws

S_t driftless α-stable subordinator $\alpha \in (0,1)$, $r \geq 1$:
$\left(J_1^{(r+1)}/S_1^{(r)}, J_1^{(r+2)}/S_1^{(r)}, \ldots \right)$ has $\mathrm{PD}_\alpha^{(r)}$ distribution. (Ipsen, Maller (2017+)) Point process approach:

$$\mathbb{B}^{(r)} = \sum_{i=1}^{\infty} \delta_{R_r(i)}, \quad R_r(i) = J_1^{(r+i)}/J_1^{(r)}.$$

Theorem (Ipsen, Maller (2017+))

$$\mathbf{E} \exp \left\{ - \int_0^1 f \mathrm{d}\mathbb{B}^{(r)} \right\} = \left(1 + \int_0^1 (1 - e^{-f(x)}) \alpha x^{-\alpha-1} \mathrm{d}x \right)^{-r}$$

Palm distribution, explicit formula for joint distribution of the size biased reordering, ...

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Convergence of point processes

S_t driftless α-stable subordinator (jumps up to time 1)

$$\mathbb{B}^{(r)} = \sum_{i=1}^{\infty} \delta_{R_r(i)}, \quad R_r(i) = J_1^{(r+i)}/J_1^{(r)}.$$

V_t is a subordinator with Lévy measure Λ,

$$\mathbb{D}_t^{(r,r+n)} = \sum_{i=r+1}^{\infty} \delta_{Q_r(i)}, \quad Q_r(i) = \Delta_t^{(i)}/\Delta_t^{(r+n)}.$$

Theorem (Ipsen, Maller, Resnick (2017+))

If $\overline{\Lambda} \in \mathcal{RV}_{-\alpha}$, then

$$\mathbb{D}_t^{(r,r+n)} \xrightarrow{\mathcal{D}} \sum_{i=1}^{\infty} \delta_{R_r(i)}, \quad R_r(i) = J_1^{(r+i)}/J_1^{(r+n)}$$

in the space of point measures with the vague topology.

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Theorem (Ipsen, Maller, Resnick (2017+))

Whenever $\overline{\Lambda}$ is regularly varying with parameter $-\alpha \in (-1, 0)$,

$$\left(\frac{\Delta_t^{(r+1)}}{\Delta_t^{(r)}}, \ldots, \frac{\Delta_t^{(r+n)}}{\Delta_t^{(r+n-1)}} \right) \xrightarrow{\mathcal{D}} (Y_r, \ldots, Y_{r+n-1})$$

where $Y_r, \ldots, Y_{r+n-1}$ are independent, Y_s has beta$(k\alpha, 1)$ distribution.

Converse

Theorem (Ipsen, Maller, Resnick (2017+))

$\Delta_t^{(k+r)}/\Delta_t^{(k)}$ converges in distribution as $t \downarrow 0$ to Y_k iff one of the following holds:

(i) *$\overline{\Lambda}$ is regularly varying at 0 with parameter $-\alpha \in [-1, 0)$, in which case $Y_{k,r} \in (0, 1)$;*

(ii) *$\overline{\Lambda}$ is slowly varying at 0, in which case $Y_{k,r} = 0$ a.s.*

Outline

Trimmed process / jumps

Theorem (K–Mason 2014)

$V_t^{(k)}/\Delta_t^{(k+1)}$ converges in distribution to W_k as $t \downarrow 0$ iff one of the following holds:

(i) *$\overline{\Lambda}$ is regularly varying at 0 with parameter $-\alpha$, $\alpha \in (0,1)$, in which case $W_k \in (1,\infty)$;*

(ii) *$\overline{\Lambda}$ is slowly varying at 0, in which case $W_k = 1$ a.s.;*

(iii) *condition*

$$\frac{x\overline{\Lambda}(x)}{\int_0^x u\Lambda(\mathrm{d}u)} = 0 \quad \text{as } x \downarrow 0$$

holds, in which case $W_k = \infty$ a.s.

Remark

- Buchmann, Fan and Maller (2016): (ii) and (iii) for Lévy processes without a normal components.
- Ipsen, Maller (2017+): more general limit theorems for Lévy processes in the domain of attraction of a stable law.

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IID counterpart

limit of order statistics:

- Arov, Bobrov (1960) sufficiency
- Smid, Stam (1975), Bingham, Teugels (1979) necessity

sum / max:

- $k = 0$ (no trimming): Darling (1952) sufficiency part, and Breiman (1965) necessity part
- Teugels (1982): sufficiency part with general k

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The representation

$\omega_1, \omega_2, \ldots$ iid exponential(1) random variables, and $\Gamma_n = \omega_1 + \ldots + \omega_n$. Put

$$\varphi(s) = \sup\{y : \overline{\Lambda}(y) > s\} = \overline{\Lambda}^{\leftarrow}(s).$$

$$V_t \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi\left(\frac{\Gamma_i}{t}\right).$$

$$\left(\Delta_t^{(1)}, \Delta_t^{(2)}, \ldots\right) \stackrel{\mathcal{D}}{=} (\varphi(\Gamma_1/t), \varphi(\Gamma_2/t), \ldots)$$

Buchmann, Fan, Maller (2016), LePage, Woodroofe, Zinn (1981), Rosiński (2001)

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Proof – main idea

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$$\frac{V_t^{(k)}}{\Delta_t^{(k+1)}} \stackrel{\mathcal{D}}{=} \frac{\sum_{i=k+1}^{\infty} \varphi(\Gamma_i/t)}{\varphi(\Gamma_{k+1}/t)}.$$

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$$\frac{V_t^{(k)}}{\Delta_t^{(k+1)}} \stackrel{\mathcal{D}}{=} \frac{\sum_{i=k+1}^{\infty} \varphi(\Gamma_i/t)}{\varphi(\Gamma_{k+1}/t)}.$$

Given $\Gamma_{k+1} = s$, $(\Gamma_{k+2}, \Gamma_{k+3}, \ldots)$ is a homogeneous Poisson point process on (s, ∞), thus

$$\sum_{i=k+2}^{\infty} \varphi(S_i/t) = \sum_{i=k+2}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i - s}{t}\right)$$

$$\stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i}{t}\right)$$

$$= \sum_{i=1}^{\infty} \varphi_{s/t}(S_i/t),$$

where $\varphi_s(x) = \varphi(s + x)$.

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$$\mathbf{E}e^{-\lambda \frac{V_t^{(k)}}{m_t^{(k+1)}}} = \mathbf{E}e^{-\lambda \frac{\sum_{i=k+1}^{\infty} \varphi(S_i/t)}{\varphi(S_{k+1}/t)}}$$

$$= \int_0^\infty \frac{s^k}{k!} e^{-s} \left[e^{-\lambda} \mathbf{E} e^{-\frac{\lambda}{\varphi(s/t)} \sum_{i=1}^{\infty} \varphi_{s/t}(S_i/t)} \right] \mathrm{d}s$$

$$= e^{-\lambda} \int_0^\infty \frac{s^k}{k!} e^{-s} \exp\left\{ -t \int_{s/t}^\infty \left[1 - e^{-\frac{\lambda}{\varphi(s/t)} \varphi(x)} \right] \mathrm{d}x \right\} \mathrm{d}s$$

$$= \frac{t^{k+1}}{k!} e^{-\lambda} \int_0^\infty u^k \exp\left\{ -t \left(u + \int_u^\infty \left[1 - e^{-\lambda \frac{\varphi(x)}{\varphi(u)}} \right] \mathrm{d}x \right) \right\} \mathrm{d}u.$$

Then use the Tauberian theorem 3-times.

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St. Petersburg paradox

Nicolaus Bernoulli (1713): Paul's gain X, then

$$\mathbf{P}\{X = 2^k\} = \frac{1}{2^k}, \quad k = 1, 2, \ldots$$

What is the fair price?
Paradox:

$$\mathbf{E}(X) = \sum_{k=1}^{\infty} 2^k \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$$

$$\text{but } \mathbf{P}\{X > 40\} = 2^{-5} = 0.03125$$

'there ought not be a sane man who would not happily sell his chance for forty ducats' – Nicolaus Bernoulli

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CLT

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$$\frac{S_n - c_n}{a_n} \xrightarrow{\mathcal{D}} ?$$

Doeblin–Gnedenko criterion:

$$\mathbf{P}\{X \le x\} = \begin{cases} 0, & \text{for } x < 2, \\ 1 - 2^{-\lfloor \log_2 x \rfloor} = 1 - \frac{2^{\{\log_2 x\}}}{x}, & \text{for } x \ge 2, \end{cases}$$

$2^{\{\log_2 x\}}$ is not slowly varying ($\{\cdot\}$ fractional part)

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CLT

$$X_1, X_2, \ldots \text{ iid St.Petersburg rv's}, S_n = \sum_{i=1}^{n} X_i,$$

$$\frac{S_n - c_n}{a_n} \xrightarrow{\mathcal{D}} ?$$

Doeblin–Gnedenko criterion:

$$\mathbf{P}\{X \le x\} = \begin{cases} 0, & \text{for } x < 2, \\ 1 - 2^{-\lfloor \log_2 x \rfloor} = 1 - \dfrac{2^{\{\log_2 x\}}}{x}, & \text{for } x \ge 2, \end{cases}$$

$2^{\{\log_2 x\}}$ is not slowly varying ($\{\cdot\}$ fractional part) $\Rightarrow$ there is no limit theorem for $\frac{S_n - c_n}{a_n}$ for any choice of a_n, c_n.

There is on subsequences!

Theorem (Martin-Löf (1985))

$$\frac{S_{2^n}}{2^n} - n \xrightarrow{\mathcal{D}} W\,, \quad as\ n \to \infty\,.$$

W semistable rv. Moreover, convergence holds on subsequences $n_k = \lfloor \gamma 2^k \rfloor$, $\gamma \in (1/2, 1]$.

Csörgő & Dodunekova (1991): convergence only on these subsequences

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Merging

Theorem (Csörgő (2002))

$$\sup_{x\in\mathbb{R}} \left| \mathbf{P}\left\{ \frac{S_n}{n} - \log_2 n \le x \right\} - G_{\gamma_n}(x) \right| \to 0, \quad \textit{as } n \to \infty,$$

where

$$\gamma_n = \frac{n}{2^{\lceil \log_2 n \rceil}}.$$

The limit

Characteristic function of W_γ, $\gamma \in (1/2, 1]$,

$$\mathbf{E}\left(e^{\mathrm{i}tW_\gamma}\right) = \exp\left(\mathrm{i}ta + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right)\mathrm{d}R_\gamma(x)\right),$$

with right-hand-side Lévy function

$$R_\gamma(x) = -\frac{\gamma}{2^{\lfloor \log_2(\gamma x)\rfloor}} = -\frac{2^{\{\log_2(\gamma x)\}}}{x}, \quad x > 0.$$

(semistable laws, Lévy)

The maximum

For $j \in \mathbb{Z}$ and $\gamma \in [1/2, 1]$ introduce the notation

$$p_{j,\gamma} = e^{-\gamma 2^{-j}} \left(1 - e^{-\gamma 2^{-j}}\right), \quad \gamma_n = \frac{n}{2^{\lceil \log_2 n \rceil}}$$

Lemma

$$\sup_{j \in \mathbb{Z}} \left| \mathbf{P}\left\{ X_n^* = 2^{\lceil \log_2 n \rceil + j} \right\} - p_{j,\gamma_n} \right| = O(n^{-1}).$$

In particular for any $j \in \mathbb{Z}$, as $n \to \infty$

$$\mathbf{P}\left\{ X_n^* = 2^{\lceil \log_2 n \rceil + j} \right\} \sim e^{-\gamma_n 2^{-j}} \left(1 - e^{-\gamma_n 2^{-j}}\right).$$

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Typical maximum

Proposition (Gut & Martin-Löf (2016))

Conditionally on $X_n^* = 2^{\lceil \log_2 n \rceil + j}$, $j \in \mathbb{Z}$,

$$\#\{j : j \le n, X_j = X_n^*\} \xrightarrow{\mathcal{D}} M_{j,\gamma_n} \quad \text{(in the merging sense)},$$

where $M_{j,\gamma} \sim \text{Poisson}(2^{-j}\gamma)$ *conditioned on not being zero.*

Outline

Conditioning on typical maximum

Proposition (Berkes–Györfi–K (2016))

For $j \in \mathbb{Z}$ we have

$$\left| \mathbf{P} \left\{ \frac{S_n}{n} - \log_2 n \leq x | X_n^* = 2^{\lceil \log_2 n \rceil + j} \right\} - \widetilde{G}_{j,\gamma_n}(x) \right| \to 0,$$

where

$$\widetilde{G}_{j,\gamma}(x) = \sum_{m=1}^{\infty} G_{j-1,\gamma} \left(x - m \frac{2^j}{\gamma} \right) \frac{(2^{-j}\gamma)^m}{m!} (e^{2^{-j}\gamma} - 1)^{-1}.$$

$G_{j,\gamma}$

$$G_{j,\gamma}(x) = \mathbf{P}(W_{j,\gamma} \leq x)$$

$$\varphi_{j,\gamma}(t) = \mathbf{E}e^{\mathrm{i}tW_{j,\gamma}} = \exp\left[\mathrm{i}tu_{j,\gamma} + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \mathrm{i}tx\right)\mathrm{d}L_{j,\gamma}(x)\right],$$

with

$$L_{j,\gamma}(x) = \begin{cases} \gamma 2^{-j} - \frac{2^{\{\log_2(\gamma x)\}}}{x}, & \text{for } x < 2^j\gamma^{-1}, \\ 0, & \text{for } x \geq 2^j\gamma^{-1}, \end{cases}$$

Corollary

Theorem (Gut & Martin-Löf (2016))

For any $\gamma \in [1/2, 1]$

$$G_\gamma(x) = \sum_{j=-\infty}^{\infty} \widetilde{G}_{j,\gamma}(x) e^{-\gamma 2^{-j}} \left(1 - e^{-\gamma 2^{-j}}\right).$$

This is equivalent to the distributional representation

$$W_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma - 1,\gamma} + M_{Y_\gamma,\gamma} 2^{Y_\gamma} \gamma^{-1},$$

where $(W_{j,\gamma})_{j\in\mathbb{Z}}$, $(M_{j,\gamma})_{j\in\mathbb{Z}}$ and Y_γ are independent, $Y_\gamma \sim (p_{j,\gamma})_{j\in\mathbb{Z}}$, $M_{j,\gamma} \sim \text{Poisson}(\gamma 2^{-j})$ conditioned on not being 0.

Corollary

Theorem (Gut & Martin-Löf (2016))

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Buchmann, Fan & Maller (2016) result

Lévy process setup: W_γ is a semistable Lévy process at time 1.

$$W_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma - 1,\gamma} + M_{Y_\gamma,\gamma} 2^{Y_\gamma} \gamma^{-1},$$

The value $2^{Y_\gamma}/\gamma$ corresponds to the maximum jump, $M_{Y_\gamma,\gamma}$ is the number of the maximum jumps, and $W_{Y_\gamma - 1,\gamma}$ has the law of the Lévy process conditioned on that the maximum jump is strictly less than $2^{Y_\gamma}/\gamma$.

This kind of distributional representations for general Lévy processes were obtained by Buchmann, Fan & Maller (2016).

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Outline

Notation

$X, X_1, X_2, \ldots$ iid St. Petersburg rv's
$X_{1n} \leq X_{2n} \leq \ldots \leq X_{nn}$ ordered sample of $X_1, X_2, \ldots, X_n$.
r-trimmed sum: $S_{n,r} = \sum_{k=1}^{n-r} X_{kn}$.

$\omega_1, \omega_2, \ldots$ iid Exp(1), $\Gamma_k = \omega_1 + \ldots + \omega_k$

Theorem (Berkes–Györfi–K (2016))

Let $n_k = \lfloor \gamma 2^k \rfloor$*, for some* $\gamma \in (1/2, 1]$*. Then for any* $r \geq 0$

$$\frac{1}{n_k} S_{n_k,r} - a_{n_k,\gamma}^{(r)} \xrightarrow{\mathcal{D}} Y_{r,\gamma} = \sum_{k=r+1}^{\infty} \gamma^{-1} \left(2^{-\lfloor \log_2 \Gamma_k/\gamma \rfloor} - 2^{-\lfloor \log_2 k/\gamma \rfloor} \right),$$

with centering sequence

$$a_{n,\gamma}^{(r)} = \gamma^{-1} \sum_{j=r+1}^{n} 2^{-\lfloor j/\gamma \rfloor}.$$

Proof (sketch)

Quantile method & LePage, Woodroofe, Zinn idea.

Quantile representation: $(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} (Q(U_{1n}), \ldots, Q(U_{nn}))$, where $F^{-1}(s) = Q(s) = \inf\{x : s \leq F(x)\}$

$$Q(s) = \begin{cases} 2, & s = 0, \\ 2^{\lceil -\log_2(1-s) \rceil} = \frac{2^{\{\log_2(1-s)\}}}{1-s}, & s \in (0,1). \end{cases}$$

$(\omega_i)_{i \in \mathbb{N}}$ iid Exp(1), $\Gamma_n = \omega_1 + \ldots + \omega_n$. For n fix

$$(U_{1n}, U_{2n}, \ldots, U_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{\Gamma_1}{\Gamma_{n+1}}, \frac{\Gamma_2}{\Gamma_{n+1}}, \ldots, \frac{\Gamma_n}{\Gamma_{n+1}} \right),$$

where U's are ordered sample of n iid Uniform(0, 1).

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Proof

$\Psi(x) = 2^{\{\log_2 x\}}$ (grows linearly from 1 to 2 in each $[2^j, 2^{j+1})$).
$Q(1-s) = \Psi(s)/s$

$$(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{\Gamma_{n+1}}{\Gamma_1} \Psi(\Gamma_1/\Gamma_{n+1}), \ldots, \frac{\Gamma_{n+1}}{\Gamma_n} \Psi(\Gamma_n/\Gamma_{n+1}) \right)$$

SLLN $\Gamma_{n+1}/n \to 1$ a.s.

$$X^*_{j,n} = \frac{n}{\Gamma_j} \Psi\left(\frac{\Gamma_j}{n}\right)(1 + o(1)) \qquad \text{a.s.}$$

$$\Psi(\Gamma_j/n) = \Psi(\Gamma_j/\gamma_n)$$

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LePage, Woodroofe & Zinn (1981)

$Y, Y_1, Y_2, \ldots$ iid, ≥ 0, $Y \in D(\alpha)$, S_n partial sum,
$(S_n - nb_n)/a_n \to S$. $Y_{1,n} \geq Y_{2,n} \geq \ldots \geq Y_{n,n}$

$$S = \sum_{k=1}^{\infty} \left(\Gamma_k^{-1/\alpha} - \mathbf{E}\Gamma_k^{-1/\alpha} I(\Gamma_k^{-1/\alpha} < 1) \right),$$

where $\omega_1, \omega_2, \ldots$ are iid Exp(1), $\Gamma_k = \omega_1 + \ldots + \omega_k$. Moreover,

$$\left(\frac{S_n - nb_n}{a_n}, \left(\frac{Y_{1,n}}{a_n}, \ldots, \frac{Y_{n,n}}{a_n} \right) \right) \xrightarrow{\mathcal{D}} \left(S, (\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \ldots) \right).$$

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On the centering

For any $\gamma \in (1/2, 1]$, $n_k = \lfloor \gamma 2^k \rfloor$,

$$a^{(0)}_{n_k,\gamma} - \log_2 n_k \to 2 - \frac{1}{\gamma} \sum_{k=1}^{\infty} \frac{k\varepsilon_k}{2^k} - \log_2 \gamma = \xi(\gamma),$$

where $\gamma = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$.

Steinhaus' resolution of the St. Petersburg paradox (Csörgő & Simons 1993)

ξ is right-continuous, left-continuous except at dyadic rationals greater than $1/2$ and has unbounded variation (Csörgő & Simons 1993); the Hausdorff and box-dimension of the graph of ξ is 1 (Kern & Wedrich 2014).

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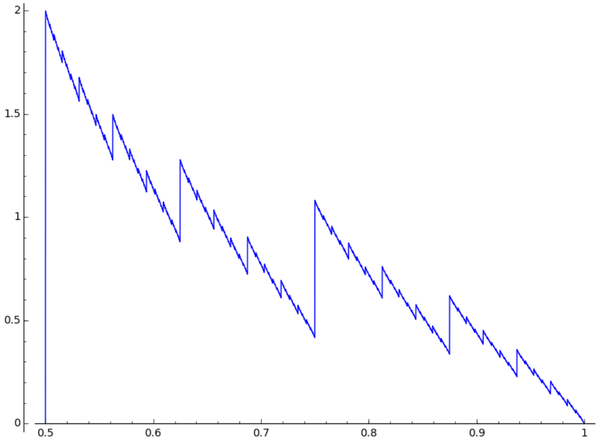
$\xi(\gamma)$



Outline

Tail of the trimmed limit

$$Y_{r,\gamma} = \sum_{k=r+1}^{\infty} \gamma^{-1} \left(2^{-\lfloor \log_2 \Gamma_k/\gamma \rfloor} - 2^{-\lfloor \log_2 k/\gamma \rfloor} \right), \quad A_{r,\gamma} = \gamma^{-1} \sum_{k=1}^{r} 2^{\lfloor k/\gamma \rfloor}$$

Theorem (Berkes–Györfi–K (2016))

$$\mathbf{P}\{ Y_{r,\gamma} > x \} \sim \frac{2^{\{\log_2(\gamma x)\}(r+1)}}{(r+1)!\, x^{r+1}} \Bigg[2^{-r-1} + (2^{r+1} - 1) \times \sum_{\ell=0}^{1} 2^{-\ell(r+1)} \mathbf{P} \left\{ Y_{0,\gamma} + A_{r,\gamma} > x \left(1 - 2^{\ell - \{\log_2(\gamma x)\}} \right) \right\} \Bigg].$$

Untrimmed case

$$\mathbf{P}\{Y_{0,\gamma} > x\} \sim \frac{2^{\{\log_2(\gamma x)\}}}{x} \times \left[2^{-1} + \sum_{\ell=0}^{1} 2^{-\ell}\mathbf{P}\left\{Y_{0,\gamma} > x\left(1 - 2^{\ell-\{\log_2(\gamma x)\}}\right)\right\}\right].$$

Exactly the tail of the Lévy measure appears

$$\frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} \sim 2^{-1} + \sum_{\ell=0}^{1} 2^{-\ell}\mathbf{P}\left\{Y_{0,\gamma} > x\left(1 - 2^{\ell-\{\log_2(\gamma x)\}}\right)\right\}\right].$$

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Properties of the r-trimmed limit

$$\frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} \sim 2^{-1} + \sum_{\ell=0}^{1} 2^{-\ell}\mathbf{P}\left\{Y_{0,\gamma} > x\left(1 - 2^{\ell-\{\log_2(\gamma x)\}}\right)\right\}\Bigg].$$

$$2^{-1} + 2^{-1}\mathbf{P}\{Y_{0,\gamma} > 0\} = \liminf_{x\to\infty} \frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} < \limsup_{x\to\infty} \frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} = 1 + \mathbf{P}\{Y_{0,\gamma} > 0\}$$

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$$< \limsup_{x\to\infty} \frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} = 1 + \mathbf{P}\{Y_{0,\gamma} > 0\}$$

For any $\delta \in (0, 1/2)$ we have

$$\lim_{x \to \infty, \delta < \{\log_2(\gamma x)\} < 1-\delta} \mathbf{P}\{Y_{r,\gamma} > x\} \frac{x}{2^{\{\log_2(\gamma x)\}}} = 1 \,.$$

In the untrimmed case ($r = 0$) for $\gamma = 1$

$$\mathbf{P}\{Y_{0,1} > 2^m + c\} \sim 2^{-m} \left[1 + \mathbf{P}\{Y_{0,1} > c\}\right], \quad \text{as } m \to \infty,$$

(Martin-Löf 1985).

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Watanabe & Yamamuro (2012) result

For general semistable distributions:

$$\lim_{n\to\infty} 2^n \mathbf{P}\{W_1 > x2^n\} = -R_1(x) + [R_1(x-) - R_1(x)]\mathbf{P}\{W_1 > 0\}$$

$$C_* = \liminf_{x\to\infty} \frac{\mathbf{P}\{W > x\}}{-R(x)} \le \limsup_{x\to\infty} \frac{\mathbf{P}\{W > x\}}{-R(x)} = C^*,$$

with

$$C_* = 1 - (1 - Q^{-1})\mathbf{P}\{W < 0\}, \ C^* = Q + (Q-1)\mathbf{P}\{W < 0\},$$

and $Q = \sup_{x\in[1,2]} R(x-)/R(x)$.