A note on the Kesten–Grincevičius–Goldie theorem

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Probabilistic Aspects of Harmonic Analysis

Outline

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Perpetuity equation

$$X \stackrel{\mathcal{D}}{=} AX + B,$$

where (A, B) and X on the right-hand side are independent.
Assume $\mathbf{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$, $A \neq 1$, and that $\log A$ conditioned on being nonzero is nonarithmetic.

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Applications

Actuarial application

$$B_1 + A_1 B_2 + A_1 A_2 B_2 + \dots$$

Financial mathematics: ARCH models and perpetuities (Embrechts & Klüppelberg & Mikosch); Branching processes in random environment, ...

Applications II

Exponential functional of Lévy processes:

$$J = \int_0^\infty e^{\xi_t} \mathrm{d}t$$

Carmona & Petit & Yor (2001); Bertoin & Yor (2005): survey; Maulik, Zwart, Kuznetsov, Pardo, Patie, Savov, Rivero, Behme, Lindner, Maller, ...

If (ξ_t) has finite jump activity and 0 drift then conditioning on its first jump time one has the perpetuity equation

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with B being an exponential random variable, independent of A, and the jump size is $\log A$.

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Applications III (self-advertising)

Random iterative geometric structures: K regular d-dimensional simplex with centroid $(0, 0, \ldots, 0)$ and vertices $(e_0, e_1, \ldots, e_d)$, $e_0 = (1, 0, \ldots, 0)$.
$K_0 = K$, p_{n+1} uniformly distributed random point in K_n, and $K_{n+1} = K_n \cap (p_{n+1} + K)$.
Clearly $\{K_n\}$ is a nested sequence of regular simplexes, which converges to a regular simplex.
The barycentric coordinates of the limiting simplex satisfy a d-dimensional perpetuity equation $\Rightarrow$ have $\mathcal{D}(d/(d+1), \ldots, d/(d+1))$ distribution. (Ambrus & K & Vígh (2011); Hitczenko & Letac (2014); K & Vígh (2016))

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Existence

$$X \stackrel{\mathcal{D}}{=} AX + B$$

If $\mathbf{E} \log A < 0$, $\mathbf{E} \log_+ |B| < \infty$, then there is a unique solution.
For NASC see Goldie, Maller (2001).

Tail asymptotic: heavy tails

$$X \stackrel{\mathcal{D}}{=} AX + B$$

Theorem (Kesten (1973))

If $\mathbf{E}|A|^\kappa = 1, \mathbf{E}|A|^\kappa \log_+ |A| < \infty$, $\mathbf{E}|B|^\kappa < \infty$ *then*

$$\mathbf{P}\{X > x\} \sim c_+ x^{-\kappa} \text{ and } \mathbf{P}\{X < -x\} \sim c_- x^{-\kappa} \text{ as } x \to \infty.$$

Goldie (1991) simplified proof (for more general equations), based on Grincevičius (1975)

Where is the slowly varying function $\ell(x)$ from the asymptotics?

$$\mathbf{P}\{X > x\} \sim \frac{\ell(x)}{x^\kappa}.$$

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Tail asymptotic: heavy tails II

$$X \stackrel{\mathcal{D}}{=} AX + B$$

Theorem (Grincevičius (1975), Grey (1994))

If $A \geq 0$, $\mathbf{E}A^{\kappa} < 1$, $\mathbf{E}A^{\kappa+\epsilon} < \infty$ then the tail of X is regularly varying with parameter $-\kappa$ if and only if the tail of B is.

That is, the regular variation of X is either caused by A alone, or by B alone.

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Tail asymptotics: light tails

If $\mathbf{P}\{|A| > 1\} > 0$ then the tail decreases at least polynomially (Goldie & Grübel, 1996). Can even be slowly varying: Dyszewski (2016)

Theorem (Goldie & Grübel (1996))

X has at least exponential tail under the assumption $|A| \leq 1$.

See also Hitczenko & Wesołowski 2009;
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Always assume

$$X \stackrel{\mathcal{D}}{=} AX + B,$$

$A \geq 0$, $\mathbf{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$, $A \not\equiv 1$, and that $\log A$

conditioned on being nonzero is nonarithmetic, $\mathbf{E}|B|^{\nu} < \infty$ for some $\nu > \kappa > 0$.

$\mathbf{E}A^\kappa = 1$

Assume that $\mathbf{E}A^\kappa = 1$, $\kappa > 0$. Put $F_\kappa(x) = \int_{-\infty}^x e^{\kappa y} F(\mathrm{d}y)$, $\log A \sim F$, and assume $\overline{F}_\kappa(x) = \ell(x) x^{-\alpha}$, $\alpha \in (0,1)$. That is $\mathbf{E}_\kappa \log A = \infty$!

The truncated expectation

$$m(x) = \int_0^x [F_\kappa(-u) + \overline{F}_\kappa(u)]\mathrm{d}u \sim \int_0^x \overline{F}_\kappa(u)\mathrm{d}u \sim \frac{\ell(x) x^{1-\alpha}}{1-\alpha}.$$

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$\mathbf{E}A^{\kappa} = 1$

Assume (Caravenna–Doney condition)

$$\lim_{\delta \to 0} \limsup_{x \to \infty} x \overline{F}_{\kappa}(x) \int_1^{\delta x} \frac{1}{y \overline{F}_{\kappa}(y)^2} F_{\kappa}(x - \mathrm{d}y) = 0.$$

Theorem (K)

If the assumptions above are satisfied then

$$\lim_{x \to \infty} m(\log x) x^{\kappa} \mathbf{P}\{X > x\} = C_{\alpha} \frac{1}{\kappa} \mathbf{E}[(AX + B)_+^{\kappa} - (AX)_+^{\kappa}],$$

$$\lim_{x \to \infty} m(\log x) x^{\kappa} \mathbf{P}\{X \le -x\} = C_{\alpha} \frac{1}{\kappa} \mathbf{E}[(AX + B)_-^{\kappa} - (AX)_-^{\kappa}].$$

Moreover, $\mathbf{E}[(AX + B)_+^{\kappa} - (AX)_+^{\kappa}] + \mathbf{E}[(AX + B)_-^{\kappa} - (AX)_-^{\kappa}] > 0.$

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Moreover, $\mathbf{E}[(AX+B)_+^{\kappa}-(AX)_+^{\kappa}]+\mathbf{E}[(AX+B)_-^{\kappa}-(AX)_-^{\kappa}]>0.$

Comments

Theorem is stated as a conjecture/open problem by Iksanov 2007.

The conditions of the theorem are stated in terms of F_{κ}. If

$$e^{\kappa x}\overline{F}(x) = \frac{\alpha\,\ell(x)}{\kappa\, x^{\alpha+1}}$$

with a slowly varying ℓ then $F_{\kappa} \in D(\alpha)$.

The Caravenna–Doney condition

$$\lim_{\delta\to 0}\limsup_{x\to\infty} x\overline{F}_{\kappa}(x)\int_1^{\delta x}\frac{1}{y\overline{F}_{\kappa}(y)^2}F_{\kappa}(x-\mathrm{d}y) = 0$$

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Comments II

X is closely related to the maximum $M = \max\{0, S_1, S_2, \ldots\}$ of the RW $S_n = \log A_1 + \log A_2 + \ldots + \log A_n$, $\log A_1, \log A_2, \ldots$ iid $\log A$ ($\mathbf{E}A^{\kappa} = 1$ implies that $\mathbf{E} \log A < 0$, so M is a.s. finite).
Korshunov (2005)

$$\lim_{x \to \infty} \mathbf{P}\{M > x\} e^{\kappa x} m(x) = c.$$

In specific cases this result is equivalent to our theorem. Let $(\xi_t)_{t\geq 0}$ be a nonmonotone Lévy process, $J = \int_0^\infty e^{\xi_t} \mathrm{d}t$, and $\overline{\xi}_\infty = \sup_{t\geq 0} \xi_t$. Arista and Rivero (2015) showed that $\mathbf{P}\{J > x\} \in \mathcal{RV}_{-\alpha}$ iff $\mathbf{P}\{e^{\overline{\xi}_\infty} > x\} \in \mathcal{RV}_{-\alpha}$.

If (ξ_t) has finite jump activity and 0 drift then conditioning on its first jump

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Comments III

Rivero (2007): Let $(\sigma_t)_{t\geq 0}$ be a nonlattice subordinator, such that $\mathbf{E}e^{\kappa\sigma_1} < \infty$ and $m(x) = \mathbf{E}I(\sigma_1 > x)e^{\kappa\sigma_1}$ is regularly varying with index $-\alpha \in (-1/2, -1)$. Consider the Lévy process $(\xi_t)_{t\geq 0}$ obtained by killing σ at ζ, an independent exponential time with parameter $\log \mathbf{E}e^{\kappa\sigma_1}$. Then for $J = \int_0^{\zeta} e^{\xi_t} \mathrm{d}t$
$\lim_{x\to\infty} m(\log x)x^{\kappa}\mathbf{P}\{J > x\} = c$.

Proof I

$$X \stackrel{\mathcal{D}}{=} AX + B,$$

$$\mathbf{P}\{X > e^x\} = [\mathbf{P}\{AX + B > e^x\} - \mathbf{P}\{AX > e^x\}] + \mathbf{P}\{AX > e^x\}$$

$\psi(x) = e^{\kappa x}(\mathbf{P}\{AX+B > e^x\} - \mathbf{P}\{AX > e^x\}),\ f(x) = e^{\kappa x}\mathbf{P}\{X > e^x\}$

using that X and A are independent

$f(x) = \psi(x) + A^\kappa e^{\kappa(x-\log A)}\mathbf{P}\{X > e^{x-\log A}\} = \psi(x) + \mathbf{E}f(x - \log A)A^\kappa.$

Under the measure $\mathbf{P}_\kappa\{\log A \in C\} = \mathbf{E}[I(\log A \in C)A^\kappa]$

$$f(x) = \psi(x) + \mathbf{E}_\kappa f(x - \log A).$$

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Proof II

$$f(x) = \psi(x) + \mathbf{E}_\kappa f(x - \log A).$$

We have

$$f(x) = \int_{\mathbb{R}} \psi(x-y) U(\mathrm{d}y),$$

where $U(x) = \sum_{n=0}^{\infty} F_\kappa^{*n}(x)$. If $\mathbf{E}_\kappa \log A < \infty$ then, from the renewal theorem

$$\lim_{x\to\infty} f(x) = m^{-1} \int_{\mathbb{R}} \psi(y) \mathrm{d}y,$$

which is the KGG theorem. In our case under $\mathbf{P}_\kappa$ $\log A \in D(\alpha)$, so $\mathbf{E}_\kappa \log A = \infty$.

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Infinite mean renewal theorems

Infinite mean analogue of SRT

$$\lim_{x\to\infty} m(x)[U(x+h) - U(x)] = hC_\alpha, \quad \forall h > 0.$$

Infinite mean SRT: Garsia & Lamperti (1963), Erickson (1970): for $\alpha \in (1/2, 1]$ assumption $H \in D(\alpha)$ implies SRT; for $\alpha \leq 1/2$ further assumptions are needed.
NASC for nonnegative random variables was given independently by Caravenna (2015+) and Doney (2015+):

$$\lim_{\delta\to 0} \limsup_{x\to\infty} x\overline{H}(x) \int_1^{\delta x} \frac{1}{y\overline{H}(y)^2} H(x - \mathrm{d}y) = 0.$$

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NASC for the regular variation of X?

$$X \in \mathcal{RV}_{-\kappa} \Rightarrow \mathbf{E}A^{\kappa} = 1\,?$$

If $X \in \mathcal{RV}_{-\kappa}$ then $\mathbf{E}|X|^p < \infty$ for all $p < \kappa$ and $\mathbf{E}|X|^p = \infty$ for all $p > \kappa$.

Theorem (Alsmeyer & Iksanov & Rösler (2009))

$\mathbf{E}|X|^p < \infty$ iff $\mathbf{E}A^p < 1$ and $\mathbf{E}|B|^p < \infty$.

Thus $X \in \mathcal{RV}_{-\kappa}$ implies $\mathbf{E}A^{\kappa} \le 1$. Can it be < 1?

Theorem (K)

Yes.

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Yes.

NASC for the regular variation of X?

$$X \in \mathcal{RV}_{-\kappa} \Rightarrow \mathbf{E}A^\kappa = 1\,?$$

If $X \in \mathcal{RV}_{-\kappa}$ then $\mathbf{E}|X|^p < \infty$ for all $p < \kappa$ and $\mathbf{E}|X|^p = \infty$ for all $p > \kappa$.

Theorem (Alsmeyer & Iksanov & Rösler (2009))

$\mathbf{E}|X|^p < \infty$ *iff* $\mathbf{E}A^p < 1$ *and* $\mathbf{E}|B|^p < \infty$.

Thus $X \in \mathcal{RV}_{-\kappa}$ implies $\mathbf{E}A^\kappa \leq 1$. Can it be < 1?

Theorem (K)

Yes.

Assume $\mathbf{E}A^\kappa = \theta < 1$ for some $\kappa > 0$, and $\mathbf{E}A^t = \infty$ for any $t > \kappa$.

$$F_\kappa(x) = \theta^{-1} \int_{-\infty}^{x} e^{\kappa y} F(\mathrm{d}y).$$

The assumption $\mathbf{E}A^t = \infty$ for all $t > \kappa$ means that F_κ is heavy-tailed.
To analyze the asymptotic behavior of the resulting defective renewal equation we use the techniques and results developed by Asmussen, Foss and Korshunov (2003).

Locally subexponential distributions

For some $T \in (0, \infty]$ let $\Delta = (0, T]$. For a df H we put $H(x + \Delta) = H(x + T) - H(x)$. A df H on $\mathbb{R}$ is in the class $\mathcal{L}_\Delta$ if $H(x + t + \Delta)/H(x + \Delta) \to 1$ uniformly in $t \in [0, 1]$, and it belongs to the class of Δ-subexponential distributions, $H \in \mathcal{S}_\Delta$, if $H(x + \Delta) > 0$ for x large enough, $H \in \mathcal{L}_\Delta$, and $(H * H)(x + \Delta) \sim 2H(x + \Delta)$. If $H \in \mathcal{S}_\Delta$ for every $T > 0$ then it is called *locally subexponential*, $H \in \mathcal{S}_{loc}$.

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Or assume simply that F_κ is a nice subexponential distribution.

Theorem (K)

Assume $\mathbf{E}A^\kappa = \theta < 1$*, and* F_κ *is a nice subexponential distribution. Then*

$$\lim_{x\to\infty} g(\log x)^{-1} x^\kappa \mathbf{P}\{X > x\} = \frac{\theta}{(1-\theta)^2\kappa}\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa],$$

$$\lim_{x\to\infty} g(\log x)^{-1} x^\kappa \mathbf{P}\{X \le -x\} = \frac{\theta}{(1-\theta)^2\kappa}\mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa],$$

where $g(x) = F_\kappa(x+1) - F_\kappa(x)$*. Moreover,*
$\mathbf{E}[(AX+B)_+^\kappa - (AX)_+^\kappa] + \mathbf{E}[(AX+B)_-^\kappa - (AX)_-^\kappa] > 0.$

Note that $g(\log x)$ is slowly varying.

Comment

In the Pareto case, $\overline{F}_\kappa(x) = c\,x^{-\beta}$, then $g(x) \sim c\beta x^{-\beta-1}$, and so $\mathbf{P}\{X > x\} \sim c' x^{-\kappa}(\log x)^{-\beta-1}$. In the lognormal case, $F_\kappa(x) = \Phi(\log x)$, with Φ being the standard normal df, $\mathbf{P}\{X > x\} \sim cx^{-\kappa} e^{-(\log\log x)^2/2} / \log x$, $c > 0$. For Weibull tails $\overline{F}_\kappa(x) = e^{-x^\beta}$, $\beta \in (0,1)$, we obtain $\mathbf{P}\{X > x\} \sim cx^{-\kappa}(\log x)^{\beta-1} e^{-(\log x)^\beta}$. Note that $\mathbf{E}|X|^\kappa < \infty$, so $\int_0^\infty x^{\kappa-1}\overline{F}_\kappa(x)\mathrm{d}x < \infty$.

Outline

Goldie's unified approach

Goldie obtained tail asymptotics for more general random equations. Consider the equation

$$X \stackrel{\mathcal{D}}{=} AX \vee B,$$

where $a \vee b = \max\{a, b\}$, $A \geq 0$ and (A, B) and X on the right-hand side are independent.
If $B \equiv 1$ then $\log X = M$, where $M = \max\{0, S_1, S_2, \ldots\}$, and $S_n = \log A_1 + \log A_2 + \ldots + \log A_n$, where $\log A_1, \log A_2, \ldots$ are iid $\log A$.

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Theorem (Goldie (1991))

If $\mathbf{E}A^\kappa = 1, \mathbf{E}A^\kappa \log_+ A < \infty$ *then there is a unique solution* X*, and* $\mathbf{P}\{X > x\} \sim cx^{-\kappa}$*.*

Theorem (K)

Assume $\mathbf{E}A^\kappa = 1$*,* $F_\kappa \in D(\alpha)$*, and the Caravenna–Doney condition holds. Then*

$$\lim_{x\to\infty} m(\log x) x^\kappa \mathbf{P}\{X > x\} = C_\alpha \frac{1}{\kappa} \mathbf{E}[(AX_+ \vee B_+)^\kappa - (AX_+)^\kappa].$$

For $B \equiv 1$ we get back Korshunov's result.

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$\mathbf{E}A^\kappa < 1$

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Assume $\mathbf{E}A^\kappa < 1$, and F_κ is a nice subexponential distribution.

$$\lim_{x\to\infty} g(\log x)^{-1} x^\kappa \mathbf{P}\{X > x\} = \frac{\theta}{(1-\theta)^2\kappa}\mathbf{E}[(AX_+ \vee B_+)^\kappa - (AX_+)^\kappa],$$

where $g(x) = F_\kappa(x+1) - F_\kappa(x)$.

In the special case $B \equiv 1$ we have the following.

Corollary

$S_n = \log A_1 + \log A_2 + \ldots + \log A_n$, $M = \max\{0, S_1, S_2, \ldots\}$. *Then*

$$\mathbf{P}\{M > x\} \sim c g(x) e^{-\kappa x},$$

where $g(x) = F_\kappa(x+1) - F_\kappa(x)$.

$\mathbf{E}A^{\kappa} < 1$

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