A note on asymptotics of linear combinations of iid random variables

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Abstract
Let $X_1, X_2, \ldots$ be iid random variables, and let $\boldsymbol{a}_n = (a_{1,n}, \ldots, a_{n,n})$ be an arbitrary sequence of weights. We investigate the asymptotic distribution of the linear combination $S_{\boldsymbol{a}_n} = a_{1,n}X_1 + \cdots + a_{n,n}X_n$ under the natural negligibility condition $\lim_{n\to\infty} \max\{|a_{k,n}| : k = 1, \ldots, n\} = 0$. We prove that if $S_{\boldsymbol{a}_n}$ is asymptotically normal for a weight sequence $\boldsymbol{a}_n$, in which the components are of the same magnitude, then the common distribution belongs to $\mathbb{D}(2)$.

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1 Introduction

Let $X, X_1, X_2, \ldots$ be iid random variables with the common distribution function $F(x) = \mathbf{P}\{X \leq x\}$. For each $n \in \mathbb{N} = \{1, 2, \ldots\}$ consider the random variable

$$S_{\boldsymbol{a}_n} = a_{1,n}X_1 + a_{2,n}X_2 + \cdots + a_{n,n}X_n\,,$$

where $\boldsymbol{a}_n = (a_{1,n}, \ldots, a_{n,n})$ is an arbitrary sequence of weights. We investigate the asymptotic behavior of the weighted sum $S_{\boldsymbol{a}_n}$, therefore it is reasonable to assume that each component is asymptotically negligible, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{1\leq k\leq n} \mathbf{P}\{|a_{k,n}X_k| \geq \varepsilon\} = \lim_{n\to\infty} \mathbf{P}\{|X| \geq \varepsilon/\overline{a}_n\} = 0,$$

where $\overline{a}_n = \max\{|a_{k,n}| : k = 1, 2, \ldots, n\}$, which holds, if and only if $\overline{a}_n \to 0$, as $n \to \infty$. Therefore from now on we assume that $\overline{a}_n \to 0$. In the followings asymptotic relations are meant as $n \to \infty$, unless otherwise specified.

Since the possible limiting distributions of $S_{\boldsymbol{a}_n}$ are necessarily infinitely divisible, we need the well-known representation of their characteristic functions. Let Y be an infinitely divisible real random variable with characteristic function $\phi(t) = \mathbf{E}(\mathrm{e}^{\mathrm{i}tY})$ in its Lévy form ([4] p. 70), given for each $t \in \mathbb{R}$ by

$$\phi(t) = \exp\left\{\mathrm{i}t\theta - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \beta_t(x)\,\mathrm{d}L(x) + \int_{0}^{\infty} \beta_t(x)\,\mathrm{d}R(x)\right\},$$

where

$$\beta_t(x) = \mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}$$

and the constants $\theta \in \mathbb{R}$ and $\sigma \geq 0$ and the functions $L(\cdot)$ and $R(\cdot)$ are uniquely determined: $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $\lim_{x\to-\infty} L(x) = L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0, \infty)$ with $\lim_{x\to\infty} R(x) = R(\infty) = 0$, such that $\int_{-\varepsilon}^{0} x^2 \mathrm{d}L(x) + \int_0^{\varepsilon} x^2 \mathrm{d}R(x) < \infty$ for every $\varepsilon > 0$.

As usual, we say that the distribution F is in the *domain of attraction* of the α-stable law W, $\alpha \in (0, 2]$, written $F \in \mathbb{D}(\alpha)$, if for some centering and norming sequence A_n and C_n

$$\frac{1}{C_n}\Big[\sum_{k=1}^{n} X_k - A_n\Big] \xrightarrow{\mathcal{D}} W\,,$$

where $X_1, X_2, \ldots$ are iid random variables with distribution function F. Moreover, F is in the domain of *partial* attraction of the infinitely divisible random variable W, written $F \in \mathbb{D}_p(W)$, if there exist a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$, and centering and norming sequence A_n, C_n, such that the convergence takes place along k_n, that is

$$\frac{1}{C_n}\Big[\sum_{k=1}^{k_n} X_k - A_n\Big] \xrightarrow{\mathcal{D}} W\,. \tag{1}$$

For an α-stable W we write $\mathbb{D}_p(\alpha)$ instead of $\mathbb{D}_p(W)$.

The motivation for considering linear combinations of iid random variables comes from a recent result in [7] in connection with semistable laws.

An infinitely divisible law is called semistable, if and only if either it is normal, or in its Lévy representation the normal component $\sigma = 0$ and the spectral functions can be written in the form $L(x) = M_L(x)/|x|^{\alpha}$, $x < 0$, and

$R(x) = -M_R(x)/x^\alpha$, $x > 0$, for some $\alpha \in (0, 2)$, where M_L and M_R are nonnegative bounded functions $M_L(\cdot)$ on $(-\infty, 0)$ and $M_R(\cdot)$ on $(0, \infty)$, one of which has strictly positive infimum and the other one either has strictly positive infimum or is identically zero, and $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x > 0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x < 0$, with the same period $c > 1$. The following result of Kruglov [9] shows the probabilistic meaning of the semistability and c. If (1) holds for some $F(\cdot)$ along some $\{k_n\}$ for which $\lim_{n\to\infty} k_{n+1}/k_n = c$ for some $c \in (1, \infty)$, then the limit distribution is necessarily semistable and, when the exponent $\alpha < 2$, the common multiplicative period of $M_R(\cdot)$ and $M_L(\cdot)$ is the $c^{1/\alpha}$ from the latter growth condition on $\{k_n\}$. In this case F is in the domain of *geometric partial* attraction of the semistable law W, $F \in \mathbb{D}_{gp}(W)$. The converse of the result above is also true, that is for any semistable law W, $\mathbb{D}_{gp}(W) \neq \emptyset$. For more background about semistable laws we refer to [10].

The above mentioned result is the following corollary in [7]:

Corollary. *For any semistable law W and for any distribution function $F \in \mathbb{D}_{gp}(W)$, there exist an $\{\boldsymbol{a}_n\}_{n=1}^\infty$ sequence of weights and a well determined centering sequence A_n, such that $S_{\boldsymbol{a}_n} - A_n \xrightarrow{\mathcal{D}} W$. Moreover, in each row of the weight sequence there are only two different components $a_{n,1} \geq a_{n,2}$, and $\lim_{n\to\infty} a_{n,1}/a_{n,2} = c$, where c comes from the representation of W.*

Classical theory says that if limiting distribution exists for a uniform weight sequence, that is when each components in a row are equal, then it must be stable. As an essence of semistability, this corollary claims that semistable limiting distributions can be achieved by weight sequences that consist of only two different components. We will see the contrast of this corollary and Theorem 6.

The prototypes of random variables from the domain of geometric partial attraction of a semistable laws are the so-called St. Petersburg games. Since these games are our base motivations for working with special types of linear combinations, we spell out the details. For $\alpha \in (0, 2)$ and $p \in (0, 1)$ consider the St. Petersburg(α, p) game, where Peter, the banker, tosses a possibly biased coin until it lands 'heads' and pays r^k ducats to Paul, if this happens on the k^{th} toss, where $r = 1/q$, $q = 1 - p$, and p is the probability of 'heads' at each throw. If there are n players, Paul$_1$, Paul$_2$, ..., Paul$_n$, each of them playing exactly one game, then before they play, they may agree to use a *pooling strategy* $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$, where the components are nonnegative and add to unity. If X_k denotes the gain of Paul$_k$, $k = 1, 2, \ldots, n$, under this strategy Paul$_1$ receives $p_{1,n}^{1/\alpha}X_1 + p_{2,n}^{1/\alpha}X_2 + \cdots + p_{n,n}^{1/\alpha}X_n$ ducats, Paul$_2$ receives $p_{n,n}^{1/\alpha}X_1 + p_{1,n}^{1/\alpha}X_2 + \cdots + p_{n-1,n}^{1/\alpha}X_n$ ducats, ..., Paul$_n$ receives

$p_{2,n}^{1/\alpha} X_1 + p_{3,n}^{1/\alpha} X_2 + \cdots + p_{1,n}^{1/\alpha} X_n$ ducats. Consider the centered version of Paul$_1$'s gain

$$S_{\boldsymbol{p}_n}^{\alpha,p} = \sum_{k=1}^{n} p_{k,n}^{1/\alpha} X_k - \frac{p}{q} H_{\alpha,p}(\boldsymbol{p}_n),$$

where the constant $H_{\alpha,p}(\boldsymbol{p}_n)$ depends only on the strategy. Under the negligibility assumption it is possible to prove a merging asymptotic expansion for arbitrary sequence of strategies. But what is more important from our point of view, it turns out that if the components are integer powers of q, then a usual limit theorem holds:

$$S_{\boldsymbol{p}_n}^{\alpha,p} \xrightarrow{\mathcal{D}} W_\alpha,$$

where W_α is a well-determined semistable random variable with characteristic exponent α. In this case the merging asymptotic expansions reduce to usual asymptotic expansions, so it is possible to determine the rate of convergence. These results rely on deep Fourier analysis, and on the specific forms of the St. Petersburg distribution. For more details see [2] and [1].

2 Results

Our starting point is Theorem 25.1 in [4]. Let $\{Y_{1,n}, Y_{2,n}, \ldots, Y_{n,n}\}_{n=1}^{\infty}$ be an an infinite array of asymptotically negligible, rowwise independent random variables, with distribution functions $F_{k,n}(x) = \mathbf{P}\{Y_{k,n} \leq x\}$, $x \in \mathbb{R}$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots, n$. Then the random variable $\sum_{k=1}^{n} Y_{k,n} - A_n$, for an appropriate numerical sequence A_n, converges in distribution to a nondegenerate random variable W, with Lévy functions L and R, and normal component σ, if and only if

$$\begin{aligned} &\sum_{k=1}^{n} F_{k,n}(x) \to L(x), \ x < 0, \ x \in C_L\,, \\ &\sum_{k=1}^{n} (F_{k,n}(x) - 1) \to R(x), \ x > 0, \ x \in C_R\,, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \sum_{k=1}^{n} \left\{ \int_{|x| \leq \varepsilon} x^2 \, \mathrm{d}F_{k,n}(x) - \Big(\int_{|x| \leq \varepsilon} x \, \mathrm{d}F_{k,n}(x) \Big)^2 \right\} \\ = \ &\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{n} \left\{ \int_{|x| \leq \varepsilon} x^2 \, \mathrm{d}F_{k,n}(x) - \Big(\int_{|x| \leq \varepsilon} x \, \mathrm{d}F_{k,n}(x) \Big)^2 \right\} = \sigma^2, \end{aligned} \tag{3}$$

where for a real function f, C_f denotes its continuity points.

The Lévy functions of the normal distribution are identically 0. Adding the two equations in (2) and using Theorem 26.2 in [4] we obtain, that $S_{\boldsymbol{a}_n} - A_n \xrightarrow{\mathcal{D}} Z \sim N(0,1)$ for some appropriate A_n, if and only if for every $\varepsilon > 0$

$$\begin{aligned}
&\sum_{k=1}^{n} \int_{|a_{k,n}X_k|>\varepsilon} \mathrm{d}\mathbf{P} \to 0, \quad \text{and} \\
&\sum_{k=1}^{n} \left\{ \int_{|a_{k,n}X_k|<\varepsilon} a_{k,n}^2 X_k^2 \,\mathrm{d}\mathbf{P} - \Big(\int_{|a_{k,n}X_k|<\varepsilon} a_{k,n}X_k \,\mathrm{d}\mathbf{P} \Big)^2 \right\} \to 1\,.
\end{aligned}$$

It follows immediately from this form that $\sum_{k=1}^{n} a_{k,n}X_k - A_n \xrightarrow{\mathcal{D}} N(0,1)$ if and only if $\sum_{k=1}^{n} |a_{k,n}| X_k - A_n \xrightarrow{\mathcal{D}} N(0,1)$.

In the simplest case, when X has finite variance we obtain the following characterization of convergence:

Theorem 1 *Let $X, X_1, X_2, \ldots$ be iid random variables with finite variance, and put $\mu = \mathbf{E}(X)$. Then $\overline{a}_n \to 0$ and*

$$\sum_{k=1}^{n} a_{k,n}(X_k - \mu) \xrightarrow{\mathcal{D}} N(0,1)\,,$$

if and only if $\sum_{k=1}^{n} a_{k,n}^2 \to 1/\mathrm{Var}(X)$.

All the proofs are placed in the next section.

Asymptotic normality of linear combinations is closely related to the following problem: Let $(R_{\nu,1}, R_{\nu,2}, \ldots, R_{\nu,N_\nu})$ be a random vector, which takes on the $N_\nu!$ permutations of $(1, \ldots, N_\nu)$ with equal probabilities. Consider $\{b_{\nu,i} \,:\, 1 \le i \le N_\nu,\ \nu \le 1\}$ and $\{a_{\nu,i} \,:\, 1 \le i \le N_\nu,\ \nu \le 1\}$ two double sequence of real numbers. Hájek [5] gave necessary and sufficient condition for the asymptotic normality of the random sum $\sum_{i=1}^{N_\nu} b_{\nu,i}\, a_{\nu,R_{\nu,i}}$. During the proof of his main theorem, as a corollary he obtains Theorem 1 here, though the theorem is not stated explicitly.

The sufficiency part –in case of nonnegative weights– is stated explicitly in [8] in a more general setup, when $\{X_n\}_{n=1}^{\infty}$ is a sequence of linearly negative quadrant dependent random variables.

In the special case, when each weight sequence is a normalized pooling strategy, as a consequence of Theorem 1 we obtain the following corollary. Recall that $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ is a strategy if its components are nonnegative and add to unity.

Corollary 2 *Let $X_1, X_2, \ldots$ be iid random variables with 0 mean and finite variance. Then for a sequence of strategies $\{\boldsymbol{p}_n\}$, there exists a normalizing sequence c_n, such that*

$$\frac{1}{c_n}\sum_{k=1}^{n} p_{k,n} X_k \xrightarrow{\mathcal{D}} N(0,1)$$

and $\overline{p}_n / c_n \to 0$, if and only if

$$\frac{\overline{p}_n}{\sqrt{\sum_{k=1}^{n} p_{k,n}^2}} \to 0\,,$$

and in this case $c_n = \sqrt{\mathrm{Var}(X)\sum_{k=1}^{n} p_{k,n}^2}$.

Consider an other special case of the weight sequences. Let $X_1, X_2, \ldots$ be iid random variables with $\mathbf{E}(X) = 0$ and $\mathbf{E}(X^2) = 1$. Let $\{w_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $w_k \neq 0$ for all k, and put $W_n = w_1^2 + \cdots + w_n^2$. The weight sequence now is $\boldsymbol{a}_n = (w_1/\sqrt{W_n}, \ldots, w_n/\sqrt{W_n})$. Easy computation shows that in this particular case asymptotic negligibility $\overline{a}_n \to 0$ holds if and only if $W_n \to \infty$ and $w_n^2/W_n \to 0$. With no more moment assumptions on X, Fisher [3] proved that $S_{\boldsymbol{a}_n} \xrightarrow{\mathcal{D}} N(0,1)$, if $W_n \to \infty$ and $\limsup_{t\to\infty} \#\{n : W_n/w_n^2 < t\}/t < \infty$, where $\#A$ stands for the cardinality of a set A. It is easy to show that these conditions imply asymptotic negligibility, but the converse is not true. Later Weber [11] found sufficient conditions for $S_{\boldsymbol{a}_n} \xrightarrow{\mathcal{D}} N(0,1)$, with higher moment assumptions, and these assumptions also imply asymptotic negligibility. We note that these arithmetic assumptions on the weight sequence are necessary in the case of law of large numbers; see [6] and [12]. Therefore it is very interesting that the necessary and sufficient condition for the asymptotic normality is so simple: as a corollary of Theorem 1 we obtain that in this case asymptotic negligibility implies distributional convergence:

Corollary 3 *Let $X_1, X_2, \ldots$, $\{w_n\}_{n=1}^{\infty}$, $\{W_n\}_{n=1}^{\infty}$ and $\boldsymbol{a}_n$ be as above. If $W_n \to \infty$ and $w_n^2/W_n \to 0$, then $S_{\boldsymbol{a}_n} \xrightarrow{\mathcal{D}} N(0,1)$.*

Now assume that the variance is infinite. In this case assumption (3), especially in the normal case, becomes simpler, because by [4] p.173

$$\left[\int_{-x}^{x} y \,\mathrm{d}F(y)\right]^2 = o(1)\int_{-x}^{x} y^2 \,\mathrm{d}F(y),$$

where $o(1) \to 0$ as $x \to \infty$.

According to the next result if the common distribution function F belongs to the domain of attraction of a stable law, then the only possible limit is that stable.

Theorem 4 *Assume that $F \in \mathbb{D}(\alpha)$, $\alpha \in (0, 2]$. If for some weight sequence $\boldsymbol{a}_n$ and centering sequence A_n, $S_{\boldsymbol{a}_n} - A_n \xrightarrow{\mathcal{D}} W$, where W is a nondegenerate random variable, then W is necessarily α-stable.*

We investigate a particular converse of the theorem above. What can we say about the random variable X, if for some sequence $\boldsymbol{a}_n$ the limit distribution exists, and it is normal?

Theorem 5 *Let $X_1, X_2, \dots$ be iid random variables with common distribution function F. If there exists a weight sequence $\boldsymbol{a}_n$ and a centering numerical sequence A_n, such that $S_{\boldsymbol{a}_n} - A_n \xrightarrow{\mathcal{D}} N(0, 1)$, then $F \in \mathbb{D}_p(2)$.*

In a certain sense, according to the latter theorem the distributional convergence through linear combinations is not more general, than along subsequences. The converse is trivially true. Indeed, assume that for a given subsequence $\{k_n\}$ the distributional convergence (1) holds. Then we can define the weight sequence $\boldsymbol{a}_n = (1/C_j, \dots, 1/C_j, 0, \dots, 0)$, if $k_j \leq n < k_{j+1}$, where the number of C_j-s is k_j. Now, obviously $S_{\boldsymbol{a}_n} - A_j/C_j \xrightarrow{\mathcal{D}} W$. To exclude such trivial cases we introduce the following notion. We call a weight sequence $\{\boldsymbol{a}_n\}_{n=1}^{\infty}$ *balanced* if

$$\liminf_{n \to \infty} \frac{\min\{|a_{k,n}| \, : \, k = 1, \dots, n\}}{\max\{|a_{k,n}| \, : \, k = 1, \dots, n\}} > 0 \, .$$

Roughly speaking this means that each component is important. We note that this is the same as the definition of balanced strategies in [7].

The next theorem says that convergence to a normal through a balanced weight sequence implies convergence through the whole sequence of integers.

Theorem 6 *Let $\boldsymbol{a}_n$ be a balanced weight sequence and A_n a centering sequence, such that $S_{\boldsymbol{a}_n} - A_n \xrightarrow{\mathcal{D}} N(0, 1)$. Then $F \in \mathbb{D}(2)$.*

It is important to note that in general the two types of convergence are very different. According to the Corollary in [7] (and in Section 1 here) if $F \in \mathbb{D}_{gp}(W)$, for a nondegenerate semistable law W, then there is a balanced weight sequence $\boldsymbol{a}_n$, which contains only two different components, and for which $S_{\boldsymbol{a}_n} - A_n \xrightarrow{\mathcal{D}} W$, where A_n is well determined. However, in this case F is not necessarily contained in the domain of attraction of any stable law. We also note that Megyesi [10] proved for any stable law W that its domain of geometric partial attraction and its domain of attraction coincide. These results show similarity between convergence along a geometric subsequence, and convergence through balanced weight sequence.

There is an interesting problem in connection with such weight sequences: What is the class of infinitely divisible random variables, whose distribution can be obtained as the limit distribution of linear combinations of iid variables with balanced weight sequences? We do not even know whether nonsemistable limits of this type exist or not.

The validity of Theorem 5 in the general infinitely divisible case, and the validity Theorem 6 in the general α-stable case are also interesting open problems.

3 Proofs

Proof of Theorem 1. As we have seen before Theorem 1 we may assume that the weights are nonnegative. In this case $F_{k,n}(x) = \mathbf{P}\{a_{k,n}X \leq x\} = F(x/a_{k,n})$. We spell out the conditions again: there is asymptotic normality if and only if

$$\textstyle\sum_{k=1}^{n} \Big[F(-x/a_{k,n}) + 1 - F(x/a_{k,n})\Big] \to 0 \quad \text{for every } x > 0\,, \text{ and} \tag{4}$$

$$\textstyle\sum_{k=1}^{n} a_{k,n}^2 \left\{ \displaystyle\int_{|x|\leq\varepsilon/a_{k,n}} x^2 \,\mathrm{d}F(x) - \Big(\int_{|x|\leq\varepsilon/a_{k,n}} x \,\mathrm{d}F(x) \Big)^2 \right\} \to 1\,, \tag{5}$$

for every $\varepsilon > 0$.

We may assume that $\mathbf{E}(X) = 0$. Since $\overline{a}_n \to 0$, each term in (5) tends to $\mathrm{Var}(X)$. Thus the validity of (5) is equivalent to $\lim_{n\to\infty} \sum_{k=1}^{n} a_{k,n}^2 = 1/\mathrm{Var}(X)$. Moreover in this case (4) also holds. Indeed, $\int_{\mathbb{R}\setminus[-x,x]} y^2 \,\mathrm{d}F(y) \geq x^2(F(-x) + 1 - F(x))$, and since the left side tends to 0 as $x \to \infty$, we have

$$\sum_{k=1}^{n} \Big[F(-x/a_{k,n}) + 1 - F(x/a_{k,n})\Big] = \sum_{k=1}^{n} \frac{a_{k,n}^2}{x^2}\, o(1) \to 0\,,$$

where $o(1) \to 0$, as $n \to \infty$, proving (4), and thus the statement. ■

Proof of Corollary 2. Necessity. According to Theorem 1 asymptotic normality implies $\sum_{k=1}^{n} p_{k,n}^2/c_n^2 \to \mathrm{Var}(X)^{-1}$ and since $\sum_{k=1}^{n} p_{k,n}^2 \leq \overline{p}_n \leq 1$, we get that c_n is bounded. Therefore $\overline{p}_n/c_n \to 0$ implies $\overline{p}_n \to 0$, and hence

$c_n \to 0$ too. Since $c_n \sim \sqrt{\mathrm{Var}(X) \sum_{k=1}^n p_{k,n}^2}$ [for numerical sequences we write $a_n \sim b_n$ if $a_n/b_n \to 1$], we obtain

$$\lim_{n\to\infty} \frac{\overline{p}_n}{\sqrt{\sum_{k=1}^n p_{k,n}^2}} = 0\,,$$

as claimed.

Sufficiency. Put $c_n = \sqrt{\mathrm{Var}(X) \sum_{k=1}^n p_{k,n}^2}$ for the norming sequence. Then $\sum_{k=1}^n p_{k,n}^2/c_n^2 = \mathrm{Var}(X)^{-1}$ and $\overline{a}_n = \overline{p}_n / \sqrt{\mathrm{Var}(X) \sum_{k=1}^n p_{k,n}^2} \to 0$, so by Theorem 1 the statement follows. ■

Proof of Theorem 4. First consider the case $\alpha = 2$. It is well known that $F \in \mathbb{D}(2)$ if and only if

$$\lim_{x\to\infty} \frac{x^2\,[F(-x) + 1 - F(x)]}{\int_{|y|\le x} y^2\,\mathrm{d}F(y)} = 0\,. \tag{6}$$

By (6) we have

$$\sum_{k=1}^n \Big[F(-x/|a_{k,n}|) + 1 - F(x/|a_{k,n}|)\Big] = o(1)\frac{1}{x^2}\sum_{k=1}^n \int_{|y|\le x/|a_{k,n}|} a_{k,n}^2\, y^2\,\mathrm{d}F(y)\,,$$

where $o(1)$ is meant as $o(1) \to 0$, if $n \to \infty$. By (3) the sum after $o(1)$ on the right-hand side of the equality is bounded for x small enough, and using (2) it is easy to see that it is bounded for all $x > 0$. Thus the right-hand side goes to 0. Since the left-hand side converge to $L(x) - R(x)$, where L and R are the Lévy functions as in (2), we obtain that both Lévy functions are identically 0, which means that the limit distribution is necessarily normal.

Now let $\alpha < 2$. Suppose that the components are nonnegative. According to the characterization of $\mathbb{D}(\alpha)$, $F \in \mathbb{D}(\alpha)$ if and only if

$$\frac{F(-x)}{1-F(x)} \to c, \quad \text{as } x\to\infty,\ c\in[0,\infty], \text{ and} \tag{7}$$

$$\frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx)} \to k^\alpha, \quad \text{as } x\to\infty, \text{ for each } k>0\,. \tag{8}$$

Consider the case when $0 < c < \infty$. Then by (7) $F(-x) = (c+o(1))[1-F(x)]$, and hence

$$\frac{[1-F(x)](1+c+o(1))}{[1-F(kx)](1+c+o(1))} \to k^\alpha$$

as $x \to \infty$, where now $o(1) \to 0$ as $x \to \infty$. This implies

$$\frac{1 - F(x)}{1 - F(kx)} \to k^\alpha \text{ and similarly } \frac{F(-x)}{F(-kx)} \to k^\alpha ,$$

as $x \to \infty$. Applying this now for fix x and $n \to \infty$ we obtain $1 - F(1/a_{k,n}) = (x^\alpha + o(1))[1 - F(x/a_{k,n})]$, where now $o(1) \to 0$ as $n \to \infty$. Summing from 1 to n

$$\sum_{k=1}^{n} \Big[1 - F(1/a_{k,n})\Big] = \sum_{k=1}^{n} \big(x^\alpha + o(1)\big) \Big[1 - F(x/a_{k,n})\Big] .$$

By (2) the left side converges to $-R(1)$, the right converges to $-x^\alpha R(x)$, which means that $R(x) = R(1)/x^\alpha$, $x < 0$. Similarly $L(x) = L(1)/x^\alpha$, $x > 0$. Now the proof of $\sigma = 0$ is routine. These together implies that the limit is stable.

If $c = 0$ or $c = \infty$ the ideas are the same. The only different that in this case one of the Lévy functions vanishes.

If the components are arbitrary, then handling the positive and negative weights separately, the proof is a trivial modification of the above special case. ∎

Proof of Theorem 5. As before we may assume that the weights are nonnegative. Suppose indirectly that $X \notin \mathbb{D}_p(2)$. By the well-known characterization this means that

$$\liminf_{x \to \infty} \frac{x^2 \big[F(-x) + 1 - F(x)\big]}{\int_{|y| \le x} y^2 \, \mathrm{d}F(y)} > 0 .$$

Choose $a > 0$, which is smaller than the lim inf above. Hence if x is large enough, we have

$$x^2 \Big[F(-x) + 1 - F(x)\Big] > a \int_{|y| \le x} y^2 \, \mathrm{d}F(x) .$$

Since $\overline{a}_n \to 0$ we obtain

$$\sum_{k=1}^{n} [F(-x/a_{k,n}) + 1 - F(x/a_{k,n})] \ge a \sum_{k=1}^{n} \frac{a_{k,n}^2}{x^2} \int_{|y| \le x/a_{k,n}} y^2 \, \mathrm{d}F(y) .$$

By (4) the left-hand side goes to 0, so the right-hand side also does, which implies by (5) that $\sigma = 0$. The contradiction proves the statement. ∎

Proof of Theorem 6. We assume as before that the weight sequence is nonnegative. If $\mathbf{E}(X^2) < \infty$ then the statement is obvious, therefore we suppose that the variance is infinite. In this case, as we mentioned before, the second term in (5) is superfluous. The definition of balancedness implies that there exists $K > 1$, such that $\overline{a}_n / a_{k,n} < K$, for each n and $k = 1, \dots, n$. Then writing $1/K$ instead of x in (4) we obtain

$$\sum_{k=1}^{n} \int_{|y| > \frac{1}{K a_{k,n}}} \mathrm{d}F(y) \geq \sum_{k=1}^{n} \int_{|y| > 1/\overline{a}_n} \mathrm{d}F(y) = n \int_{|y| > 1/\overline{a}_n} \mathrm{d}F(y) \,,$$

and since the left side tends to 0, so does the right.

Rewriting the left side of (5), without the second term, we have for $\varepsilon = 1$

$$\begin{aligned} \sum_{k=1}^{n} a_{k,n}^2 \int_{|y| < 1/a_{k,n}} y^2 \,\mathrm{d}F(y) \;&=\; \sum_{k=1}^{n} a_{k,n}^2 \int_{|y| < 1/\overline{a}_n} y^2 \,\mathrm{d}F(y) \\ &\quad + \sum_{k=1}^{n} a_{k,n}^2 \int_{1/a_{k,n} \geq |y| \geq 1/\overline{a}_n} y^2 \,\mathrm{d}F(y) \,, \end{aligned}$$

and for the remainder term

$$\begin{aligned} \sum_{k=1}^{n} a_{k,n}^2 \int_{1/a_{k,n} \geq |y| \geq 1/\overline{a}_n} y^2 \,\mathrm{d}F(y) \;&\leq\; \sum_{k=1}^{n} \int_{1/a_{k,n} \geq |y| \geq 1/\overline{a}_n} \mathrm{d}F(y) \\ &\leq\; n \int_{|y| \geq 1/\overline{a}_n} \mathrm{d}F(y) \,, \end{aligned}$$

which tends to 0. This means that

$$\sum_{k=1}^{n} a_{k,n}^2 \int_{|y| \leq 1/\overline{a}_n} y^2 \,\mathrm{d}F(y) \to 1$$

as $n \to \infty$. Finally, since $n\overline{a}_n^2 / K^2 \leq \sum_{k=1}^{n} a_{k,n}^2 \leq n\overline{a}_n^2$, we obtain

$$1 \leq \liminf_{n \to \infty} n \, \overline{a}_n^2 \int_{|y| \leq \frac{1}{\overline{a}_n}} y^2 \,\mathrm{d}F(y) \leq \limsup_{n \to \infty} n \, \overline{a}_n^2 \int_{|y| \leq \frac{1}{\overline{a}_n}} y^2 \,\mathrm{d}F(y) \leq K^2 \,.$$

From this boundedness we show that $F \in \mathbb{D}(2)$, with the same idea as in [4] p. 181. Put $\chi(x) = \int_{|y| > x} \mathrm{d}F(y)$ and $H(x) = \int_{|y| < x} y^2 \,\mathrm{d}F(y) / x^2$. Now $\overline{a}_n \to 0$ implies that for each x large enough we can find $n \in \mathbb{N}$ such that $1/\overline{a}_n < x \leq 1/\overline{a}_{n+1}$. Then clearly $\chi(x) \leq \chi(1/\overline{a}_n)$ and $H(x) \geq H(1/\overline{a}_{n+1}) - \chi(1/\overline{a}_n)$. Thus

$$\frac{\chi(x)}{H(x)} \leq \frac{n\chi(1/\overline{a}_n)}{nH(1/\overline{a}_{n+1}) - n\chi(1/\overline{a}_n)} \,.$$

We have just seen above that $n\chi(1/\overline{a}_n) \to 0$ and $nH(1/\overline{a}_n)$ is bounded, thus $\chi(x)/H(x) \to 0$, as $x \to \infty$, which is exactly the same as (6), that is $F \in \mathbb{D}(2)$ as claimed. ∎

References

[1] Csörgő, S., Kevei, P., 2008. Merging asymptotic expansion for cooperative gamblers in generalized St. Petersburg games. *Acta Math. Hungar.*, **121 (1–2)**, 119–156.

[2] Csörgő, S., Simons, G., 2006. Pooling strategies for St. Petersburg gamblers. *Bernoulli* **12**, 971–1002.

[3] Fisher, E., 1992. A Skorohod representation and an invariance principle for sums of weighted iid random variables. *Rocky Mount. J. Math.* **22**, 169–179.

[4] Gnedenko, B. V., Kolmogorov, A. N. 1954. *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Reading, Massachusetts.

[5] Hájek, J., 1961. Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.* **32**, 506–523.

[6] Jamison, B., Orey, S., Pruitt, W., 1965. Convergence of weighted averages of independent random variables. *Z. Wahrsch. Gebiete* **4**, 40–44.

[7] Kevei, P., Csörgő, S., 2009. Merging of linear combinations to semistable laws. *J. Theor. Probab.*, **22**, 772–790.

[8] Kim, H. C., Kim T. S., 2005. A central limit theorem for general weighted sum of lnqd random variables and its application. *Commun. Korean Math. Soc.* **20**, No. 3, pp. 531–538.

[9] Kruglov, V. M., 1972. On the extensions of the class of stable distributions. *Theory Probab. Appl.* **17**, 685–694.

[10] Megyesi, Z., 2000. A probabilistic approach to semistable laws and their domains of partial attraction. *Acta Sci. Math.* (Szeged) **66**, 403–434.

[11] Weber, M., 2006. A weighted central limit theorem. *Stat. Probab. Letters* **76**, 1482–1487.

[12] Wright, F. T., Platt, R. D., Robertson T., 1977. A strong law for weighted averages of independent, identically distributed random variables with arbitrarily heavy tails. *Annals of Probability*, Vol. 5, No. 4, pp. 586–590.