On the Breiman conjecture

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Breiman 1965

Coin tossing $\longrightarrow$ random walk $S_1, S_2, \ldots$.
Put $Y_1, Y_2, \ldots$ the interarrival times between the zeros of $S_1, S_2, \ldots$.
$X, X_1, X_2, \ldots$ iid $\mathbf{P}\{X = 0\} = \frac{1}{2} = \mathbf{P}\{X = 1\}$.

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

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Arc-sine law

In this case:

$$\lim_{n\to\infty} \mathbf{P}\{T_n \leq x\} = \frac{2}{\pi} \arcsin \sqrt{x}$$

In general

$Y, Y_1, Y_2, \ldots$ non-negative iid rv's with df G

$X, X_1, X_2, \ldots$ iid with df F, independent from $Y, Y_1, Y_2, \ldots$, $\mathbf{E}|X| < \infty$

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Remark

If $\mathbf{E}Y < \infty$, then

$$\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} Y_i} = \frac{\frac{\sum_{i=1}^{n} X_i Y_i}{n}}{\frac{\sum_{i=1}^{n} Y_i}{n}} \xrightarrow{\text{a.s.}} \mathbf{E}X.$$

$\mathbf{E}|X| < \infty$ implies (T_n) is tight.

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Theorem (Breiman, 1965)

If T_n converges in distribution for every F, and the limit is non-degenerate for at least one F, then $Y \in D(\beta)$, for some $\beta \in [0, 1)$.

Conjecture (Breiman)

If T_n has a non-degenerate limit for some F, then $Y \in D(\beta)$ for some $\beta \in [0, 1)$.

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On some limit theorems similar to the arc-sin law
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$D(\beta)$

Domain of attraction of an β-stable law:

$$Y \in D(\beta) \Leftrightarrow 1 - G(x) = \frac{\ell(x)}{x^\beta},$$

where ℓ is slowly varying ($\ell(\lambda x)/\ell(x) \to 1$ for any $\lambda > 0$ as $x \to \infty$).

$D(0)$

$Y \in D(0)$ if $1 - G(x)$ is slowly varying

in which case (Darling, 1952)

$$\frac{\max\{Y_i : i = 1, 2, \ldots, n\}}{\sum_{i=1}^{n} Y_i} \xrightarrow{\mathbf{P}} 1$$

and so

$$\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} Y_i} \xrightarrow{\mathcal{D}} X$$

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Limits

Theorem (Breiman)

Assume that $Y \in D(\beta)$, $\beta \in (0,1)$, *and* $\mathbf{E}|X|^{\beta+\varepsilon} < \infty$, *for some* $\varepsilon > 0$. *Then* $T_n \xrightarrow{\mathcal{D}} T$, *where*

$$\mathbf{P}\{T \le x\} = \frac{1}{2} + \frac{1}{\pi\beta} \arctan \left[\frac{\int |u-x|^\beta \mathrm{sgn}(x-u) F(\mathrm{d}u)}{\int |u-x|^\beta F(\mathrm{d}u)} \tan \frac{\pi\beta}{2} \right].$$

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$\mathbf{E}|X|^{2+\delta} < \infty$

Theorem (Mason & Zinn, 2005)

Assume that $\mathbf{E}|X|^{2+\delta} < \infty$*. Then* $T_n \to R$*, where* R *is non-degenerate, iff* $Y \in D(\beta)$*,* $\beta \in [0, 1)$*.*

Studentization

Other type of self-normalization (Logan & Mallows & Rice & Shepp, 1973):

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{\sum_{i=1}^{n} X_i^2}},$$

$X, X_1, X_2, \ldots$ iid. Student's T-statistic:

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}}$$

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Conjecture (Logan & Mallows & Rice & Shepp, 1973)

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{\sum_{i=1}^{n} X_i^2}} \xrightarrow{\mathcal{D}} W,$$

where $\mathbf{P}\{|W| = 1\} < 1$*, iff* $X \in D(\alpha)$*,* $\alpha \in (0, 2]$*; if* $\alpha > 1$*,* $\mathbf{E}X = 0$*; if* $\alpha = 1$*,* $X \in D$*(Cauchy).*

Giné & Götze & Mason (1997): W is standard normal iff $X \in D(2)$ and $\mathbf{E}X = 0$

Chistyakov & Götze (2004): in general

Outline

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$\mathbf{E}|X| < \infty$.

$$\phi_X(t) = \mathbf{E}e^{\mathrm{i}tX}$$

Theorem (K – Mason)

Assume that for some $\mathbf{E}X = 0$, $1 < \alpha \leq 2$, *positive slowly varying function* L *at zero and* $c > 0$,

$$\frac{-\log\left(\Re\phi_X(t)\right)}{|t|^{\alpha} L\left(|t|\right)} \to c, \ \text{as } t \to 0.$$

Whenever

$$\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} Y_i} \xrightarrow{\mathcal{D}} W \quad (W \text{ nondegenerate})$$

then $Y \in D(\beta)$ *for some* $\beta \in [0, 1)$.

What does this condition mean?

As $-\log \Re\phi_X(t) \sim 1 - \Re\phi_X(t)$, $t \to 0$,

$$\frac{-\log\left(\Re\phi_X(t)\right)}{|t|^\alpha L(|t|)} \to c \Leftrightarrow \frac{1 - \Re\phi_X(t)}{|t|^\alpha L(|t|)} \to c.$$

For $\alpha < 2$ this holds iff (Pitman)

$$\mathbf{P}\{|X| > x\} \sim L(1/x)x^{-\alpha} c\Gamma(\alpha)\frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)$$

If $\mathbf{E}X = 0$ and $X \in D(\alpha)$ then this condition is satisfied.
Also if $\mathbf{E}X = 0$, $\mathbf{E}X^2 < \infty$ then the condition of the theorem is satisfied ($\alpha = 2$, $c = \sigma^2/2$).

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Proposition

Assume that the assumptions of the theorem hold. Then for some $0 < \gamma \leq 1$

$$\mathbf{E}\frac{\sum_{i=1}^{n} Y_i^{\alpha}}{\left(\sum_{i=1}^{n} Y_i\right)^{\alpha}} \to \gamma. \qquad (\star)$$

Proposition

If $(\star)$ holds with some $\gamma \in (0, 1]$ then $Y \in D(\beta)$, for some $\beta \in [0, 1)$, where $-\beta \in (-1, 0]$ is the unique solution of

$$\mathrm{Beta}(\alpha - 1, 1 - \beta) = \frac{\Gamma(\alpha - 1)\Gamma(1 - \beta)}{\Gamma(\alpha - \beta)} = \frac{1}{\gamma(\alpha - 1)}.$$

In particular, $Y \in D(0)$ for $\gamma = 1$. Conversely, if $Y \in D(\beta)$, $0 \le \beta < 1$, then $(\star)$ holds with

$$\gamma = \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(1 - \beta)} = \frac{1}{(\alpha - 1)\mathrm{Beta}(\alpha - 1, 1 - \beta)}.$$

Extension of a result by Fuchs, Joffe and Teugels (2001), where $\alpha = 2$.

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For $\alpha = 2$ this gives $1-\gamma = \beta$.

$$\mathbf{E}\frac{\sum_{i=1}^n Y_i^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha} = n\mathbf{E}\frac{Y_1^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha}$$
$$= \frac{n}{\Gamma(\alpha)}\mathbf{E}\int_0^\infty Y_1^\alpha e^{-t\sum_{i=1}^n Y_i} t^{\alpha-1}\mathrm{d}t$$
$$= \frac{n}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathbf{E}\left(e^{-tY_1}Y_1^\alpha\right)\left(\mathbf{E}e^{-tY_1}\right)^{n-1}\mathrm{d}t$$
$$=: \frac{n}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\phi_\alpha(t)\phi_0(t)^{n-1}\mathrm{d}t.$$

Note that for $\alpha = 2$ we have $\phi_\alpha = \phi_0''$.

$$s\int_0^\infty t^{\alpha-1}\phi_\alpha(t)e^{s\log\phi_0(t)}\mathrm{d}t \to \gamma\Gamma(\alpha), \quad s\to\infty.$$

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$$\phi_\alpha(t) = \mathbf{E}e^{-tY}Y^\alpha, \quad \phi_0(t) = \mathbf{E}e^{-tY}$$

By Karamata's Tauberian theorem

$$\lim_{t\to 0} \frac{\int_0^t y^{\alpha-1}\phi_\alpha(y)\mathrm{d}y}{1-\phi_0(t)} = \gamma\Gamma(\alpha).$$

After some further calculation

$$t^{\alpha-1}\frac{\int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}\mathrm{d}u}{\int_0^\infty \overline{G}(u)e^{-ut}\mathrm{d}u} \to \gamma\Gamma(\alpha), \text{ as } t \searrow 0.$$

$$u^{1-\alpha}e^{-ut} = \frac{1}{\Gamma(\alpha-1)}\int_0^\infty y^{\alpha-2}e^{-(y+t)u}\mathrm{d}y,$$

which holds for $u > 0$ and $\alpha \in (1, 2]$. *Weyl-transform*, or *Weyl-fractional integral* of the function e^{-ut}.

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We obtain

$$\int_1^\infty (u-1)^{\alpha-2} u^{-\alpha} \frac{g_\infty(x/u)}{g_\infty(x)} \mathrm{d}u = \frac{k \overset{M}{*} g_\infty(x)}{g_\infty(x)} \to [\gamma(\alpha-1)]^{-1}$$

with

$$g_\infty(x) = \int_0^\infty \overline{G}(ux) u^{\alpha-1} e^{-u} \mathrm{d}u.$$

$$k \overset{M}{*} h(x) = \int_0^\infty h(x/u) k(u)/u \mathrm{d}u$$

Mellin-convolution of h and k.
Drasin-Shea theorem implies that $g_\infty(x)$ is regularly varying at infinity with index $0 \geq \rho > -1$.

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Notation

$\mathrm{id}(a, b, \nu)$ infinitely divisible distribution on $\mathbb{R}^d$ with characteristic exponent

$$\mathrm{i}u'b - \frac{1}{2}u'au + \int \left(\mathrm{e}^{\mathrm{i}u'x} - 1 - \mathrm{i}u'xI(|x| \leq 1)\right)\nu(\mathrm{d}x),$$

where $b \in \mathbb{R}^d$, $a \in \mathbb{R}^{d \times d}$ is a positive semidefinite matrix and ν is the Lévy measure.

Theorem (K & Mason, 2012)

If along a subsequence $\{n'\}$

$$\frac{1}{a_{n'}}\sum_{i=1}^{n'} Y_i \xrightarrow{\mathcal{D}} W_2, \text{ as } n' \to \infty,$$

where $W_2 \sim \mathrm{id}(0, b, \Lambda)$, *then*

$$\left(\frac{\sum_{i=1}^{n'} X_i Y_i}{a_{n'}}, \frac{\sum_{i=1}^{n'} Y_i}{a_{n'}}\right) \xrightarrow{\mathcal{D}} (W_1, W_2),\ n' \to \infty,$$

where $(W_1, W_2) \sim \mathrm{id}(\mathbf{0}, \mathbf{b}, \Pi)$

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Theorem (K & Mason, 2012)

i.e. its characteristic function

$$\Psi(\theta_1,\theta_2) = \mathbf{E}\mathrm{e}^{\mathrm{i}(\theta_1 W_1+\theta_2 W_2)} = \exp\Big\{\mathrm{i}(\theta_1 b_1 + \theta_2 b_2) + \int_0^\infty\int_{-\infty}^\infty \Big(\mathrm{e}^{\mathrm{i}(\theta_1 x+\theta_2 y)} - 1 - (\mathrm{i}\theta_1 x + \mathrm{i}\theta_2 y)\,\mathbf{1}_{\{x^2+y^2\le 1\}}\Big) F(\mathrm{d}x/y)\,\Lambda(\mathrm{d}y)\Big\}.$$

$$H(x) = \mathbf{P}\left\{\frac{W_1}{W_2}\le x\right\} = \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Im\mathrm{m}\Psi(u,-ux)}{u}\mathrm{d}u.$$

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Feller class

$\xi, \xi_1, \ldots$ iid with df F, $S_n = \sum_{i=1}^n \xi_i$. F is in the *centered Feller class*, if there exists B_n, such that every subsequence n' has a further subsequence n'', such that

$$\frac{S_{n''}}{B_{n''}} \xrightarrow{\mathcal{D}} W,$$

where W is non-degenerate.

Theorem (Feller (1966), Maller (1979))

Y is in the centered Feller class, iff

$$\limsup_{x \to \infty} \frac{x^2 \mathbf{P}\{|Y| > x\} + x|\mathbf{E}YI(|Y| \le x)|}{\mathbf{E}[Y^2 I(|Y| \le x)]} < \infty.$$

Feller class

$\xi, \xi_1, \dots$ iid with df F, $S_n = \sum_{i=1}^{n} \xi_i$. F is in the *centered Feller class*, if there exists B_n, such that every subsequence n' has a further subsequence n'', such that

$$\frac{S_{n''}}{B_{n''}} \xrightarrow{\mathcal{D}} W,$$

where W is non-degenerate.

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Surprising result

Theorem (K & Mason, 2012)

The subsequential limit distributions of

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

are continuous for all X with finite expectation if and only if $Y \in \mathcal{F}_c$.

Outline

Towards Lévy processes

$$(W_1, W_2) \stackrel{\mathcal{D}}{=} (a_1 + U, a_2 + V),$$

where $(a_1, a_2) = \left(\left(b - \int_0^1 x\Lambda(\mathrm{d}x)\right)\mathbf{E}X, b - \int_0^1 x\Lambda(\mathrm{d}x)\right)$

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(\theta_1 U + \theta_2 V)} = \exp\left\{\int_0^\infty \int_{-\infty}^\infty \left(\mathrm{e}^{\mathrm{i}(\theta_1 x + \theta_2 y)} - 1\right) F(\mathrm{d}x/y)\,\Lambda(\mathrm{d}y)\right\}$$

Under the conditions of the theorem

$$\left(\frac{\sum_{1\le i\le n't} X_i Y_i}{a_{n'}}, \frac{\sum_{1\le i\le n't} Y_i}{a_{n'}}\right)_{t>0} \xrightarrow{\mathcal{D}} (a_1 t + U_t, a_2 t + V_t)_{t>0},$$

where (U_t, V_t), $t \ge 0$, is the corresponding Lévy process.

$$\frac{U_t}{V_t} \xrightarrow{\mathcal{D}} ?, \; t \to 0 \; \text{ or } \; t \to \infty$$

Kevei, P, Mason, D.M.
Randomly Weighted Self-normalized Lévy Processes
Stochastic Processes and their Applications, **123** (2) 2013, 490–522.