Tail probabilities of St. Petersburg sums, trimmed sums, and their limit

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Conference on Ambit Fields and Related Topics
Aarhus, Denmark

Outline

St. Petersburg game
- Sum
- Maximum

Conditioning on the maximum
- Number of maximum terms
- Conditional limit results

Trimmed sums
- Finite number of summands
- Properties of the r-trimmed limit

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St. Petersburg paradox

Nicolaus Bernoulli (1713): Paul's gain X, then

$$\mathbf{P}\{X = 2^k\} = \frac{1}{2^k}, \quad k = 1, 2, \ldots$$

What is the fair price?
Paradox:

$$\mathbf{E}(X) = \sum_{k=1}^{\infty} 2^k \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$$

$$\text{but } \mathbf{P}\{X > 40\} = 2^{-5} = 0.03125$$

'there ought not be a sane man who would not happily sell his chance for forty ducats' – Nicolaus Bernoulli

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LLN

$X_1, X_2, \dots$ iid St. Petersburg rv's $S_n = \sum_{k=1}^n X_k$

Theorem (Feller (1945))

$$\frac{S_n}{n \log_2 n} \xrightarrow{\mathbf{P}} 1$$

There are no strong laws!

Theorem (Adler (1990), Chow & Robbins (1961))

$$\liminf_{n\to\infty} \frac{S_n}{n \log_2 n} = 1 \quad a.s., \quad \limsup_{n\to\infty} \frac{S_n}{n \log_2 n} = \infty \quad a.s.$$

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CLT

$$\frac{S_n - c_n}{a_n} \xrightarrow{\mathcal{D}} ?$$

Doeblin–Gnedenko criterion:

$$\mathbf{P}\{X \leq x\} = \begin{cases} 0, & \text{for } x < 2, \\ 1 - 2^{-\lfloor \log_2 x \rfloor} = 1 - \frac{2^{\{\log_2 x\}}}{x}, & \text{for } x \geq 2, \end{cases}$$

$2^{\{\log_2 x\}}$ is not slowly varying ($\{\cdot\}$ fractional part)

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$2^{\{\log_2 x\}}$ is not slowly varying ($\{\cdot\}$ fractional part) $\Rightarrow$ there is no limit theorem for $\frac{S_n - c_n}{a_n}$ for any choice of a_n, c_n.

There is on subsequences!

Theorem (Martin-Löf (1985))

$$\frac{S_{2^n}}{2^n} - n \xrightarrow{\mathcal{D}} W\,, \quad \textit{as } n \to \infty\,.$$

W semistable rv. Moreover, convergence holds on subsequences $n_k = \lfloor \gamma 2^k \rfloor$, $\gamma \in (1/2, 1]$.

Theorem (Csörgő & Dodunekova (1991))

$\frac{S_{n_k}}{n_k} - \log_2 n_k$ *converges in distribution if and only if*

$$\gamma_{n_k} = \frac{n_k}{2^{\lceil \log_2 n_k \rceil}} \longrightarrow \gamma \in (1/2, 1]\,.$$

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Merging

Theorem (Csörgő (2002))

$$\sup_{x\in\mathbb{R}} \left| \mathbf{P}\left\{ \frac{S_n}{n} - \log_2 n \le x \right\} - G_{\gamma_n}(x) \right| \to 0, \quad as\ n \to \infty$$

The limit

Characteristic function of W_γ, $\gamma \in (1/2, 1]$,

$$\mathbf{E}\left(e^{\mathrm{i}tW_\gamma}\right) = \exp\left(\mathrm{i}ta + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) \mathrm{d}R_\gamma(x)\right),$$

with right-hand-side Lévy function

$$R_\gamma(x) = -\frac{\gamma}{2^{\lfloor \log_2(\gamma x) \rfloor}} = -\frac{2^{\{\log_2(\gamma x)\}}}{x}, \quad x > 0.$$

(semistable laws)

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Trimmed LLN

$X_1, X_2, \ldots$ iid St. Petersburg rv's,

$$S_n = X_1 + \ldots + X_n \quad \text{and} \quad X_n^* = \max_{1 \leq i \leq n} X_i$$

Theorem (Csörgő and Simons (1996))

$$\lim_{n \to \infty} \frac{S_n - X_n^*}{n \log_2 n} = 1 \quad a.s.$$

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Merging again

For $\gamma \in (1/2, 1]$ ($\approx$ Fréchet)

$$H_\gamma(x) = \begin{cases} 0, & \text{for } x \le 0, \\ \exp\left(-\gamma 2^{-\lfloor \log_2(\gamma x) \rfloor}\right), & \text{for } x > 0. \end{cases}$$

Theorem (Berkes, Csáki & Csörgő (1999))

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}\left\{ \frac{X_n^*}{n} \le x \right\} - H_{\gamma_n}(x) \right| = O(n^{-1}), \quad \textit{as } n \to \infty.$$

Typical value: $X_n^* \approx 2^{\lceil \log_2 n \rceil + j}$, $j \in \mathbb{Z}$.

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Maximum and sum

Theorem (Darling (1952), Breiman (1965))

$Y, Y_1, Y_2, \ldots$ iid ≥ 0.

$$\frac{\max_{i \leq n} Y_i}{\sum_{i=1}^n Y_i} \xrightarrow{\mathcal{D}} Z,$$

with Z nondegenerate, iff $Y \in D(\alpha)$, $\alpha \in (0,1)$; $Z = 1$ iff $\mathbf{P}\{Y > y\}$ is slowly varying, and $Z = 0$ iff $\sqrt{Y} \in D(2)$.

St.Petersburg case:

$$\frac{X_n^*}{S_n} \xrightarrow{\mathbf{P}} 0.$$

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Number of maximum terms

Joint with Gábor Fukker and László Györfi.

For $j \in \mathbb{Z}$ and $\gamma \in [1/2, 1]$ introduce the notation

$$p_{j,\gamma} = e^{-\gamma 2^{-j}} \left(1 - e^{-\gamma 2^{-j}}\right), \quad \gamma_n = \frac{n}{2^{\lceil \log_2 n \rceil}}$$

Lemma

$$\sup_{j \in \mathbb{Z}} \left| \mathbf{P}\left\{X_n^* = 2^{\lceil \log_2 n \rceil + j}\right\} - p_{j,\gamma_n} \right| = O(n^{-1}).$$

In particular for any $j \in \mathbb{Z}$, as $n \to \infty$

$$\mathbf{P}\left\{X_n^* = 2^{\lceil \log_2 n \rceil + j}\right\} \sim e^{-\gamma_n 2^{-j}} \left(1 - e^{-\gamma_n 2^{-j}}\right).$$

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Small maximum

Put $N_n = |\{k : 1 \leq k \leq n, X_k = X_n^*\}|$.

Proposition

Conditionally on $X_n^* = 2^{k_n}$, where $\log_2 n - k_n \to \infty$

$$\frac{N_n - \mathbf{E}[N_n | X_n^* = 2^{k_n}]}{\sqrt{\mathbf{Var}(N_n | X_n^* = 2^{k_n})}} \xrightarrow{\mathcal{D}} N(0,1), \quad as\ n \to \infty.$$

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Typical maximum

Proposition (Gut & Martin-Löf (2016))

Conditionally on $X_n^* = 2^{\lceil \log_2 n \rceil + j}$, $j \in \mathbb{Z}$,

$$N_n \xrightarrow{\mathcal{D}} M_{j,\gamma_n} \quad \textit{(in the merging sense)},$$

where $M_{j,\gamma} \sim$ *Poisson*$(2^{-j}\gamma)$ *conditioned on not being zero.*

Large maximum

Proposition

While, if $k_n - \log_2 n \to \infty$ *then conditionally on* $X_n^* = 2^{k_n}$

$$N_n \xrightarrow{\mathbf{P}} 1, \quad as \;\; n \to \infty.$$

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Notation

$$\mathbf{P}\left\{X = 2^i | X \leq 2^k\right\} = 2^{-i}/(1 - 2^{-k})$$

$$F_k(x) = \mathbf{P}\left\{X \leq x | X \leq 2^k\right\} = \begin{cases} \frac{1}{1-2^{-k}}\left[1 - \frac{2^{\{\log_2 x\}}}{x}\right], & x \in [2, 2^k], \\ 1, & x > 2^k. \end{cases}$$

$X^{(k)}, X_1^{(k)}, \ldots,$ are iid F_k, and

$$S_n^{(k)} = X_1^{(k)} + \ldots + X_n^{(k)}$$

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Conditioning on small maximum

Proposition

Given that $X_n^* = 2^{k_n}$, $k_n \geq 2$, *such that* $\log_2 n - k_n \to \infty$

$$\frac{S_n - \mathbf{E}[S_n | X_n^* = 2^{k_n}]}{\sqrt{\mathbf{Var}\left(S_n | X_n^* = 2^{k_n}\right)}} \xrightarrow{\mathcal{D}} N(0,1).$$

Conditioning on typical maximum

Proposition

$$\frac{S_{n_k}^{(\lceil \log_2 n_k \rceil + j)}}{n_k} - \log_2 n_k$$

converges in distribution iff $\gamma_{n_k} \to \gamma$. *The limit* $W_{j,\gamma}$

$$\varphi_{j,\gamma}(t) = \mathbf{E} e^{\mathrm{i}tW_{j,\gamma}} = \exp\left[\mathrm{i}tu_{j,\gamma} + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \mathrm{i}tx\right) \mathrm{d}L_{j,\gamma}(x)\right],$$

with

$$L_{j,\gamma}(x) = \begin{cases} \gamma 2^{-j} - \frac{2^{\{\log_2(\gamma x)\}}}{x}, & \text{for } x < 2^j \gamma^{-1}, \\ 0, & \text{for } x \geq 2^j \gamma^{-1}, \end{cases}$$

Conditioning on typical maximum

Proposition

For $j \in \mathbb{Z}$ we have

$$\left| \mathbf{P}\left\{ \frac{S_n}{n} - \log_2 n \le x | X_n^* = 2^{\lceil \log_2 n \rceil + j} \right\} - \widetilde{G}_{j,\gamma_n}(x) \right| \to 0,$$

where

$$\widetilde{G}_{j,\gamma}(x) = \sum_{m=1}^{\infty} G_{j-1,\gamma}\left(x - m\frac{2^j}{\gamma} \right) \frac{(2^{-j}\gamma)^m}{m!}(e^{2^{-j}\gamma} - 1)^{-1}.$$

Corollary

Theorem (Gut & Martin-Löf (2016))

For any $\gamma \in [1/2, 1]$

$$G_\gamma(x) = \sum_{j=-\infty}^{\infty} \widetilde{G}_{j,\gamma}(x) e^{-\gamma 2^{-j}} \left(1 - e^{-\gamma 2^{-j}}\right).$$

This is equivalent to the distributional representation

$$W_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma - 1,\gamma} + M_{Y_\gamma,\gamma} 2^{Y_\gamma} \gamma^{-1},$$

where $(W_{j,\gamma})_{j\in\mathbb{Z}}$, $(M_{j,\gamma})_{j\in\mathbb{Z}}$ and Y_γ are independent, $Y_\gamma \sim (p_{j,\gamma})_{j\in\mathbb{Z}}$, $M_{j,\gamma} \sim \text{Poisson}(\gamma 2^{-j})$ conditioned on not being 0.

Corollary

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Buchmann, Fan & Maller (2016) result

Lévy process setup: W_γ is a semistable Lévy process at time 1.

$$W_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma - 1, \gamma} + M_{Y_\gamma, \gamma} 2^{Y_\gamma} \gamma^{-1},$$

The value $2^{Y_\gamma}/\gamma$ corresponds to the maximum jump, $M_{Y_\gamma,\gamma}$ is the number of the maximum jumps, and $W_{Y_\gamma-1,\gamma}$ has the law of the Lévy process conditioned on that the maximum jump is strictly less than $2^{Y_\gamma}/\gamma$.

This kind of distributional representations for general Lévy processes were obtained by Buchmann, Fan & Maller (2016).

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This kind of distributional representations for general Lévy processes were obtained by Buchmann, Fan & Maller (2016).

Conditioning on large maximum

Proposition

Assume that $k_n - \log_2 n \to \infty$. *Given that* $X_n^* = 2^{k_n}$

$$\frac{S_n}{X_n^*} - A_n \xrightarrow{\mathbf{P}} 1,$$

where

$$A_n = \frac{nk_n}{2^{k_n}}.$$

Conditioning on the maximum

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Subexponential distributions

$Y, Y_1, Y_2, \ldots$ iid ≥ 0, G. $\overline{G}(x) = 1 - G(x)$.

G is *subexponential*, $G \in \mathcal{S}$,

$$\lim_{x\to\infty} \frac{\overline{G*G}(x)}{\overline{G}(x)} = 2,$$

Characterizing property of $\mathcal{S}$: for any $n \geq 1$

$$\lim_{x\to\infty} \frac{\mathbf{P}\{Y_1 + \ldots + Y_n > x\}}{\mathbf{P}\{\max\{Y_i : i = 1, 2, \ldots, n\} > x\}} = 1,$$

equivalently

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O-subexponential distributions

Goldie (1978): St. Petersburg distribution F is not subexponential.

$$2 = \liminf_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} < \limsup_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 4.$$

G is *O-subexponential* (Klüppelberg, 1990), $G \in \mathcal{OS}$, if

$$\ell^*(G) := \limsup_{x\to\infty} \frac{\overline{G * G}(x)}{\overline{G}(x)} < \infty.$$

always $\liminf \geq 2$; $= 2$ for heavy-tailed (Foss & Korshunov 2007)

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always $\liminf \geq 2$; $= 2$ for heavy-tailed (Foss & Korshunov 2007)

O-subexponential distributions

Goldie (1978): St. Petersburg distribution F is not subexponential.

$$2 = \liminf_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} < \limsup_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 4.$$

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Shimura and Watanabe (2005): $G \in \mathcal{OS}$, $\forall \varepsilon > 0$, $\exists c > 0$,

$$\frac{\overline{G^{n*}}(x)}{\overline{G}(x)} \leq c(\ell^*(G) - 1 + \varepsilon)^n.$$

Notation

$X, X_1, X_2, \ldots$ iid St. Petersburg rv's
$X_{1n} \leq X_{2n} \leq \ldots \leq X_{nn}$ ordered sample of $X_1, X_2, \ldots, X_n$.
r-trimmed sum: $S_{n,r} = \sum_{k=1}^{n-r} X_{kn}$.

Tail of the sums

Theorem

As n, r fix, $x \to \infty$

$$\mathbf{P}\{S_{n,r} > x\} \sim \frac{2^{(r+1)\{\log_2 x\}}}{x^{r+1}} \binom{n}{r+1} \times \left(1 + \mathbf{P}\left\{S_{n-r-1} > x(1 - 2^{-\{\log_2 x\}})\right\}(2^{r+1} - 1)\right).$$

In particular, for any $0 < \delta < 1$,

$$\lim_{x\to\infty, \{\log_2 x\} > \delta} \mathbf{P}\{S_{n,r} > x\} \frac{x^{r+1}}{2^{(r+1)\{\log_2 x\}}} = \binom{n}{r+1}.$$

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$r = 0$

sumdf16iter10m.png

Almost subexponential

Untrimmed case:

$$\mathbf{P}\{S_n > x\} \sim \frac{2^{\{\log_2 x\}}}{x} n \left(1 + \mathbf{P}\left\{S_{n-1} > x(1 - 2^{-\{\log_2 x\}})\right\}\right)$$

from which

$$n = \liminf_{x\to\infty} x\mathbf{P}\left\{S_n > x\right\} < \limsup_{x\to\infty} x\mathbf{P}\left\{S_n > x\right\} = 2n.$$

Since $x\mathbf{P}\{X > x\} = 2^{\{\log_2 x\}}$, $x \geq 2$, we have

$$\lim_{x\to\infty, \{\log_2 x\}\geq\delta} \frac{\mathbf{P}\{S_n > x\}}{\mathbf{P}\{X > x\}} = n.$$

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Outline

St. Petersburg game
- Sum
- Maximum

Conditioning on the maximum
- Number of maximum terms
- Conditional limit results

Trimmed sums
- Finite number of summands
- Properties of the r-trimmed limit

$E_k, k = 1, 2, \ldots$ iid Exp(1), $Z_k = E_1 + \ldots + E_k$

Theorem

Let $n_k = \lfloor \gamma 2^k \rfloor$, for some $\gamma \in (1/2, 1]$. Then for any $r \geq 0$

$$\frac{1}{n_k} S_{n_k,r} - a^{(r)}_{n_k,\gamma} \xrightarrow{\mathcal{D}} Y_{r,\gamma} = \sum_{k=r+1}^{\infty} \gamma^{-1} \left(2^{-\lfloor \log_2 Z_k/\gamma \rfloor} - 2^{-\lfloor \log_2 k/\gamma \rfloor} \right),$$

with centering sequence

$$a^{(r)}_{n,\gamma} = \gamma^{-1} \sum_{j=r+1}^{n} 2^{-\lfloor j/\gamma \rfloor}.$$

Proof (sketch)

Quantile method & LePage, Woodroofe, Zinn idea.

Quantile representation: $(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} (Q(U_{1n}), \ldots, Q(U_{nn}))$, where $F^{-1}(s) = Q(s) = \inf\{x : s \le F(x)\}$

$$Q(s) = \begin{cases} 2, & s = 0, \\ 2^{\lceil -\log_2(1-s) \rceil} = \frac{2^{\{\log_2(1-s)\}}}{1-s}, & s \in (0,1). \end{cases}$$

$(E_i)_{i \in \mathbb{N}}$ iid Exp(1), $Z_n = E_1 + \ldots + E_n$. For n fix

$$(U_{1n}, U_{2n}, \ldots, U_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{Z_1}{Z_{n+1}}, \frac{Z_2}{Z_{n+1}}, \ldots, \frac{Z_n}{Z_{n+1}} \right),$$

where U's are ordered sample of n iid Uniform(0, 1).

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Proof

$\Psi(x) = 2^{\{\log_2 x\}}$ (grows linearly from 1 to 2 in each $[2^j, 2^{j+1})$).
$Q(1-s) = \Psi(s)/s$

$$(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{Z_{n+1}}{Z_1} \Psi(Z_1/Z_{n+1}), \ldots, \frac{Z_{n+1}}{Z_n} \Psi(Z_n/Z_{n+1}) \right)$$

SLLN $Z_{n+1}/n \to 1$ a.s.

$$X^*_{j,n} = \frac{n}{Z_j} \Psi \left(\frac{Z_j}{n} \right) (1 + o(1)) \qquad \text{a.s.}$$

$$\Psi(Z_j/n) = \Psi(Z_j/\gamma_n)$$

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LePage, Woodroofe & Zinn (1981)

$Y, Y_1, Y_2, \ldots$ iid, ≥ 0, $Y \in D(\alpha)$, S_n partial sum, $(S_n - nb_n)/a_n \to S$. $Y_{1,n} \geq Y_{2,n} \geq \ldots \geq Y_{n,n}$

$$S = \sum_{k=1}^{\infty} \left(Z_k^{-1/\alpha} - \mathbf{E} Z_k^{-1/\alpha} I(Z_k^{-1/\alpha} < 1) \right),$$

where $E_1, E_2, \ldots$ are iid Exp(1), $Z_k = E_1 + \ldots + E_k$. Moreover,

$$\left(\frac{S_n - nb_n}{a_n}, \left(\frac{Y_{1,n}}{a_n}, \ldots, \frac{Y_{n,n}}{a_n} \right) \right) \xrightarrow{\mathcal{D}} \left(S, (Z_1^{-1/\alpha}, Z_2^{-1/\alpha}, \ldots) \right).$$

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On the centering

For any $\gamma \in (1/2, 1]$, $n_k = \lfloor \gamma 2^k \rfloor$,

$$a_{n_k,\gamma}^{(0)} - \log_2 n_k \to 2 - \frac{1}{\gamma} \sum_{k=1}^{\infty} \frac{k \varepsilon_k}{2^k} - \log_2 \gamma = \xi(\gamma),$$

where $\gamma = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$.

Steinhaus' resolution of the St. Petersburg paradox (Csörgő & Simons 1993)

ξ is right-continuous, left-continuous except at dyadic rationals greater than $1/2$ and has unbounded variation (Csörgő & Simons 1993); the Hausdorff and box-dimension of the graph of ξ is 1 (Kern & Wedrich 2014).

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Properties of the r-trimmed limit

$\xi(\gamma)$

xi.png

Tail of the trimmed limit

$$A_{r,\gamma} = \gamma^{-1} \sum_{k=1}^{r} 2^{\lfloor k/\gamma \rfloor}$$

Theorem

$$\mathbf{P}\{ Y_{r,\gamma} > x \} \sim \frac{2^{\{\log_2(\gamma x)\}(r+1)}}{(r+1)!\, x^{r+1}} \Bigg[2^{-r-1} + (2^{r+1} - 1) \times \sum_{\ell=0}^{1} 2^{-\ell(r+1)} \mathbf{P} \left\{ Y_{0,\gamma} + A_{r,\gamma} > x \left(1 - 2^{\ell - \{\log_2(\gamma x)\}} \right) \right\} \Bigg].$$

Untrimmed case

$$\mathbf{P}\{Y_{0,\gamma} > x\} \sim \frac{2^{\{\log_2(\gamma x)\}}}{x} \times \left[2^{-1} + \sum_{\ell=0}^{1} 2^{-\ell} \mathbf{P}\left\{Y_{0,\gamma} > x\left(1 - 2^{\ell - \{\log_2(\gamma x)\}}\right)\right\}\right].$$

Exactly the tail of the Lévy measure appears

$$\frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} \sim 2^{-1} + \sum_{\ell=0}^{1} 2^{-\ell} \mathbf{P}\left\{Y_{0,\gamma} > x\left(1 - 2^{\ell - \{\log_2(\gamma x)\}}\right)\right\}\right].$$

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$$< \limsup_{x\to\infty} \frac{\mathbf{P}\{Y_{0,\gamma} > x\}}{-R_\gamma(x)} = 1 + \mathbf{P}\{Y_{0,\gamma} > 0\}$$

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For any $\delta \in (0, 1/2)$ we have

$$\lim_{x\to\infty, \delta < \{\log_2(\gamma x)\} < 1-\delta} \mathbf{P}\{Y_{r,\gamma} > x\} \frac{x}{2^{\{\log_2(\gamma x)\}}} = 1 \,.$$

In the untrimmed case ($r = 0$) for $\gamma = 1$

$$\mathbf{P}\{Y_{0,1} > 2^m + c\} \sim 2^{-m}\left[1 + \mathbf{P}\{Y_{0,1} > c\}\right], \quad \text{as } m \to \infty,$$

(Martin-Löf 1985).

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Watanabe & Yamamuro (2012) result

For general semistable distributions:

$$\lim_{n\to\infty} 2^n \mathbf{P}\{W_1 > x2^n\} = -R_1(x) + [R_1(x-) - R_1(x)]\mathbf{P}\{W_1 > 0\}$$

$$C_* = \liminf_{x\to\infty} \frac{\mathbf{P}\{W > x\}}{-R(x)} \leq \limsup_{x\to\infty} \frac{\mathbf{P}\{W > x\}}{-R(x)} = C^*,$$

with

$$C_* = 1 - (1 - Q^{-1})\mathbf{P}\{W < 0\}, \; C^* = Q + (Q-1)\mathbf{P}\{W < 0\},$$

and $Q = \sup_{x\in[1,2]} R(x-)/R(x)$.

‘the modern student will hardly understand the mysterious discussions of this “paradox” ’– Feller

A natural example, which is not in the domain of attraction of any stable law, but it is in the domain of geometric partial attraction of a semistable law.
Not subexponential, but tractable tail behavior.

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