MERGING ASYMPTOTIC EXPANSIONS FOR COOPERATIVE GAMBLERS IN GENERALIZED ST. PETERSBURG GAMES

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Abstract. Merging asymptotic expansions are established for the distribution functions of suitably centered and normed linear combinations of winnings in a full sequence of generalized St. Petersburg games, where a linear combination is viewed as the share of any one of n cooperative gamblers who play with a pooling strategy. The expansions are given in terms of Fourier – Stieltjes transforms and are constructed from suitably chosen members of the classes of subsequential semistable infinitely divisible asymptotic distributions for the total winnings of the n players and from their pooling strategy, where the classes themselves are determined by the two parameters of the game. For all values of the tail parameter, the expansions yield best possible rates of uniform merge. Surprisingly, it turns out that for a subclass of strategies, not containing the averaging uniform strategy, our merging approximations reduce to asymptotic expansions of the usual type, derived from a proper limiting distribution. The Fourier – Stieltjes transforms are shown to be numerically invertible in general and it is also demonstrated that the merging expansions provide excellent approximations even for very small n.

1. Introduction

Peter offers to let Paul toss a possibly biased coin repeatedly until it lands heads and pays him $r^{k/\alpha}$ ducats if this happens on the k^{th} toss, $k \in \mathbb{N} = \{1, 2, \ldots\}$, where $r = 1/q$ for $q = 1 - p$, and $p \in (0, 1)$ is the probability of heads on each throw, while $\alpha > 0$ is a payoff parameter. Thus if X denotes Paul's winning in this generalized St. Petersburg(α, p) game, then $\boldsymbol{P}\{X = r^{k/\alpha}\} = q^{k-1}p$, $k \in \mathbb{N}$.

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Put $\lfloor y \rfloor = \max\{k \in \mathbb{Z}\colon\ k \le y\}$ and $\lceil y \rceil = \min\{k \in \mathbb{Z}\colon\ k \ge y\} = -\lfloor -y \rfloor$ for the usual integer part and 'ceiling' and $\langle y \rangle = y - \lfloor y \rfloor = y + \lceil -y \rceil$ for the fractional part of a number $y \in \mathbb{R}$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{R}$ is the real line. Then the generalized St. Petersburg distribution function of a single gain is

$$(1)\qquad F_{\alpha,p}(x) = \boldsymbol{P}\{X \le x\} = \begin{cases} 0, & \text{if } x < r^{1/\alpha}, \\ 1 - q^{\lfloor \alpha \log_r x \rfloor} = 1 - \frac{r^{\langle \alpha \log_r x \rangle}}{x^\alpha}, & \text{if } x \ge r^{1/\alpha}, \end{cases}$$

where $\log_r$ stands for the logarithm to the base r. We see that the payoff parameter $\alpha > 0$ is in fact a tail parameter of the distribution. In particular, $\boldsymbol{E}(X^\alpha) = \infty$, but $\boldsymbol{E}(X^\beta) = p/(q^{\beta/\alpha} - q)$ is finite for $\beta \in (0, \alpha)$, so for $\alpha > 2$ Paul's gain X has a finite variance and, as pointed out in [1], even for $\alpha = 2$ the St. Petersburg(α, p) distribution is in the domain of attraction of the normal law. Hence for the problems to be entertained in this paper the case $\alpha \ge 2$ is either not interesting or at least substantially different from the more difficult case $\alpha < 2$. Therefore, just as in [1] and [3] we assume throughout this paper that $\alpha \in (0, 2)$. Of course, the most interesting case of this is when $\alpha \le 1$, for which the mean is infinite.

To get to the problems in this paper, we consider independent copies of Paul's gain X in a single game, that is, we let $X_1, X_2, \ldots$ denote independent St. Petersburg(α, p) random variables defined on a probability space $(\Omega, \mathcal{A}, \boldsymbol{P})$. The main question is of course that of the 'fair' price to be payed to Peter for the cumulative winnings $S_n = X_1 + \cdots + X_n$ in a given large number $n \in \mathbb{N}$ of independent games. Subsequent to the initial steps taken in [12] and [5], this question may be answered by results in [1] and [3]. Once this price is agreed upon to the mutual satisfaction of the two sides, it is wholly indifferent to Peter whether the other side is our old Paul playing n games in a row, or a company of n gamblers, Paul$_1$, Paul$_2$, …, Paul$_n$, each playing exactly one game with respective individual winnings $X_1, X_2, \ldots, X_n$, and cooperating among themselves. This latter scenario was first considered in [7] and [9] for the classical case $(\alpha, p) = (1, 1/2)$, and then in [8] and [11] for St. Petersburg$(1, p)$ games for a general $p \in (0, 1)$. In general, for any St. Petersburg(α, p) game, a *pooling strategy* $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, to which all players agree before any of them plays, is such that $p_{1,n}, \ldots, p_{n,n} \ge 0$ and $\sum_{k=1}^n p_{k,n} = 1$. Under this strategy, Paul$_1$ is to receive the amount $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$, Paul$_2$ is to receive the amount $p_{n,n}X_1 + p_{1,n}X_2 + \cdots + p_{n-1,n}X_n$, Paul$_3$ is to receive the amount $p_{n-1,n}X_1 + p_{n,n}X_2 + p_{1,n}X_3 + \cdots + p_{n-2,n}X_n$, …, and Paul$_n$ is to receive the amount of $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{n,n}X_{n-1} + p_{1,n}X_n$ ducats. Under these rotating assignments of weights, every bit of all of the individual winnings is paid out and the strategy is fair to every Paul since their pooled winnings have the same distribution. The surprising genuine benefit of this kind of sharing for each of the gamblers has been demonstrated for all $n \geq 2$ when $\alpha = 1$, first for the classical case $p = 1/2$ in [9], and then for all $p \in (0,1)$ in [11]. Since in this paper we are interested in the asymptotic distribution of such linear combinations, we need to introduce some limiting quantities.

For the bias parameter $p \in (0,1)$, the payoff or tail parameter $\alpha \in (0,2)$ and a third parameter $\gamma \in (q,1]$, consider the infinitely divisible random variable

$$W_\gamma^{\alpha,p} = \frac{1}{\gamma^{1/\alpha}} \left\{ \sum_{m=0}^{-\infty} r^{m/\alpha} \left[Y_m^{p,\gamma} - \frac{p\gamma}{qr^m} \right] + \sum_{m=1}^{\infty} r^{m/\alpha} Y_m^{p,\gamma} \right\} + s_\gamma^{\alpha,p}, \tag{2}$$

where $\ldots, Y_{-2}^{p,\gamma}, Y_{-1}^{p,\gamma}, Y_0^{p,\gamma}, Y_1^{p,\gamma}, Y_2^{p,\gamma}, \ldots$ are independent random variables such that

$$\boldsymbol{P}\{Y_m^{p,\gamma} = k\} = \frac{(pr\gamma q^m)^k}{k!} \, \mathrm{e}^{-pr\gamma q^m}, \quad k = 0,1,2,\ldots,$$

that is, $Y_m^{p,\gamma}$ has the Poisson distribution with mean $pr\gamma q^m = p\gamma/(qr^m)$, $m \in \mathbb{Z}$, and where

$$s_\gamma^{\alpha,p} = \begin{cases} -\frac{p\gamma^{(\alpha-1)/\alpha}}{q^{1/\alpha}-q} = \frac{p}{q-q^{1/\alpha}} \, \frac{1}{\gamma^{(1-\alpha)/\alpha}}\,, & \text{if } \alpha \neq 1, \\ -\frac{p}{q} \, \log_r \gamma = \frac{p}{q} \, \log_r \frac{1}{\gamma}\,, & \text{if } \alpha = 1. \end{cases}$$

Let $G_{\alpha,p,\gamma}(x) = \boldsymbol{P}\{W_\gamma^{\alpha,p} \leq x\}$, $x \in \mathbb{R}$, denote its distribution function. As derived in [1], pp. 821–823, its characteristic function is

$$\boldsymbol{g}_{\alpha,p,\gamma}(t) = \boldsymbol{E}\big(\mathrm{e}^{\mathrm{i}tW_\gamma^{\alpha,p}}\big) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}G_{\alpha,p,\gamma}(x) = \mathrm{e}^{y_\gamma^{\alpha,p}(t)}, \quad t \in \mathbb{R}, \tag{3}$$

where

$$\begin{aligned} y_\gamma^{\alpha,p}(t) &= \mathrm{i}ts_\gamma^{\alpha,p} + \sum_{l=0}^{-\infty} \left(\exp\left\{ \frac{\mathrm{i}tr^{\frac{l}{\alpha}}}{\gamma^{\frac{1}{\alpha}}} \right\} - 1 - \frac{\mathrm{i}tr^{\frac{l}{\alpha}}}{\gamma^{\frac{1}{\alpha}}} \right) \frac{p\gamma}{qr^l} + \sum_{l=1}^{\infty} \left(\exp\left\{ \frac{\mathrm{i}tr^{\frac{l}{\alpha}}}{\gamma^{\frac{1}{\alpha}}} \right\} - 1 \right) \frac{p\gamma}{qr^l} \\ &= \exp\left\{ \mathrm{i}t\big[s_\gamma^{\alpha,p} + u_\gamma^{\alpha,p}\big] + \int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2} \right) \mathrm{d}R_\gamma^{\alpha,p}(x) \right\} \end{aligned} \tag{4}$$

with the finite constant

$$u_\gamma^{\alpha,p} = \frac{p\gamma^{(\alpha+1)/\alpha}}{q} \sum_{l=1}^{\infty} \frac{r^{(1-\alpha)l/\alpha}}{\gamma^{2/\alpha} + r^{2l/\alpha}} - \frac{p\gamma^{(\alpha-1)/\alpha}}{q} \sum_{l=0}^{\infty} \frac{1}{\gamma^{2/\alpha} r^{(3-\alpha)l/\alpha} + r^{(1-\alpha)l/\alpha}}$$

and right-hand-side Lévy function

$$R_\gamma^{\alpha,p}(x) = -\gamma q^{\lfloor \log_r(\gamma x^\alpha) \rfloor} = -\frac{\gamma}{r^{\lfloor \log_r(\gamma x^\alpha) \rfloor}} = -\frac{r^{\langle \log_r(\gamma x^\alpha) \rangle}}{x^\alpha}\,, \quad x > 0.$$

The integral form of the exponent of the characteristic function immediately implies that for every $p \in (0,1)$ and $\gamma \in (q,1]$ the infinitely divisible distribution of $W_\gamma^{\alpha,p}$ is semistable with exponent α; for the theory of semistable distributions required here we refer to [13], [6] and [4]. It follows that $G_{\alpha,p,\gamma}(\cdot)$ is infinitely many times differentiable and by classical results of Kruglov, recently exposed in [2], $\boldsymbol{E}(|W_\gamma^{\alpha,p}|^\alpha) = \infty$, but, for all $p \in (0,1)$ and $\gamma \in (q,1]$, the absolute moment

$$\text{(5)}\quad \boldsymbol{E}\Big(|W_\gamma^{\alpha,p}|^\beta\Big) = \int_{-\infty}^{\infty} |x|^\beta \,\mathrm{d}G_{\alpha,p,\gamma}(x) = \int_{-\infty}^{\infty} |x|^\beta g_{\alpha,p,\gamma}(x)\,\mathrm{d}x < \infty \ \text{ if } \ \beta \in (0,\alpha)$$

with the density function $g_{\alpha,p,\gamma}(\cdot) = G'_{\alpha,p,\gamma}(\cdot) = G^{(1)}_{\alpha,p,\gamma}(\cdot)$.

To motivate the above, first note that the function $x \mapsto r^{\langle \alpha \log_r x\rangle}$ in (1) is not slowly varying at infinity, and hence it follows by the classical Doeblin – Gnedenko criterion that $F_{\alpha,p}(\cdot)$ in (1) is not in the domain of attraction of any (stable) distribution, that is, the cumulative winnings S_n cannot be centered and normalized to have a proper limiting distribution as $n \to \infty$ over the entire sequence $\mathbb{N}$ of natural numbers. However, it turned out in [12] and [5] that asymptotic distributions do exist along subsequences of $\mathbb{N}$ when $\alpha = 1$ and $p = 1/2$. In fact, subsequential limiting distributions exist for all $\alpha \in (0,2)$ and $p \in (0,1)$ for the sequence

$$\text{(6)}\qquad F_n^{\alpha,p}(x) = \boldsymbol{P}\bigg\{\frac{S_n - c_n^{\alpha,p}}{n^{1/\alpha}} \le x\bigg\}, \quad \text{where} \quad c_n^{\alpha,p} = \begin{cases} \frac{p\,n}{q^{1/\alpha}-q}\,, & \text{if } \ \alpha \neq 1, \\ \frac{p}{q}\, n \log_r n\,, & \text{if } \ \alpha = 1, \end{cases}$$

and are regulated by the position parameter

$$\text{(7)}\qquad \gamma_n = \frac{n}{r^{\lceil \log_r n\rceil}} \in (q,1],$$

which describes the location of $n = \gamma_n r^{\lceil \log_r n\rceil} \in \mathbb{N}$ between two consecutive powers of $r = 1/q$. As an extension of one of the results in [5] it can be shown that for any given subsequence $\{n_k\}_{k=1}^\infty$ of $\mathbb{N}$, the sequence $F_{n_k}^{\alpha,p}(\cdot)$ converges weakly as $k \to \infty$ if and only if $\gamma_{n_k} \xrightarrow{\textbf{cir}} \gamma$ for some $\gamma \in (q,1]$, where we write $\gamma_{n_k} \xrightarrow{\textbf{cir}} \gamma$ if $\lim_{k\to\infty} \gamma_{n_k} = \gamma$ for $\gamma \in (q,1]$, but we also write $\gamma_{n_k} \xrightarrow{\textbf{cir}} 1$ if either $\lim_{k\to\infty} \gamma_{n_k} = q$, or the sequence $\{\gamma_{n_k}\}_{k=1}^\infty$ has exactly two limit points, q and 1. If this circular convergence $\gamma_{n_k} \xrightarrow{\textbf{cir}} \gamma$ takes place for some $\gamma \in (q,1]$, as $k \to \infty$, then $\lim_{k\to\infty} \sup_{x\in\mathbb{R}} |F_{n_k}^{\alpha,p}(x) - G_{\alpha,p,\gamma}(x)| = 0$.

The trouble with having many asymptotic distributions is resolved by the selection of a merging approximation to $F_n^{\alpha,p}(\cdot)$ for every $n \in \mathbb{N}$ from the class $\mathcal{G}^{\alpha,p} = \big\{G_{\alpha,p,\gamma}(\cdot) \colon q < \gamma \le 1\big\}$ of subsequential limits. The selection is given by the position parameter γ_n itself in (7), and we have $\sup_{x\in\mathbb{R}} |F_n^{\alpha,p}(x) - G_{\alpha,p,\gamma_n}(x)| \to 0$, where, and throughout the paper, an asymptotic relationship is meant as $n \to \infty$ unless otherwise specified. In fact, rates of merge are derived in [1] and, finally, asymptotic expansions are established in [3] for $F_n^{\alpha,p}(\cdot) - G_{\alpha,p,\gamma_n}(\cdot)$ with uniform error terms depending on α.

Assuming $\overline{p}_n = \max\{p_{1,n}, \ldots, p_{n,n}\} \to 0$ for an infinite sequence of strategies $\{\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})\}$, our first interest in this paper is the asymptotic distribution of

$$S_{\boldsymbol{p}_n}^{\alpha,p} = \sum_{k=1}^{n} p_{k,n}^{1/\alpha} X_k - \frac{p}{q} H_{\alpha,p}(\boldsymbol{p}_n), \tag{8}$$

a particular type of linear combinations when $\alpha \neq 1$, where

$$H_{\alpha,p}(\boldsymbol{p}_n) = \begin{cases} -\frac{1}{1-q^{\frac{1}{\alpha}-1}} \sum_{k=1}^{n} p_{k,n}^{1/\alpha}, & \text{if } \alpha \neq 1, \\ \sum_{k=1}^{n} p_{k,n} \log_r \frac{1}{p_{k,n}}, & \text{if } \alpha = 1. \end{cases}$$

Even though $p_{1,n}^{1/\alpha}, \ldots, p_{n,n}^{1/\alpha}$ sum to one, and hence form a strategy only for $\alpha = 1$, it is a major technical step to come up with a merging approximation in terms of the distribution functions of the semistable random variables

$$W_{\boldsymbol{p}_n}^{\alpha,p} = \begin{cases} \sum_{k=1}^{n} p_{k,n}^{1/\alpha} W_{1,k}^{\alpha,p}, & \text{if } \alpha \neq 1, \\ \sum_{k=1}^{n} p_{k,n} W_{1,k}^{1,p} - \frac{p}{q} H_{1,p}(\boldsymbol{p}_n), & \text{if } \alpha = 1, \end{cases} \tag{9}$$

where the random variables $W_{1,1}^{\alpha,p}, W_{1,2}^{\alpha,p}, \ldots, W_{1,n}^{\alpha,p}$ are independent copies of $W_1^{\alpha,p}$, given by substituting $\gamma = 1$ in (2). The characteristic and the distribution functions will be denoted by $\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \boldsymbol{E}(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}})$ and $G_{\alpha,p,\boldsymbol{p}_n}(x) = \boldsymbol{P}\{W_{\boldsymbol{p}_n}^{\alpha,p} \leq x\}$, $t, x \in \mathbb{R}$, respectively; the ostensible notational clash with (3), the strategy $\boldsymbol{p}_n$ appearing in place of γ, will turn out to be absolutely beneficial. It is easy to see that $W_{\boldsymbol{p}_n}^{\alpha,p}$ is indeed a semistable random variable with exponent α for an arbitrary strategy $\boldsymbol{p}_n$.

In the classical case, approximations of $\boldsymbol{P}\{S_{\boldsymbol{p}_n}^{1,1/2} \leq x\}$ by $G_{1,1/2,\boldsymbol{p}_n}(x)$ were obtained in [9] with rates of merge. The main goal of the present paper is to generalize the merging asymptotic expansions in [3] to strategies, that is, to general linear combinations, such that the classical special case $\alpha = 1$, $p = 1/2$ of the expansion will yield the rates of merge in [9] and also show that those rates are not improvable. Our expansions here require certain mixed derivatives and their properties, which we now introduce, following [3] and [4]. Fix the parameters $\alpha \in (0,2)$, $p \in (0,1)$ and $\gamma \in (q,1]$, and consider for each $u > 0$ the infinitely divisible distribution function $G_{\alpha,p,\gamma}(x\,;u)$, $x \in \mathbb{R}$, that has characteristic function $\boldsymbol{g}_{\alpha,p,\gamma}(t\,;u) = \mathrm{e}^{uy_\gamma^{\alpha,p}(t)}$, that is,

$$\boldsymbol{g}_{\alpha,p,\gamma}(t\,;u) = \mathrm{e}^{uy_\gamma^{\alpha,p}(t)} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}G_{\alpha,p,\gamma}(x\,;u), \quad t \in \mathbb{R}.$$

It was shown in Lemma 4 in [3] that the partial derivatives

$$G_{\alpha,p,\gamma}^{(k,j)}(x\,;u) = \frac{\partial^{k+j}\, G_{\alpha,p,\gamma}(x\,;u)}{\partial x^k\, \partial u^j} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} (-\mathrm{i}t)^{k-1} \big[y_\gamma^{\alpha,p}(t)\big]^j\, \mathrm{e}^{uy_\gamma^{\alpha,p}(t)}\, \mathrm{d}t \tag{10}$$

are well defined at all $x \in \mathbb{R}$ and $u > 0$ for every $j \in \{0,1,2,\ldots\}$ and $k \in \mathbb{N}$, so that

$$
(11) \quad G_{\alpha,p,\gamma}^{(k,j)}(x) = \frac{\partial^{k+j}\, G_{\alpha,p,\gamma}(x\,;u)}{\partial x^k\, \partial u^j}\bigg|_{u=1}, \quad x \in \mathbb{R}, \quad \text{for} \quad j \in \{0,1,2,\ldots\}, k \in \mathbb{N},
$$

are all meaningful. Furthermore, by Lemma 6 in [3] we have the moment property

$$
(12) \qquad \int_{-\infty}^{\infty} |x|^\beta \big|G_{\alpha,p,\gamma}^{(k+1,j)}(x)\big|\,\mathrm{d}x < \infty \quad 0 \le \beta < \alpha \quad \text{for all} \quad j,k \in \{0,1,2,\ldots\},
$$

extending (5) from the case $G_{\alpha,p,\gamma}^{(1,0)}(\cdot) = G_{\alpha,p,\gamma}^{(1)}(\cdot) = G'_{\alpha,p,\gamma}(\cdot) = g_{\alpha,p,\gamma}(\cdot)$, and

$$
(13) \qquad G_{\alpha,p,\gamma}^{(k+1,j)}(\pm\infty) = \lim_{x\to\pm\infty} G_{\alpha,p,\gamma}^{(k+1,j)}(x) = 0 \quad \text{for all} \quad j,k \in \{0,1,2,\ldots\}.
$$

In particular, for every $j,k \in \{0,1,2,\ldots\}$ the function $G_{\alpha,p,\gamma}^{(k+1,j)}(\cdot)$ is Lebesgue integrable on $\mathbb{R}$, and hence

$$
G_{\alpha,p,\gamma}^{(k,j)}(x) = \int_{-\infty}^{x} G_{\alpha,p,\gamma}^{(k+1,j)}(v)\,\mathrm{d}v, \quad x \in \mathbb{R},
$$

is a function of bounded variation on the whole $\mathbb{R}$, with Fourier – Stieltjes transform

$$
(14) \qquad \begin{aligned} \boldsymbol{g}_{\alpha,p,\gamma}^{(k,j)}(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_{\alpha,p,\gamma}^{(k,j)}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, G_{\alpha,p,\gamma}^{(k+1,j)}(x)\,\mathrm{d}x \\ &= (-\mathrm{i}t)^k \big[y_\gamma^{\alpha,p}(t)\big]^j \boldsymbol{g}_{\alpha,p,\gamma}(t) = (-\mathrm{i}t)^k \big[y_\gamma^{\alpha,p}(t)\big]^j\, \mathrm{e}^{y_\gamma^{\alpha,p}(t)}, \quad t \in \mathbb{R}. \end{aligned}
$$

These results in Lemma 6 in [3] are extended in [4] to arbitrary semistable distributions of exponent $\alpha \in (0,2)$.

Theorem 1 in the next section contains the merging asymptotic expansions for the linear combinations in (8). However, these combinations are satisfactory for the n Pauls who wish to pool their individual winnings only in the case $\alpha = 1$. The equivalent Theorem 2 contains an overall satisfactory version after a simple transformation. As shown in [9] for $p = 1/2$ and in [11] in general, for $\alpha = 1$ genuine benefits of pooling realize for a fixed n if and only if every component of the pooling strategy $\boldsymbol{p}_n = (p_{1,n},\ldots,p_{n,n})$ is either an integer power of $q = 1 - p$ or zero. Surprisingly, it will turn out in Corollary 2, that for any sequence of such *admissible* strategies there is a proper limiting distribution for $S_{\boldsymbol{p}_n}^{\alpha,p}$ and its equivalent form in Theorem 2 for every α, and the merging approximations reduce to asymptotic expansions of the usual type. The example of the *best admissible strategy* in [9] for the classical case $(\alpha,p) = (1,1/2)$ is spelled out in detail. For the case $\alpha = 1$ and $p \neq 1/2$, the existence of admissible strategies for a given $n \geq 2$ and algorithms to construct them with some special properties are investigated in [11]. Numerical analysis is presented in Section 3, all the proofs are placed in Section 4.

2. The expansions

Fix any strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, and consider the position parameters $\gamma_{k,n} = 1/(p_{k,n} r^{\lceil \log_r 1/p_{k,n} \rceil}) \in (q, 1]$ for each component $k = 1, 2, \ldots, n$ for which $p_{k,n} > 0$. Roughly speaking $\gamma_{k,n} \in (q, 1]$ determines the position of $p_{k,n}$ between two consecutive powers of r. Note that for the (generally inadmissible) uniform strategy $\boldsymbol{p}_n^{\diamond} = (1/n, \ldots, 1/n)$ all the $\gamma_{k,n}$ reduce to γ_n in (7). Recalling formula (3) for the ingredients and the notation $\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \boldsymbol{E}(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}})$ at (9), for $t \in \mathbb{R}$ we introduce the complex-valued function $\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$, defined for $\alpha \neq 1$ as

$$\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) \Bigg[1 - \frac{1}{2} \sum_{k=1}^{n} p_{k,n}^2 \big[y_{\gamma_{k,n}}^{\alpha,p}(t) \big]^2 + \mathrm{i}t s_1^{\alpha,p} \sum_{k=1}^{n} p_{k,n}^{1+\frac{1}{\alpha}} \, y_{\gamma_{k,n}}^{\alpha,p}(t) + \frac{t^2}{2} \bigg\{ (s_1^{\alpha,p})^2 + \frac{p}{q - q^{2/\alpha}} \bigg\} \sum_{k=1}^{n} p_{k,n}^{\frac{2}{\alpha}} \Bigg],$$

where the constant $s_1^{\alpha,p} = p/(q - q^{1/\alpha})$ is from (2), and for $\alpha = 1$ as

$$\boldsymbol{g}_{\boldsymbol{p}_n}^{1,p}(t) = \boldsymbol{g}_{1,p,\boldsymbol{p}_n}(t) \Bigg[1 - \frac{1}{2} \sum_{k=1}^{n} p_{k,n}^2 \big[y_{\gamma_{k,n}}^{1,p}(t) \big]^2 - \mathrm{i}t \frac{p}{q} \sum_{k=1}^{n} p_{k,n}^2 \, y_{\gamma_{k,n}}^{1,p}(t) \log_r \frac{1}{p_{k,n}} + \frac{t^2}{2} \bigg\{ \frac{p^2}{q^2} \sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{q} \sum_{k=1}^{n} p_{k,n}^2 \bigg\} \Bigg].$$

For any sequence $c_{1,n}, \ldots, c_{n,n}$ of complex numbers, where $c_{k,n}$ may be formally undefined if $p_{k,n} = 0$, here and throughout we use the convention $\sum_{k=1}^{n} p_{k,n} c_{k,n} = \sum_{\{1 \le k \le n:\, p_{k,n} \neq 0\}} p_{k,n} c_{k,n}$. Consider finally the function $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ on $\mathbb{R}$ that has Fourier – Stieltjes transform $\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$, that is,

$$\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}G_{\boldsymbol{p}_n}^{\alpha,p}(x), \quad t \in \mathbb{R}. \tag{15}$$

This is meaningful because the function $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ is a sum with four terms, the first of which is the distribution function $G_{\alpha,p,\boldsymbol{p}_n}(\cdot)$, while the other three terms will turn out to be constant multiples of sums of convolutions of well-determined distribution functions and some mixed derivatives in (11). To obtain an explicit formula of this nature for $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ we need the following scaling properties of the logarithm of the characteristic function in (4), which in particular will also be useful later for proving limit theorems for admissible strategies and which in general will add to our understanding in (16) below of the merging approximation itself.

For all $p \in (0,1)$ and $\gamma \in (q,1]$ the definition in (4) immediately implies

$$\gamma\, y_1^{\alpha,p}\Big(\frac{t}{\gamma^{1/\alpha}}\Big) = \begin{cases} y_\gamma^{\alpha,p}(t), & \text{if } \alpha \neq 1, \\ y_\gamma^{1,p}(t) - \mathrm{i}ts_\gamma^{1,p}, & \text{if } \alpha = 1, \end{cases} \qquad t \in \mathbb{R}.$$

Also, lengthy but straightforward calculation shows what in fact is the semistable property of the characteristic function $\boldsymbol{g}_{\alpha,p,\gamma}(\cdot)$ in (3), which for the classical case $(\alpha,p) = (1,1/2)$ was first noticed by Martin-Löf ([12], Theorem 2), namely,

$$y_\gamma^{\alpha,p}\big(r^{\frac{m}{\alpha}}s\big) = \begin{cases} r^m y_\gamma^{\alpha,p}(s), & \text{if } \alpha \neq 1, \\ r^m y_\gamma^{1,p}(s) - \mathrm{i}sr^m m\frac{p}{q}, & \text{if } \alpha = 1, \end{cases} \qquad s \in \mathbb{R},$$

for all $m \in \mathbb{Z}$. Combining these two scaling properties we get for $\alpha \neq 1$,

$$y_1^{\alpha,p}\Big(tp_{k,n}^{1/\alpha}\Big) = p_{k,n}\, y_{\gamma_{k,n}}^{\alpha,p}(t), \quad t \in \mathbb{R},$$

and for $\alpha = 1$,

$$y_1^{1,p}(tp_{k,n}) = p_{k,n}\, y_{\gamma_{k,n}}^{1,p}(t) + \mathrm{i}t\,\frac{p}{q}\,p_{k,n}\log_r\frac{1}{p_{k,n}}, \quad t \in \mathbb{R}.$$

We claim that for all $\alpha \in (0,2)$, $p \in (0,1)$ and strategy $\boldsymbol{p}_n = (p_{1,n},\dots,p_{n,n})$ this implies the unified formula

$$\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}}\Big) = \exp\Bigg\{\sum_{k=1}^n p_{k,n}\, y_{\gamma_{k,n}}^{\alpha,p}(t)\Bigg\}, \quad t \in \mathbb{R}, \tag{16}$$

for the pertaining characteristic functions. Indeed, if $\alpha \neq 1$, then

$$\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \prod_{k=1}^n \boldsymbol{g}_{\alpha,p,1}\Big(tp_{k,n}^{1/\alpha}\Big) = \prod_{k=1}^n \mathrm{e}^{\,y_1^{\alpha,p}\left(tp_{k,n}^{1/\alpha}\right)} = \exp\Bigg\{\sum_{k=1}^n p_{k,n}\, y_{\gamma_{k,n}}^{\alpha,p}(t)\Bigg\},$$

while if $\alpha = 1$, then

$$\begin{aligned} \boldsymbol{g}_{1,p,\boldsymbol{p}_n}(t) &= \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{1,p}(\boldsymbol{p}_n)}\prod_{k=1}^n \boldsymbol{g}_{1,p,1}(tp_{k,n}) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{1,p}(\boldsymbol{p}_n)}\prod_{k=1}^n \mathrm{e}^{\,y_1^{1,p}(tp_{k,n})} \\ &= \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{1,p}(\boldsymbol{p}_n)}\exp\Bigg\{\sum_{k=1}^n\Bigg[p_{k,n}\, y_{\gamma_{k,n}}^{1,p}(t) + \mathrm{i}t\,\frac{p}{q}\,p_{k,n}\log_r\frac{1}{p_{k,n}}\Bigg]\Bigg\}, \end{aligned}$$

which, writing out the entropy $H_{1,p}(\boldsymbol{p}_n) = -\sum_{k=1}^n p_{k,n}\log_r p_{k,n}$, gives (16) also for $\alpha = 1$. Another general consequence of the scaling properties is that for all $\alpha \in (0,2)$ we can rewrite the functions $\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$ in (15) in the following simpler form

$$\begin{aligned} \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\Bigg[1 - \frac{1}{2}\sum_{k=1}^n\Big(y_1^{\alpha,p}\big(tp_{k,n}^{1/\alpha}\big)\Big)^2 + \mathrm{i}ts_1^{\alpha,p}\sum_{k=1}^n p_{k,n}^{1/\alpha}\, y_1^{\alpha,p}\big(tp_{k,n}^{1/\alpha}\big) \\ + \frac{t^2}{2}\Big\{\big(s_1^{\alpha,p}\big)^2 + \frac{p}{q-q^{2/\alpha}}\Big\}\sum_{k=1}^n p_{k,n}^{2/\alpha}\Bigg] \end{aligned} \tag{17}$$

for all $t \in \mathbb{R}$, noting also from (2) that $s_1^{1,p} = \frac{p}{q} \log 1 = 0$ for $\alpha = 1$.

Using the latter formula (17), we can now determine $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ as follows. The semistable random variable $p_{l,n}^{1/\alpha} W_1^{\alpha,p}$ has characteristic function

$$\boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tp_{l,n}^{1/\alpha} W_1^{\alpha,p}}\Big) = \mathrm{e}^{y_1^{\alpha,p}(tp_{l,n}^{1/\alpha})}, \quad t \in \mathbb{R},$$

and distribution function

$$\boldsymbol{P}\Big\{p_{l,n}^{1/\alpha} W_1^{\alpha,p} \le x\Big\} = G_{\alpha,p,1}\big(xp_{l,n}^{-1/\alpha}\big), \quad x \in \mathbb{R},$$

for all $l = 1, 2, \ldots, n$ for which $p_{l,n} > 0$. Using (14) for $G_{\alpha,p,1}^{(m,j)}(x)$ and then replacing the latter argument x by $x/p_{l,n}^{1/\alpha}$, we obtain

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}G_{m,j,l}^{\alpha,p,1}(x) = p_{l,n}^{m/\alpha} (-\mathrm{i}t)^m \left(y_1^{\alpha,p}\big(tp_{l,n}^{1/\alpha}\big)\right)^j \mathrm{e}^{y_1^{\alpha,p}(tp_{l,n}^{1/\alpha})}, \quad t \in \mathbb{R}, \tag{18}$$

where $G_{m,j,l}^{\alpha,p,1}(x) = G_{\alpha,p,1}^{(m,j)}(x/p_{l,n}^{1/\alpha})$, $x \in \mathbb{R}$, is of bounded variation, $m, j \ge 0$. Using (17) and the form $\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \mathrm{e}^{-I(\alpha=1)\mathrm{i}tp\, H_{1,p}(\boldsymbol{p}_n)/q} \prod_{k=1}^n \exp\big\{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})\big\}$ from (9), where $I(A)$ is the indicator of the event A, for $\alpha \ne 1$ we obtain

$$\begin{aligned} \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) &- \frac{1}{2} \sum_{k=1}^n \left[\big\{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})\big\}^2 \mathrm{e}^{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})} \prod_{\substack{j=1\\ j\ne k}}^n \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}\right] \\ &- s_1^{\alpha,p} \sum_{k=1}^n \left[p_{k,n}^{1/\alpha}(-\mathrm{i}t) y_1^{\alpha,p}\big(tp_{k,n}^{1/\alpha}\big) \mathrm{e}^{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})} \prod_{\substack{j=1\\ j\ne k}}^n \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}\right] \\ &- \frac{1}{2}\left[(s_1^{\alpha,p})^2 + \frac{p}{q - q^{2/\alpha}}\right] \sum_{k=1}^n \left[p_{k,n}^{2/\alpha}(-\mathrm{i}t)^2 \mathrm{e}^{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})} \prod_{\substack{j=1\\ j\ne k}}^n \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}\right] \end{aligned} \tag{19}$$

for all $t \in \mathbb{R}$, and, setting $h_p(\boldsymbol{p}_n) = -pH_{1,p}(\boldsymbol{p}_n)/q$, for $\alpha = 1$,

$$\begin{aligned} \boldsymbol{g}_{\boldsymbol{p}_n}^{1,p}(t) = \boldsymbol{g}_{1,p,\boldsymbol{p}_n}(t) &- \frac{\mathrm{e}^{\mathrm{i}th_p(\boldsymbol{p}_n)}}{2} \sum_{k=1}^n \left[\big\{y_1^{1,p}(tp_{k,n})\big\}^2 \mathrm{e}^{y_1^{1,p}(tp_{k,n})} \prod_{\substack{j=1\\ j\ne k}}^n \mathrm{e}^{y_1^{1,p}(tp_{j,n})}\right] \\ &- \frac{\mathrm{e}^{\mathrm{i}th_p(\boldsymbol{p}_n)}}{2\,q} \sum_{k=1}^n \left[p_{k,n}^2(-\mathrm{i}t)^2 \mathrm{e}^{y_1^{1,p}(tp_{k,n})} \prod_{\substack{j=1\\ j\ne k}}^n \mathrm{e}^{y_1^{1,p}(tp_{j,n})}\right]. \end{aligned} \tag{20}$$

Consider the distribution functions $F_{k,n}^{\alpha,p}(x) = \boldsymbol{P}\big\{\sum_{j=1, j\neq k}^{n} p_{j,n}^{1/\alpha} W_{1,j}^{\alpha,p} \le x\big\}$, $x \in \mathbb{R}$, where $W_{1,j}^{\alpha,p}$ are still independent copies of $W_1^{\alpha,p}$ in (2), $k = 1, \ldots, n$. Clearly, its characteristic function is

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}F_{k,n}^{\alpha,p}(x) = \prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}, \quad t \in \mathbb{R}.$$

Using the notation $[F\star G](x) = \int_{-\infty}^{\infty} F(x-y)\, \mathrm{d}G(y) = \int_{-\infty}^{\infty} G(x-y)\, \mathrm{d}F(y)$, $x \in \mathbb{R}$, for the Lebesgue – Stieltjes convolution of the functions F and G of bounded variation and writing $s_1^{\alpha,p} = p/(q - q^{1/\alpha})$ in from (2), we see by (18) and (19) that for $\alpha \neq 1$,

$$\begin{aligned} G_{\boldsymbol{p}_n}^{\alpha,p}(x) = G_{\alpha,p,\boldsymbol{p}_n}(x) &- \frac{1}{2}\sum_{k=1}^{n}\Big[G_{0,2,k}^{\alpha,p,1} \star F_{k,n}^{\alpha,p}\Big](x) \\ &- \frac{p}{q - q^{1/\alpha}}\sum_{k=1}^{n}\Big[G_{1,1,k}^{\alpha,p,1} \star F_{k,n}^{\alpha,p}\Big](x) \\ &- \frac{1}{2}\bigg\{\frac{p^2}{(q - q^{1/\alpha})^2} + \frac{p}{q - q^{2/\alpha}}\bigg\}\sum_{k=1}^{n}\Big[G_{2,0,k}^{\alpha,p,1} \star F_{k,n}^{\alpha,p}\Big](x) \end{aligned} \tag{21}$$

and by (18) and (20) that for $\alpha = 1$,

$$\begin{aligned} G_{\boldsymbol{p}_n}^{1,p}(x) = G_{1,p,\boldsymbol{p}_n}(x) &- \frac{1}{2}\sum_{k=1}^{n}\Big[F_{h_p(\boldsymbol{p}_n)} \star G_{0,2,k}^{1,p,1} \star F_{k,n}^{1,p}\Big](x) \\ &- \frac{1}{2\,q}\sum_{k=1}^{n}\Big[F_{h_p(\boldsymbol{p}_n)} \star G_{2,0,k}^{1,p,1} \star F_{k,n}^{1,p}\Big](x) \end{aligned} \tag{22}$$

for all $x \in \mathbb{R}$, where $F_c(x) = 0$ or 1, according as $x < c$ or $x \ge c$, is the degenerate distribution function of the constant $c \in \mathbb{R}$.

The formulae (21) and (22) are very complicated and in fact useless to prove anything directly; for $\alpha = 1$ the expression (22) is even misleading in the sense that it does not contain the mixed derivative $G_{1,p,\gamma}^{(1,1)}(\cdot)$ for any $\gamma \in (q, 1]$. Nevertheless, they have two important consequences. One is the immediate fact that $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ is a function of bounded variation on $\mathbb{R}$, and hence (15) is indeed meaningful for all $\alpha \in (0, 2)$, $p \in (0, 1)$ and strategy $\boldsymbol{p}_n$. The other is that we see by (13) that to prove the important properties $G_{\boldsymbol{p}_n}^{\alpha,p}(-\infty) = 0$ and $G_{\boldsymbol{p}_n}^{\alpha,p}(\infty) = 1$ it suffices to show that $G_{\alpha,p,1}^{(0,2)}(\pm\infty) = 0$. This will be done in the next section, where, in turn, these properties are the key to get a numerically manageable formula for $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$. We note that besides (21) and (22) intuitively more appealing formulae can be obtained directly by the defining formulae above (15) and by (16). Indeed, for any $u > 0$ introduce the functions $G^{(l+1,j)}_{u,\alpha,p,\gamma}(x) = G^{(l+1,j)}_{\alpha,p,\gamma}(x\,;u)$ in (10) and $G^{(l,j)}_{u,\alpha,p,\gamma}(x) = \int_{-\infty}^x G^{(l+1,j)}_{u,\alpha,p,\gamma}(y)\,\mathrm{d}y$, $x \in \mathbb{R}$, for which $\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G^{(l,j)}_{u,\alpha,p,\gamma}(x) = (-\mathrm{i}t)^l [y^{\alpha,p}_\gamma(t)]^j\,\mathrm{e}^{u\,y^{\alpha,p}_\gamma(t)}$, $t \in \mathbb{R}$, by Lemma 6 in [3], $j, l \in \{0, 1, 2, \ldots\}$, which extends (14). Also, consider the semistable distribution function $H_{\alpha,p,k}(\cdot)$, which for any $k \in \{1, \ldots, n\}$ for which $p_{k,n} > 0$ is the convolution of $G_{p_{j,n},\alpha,p,\gamma_{j,n}}(\cdot)$ for all $j \in \{1, \ldots, n\}$, $j \neq k$, for which $p_{j,n} > 0$. Then for $\alpha = 1$,

$$\begin{aligned} G^{1,p}_{\boldsymbol{p}_n}(x) = G_{1,p,\boldsymbol{p}_n}(x) &- \sum_{k=1}^n \frac{p_{k,n}^2}{2}\Big[G^{(0,2)}_{p_{k,n},1,p,\gamma_{k,n}} \star H_{1,p,k}\Big](x) \\ &+ \frac{p}{q}\sum_{k=1}^n p_{k,n}^2 \left(\log_r \frac{1}{p_{k,n}}\right)\Big[G^{(1,1)}_{p_{k,n},1,p,\gamma_{k,n}} \star H_{1,p,k}\Big](x) \\ &- \left\{\frac{p^2}{2q^2}\sum_{k=1}^n p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{2q}\sum_{k=1}^n p_{k,n}^2\right\}\Big[G^{(2,0)}_{p_{m,n},1,p,\gamma_{m,n}} \star H_{1,p,m}\Big](x) \end{aligned}$$

for all $x \in \mathbb{R}$, where $m \in \{1, \ldots, n\}$ is arbitrary as long as $p_{m,n} > 0$. It is easy to write down the analogous formula also for $\alpha \neq 1$.

Calculating directly from the corresponding special case of the formulae above (15), we point out right away for the uniform strategy $\boldsymbol{p}_n^\diamond = (1/n, \ldots, 1/n)$ that by (14) and the fact — already noticed above — that $\gamma_{k,n} = \gamma_n$ in (7) for all $k = 1, \ldots, n$, so that $\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n^\diamond}(\cdot) = \boldsymbol{g}_{\alpha,p,\gamma_n}(\cdot)$ due to (16), we obtain

$$G^{\alpha,p}_{\boldsymbol{p}_n^\diamond}(x) = \begin{cases} G_{\alpha,p,\gamma_n}(x) - \frac{G^{(0,2)}_{\alpha,p,\gamma_n}(x)}{2\,n} - \frac{pG^{(1,1)}_{\alpha,p,\gamma_n}(x)}{\left(q - q^{\frac{1}{\alpha}}\right) n^{\frac{1}{\alpha}}} - \frac{p^2 G^{(2,0)}_{\alpha,p,\gamma_n}(x)}{2\left(q - q^{\frac{1}{\alpha}}\right)^2 n^{\frac{2-\alpha}{\alpha}}} - \frac{pG^{(2,0)}_{\alpha,p,\gamma_n}(x)}{2\left(q - q^{\frac{2}{\alpha}}\right) n^{\frac{2-\alpha}{\alpha}}}, \\ G_{1,p,\gamma_n}(x) - \frac{G^{(0,2)}_{1,p,\gamma_n}(x)}{2\,n} + \frac{pG^{(1,1)}_{1,p,\gamma_n}(x)\log_r n}{q\,n} - \frac{p^2 G^{(2,0)}_{1,p,\gamma_n}(x)\log_r^2 n}{2\,q^2\,n} - \frac{G^{(2,0)}_{1,p,\gamma_n}(x)}{2\,q\,n}, \end{cases}$$

for all $x \in \mathbb{R}$, where of course the upper branch is for $\alpha \neq 1$ and the lower branch is for $\alpha = 1$. For both branches the sum of the first four terms is the function $G^{\alpha,p}_n(x)$ in the Proposition in [3], where the fifth term was missed. That the inclusion of this fifth term would be a desirable adjustment in [3], at least for $\alpha \neq 1$, was noticed by Pap [14]. Hence for any strategy $\boldsymbol{p}_n$ the definition of $G^{\alpha,p}_{\boldsymbol{p}_n}(\cdot)$ in (15) is a suitable generalization of the desired full form $G^{\alpha,p}_{\boldsymbol{p}_n^\diamond}(\cdot)$ above. Then the main result for the merging approximation of the distribution function of $S^{\alpha,p}_{\boldsymbol{p}_n}$ from (8) is the following

Theorem 1. *For any sequence of strategies* $\{\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})\}_{n\in\mathbb{N}}$,

$$\sup_{x\in\mathbb{R}} \left|\boldsymbol{P}\{S^{\alpha,p}_{\boldsymbol{p}_n} \leq x\} - G^{\alpha,p}_{\boldsymbol{p}_n}(x)\right| = \begin{cases} O\big(\overline{p}_n^2\big), & \text{if } 0 < \alpha < 1/2, \\ O\big(\overline{p}_n^{1/\alpha}\big), & \text{if } 1/2 \leq \alpha < 3/2; \\ O\big(\overline{p}_n^{(4-2\alpha)/\alpha}\big), & \text{if } 3/2 \leq \alpha < 2, \end{cases}$$

where $\overline{p}_n = \max\{p_{1,n}, \ldots, p_{n,n}\}$.

For the uniform strategy $\boldsymbol{p}_n^\diamond$, for which $S_{\boldsymbol{p}_n^\diamond}^{\alpha,p} = (S_n - c_n^{\alpha,p})/n^{1/\alpha}$ with S_n and $c_n^{\alpha,p}$ as in (6), Theorem 1 reduces to the Proposition in [3] when $\alpha \leq 1$, with the adjusted full form of $G_{\boldsymbol{p}_n^\diamond}^{\alpha,p}(\cdot)$ replacing $G_n^{\alpha,p}(\cdot)$, except for a refined statement for non-lattice random variables in the case when $1/2 < \alpha < 1$. The real effect of the adjustment to $G_{\boldsymbol{p}_n^\diamond}^{\alpha,p}(\cdot)$ is for $\alpha \in (1,2)$, where the Proposition in [3] produces a worse rate for the approximation with $G_n^{\alpha,p}(\cdot)$ which precludes a real asymptotic expansion. In fact, for $\alpha \neq 1$ Pap [14] refined the result for $S_{\boldsymbol{p}_n^\diamond}^{\alpha,p}$ to a sort of a complete asymptotic expansion, the length of it is regulated by α: the closer α is to 0 or 2, the longer the expansion may be taken. As more refined statements than those in Theorem 1 and Theorem 2 below, we could have aimed at the generalization of his complete expansion to strategies, but we did not feel that the necessarily more complicated statements could give more insight into the problem, particularly that the more complicated terms of the approximation would be hard to penetrate for a reasonable interpretation. Finally we note that for $\alpha > 1$ Pap [14] proved the expansion for $S_{\boldsymbol{p}_n^\diamond}^{\alpha,p}$ in the stronger non-uniform form with the multiplicative factor $1 + |x|$. Again, we could have aimed at an analogous form here, multiplying the deviations in Theorems 1 and 2 by $1+|x|$ before taking the supremum and keep the same order relations for $\alpha > 1$. However, in view of the tail behavior of the approximative distributions, for any given $\alpha \in (0,2)$ the useful result of this sort would be with the factor $1 + |x|^\alpha$. We conjecture that such non-uniform versions of Theorems 1 and 2 remain true; this would require new technical ideas and developments even for $\boldsymbol{p}_n^\diamond$.

As noted between (8) and (9), the sum of the weights $p_{1,n}^{1/\alpha}, \ldots, p_{n,n}^{1/\alpha}$ in $S_{\boldsymbol{p}_n}^{\alpha,p}$ adds to unity only if $\alpha = 1$, so for $\alpha \neq 1$ they cannot represent a pooling strategy. Given these weights, we transform them to obtain a pooling strategy for arbitrary α in the following way. Let $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$ be an arbitrary strategy as before and define $q_{j,n} = p_{j,n}^{1/\alpha} / \sum_{k=1}^n p_{k,n}^{1/\alpha}$, $j = 1, 2, \ldots, n$. Then $\sum_{j=1}^n q_{j,n} = 1$, and so $\boldsymbol{q}_n = (q_{1,n}, q_{2,n}, \ldots, q_{n,n})$ is also a strategy. In fact this is a one to one correspondence because, as can be seen easily, $p_{j,n} = q_{j,n}^\alpha / \sum_{k=1}^n q_{k,n}^\alpha$, $j = 1, 2, \ldots, n$. Of course, for $\alpha = 1$ this is the identity correspondence. Using this transformation we can rewrite Theorem 1 in an equivalent, more natural form. For an arbitrary strategy $\boldsymbol{q}_n = (q_{1,n}, \ldots, q_{n,n})$, let

$$T_{\boldsymbol{q}_n}^{\alpha,p} = \frac{\sum_{k=1}^n q_{k,n} X_k}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}} + \frac{p}{q - q^{1/\alpha}} \frac{1}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}} \quad \text{and} \quad V_{\boldsymbol{q}_n}^{\alpha,p} = \frac{\sum_{k=1}^n q_{k,n} W_{1,k}^{\alpha,p}}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}}$$

if $\alpha \neq 1$, while $T_{\boldsymbol{q}_n}^{1,p} = S_{\boldsymbol{q}_n}^{1,p}$ and $V_{\boldsymbol{q}_n}^{1,p} = W_{\boldsymbol{q}_n}^{1,p}$ otherwise. Notice that $\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}$

in the denominators is the ℓ_α-norm of the strategy $\boldsymbol{q}_n$. Also, for $\alpha \neq 1$ we introduce

$$\boldsymbol{h}_{\boldsymbol{q}_n}^{\alpha,p}(t) = \boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tV_{\boldsymbol{q}_n}^{\alpha,p}}\Big)\left[1 - \frac{\sum_{k=1}^n q_{k,n}^{2\alpha}\big[y_{\nu_{k,n}}^{\alpha,p}(t)\big]^2}{2\Big(\sum_{j=1}^n q_{j,n}^\alpha\Big)^2} + \frac{\mathrm{i}ts_1^{\alpha,p}\sum_{k=1}^n q_{k,n}^{1+\alpha}y_{\nu_{k,n}}^{\alpha,p}(t)}{\Big(\sum_{j=1}^n q_{j,n}^\alpha\Big)^{1+\frac{1}{\alpha}}}\right.$$
$$\left.+\frac{t^2\left((s_1^{\alpha,p})^2 + p/(q-q^{2/\alpha})\right)\sum_{k=1}^n q_{k,n}^2}{2\Big(\sum_{j=1}^n q_{j,n}^\alpha\Big)^{2/\alpha}}\right], \quad t \in \mathbb{R},$$

where $s_1^{\alpha,p} = p/(q - q^{1/\alpha})$ still and, again, just as for $\boldsymbol{p}_n$ above, the summations are only for those indices $k \in \{1, \ldots, n\}$ for which $q_{k,n} > 0$, and for such k,

$$\nu_{k,n} = \frac{\frac{1}{q_{k,n}^\alpha}\sum_{j=1}^n q_{j,n}^\alpha}{r^{\left\lceil \log_r \frac{1}{q_{k,n}^\alpha}\sum_{j=1}^n q_{j,n}^\alpha \right\rceil}} \in (q, 1].$$

For $\alpha = 1$ we see that $\nu_{k,n}$ reduces to $\gamma_{k,n}$ that corresponds to $q_{k,n} > 0$, and we simply put $\boldsymbol{h}_{\boldsymbol{q}_n}^{1,p}(t) = \boldsymbol{g}_{\boldsymbol{q}_n}^{1,p}(t)$ for all $t \in \mathbb{R}$. Now consider the function $H_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$, of bounded variation on $\mathbb{R}$, that has Fourier – Stieltjes transform $\boldsymbol{h}_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$, so that $\boldsymbol{h}_{\boldsymbol{q}_n}^{\alpha,p}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}H_{\boldsymbol{q}_n}^{\alpha,p}(x)$, $t \in \mathbb{R}$. Then we have the following

Theorem 2. *For any sequence of strategies* $\{\boldsymbol{q}_n = (q_{1,n}, \ldots, q_{n,n})\}_{n\in\mathbb{N}}$,

$$\sup_{x\in\mathbb{R}}\big|\boldsymbol{P}\{T_{\boldsymbol{q}_n}^{\alpha,p} \le x\} - H_{\boldsymbol{q}_n}^{\alpha,p}(x)\big| = \begin{cases} O\big(h_{n,\alpha}^2\big), & \text{if } 0 < \alpha < 1/2, \\ O\big(h_{n,\alpha}^{1/\alpha}\big), & \text{if } 1/2 \le \alpha < 3/2, \\ O\big(h_{n,\alpha}^{(4-2\alpha)/\alpha}\big), & \text{if } 3/2 \le \alpha < 2, \end{cases}$$

where $h_{n,\alpha} = \overline{q}_n^\alpha / \sum_{k=1}^n q_{k,n}^\alpha$.

While formally these conditions are not required, Theorem 1 of course gives asymptotic results only when $\overline{p}_n \to 0$, while Theorem 2 works for a given α only if $h_{n,\alpha} \to 0$. This second condition is needed because, in general, the conditions $\overline{p}_n \to 0$ and $\overline{q}_n \to 0$ are independent in the sense that neither of them implies the other; of course, $h_{n,1} = \overline{q}_n$. This can be seen through suitably constructed examples.

Rates of merge with the distribution functions $H_{\alpha,p,\boldsymbol{q}_n}(x) = \boldsymbol{P}\{V_{\boldsymbol{q}_n}^{\alpha,p} \le x\}$, $x \in \mathbb{R}$, implying that in Theorem 4 in [9], are contained in the following

Corollary 1. *If* $\{\boldsymbol{q}_n = (q_{1,n}, \ldots, q_{n,n})\}_{n\in\mathbb{N}}$ *is a sequence of strategies for which* $h_{n,\alpha} \to 0$, *then for every* $\varepsilon > 0$ *there is a threshold* $n_* = n_*(\varepsilon, \alpha, p) \in \mathbb{N}$ *such* *that*

$$\sup_{x\in\mathbb{R}}\big|\boldsymbol{P}\{T^{\alpha,p}_{\boldsymbol{q}_n}\le x\}-H_{\alpha,p,\boldsymbol{q}_n}(x)\big|\le\begin{cases}(1+\varepsilon)\,K(\alpha,p)\,h_{n,\alpha}, & \text{if } 0<\alpha<1,\\ (1+\varepsilon)\,K(1,p)\,\overline{q}_n\log_r^2\frac{1}{\overline{q}_n}, & \text{if } \alpha=1,\\ (1+\varepsilon)\,K(\alpha,p)\,h_{n,\alpha}^{(2-\alpha)/\alpha}, & \text{if } 1<\alpha<2,\end{cases}$$

whenever $n\ge n_*$, *where the constants are*

$$K(\alpha,p)=\begin{cases}\frac{C_7^2}{2\pi\alpha C_1^2}, & \text{if } 0<\alpha<1,\\ \frac{p^2}{2q^2\pi C_1^2}, & \text{if } \alpha=1,\\ \Big\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\Big\}\frac{\Gamma(2/\alpha)}{2\pi\alpha C_1^{2/\alpha}}, & \text{if } 1<\alpha<2,\end{cases}$$

where $\Gamma(u)=\int_0^\infty x^{u-1}\,\mathrm{e}^{-u}\,\mathrm{d}u$, $u>0$, *is the usual gamma function, in which*

$$C_1=C_1(\alpha,p)=\Big(\frac{2}{\pi}\Big)^\alpha\frac{pq^{(2-\alpha)/\alpha}}{q-q^{2/\alpha}}$$

and, for $\alpha<1$,

$$C_7=C_7(\alpha,p)=\frac{2^{1-\alpha}}{q}+\frac{2^{1-\alpha}p}{q-q^{1/\alpha}}.$$

Theorem 2 itself also implies that the order of these rates in $h_{n,\alpha}$ is optimal.

The admissibility condition is difficult to formulate in the context of the $\boldsymbol{q}_n$ weights of Theorem 2, so in this regard we focus only on Theorem 1. Since all nonzero members $p_{k,n}$ of an admissible strategy are integer powers of q, the corresponding $\gamma_{k,n}=1$, $k=1,2,\ldots,n$. Hence by (16) for any admissible strategy $\boldsymbol{p}_n$ the distributional equality $W^{\alpha,p}_{\boldsymbol{p}_n}\overset{\mathcal{D}}{=}W^{\alpha,p}_1$ holds for the random variable $W^{\alpha,p}_1$ in (2), and the functions $\boldsymbol{g}^{\alpha,p}_{\boldsymbol{p}_n}(t)$ in (15) may be written in the following simpler form: for $\alpha\ne 1$,

$$\boldsymbol{g}^{\alpha,p}_{\boldsymbol{p}_n}(t)=\mathrm{e}^{y_1^{\alpha,p}(t)}-\big[y_1^{\alpha,p}(t)\big]^2\mathrm{e}^{y_1^{\alpha,p}(t)}\frac{1}{2}\sum_{k=1}^n p_{k,n}^2-(-\mathrm{i}t)y_1^{\alpha,p}(t)\,\mathrm{e}^{y_1^{\alpha,p}(t)}\frac{p\sum_{k=1}^n p_{k,n}^{1+\frac{1}{\alpha}}}{q-q^{1/\alpha}}$$
$$-(-\mathrm{i}t)^2\mathrm{e}^{y_1^{\alpha,p}(t)}\Big\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\Big\}\frac{1}{2}\sum_{k=1}^n p_{k,n}^{2/\alpha},$$

and for $\alpha=1$,

$$\boldsymbol{g}^{1,p}_{\boldsymbol{p}_n}(t)=\mathrm{e}^{y_1^{1,p}(t)}-\big[y_1^{1,p}(t)\big]^2\,\mathrm{e}^{y_1^{1,p}(t)}\frac{1}{2}\sum_{k=1}^n p_{k,n}^2-\mathrm{i}ty_1^{1,p}(t)\,\mathrm{e}^{y_1^{1,p}(t)}\frac{p}{q}\sum_{k=1}^n p_{k,n}^2\log_r\frac{1}{p_{k,n}}$$
$$-\frac{(-\mathrm{i}t)^2\mathrm{e}^{y_1^{1,p}(t)}}{2}\Big\{\frac{p^2}{q^2}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}+\frac{1}{q}\sum_{k=1}^n p_{k,n}^2\Big\}.$$

Thus for any admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$ by (14) we have for $\alpha \neq 1$,

$$G_{\boldsymbol{p}_n}^{\alpha,p}(x) = G_{\alpha,p,1}(x) - G_{\alpha,p,1}^{(0,2)}(x)\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2 - G_{\alpha,p,1}^{(1,1)}(x)\frac{p}{q-q^{1/\alpha}}\sum_{k=1}^{n} p_{k,n}^{1+\frac{1}{\alpha}}$$
$$- G_{\alpha,p,1}^{(2,0)}(x)\left\{\frac{p^2}{(q-q^{1/\alpha})^2} + \frac{p}{q-q^{2/\alpha}}\right\}\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^{\frac{2}{\alpha}},$$

and for $\alpha = 1$,

$$G_{\boldsymbol{p}_n}^{1,p}(x) = G_{1,p,1}(x) - G_{1,p,1}^{(0,2)}(x)\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2 + G_{1,p,1}^{(1,1)}(x)\frac{p}{q}\sum_{k=1}^{n} p_{k,n}^2 \log_r \frac{1}{p_{k,n}}$$
(23)
$$- G_{1,p,1}^{(2,0)}(x)\frac{1}{2}\left\{\frac{p^2}{q^2}\sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{q}\sum_{k=1}^{n} p_{k,n}^2\right\}$$

for all $x \in \mathbb{R}$. Therefore, in the admissible case there exists a proper limiting distribution, and moreover we have real asymptotic expansions attached to this asymptotic distribution. Concentrating on the dominant terms in Theorem 1, we obtain the following

Corollary 2. *For any sequence $\{\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})\}_{n\in\mathbb{N}}$ of admissible strategies, for $\alpha \in (0,1)$,*

$$\sup_{x\in\mathbb{R}}\left|\boldsymbol{P}\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\} - \left[G_{\alpha,p,1}(x) - G_{\alpha,p,1}^{(0,2)}(x)\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2\right]\right|$$
$$= \begin{cases} O(\overline{p}_n^{\,2}), & \text{if } 0 < \alpha < 1/2, \\ O(\overline{p}_n^{\,1/\alpha}), & \text{if } 1/2 \le \alpha < 1; \end{cases}$$

for $\alpha = 1$,

$$\sup_{x\in\mathbb{R}}\left|\boldsymbol{P}\{S_{\boldsymbol{p}_n}^{1,p} \le x\} - \left[G_{1,p,1}(x) + G_{1,p,1}^{(1,1)}(x)\frac{p}{q}\sum_{k=1}^{n} p_{k,n}^2 \log_r \frac{1}{p_{k,n}}\right.\right.$$
$$\left.\left. - G_{1,p,1}^{(2,0)}(x)\frac{p^2}{2q^2}\sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}}\right]\right| = O(\overline{p}_n);$$

and for $\alpha \in (1,2)$,

$$\sup_{x\in\mathbb{R}}\left|\boldsymbol{P}\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\} - \left[G_{\alpha,p,1}(x) - G_{\alpha,p,1}^{(2,0)}(x)\left\{\frac{p^2}{(q-q^{1/\alpha})^2} + \frac{p}{q-q^{2/\alpha}}\right\}\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^{2/\alpha}\right]\right|$$
$$= \begin{cases} O(\overline{p}_n^{\,1/\alpha}), & \text{if } 1 < \alpha < 3/2, \\ O(\overline{p}_n^{\,(4-2\alpha)/\alpha}), & \text{if } 3/2 \le \alpha < 2. \end{cases}$$

For each $n \in \mathbb{N}$ the best admissible strategy for the classical case $(\alpha, p) = (1, 1/2)$, found in [9], is the following:

$$\boldsymbol{p}_n^* = (p_{1,n}^*, \ldots, p_{n,n}^*) = (2p_n^*, \ldots, 2p_n^*, p_n^*, \ldots, p_n^*) \quad \text{with} \quad p_n^* = \frac{1}{2^{\lceil \log_2 n \rceil}} = \frac{\gamma_n}{n},$$

where the number of the p_n^* components is $2n - 2^{\lceil \log_2 n \rceil}$ and the number of the $2p_n^*$ components is $2^{\lceil \log_2 n \rceil} - n$. Calculating the coefficients in (23), we obtain $G_{\boldsymbol{p}_n^*}^{1,1/2}(x) = G_{1,1/2,1}(x) - a_n G_{1,1/2,1}^{(0,2)}(x) + b_n G_{1,1/2,1}^{(1,1)}(x) - c_n G_{1,1/2,1}^{(2,0)}(x)$, $x \in \mathbb{R}$, and $\sup_{x\in\mathbb{R}} \left| \boldsymbol{P}\{S_{\boldsymbol{p}_n^*}^{1,1/2} \le x\} - \left[G_{1,1/2,1}(x) + b_n G_{1,1/2,1}^{(1,1)}(x) - d_n G_{1,1/2,1}^{(2,0)}(x)\right]\right| = O(1/n)$ as a special case of Corollary 2, where $\gamma_n = n/2^{\lceil \log_2 n \rceil}$ oscillates in $(1/2, 1]$,

$$a_n = \frac{3 \cdot 2^{\lceil \log_2 n \rceil} - 2n}{2^{2\lceil \log_2 n \rceil + 1}} = \frac{\frac{3}{2}\gamma_n - \gamma_n^2}{n},$$

$$b_n = \frac{(3 \cdot 2^{\lceil \log_2 n \rceil} - 2n)\lceil \log_2 n \rceil - 4(2^{\lceil \log_2 n \rceil} - n)}{2^{2\lceil \log_2 n \rceil}} = \frac{(3\gamma_n - 2\gamma_n^2)\log_2 \frac{n}{\gamma_n} - 4(\gamma_n - \gamma_n^2)}{n}$$

$$\begin{aligned} d_n &= \frac{(3 \cdot 2^{\lceil \log_2 n \rceil} - 2n)\lceil \log_2 n \rceil^2 - 4(2^{\lceil \log_2 n \rceil} - n)(2\lceil \log_2 n \rceil - 1)}{2^{2\lceil \log_2 n \rceil + 1}} \\ &= \frac{(\frac{3}{2}\gamma_n - \gamma_n^2)\log_2^2 \frac{n}{\gamma_n} - 2(\gamma_n - \gamma_n^2)(2\log_2 \frac{n}{\gamma_n} - 1)}{n}, \end{aligned}$$

and

$$c_n = d_n + \frac{6 \cdot 2^{\lceil \log_2 n \rceil} - 4n}{2^{2\lceil \log_2 n \rceil + 1}} = d_n + \frac{3\gamma_n - 2\gamma_n^2}{n}.$$

Also, since in the proof of Corollary 1 we show for all $p \in (0,1)$ and all $\{\boldsymbol{p}_n\}$ for which $\overline{p}_n \to 0$ that for every $\varepsilon > 0$ there is an $n_*(\varepsilon, p) \in \mathbb{N}$ such that for $n \ge n_*(\varepsilon, p)$,

$$\sup_{x\in\mathbb{R}} \left| \boldsymbol{P}\left\{S_{\boldsymbol{p}_n}^{1,p} \le x\right\} - G_{1,p,\boldsymbol{p}_n}(x)\right| \le (1+\varepsilon)\frac{p^2}{q^2 \pi C_1^2} \sum_{k=1}^n \frac{p_{k,n}^2}{2} \log_r^2 \frac{1}{p_{k,n}},$$

and since the last sum for $\boldsymbol{p}_n = \boldsymbol{p}_n^*$ is exactly d_n, for which the asymptotic equality

$$d_n \sim \frac{\gamma_n(3 - 2\gamma_n)}{2} \frac{\log_2^2 n}{n},$$

is satisfied, where we write $x_n \sim y_n$ if $x_n/y_n \to 1$, substituting $C_1(1, 1/2) = 2/\pi$ we obtain

$$\sup_{x\in\mathbb{R}} \left| \boldsymbol{P}\left\{S_{\boldsymbol{p}_n^*}^{1,1/2} \le x\right\} - G_{1,1/2,1}(x)\right| \le (1+\varepsilon)\frac{\pi\gamma_n(3 - 2\gamma_n)}{8} \frac{\log_2^2 n}{n}$$

whenever $n \ge n_*(\varepsilon)$, a slightly better bound than the one in (34) in [9].

3. Numerical computations

The merging semistable approximations are described only through their characteristic functions and their mathematical properties are inferred either through Fourier-analytic methods or by special representations, such as that in (2). The same is even more true for the derivatives featured in our expansions, for which the only conceivable tool appears to be the Fourier method. For the purpose of numerical investigation of the expansions we use what we call the *extended Gil-Pelaez – Rosén inversion formula*, which says the following. Let $H(\cdot)$ be a function of bounded variation on $\mathbb{R}$, consider its total variation function $V_H(x) = \sup\{\sum_{j=1}^{n} |H(x_j) - H(x_{j-1})| \, : -\infty < x_0 < x_1 < \cdots < x_n \le x, n \in \mathbb{N}\}$ and let $\boldsymbol{h}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}H(x)$ be its Fourier – Stieltjes transform, $t \in \mathbb{R}$. If the logarithmic moment $\int_{-\infty}^{\infty} \log(1 + |x|) \, \mathrm{d}V_H(x) < \infty$, then

$$\frac{H(x+0) - H(x-0)}{2} = \frac{H(\infty) - H(-\infty)}{2} - \frac{1}{\pi} \lim_{T\to\infty} \int_0^T \frac{\Im\mathfrak{m}\left\{\mathrm{e}^{-\mathrm{i}tx}\boldsymbol{h}(t)\right\}}{t} \, \mathrm{d}t$$

for every $x \in \mathbb{R}$, where $H(\pm\infty) = \lim_{x\to\pm\infty} H(x)$. Gil-Pelaez [10] proved this for distribution functions without the logarithmic moment condition, in which case the integral is also improper Riemann at zero. Eleven years later Rosén [16] independently proved the same formula also for a distribution function H, for which $H(\infty) - H(-\infty) = 1 - 0 = 1$, showing in particular that under the logarithmic moment condition the integral exists as a proper Lebesgue integral on $(0, T]$ for all $T > 0$. A trivial modification of Rosén's proof gives the extended form above.

The Gil-Pelaez – Rosén formula is clearly applicable to the distribution function $G_{\alpha,p,\boldsymbol{p}_n}(\cdot)$. In order to use the formula for $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ we claim that $G_{\boldsymbol{p}_n}^{\alpha,p}(\infty) = 1$ and $G_{\boldsymbol{p}_n}^{\alpha,p}(-\infty) = 0$ for every $\alpha \in (0,2)$, $p \in (0,1)$ and strategy $\boldsymbol{p}_n$. As already noted in the previous section, by (13), (21) and (22) it suffices to show that $G_{\alpha,p,\gamma}^{(0,2)}(\pm\infty) = 0$ for all $\gamma \in (q,1]$. We know that $G_{\alpha,p,\gamma}^{(0,2)}(x) = \int_{-\infty}^{x} G_{\alpha,p,\gamma}^{(1,2)}(y)\mathrm{d}y$, $x \in \mathbb{R}$, for the integrable function $G_{\alpha,p,\gamma}^{(1,2)}(\cdot)$, thus $G_{\alpha,p,\gamma}^{(0,2)}(-\infty) = 0$ and $G_{\alpha,p,\gamma}^{(0,2)}(\cdot)$ is of bounded variation. The logarithmic moment property also holds by (12), hence by the extended Gil-Pelaez – Rosén formula

$$G_{\alpha,p,\gamma}^{(0,2)}(x) = \frac{G_{\alpha,p,\gamma}^{(0,2)}(\infty)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\Im\mathfrak{m}\left\{\mathrm{e}^{-\mathrm{i}tx}\left[y_\gamma^{\alpha,p}(t)\right]^2 \mathrm{e}^{y_\gamma^{\alpha,p}(t)}\right\}}{t} \, \mathrm{d}t, \quad x \in \mathbb{R},$$

where we write the integral in this proper form since by Lemma 2 below the function $t \mapsto [y_\gamma^{\alpha,p}(t)]^2 \, \mathrm{e}^{y_\gamma^{\alpha,p}(t)}/t$ is in fact Lebesgue integrable on $(0,\infty)$. Thus the Riemann – Lebesgue lemma implies that $G_{\alpha,p,\gamma}^{(0,2)}(\infty) = G_{\alpha,p,\gamma}^{(0,2)}(\infty)/2$, and hence $G_{\alpha,p,\gamma}^{(0,2)}(\infty) = 0$

indeed. We note that the same argument shows that $G_\alpha^{(k,j)}(\infty) = 0$ for all $k, j = 0, 1, \dots$ for which $k + j > 0$ for any semistable distribution function $G_\alpha(\cdot)$ with characteristic exponent $\alpha \in (0, 2)$; these derivatives are developed in [4].

Now, applying the extended Gil-Pelaez – Rosén formula, we obtain

$$G_{\boldsymbol{p}_n}^{\alpha,p}(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\Im\mathfrak{m} \left\{ \mathrm{e}^{-\mathrm{i}tx} \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) \right\}}{t} \,\mathrm{d}t, \quad x \in \mathbb{R}.$$

Due to the mass concentrating near zero, this formula is numerically inconvenient. The problem can be overcome by the change of variables $t = \mathrm{e}^u$, which gives

$$G_{\boldsymbol{p}_n}^{\alpha,p}(x) = \frac{1}{2} - \frac{1}{\pi} \int_{-\infty}^\infty \Im\mathfrak{m} \left\{ \mathrm{e}^{-\mathrm{i}x\mathrm{e}^u} \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(\mathrm{e}^u) \right\} \mathrm{d}u, \quad x \in \mathbb{R},$$

and smears that mass on the whole negative half-line. Indeed, using Simpson's method for numerical integration, we found that for all values of the parameters and for all strategies considered in the examples below it suffices to integrate on the finite interval $[-20, 3]$. The idea of transforming variables and the whole computation for the distribution functions $G_{1,1/2,\gamma}(\cdot)$ is due to Gordon Simons. The exact same formula can be shown to produce $H_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$ from $\boldsymbol{h}_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$ in the context of Theorem 2.

It was with this method that the three examples in Figures 1–3 in [3] were obtained for the uniform averaging strategy $\boldsymbol{p}_n^\diamond = (1/n, \dots, 1/n)$ for $\alpha = 3/2, 1, 1/2$ and the respective $n = 50, 10, 7$, all with $p = 1/2$. For the six examples here we chose the same α parameters with some different, but still very small n. Figures 1, 2, 3, 6 are for the choices $\alpha = 3/2, 1, 1, 1/2$ and the strategies

$$\boldsymbol{p}_{100} = \Big(\underbrace{\frac{1}{80}, \dots, \frac{1}{80}}_{40 \text{ times}}, \underbrace{\frac{1}{120}, \dots, \frac{1}{120}}_{60 \text{ times}} \Big), \quad \boldsymbol{p}_{12}^* = \Big(\underbrace{\frac{1}{8}, \dots, \frac{1}{8}}_{4 \text{ times}}, \underbrace{\frac{1}{16}, \dots, \frac{1}{16}}_{8 \text{ times}} \Big),$$

$$\boldsymbol{p}_{12} = \Big(\underbrace{\frac{1}{10}, \dots, \frac{1}{10}}_{6 \text{ times}}, \underbrace{\frac{1}{15}, \dots, \frac{1}{15}}_{6 \text{ times}} \Big), \quad \boldsymbol{p}_8 = \Big(\underbrace{\frac{1}{6}, \dots, \frac{1}{6}}_{4 \text{ times}}, \underbrace{\frac{1}{12}, \dots, \frac{1}{12}}_{4 \text{ times}} \Big),$$

respectively; in these four cases we still chose the unbiased situation of historical interest, that is, $p = 1/2$. For the most interesting case $\alpha = 1$ of the tail or payoff parameter, for which the mean becomes infinite, we also investigated the dependence of the approximation on the bias parameter p: with $\boldsymbol{p}_{12}$ kept, Figures 4 and 5 are for the choices $p = 1/10$ and $p = 5/6$. On all six figures the solid curves depict the distribution functions $F_{\boldsymbol{p}_n}^{\alpha,p}(x) = \boldsymbol{P}\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\}$, $x \in \mathbb{R}$, which are obtained as the empirical distribution functions of 10 000 simulations of $S_{\boldsymbol{p}_n}^{\alpha,p}$. Also on all six figures the dotted curves $G_{\alpha,p,\boldsymbol{p}_n}(\cdot)$ are the merging semistable distribution functions and the dashed curves are the full approximations $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ of Theorem 1.

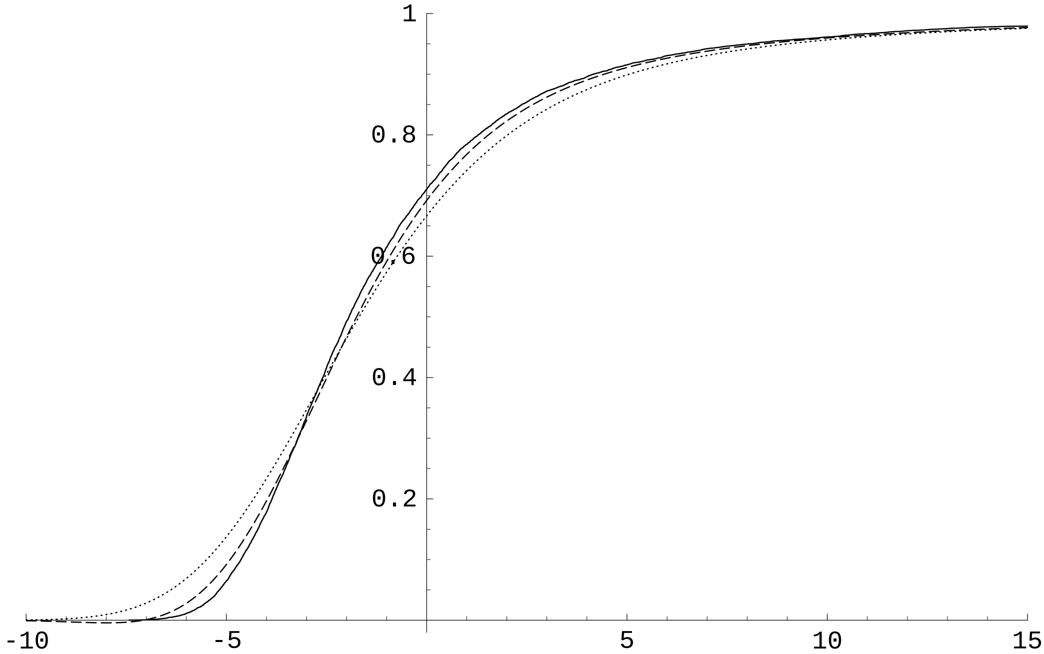

Figure 1. Solid $F^{3/2,1/2}_{\boldsymbol{p}_{100}}$, dotted $G_{3/2,1/2,\boldsymbol{p}_{100}}$, and dashed $G^{3/2,1/2}_{\boldsymbol{p}_{100}}$

For $\alpha = 3/2$ the rate of merge is $\overline{p}_n^{1/3}$ and the order of the approximation is $\overline{p}_n^{2/3}$. The very satisfactory full approximation provides a dramatic improvement.

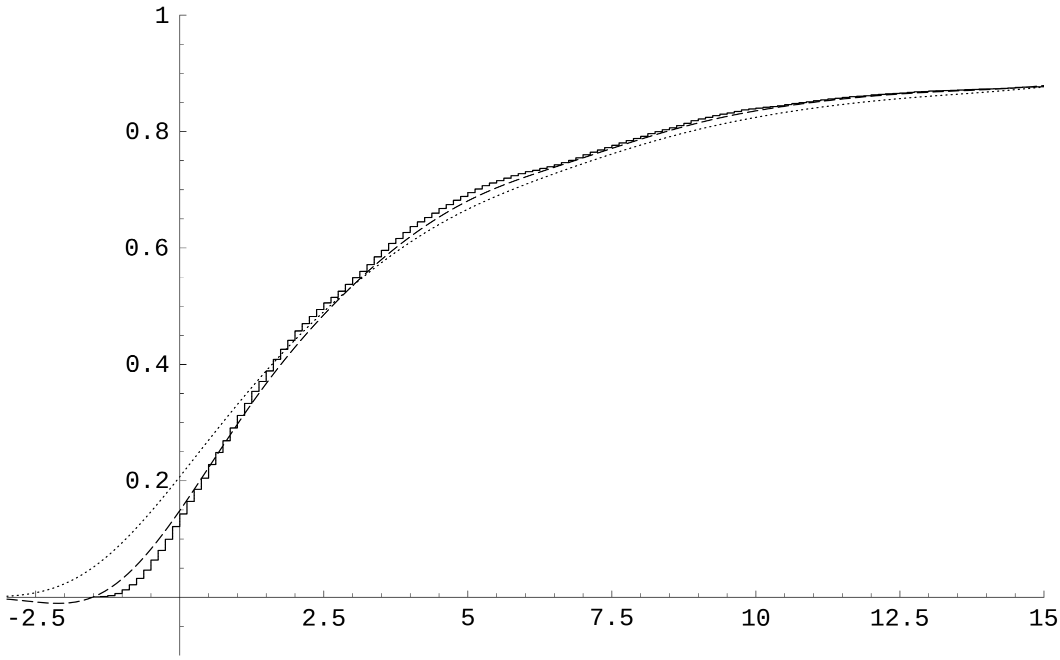

Figure 2. Solid $F^{1,1/2}_{\boldsymbol{p}^*_{12}}$, dotted $G_{1,1/2,\boldsymbol{p}^*_{12}}$, and dashed $G^{1,1/2}_{\boldsymbol{p}^*_{12}}$

For $\alpha = 1$ the rate of merge is $\overline{p}_n \log_2^2 1/\overline{p}_n$ and the order of the approximation is $\overline{p}_n$. For the best admissible strategy here, $G_{1,1/2,\boldsymbol{p}^*_n}(\cdot) \equiv G_{1,1/2,1}(\cdot)$ for all n. The following example is the exact opposite of this, but no particular difference is visible.

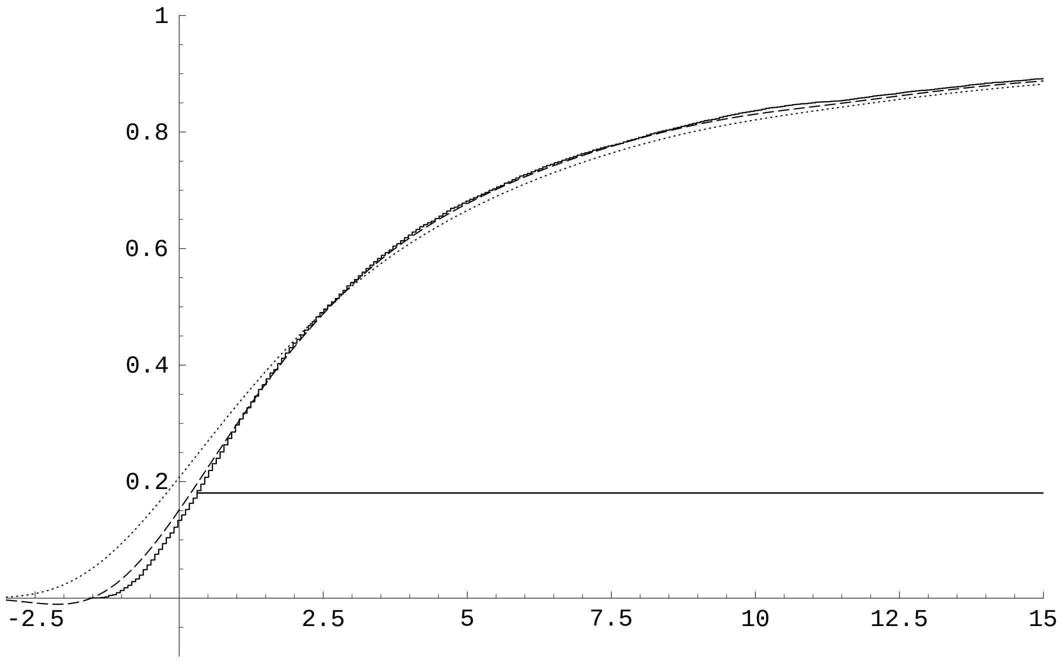

Figure 3. Solid $F^{1,1/2}_{\boldsymbol{p}_{12}}$, dotted $G_{1,1/2,\boldsymbol{p}_{12}}$, and dashed $G^{1,1/2}_{\boldsymbol{p}_{12}}$

The two different values of $\gamma_{k,12}$, $k = 1, 2, \ldots, 12$, for the strategy here, 10/16 and 15/16, differ from each other to a great extent. Roughly speaking this means that $G_{1,1/2,\boldsymbol{p}_{12}}(\cdot)$ differs from a single distribution function $G_{1,1/2,\gamma}(\cdot)$, for any γ, as much as it can. But the quality of the approximation is about the same as in Figure 2.

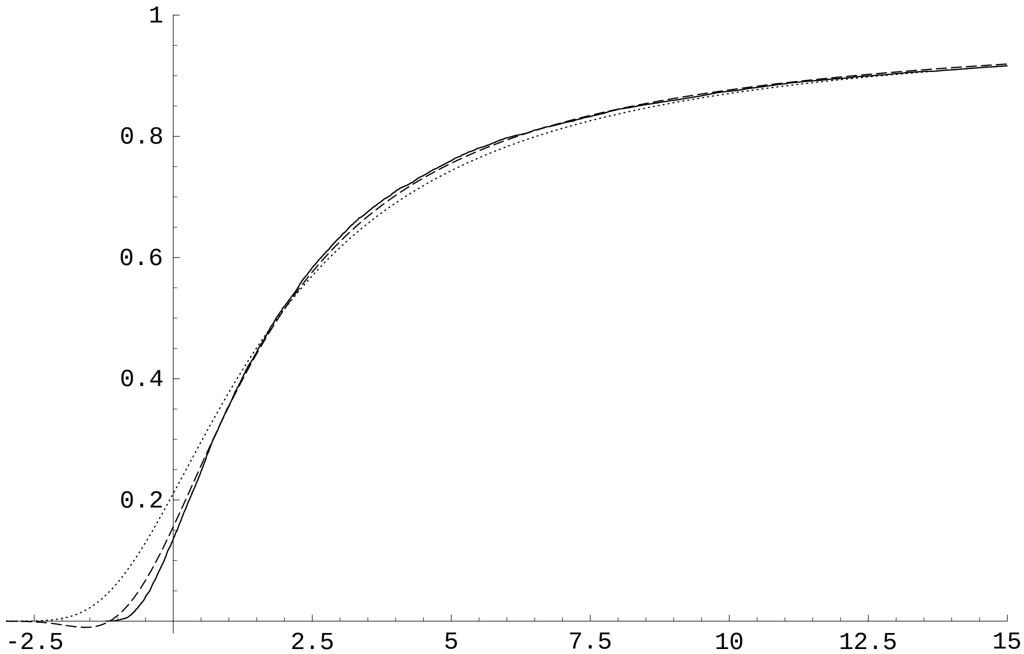

Figure 4. Solid $F^{1,1/10}_{\boldsymbol{p}_{12}}$, dotted $G_{1,1/10,\boldsymbol{p}_{12}}$, and dashed $G^{1,1/10}_{\boldsymbol{p}_{12}}$

For the present $p = 1/10$ we have $r = 10/9$, so that the gains, the powers of 10/9, increase very slowly. An easy computation shows that $F_{\boldsymbol{p}_n}^{1,1/10}(\cdot)$ has about $8 \cdot 10^{11}$ jump points in the interval $(-3, 15)$, so it seems to be continuous. As the following figure shows, for a large p the situation is the opposite.

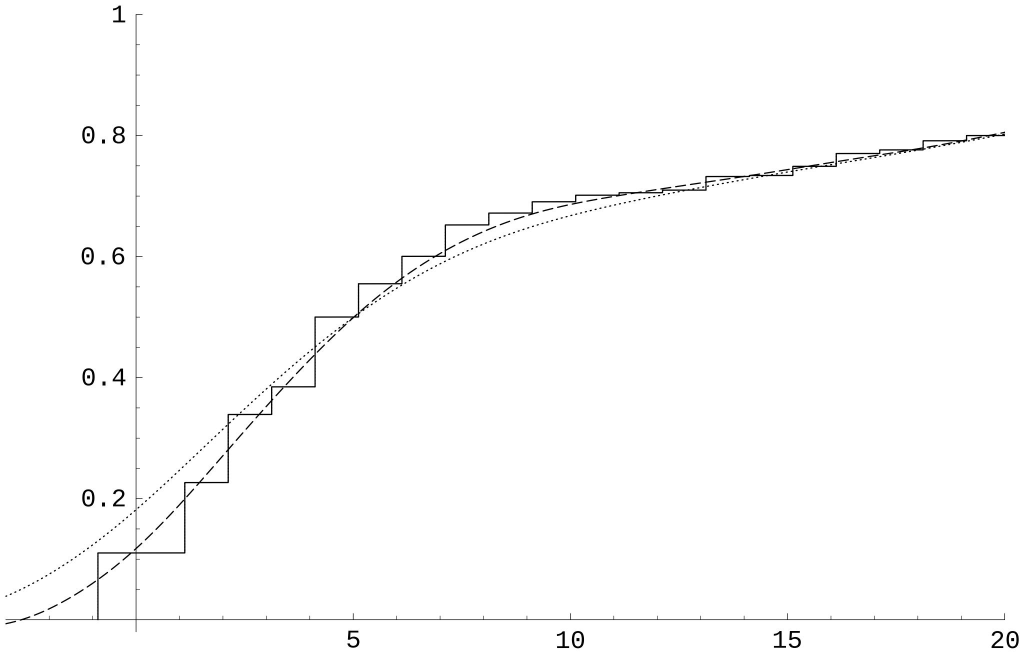

Figure 5. Solid $F_{\boldsymbol{p}_{12}}^{1,5/6}$, dotted $G_{1,5/6,\boldsymbol{p}_{12}}$, and dashed $G_{\boldsymbol{p}_{12}}^{1,5/6}$

In this case $r = 6$, so the gains increase very fast. One can easily count that $F_{\boldsymbol{p}_n}^{1,5/6}(\cdot)$ has 20 jump points in $(-3, 20)$. Thus n ought to be larger here to obtain a better approximation.

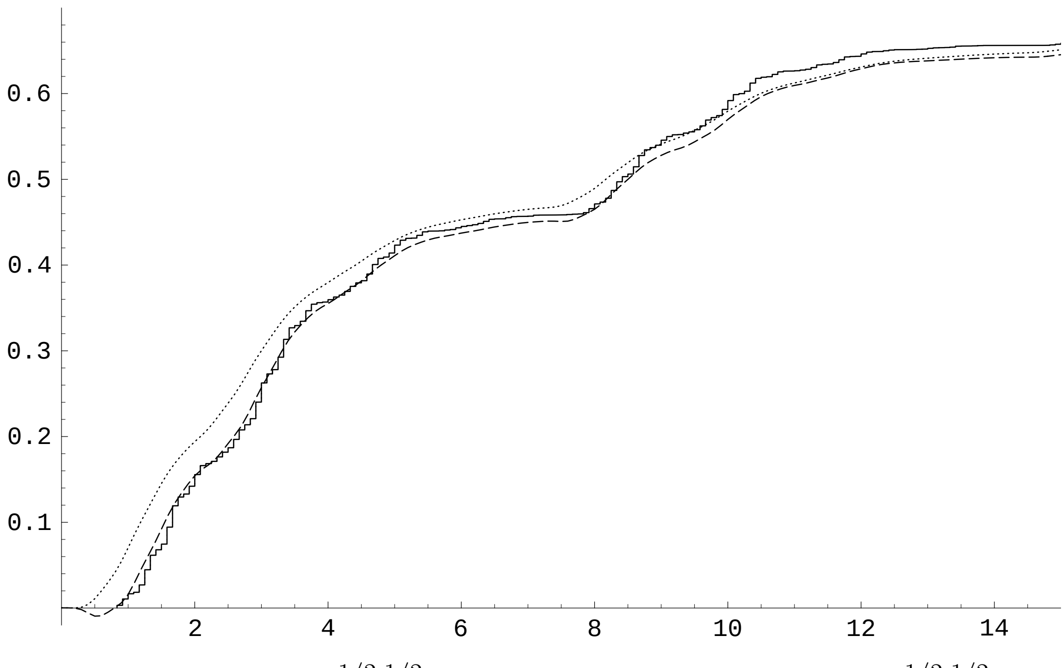

Figure 6. Solid $F_{\boldsymbol{p}_8}^{1/2,1/2}$, dotted $G_{1/2,1/2,\boldsymbol{p}_8}$, and dashed $G_{\boldsymbol{p}_8}^{1/2,1/2}$

For $\alpha = 1/2$ the rate of merge is $\overline{p}_n$, while the order of the full approximation is much better, $\overline{p}_n^2$. The precision is almost unbelievably good for even $n = 8$. We also note that despite the fact that the present strategy is not admissible, we still have $\gamma_{k,8} \equiv 3/4$ for all $k = 1, 2, \ldots, 8$, so that $G_{1/2,1/2,\boldsymbol{p}_8}(\cdot) \equiv G_{1/2,1/2,3/4}(\cdot)$ by (16).

In general we see that, extending greatly the sums from [3] to the linear combinations considered here, already the primary semistable merging approximations appear to be reasonably good, while the corresponding asymptotic expansions may be working incredibly well in a variety of different circumstances even for small n.

4. Proofs

The proof of Theorem 1 is based on Esseen's classical result (Theorem 5.2 in [15]), which we record here in a special case closest to our application.

Lemma 1. *Let F be a distribution function and G be a function of bounded variation on $\mathbb{R}$ with Fourier – Stieltjes transforms $\boldsymbol{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F(x)$ and $\boldsymbol{g}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}G(x)$, $t \in \mathbb{R}$, such that $G(-\infty) = \lim_{x\to-\infty} G(x) = 0 = F(-\infty)$ and the derivative G' of G exists and is bounded on the whole $\mathbb{R}$. Then*

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \le \frac{b}{2\pi} \int_{-T}^{T} \left| \frac{\boldsymbol{f}(t) - \boldsymbol{g}(t)}{t} \right| \mathrm{d}t + c_b \, \frac{\sup_{x\in\mathbb{R}} |G'(x)|}{T}$$

for every choice of $T > 0$ and $b > 1$, where $c_b > 0$ is a constant depending only on b, which can be given as $c_b = 4bd_b^2/\pi$, where $d_b > 0$ is the unique root d of the equation $\frac{4}{\pi} \int_0^d \frac{\sin^2 u}{u^2} \mathrm{d}u = 1 + \frac{1}{b}$.

For $j = 1, 2, 5, 6$ the constants $C_j(\alpha, p)$ below are the same as in [3] and agree with the respective constants $c_j(\alpha, p)$ in [1], while the constants $C_j(\alpha, p)$ numbered numbered with $j = 7, 8, 9$ are the same as in [3]. The following lemma is Lemma 3 in [3], the proof of the first inequality is already in [1].

Lemma 2. *Uniformly in $\gamma \in (q, 1]$,*

$$\mathfrak{Re}\, y_\gamma^{\alpha,p}(t) \le -C_1|t|^\alpha, \quad t \in \mathbb{R}, \quad \textit{where} \quad C_1 = C_1(\alpha, p) = \left(\frac{2}{\pi}\right)^\alpha \frac{pq^{(2-\alpha)/\alpha}}{q - q^{2/\alpha}} \,,$$

and

$$\left| y_\gamma^{\alpha,p}(t) \right| \le v_{\alpha,p}(|t|), \quad t \in \mathbb{R},$$

where

$$v_{\alpha,p}(s) = \begin{cases} C_7 \, s^\alpha, & \textit{if } \alpha \ne 1, \\ \left(C_7 + \frac{2p}{q} |\log_r s|\right) s, & \textit{if } \alpha = 1, \end{cases}$$

for every $s \ge 0$*, and for the constant* $C_7 = C_7(\alpha,p) > 0$ *defined as*

$$C_7(\alpha,p) = \begin{cases} \frac{2^{1-\alpha}}{q} + \frac{2^{1-\alpha}p}{q-q^{1/\alpha}}, & \text{if } \alpha < 1, \\ \frac{\max\{6\,,5+9p-8p\,\log_r 2\}}{2q}, & \text{if } \alpha = 1, \\ \frac{8p}{4^\alpha}\Big\{\frac{1}{q-q^{2/\alpha}} + \frac{1}{q-q^{(2\alpha-1)/\alpha}}\Big\}, & \text{if } \alpha > 1. \end{cases}$$

Proof of Theorem 1. The first step is to prove that the derivatives $(G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot))'$ exist and are uniformly bounded in the strategies. In fact, first we claim that $I_{j,\boldsymbol{p}_n}^{\alpha,p} := \int_{-\infty}^{\infty} |t|^j \, |\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)| \, \mathrm{d}t < \infty$ for any $j \in \{0,1,2,\ldots\}$, which, referring to (15), implies that $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ is arbitrary many times differentiable on $\mathbb{R}$. First note that by (16), Lemma 2 implies that for the characteristic function

$$|\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)| = \exp\left\{\sum_{k=1}^{n} p_{k,n}\,\Re\mathfrak{e}\, y_{\gamma_{k,n}}^{\alpha,p}(t)\right\} \le \mathrm{e}^{-C_1|t|^\alpha}, \quad t \in \mathbb{R}.$$

Proceeding for $\alpha = 1$, for which $I_j := \int_{-\infty}^{\infty} |t|^j \, \mathrm{e}^{-C_1|t|}\mathrm{d}t < \infty$, using (17), Lemma 2 and the triviality $\overline{p}_n \le 1$, we obtain

$$I_{j,\boldsymbol{p}_n}^{1,p} \le I_j + \frac{C_7^2 I_{j+2}}{2} + \frac{4C_7 p}{q}\sum_{k=1}^{n} p_{k,n}\int_0^\infty |t|^{j+1}\,\mathrm{e}^{-C_1|t|}\big|\log_r(p_{k,n}|t|)\big|(p_{k,n}|t|)\,\mathrm{d}t$$
$$+ \frac{4p^2}{q^2}\sum_{k=1}^{n} p_{k,n}\int_0^\infty |t|^{j+1}\,\mathrm{e}^{-C_1|t|}\Big\{\log_r(p_{k,n}|t|)\sqrt{p_{k,n}|t|}\Big\}^2\mathrm{d}t + \frac{I_{j+2}}{2q}.$$

Breaking the k-th integral under both sums at $1/p_{k,n}$, using that $|\log_r s| s \le l_p := (\log_r \mathrm{e})/\mathrm{e}$, $|\log_r s|\sqrt{s} \le 2l_p$, $s \in (0,1)$, and $\log_r x \le c_p x$, $x \ge 1$, for $c_p = 1/(\mathrm{e}\log r)$, where $\log = \log_\mathrm{e}$, and then extending all resulting integrals to $(0,\infty)$ again, we get

$$I_{j,\boldsymbol{p}_n}^{1,p} \le I_j + \frac{C_7^2 I_{j+2}}{2} + \frac{2C_7 p}{q}\big[l_p I_{j+1} + c_p I_{j+2}\big] + \frac{2p^2}{q^2}\big[4l_p^2 I_{j+1} + c_p^2 I_{j+4}\big] + \frac{I_{j+2}}{2q} =: M_j^{1,p}$$

for all $\boldsymbol{p}_n$. The argument is similar for $\alpha \ne 1$; in fact it is given below for $j = 0$.

Thus, writing (15) for the first derivative and using the usual Fourier inversion formula, (17) and Lemma 2 again, for $\alpha \ne 1$ we obtain

$$\Big|\big(G_{\boldsymbol{p}_n}^{\alpha,p}(x)\big)'\Big| = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)\,\mathrm{d}t\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-C_1|t|^\alpha}\Bigg[1 + \frac{1}{2}\sum_{k=1}^{n} v_{\alpha,p}^2(|t|p_{k,n}^{1/\alpha})$$
$$+ |ts_1^{\alpha,p}|\sum_{k=1}^{n} p_{k,n}^{1/\alpha} v_{\alpha,p}(|t|p_{k,n}^{1/\alpha}) + \frac{t^2}{2}\Big\{(s_1^{\alpha,p})^2 + \frac{p}{q-q^{2/\alpha}}\Big\}\sum_{k=1}^{n} p_{k,n}^{2/\alpha}\Bigg]\mathrm{d}t$$
$$\le \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-C_1 t^\alpha}\Bigg[1 + \frac{C_7^2\overline{p}_n t^{2\alpha}}{2} + C_7|s_1^{\alpha,p}|\overline{p}_n^{\,1/\alpha}\, t^{1+\alpha}$$
$$+ \frac{t^2}{2}\Big\{(s_1^{\alpha,p})^2 + \frac{p}{q-q^{2/\alpha}}\Big\}\overline{p}_n^{(2-\alpha)/\alpha}\Bigg]\mathrm{d}t \le M^{\alpha,p},$$

where the constant $M^{\alpha,p}$ is obtained upon replacing $\overline{p}_n$ by 1, and where we used the trivial inequality $\sum_{k=1}^n p_{k,n}^\beta \le \overline{p}_n^{\beta-1}$, $\beta > 1$. For $\alpha = 1$, the proof is done above, so that the bound $M^{1,p}$ on the first derivative can be taken as $M_0^{1,p}$ above.

Now we turn to the proof of the theorem, which is an extension of the proof of the Proposition in [3]; whenever possible, we use the same or analogous notation as there. We may skip some detail for $\alpha \ne 1$.

Using Esseen's inequality, we get

$$\begin{aligned}\Delta_{\boldsymbol{p}_n}^{\alpha,p} &:= \sup_{x\in\mathbb{R}} \big|\boldsymbol{P}\big\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\big\} - G_{\boldsymbol{p}_n}^{\alpha,p}(x)\big| \\ &\le \frac{b}{2\pi}\int_{-T_n^{\alpha,p}}^{T_n^{\alpha,p}} \frac{\big|\boldsymbol{E}(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}) - \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)\big|}{|t|}\,\mathrm{d}t + c_b\frac{M_{\alpha,p}}{T_n^{\alpha,p}} =: \frac{b}{2\pi}\Delta_{\boldsymbol{p}_n,1}^{\alpha,p} + c_b\,\Delta_{\boldsymbol{p}_n,2}^{\alpha,p}\,,\end{aligned}$$

where $T_n^{\alpha,p} = 2K^{1/\alpha}/\overline{p}_n^{1/\alpha}$, and on the constant $K = K_{\alpha,p} > 0$ we will introduce some restrictions as we go along. By the choice of $T_n^{\alpha,p}$ we have $\Delta_{\boldsymbol{p}_n,2}^{\alpha,p} = O(\overline{p}_n^{1/\alpha})$. The estimation of the other term requires some further notation. The characteristic functions of $S_{\boldsymbol{p}_n}^{\alpha,p}$ and $W_{\boldsymbol{p}_n}^{\alpha,p}$ can be written in the form

$$\boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}\Big) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)}\prod_{k=1}^n \boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha}X_k}\Big) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)}\prod_{k=1}^n\Big(1 + y_{k,n}^{\alpha,p}(t)\Big)$$

and

$$\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \boldsymbol{E}\left(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}}\right) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)}\prod_{k=1}^n \mathrm{e}^{z_{\alpha,p}(p_{k,n}^{1/\alpha}t)}, \quad t\in\mathbb{R},$$

where $y_{k,n}^{\alpha,p}(t) = \boldsymbol{E}\big(\exp\big\{\mathrm{i}tp_{k,n}^{1/\alpha}X_k\big\} - 1\big)$ and $z_{\alpha,p}(s) = y_1^{\alpha,p}(s) - \mathrm{i}s_1^{\alpha,p}s$, $s\in\mathbb{R}$. Notice that $z_{1,p}(s) = y_1^{1,p}(s)$. Continuing the transformations, we may write

$$\begin{aligned}\boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}\Big) &= \exp\left\{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n) + \sum_{k=1}^n \log(1 + y_{k,n}^{\alpha,p}(t))\right\} \\ &= \exp\left\{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n) + \sum_{k=1}^n z_{\alpha,p}(p_{k,n}^{1/\alpha}t) + \sum_{k=1}^n R_{k,n,1}^{\alpha,p}(t) + \sum_{k=1}^n w_{k,n}^{\alpha,p}(t)\right\} \\ &= \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\exp\left\{\sum_{k=1}^n\Big(R_{k,n,1}^{\alpha,p}(t) + w_{k,n}^{\alpha,p}(t)\Big)\right\} \\ &= \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\left[1 + \sum_{k=1}^n\Big(R_{k,n,1}^{\alpha,p}(t) + w_{k,n}^{\alpha,p}(t)\Big) + R_{n,2}^{\alpha,p}(t)\right] \\ &= \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\left[1 - \frac{1}{2}\sum_{k=1}^n (y_{k,n}^{\alpha,p}(t))^2 + R_{n,1}^{\alpha,p}(t) + R_{n,2}^{\alpha,p}(t) + R_{n,3}^{\alpha,p}(t)\right],\end{aligned}$$

where the error terms are $w_{k,n}^{\alpha,p}(t) = \log(1 + y_{k,n}^{\alpha,p}(t)) - y_{k,n}^{\alpha,p}(t)$ and

$$R_{n,1}^{\alpha,p}(t) = \sum_{k=1}^{n} R_{k,n,1}^{\alpha,p}(t) = \sum_{k=1}^{n} \left(y_{k,n}^{\alpha,p}(t) - z_{\alpha,p}(p_{k,n}^{1/\alpha} t) \right),$$

$$R_{n,2}^{\alpha,p}(t) = \sum_{l=2}^{\infty} \frac{1}{l!} \left[\sum_{k=1}^{n} w_{k,n}^{\alpha,p}(t) + R_{k,n,1}^{\alpha,p}(t) \right]^{l}, \quad R_{n,3}^{\alpha,p}(t) = \sum_{k=1}^{n} \sum_{l=3}^{\infty} (-1)^{l+1} \frac{1}{l} (y_{k,n}^{\alpha,p}(t))^{l}.$$

In general we use the simplifying convention $R_{n,j}^{\alpha,p}(t) = \sum_{k=1}^{n} R_{k,n,j}^{\alpha,p}(t)$, $j = 1, 3, 6$. Finally, using the identity $y_{k,n}^{\alpha,p}(t) = y_1^{\alpha,p}(p_{k,n}^{1/\alpha} t) - \mathrm{i}tp_{k,n}^{1/\alpha} s_1^{\alpha,p} + R_{k,n,1}^{\alpha,p}(t)$, we obtain

$$\begin{aligned}
\boldsymbol{E}\left(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}\right) &= \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t) \Bigg[1 - \frac{1}{2} \sum_{k=1}^{n} y_1^{\alpha,p}(p_{k,n}^{1/\alpha} t)^2 + \mathrm{i}ts_1^{\alpha,p} \sum_{k=1}^{n} p_{k,n}^{1/\alpha} y_1^{\alpha,p}(p_{k,n}^{1/\alpha} t) \\
&\qquad + \frac{t^2}{2} \left\{ (s_1^{\alpha,p})^2 + \frac{p}{q - q^{2/\alpha}} \right\} \sum_{k=1}^{n} p_{k,n}^{2/\alpha} + R_{n,5}^{\alpha,p}(t) \Bigg] \\
&= \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) + \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) R_{n,5}^{\alpha,p}(t) = \boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) + R_{n,7}^{\alpha,p}(t),
\end{aligned}$$

where

$$R_{n,5}^{\alpha,p}(t) = \widetilde{R}_{n,1}^{\alpha,p}(t) + R_{n,2}^{\alpha,p}(t) + R_{n,3}^{\alpha,p}(t) + R_{n,6}^{\alpha,p}(t) = R_{n,4}^{\alpha,p}(t) + R_{n,6}^{\alpha,p}(t),$$

$$\widetilde{R}_{n,1}^{\alpha,p}(t) = \sum_{k=1}^{n} \widetilde{R}_{k,n,1}^{\alpha,p}(t) = \sum_{k=1}^{n} \left[R_{k,n,1}^{\alpha,p}(t) - p_{k,n}^{2/\alpha} \frac{t^2 \, p}{2\,(q - q^{2/\alpha})} \right]$$

and

$$R_{n,6}^{\alpha,p}(t) = \sum_{k=1}^{n} R_{k,n,6}^{\alpha,p}(t) = \sum_{k=1}^{n} \left[-\frac{1}{2} R_{k,n,1}^{\alpha,p}(t)^2 - R_{k,n,1}^{\alpha,p}(t) \left\{ y_1^{\alpha,p}(tp_{k,n}^{1/\alpha}) - \mathrm{i}tp_{k,n}^{1/\alpha} s_1^{\alpha,p} \right\} \right].$$

Now we turn to the estimation of the remainder terms. By definition and (1),

$$y_{k,n}^{\alpha,p}(t) = \boldsymbol{E}\left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} X_k} - 1\right) = \int_0^{\infty} \left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} x} - 1\right) \mathrm{d}F_{\alpha,p}(x) = \sum_{l=1}^{\infty} \left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1\right) q^{l-1}\, p,$$

and by (4),

$$z_{\alpha,p}(p_{k,n}^{1/\alpha} t) = \sum_{l=0}^{-\infty} \left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1 - \mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}\right) q^{l-1} p + \sum_{l=1}^{\infty} \left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1\right) q^{l-1} p.$$

Thus, using the inequality $|e^{iu} - 1 - iu| \le u^2/2$, $u \in \mathbb{R}$ (Lemma 4 in [1]), we obtain

$$|R^{\alpha,p}_{k,n,1}(t)| = \left|y^{\alpha,p}_{k,n}(t) - z_{\alpha,p}(tp^{1/\alpha}_{k,n})\right| \le \sum_{l=0}^{-\infty} \left| \left(e^{itp^{1/\alpha}_{k,n} r^{l/\alpha}} - 1 - itp^{1/\alpha}_{k,n} r^{l/\alpha} \right) q^{l-1} p \right|$$

$$\le \frac{|t|^2 p^{2/\alpha}_{k,n}\, p}{2q} \sum_{l=0}^{\infty} q^{(\frac{2}{\alpha}-1)l} = \frac{|t|^2 p^{2/\alpha}_{k,n}\, p}{2q} \frac{1}{1-q^{\frac{2}{\alpha}-1}} = |t|^2\, C_2\, p^{2/\alpha}_{k,n},$$

where $C_2 = C_2(\alpha, p) = p/(2(q - q^{2/\alpha}))$. Since

$$\widetilde{R}^{\alpha,p}_{k,n,1}(t) = -\sum_{l=0}^{-\infty} \left(e^{itp^{1/\alpha}_{k,n} r^{l/\alpha}} - 1 - itp^{1/\alpha}_{k,n} r^{l/\alpha} - \frac{(it)^2 p^{2/\alpha}_{k,n} r^{2l/\alpha}}{2} \right) q^{l-1} p,$$

using this time the inequality $\left|e^{iu} - 1 - iu - \frac{(iu)^2}{2}\right| \le \frac{|u|^3}{6}$, $u \in \mathbb{R}$, we obtain

$$\left|\widetilde{R}^{\alpha,p}_{k,n,1}(t)\right| \le \frac{|t|^3 p^{3/\alpha}_{k,n}\, p}{6\,q} \sum_{l=0}^{\infty} q^{(\frac{3}{\alpha}-1)l} = \frac{|t|^3 p^{3/\alpha}_{k,n}\, p}{6\,q} \frac{1}{1-q^{\frac{3}{\alpha}-1}} = |t|^3\, \widetilde{C}_2\, p^{3/\alpha}_{k,n},$$

where $\widetilde{C}_2(\alpha, p) = p/(6(q - q^{3/\alpha}))$. Summing these bounds for $k = 1, \dots, n$, we get

$$\left|R^{\alpha,p}_{n,1}(t)\right| \le C_2|t|^2 \sum_{k=1}^{n} p^{2/\alpha}_{k,n} \quad \text{and} \quad \left|\widetilde{R}^{\alpha,p}_{n,1}(t)\right| \le \widetilde{C}_2|t|^3 \sum_{k=1}^{n} p^{3/\alpha}_{k,n}. \tag{24}$$

Introduce $x^{\alpha,p}_{k,n}(t) = y^{\alpha,p}_{k,n}(t)/p_{k,n}$. Then the calculation on pages 320–321 in [3], which goes back to page 837 in [1], now yields

$$|x^{\alpha,p}_{k,n}(t)| \le \begin{cases} C_5|t|^\alpha + C_6|t|p^{1/\alpha-1}_{k,n}, & \text{if } \alpha \ne 1, \\ |t| \left(r + \frac{p}{q} \log_r \frac{2}{|t|p_{k,n}} \right), & \text{if } \alpha = 1, \end{cases}$$

for $|t| \le 2q^{1/\alpha}/p^{1/\alpha}_{k,n}$, where

$$C_5 = C_5(\alpha, p) = 2^{1-\alpha}\left\{ \frac{1}{q} + \frac{p}{q - q^{1/\alpha}} \right\} \quad \text{and} \quad C_6 = C_6(\alpha, p) = \frac{p}{q^{1/\alpha} - q}.$$

Notice that $C_5 > 0$ and $C_6 < 0$ for $\alpha < 1$, so that $|x^{\alpha,p}_{k,n}(t)| \le C_5|t|^\alpha$ for $\alpha < 1$. On the other hand, $C_6 > 0$, but C_5 can be both positive and negative for $\alpha > 1$. Therefore, we need the following argument. If $|t| \le T^{\alpha,p}_n = 2K^{1/\alpha}/\overline{p}^{1/\alpha}_n$ then $|C_5||t|^\alpha \le |t||C_5||T^{\alpha,p}_n|^{\alpha-1} \le |t|p^{(1-\alpha)/\alpha}_{k,n}\{2^{\alpha-1}K^{(\alpha-1)/\alpha}|C_5|\}$, where the expression in the last pair of curly braces is < 1 if K is small enough, in which case $|x^{\alpha,p}_{k,n}(t)| \le (C_6 + 1)|t|\, p^{(1-\alpha)/\alpha}_{k,n}$. An easy monotonicity argument on the upper bounds implies that if K is small enough, then there exists $L = L_{\alpha,p} \in (0,1)$ such that $|y_{k,n}^{\alpha,p}(t)| \leq L < 1$ for t in the interval $[-T_n^{\alpha,p}, T_n^{\alpha,p}]$. Thus we have the estimates

$$|w_{k,n}^{\alpha,p}(t)| = |\log(1 + y_{k,n}^{\alpha,p}(t)) - y_{k,n}^{\alpha,p}(t)| \leq C_8 |y_{k,n}^{\alpha,p}(t)|^2,$$

and

$$|R_{k,n,3}^{\alpha,p}(t)| \leq \sum_{l=3}^{\infty} \frac{1}{l} |y_{k,n}^{\alpha,p}(t)|^l \leq C_9 |y_{k,n}^{\alpha,p}(t)|^3$$

where, by the same elementary calculations as on page 323 in [3],

$$C_8 = C_8(\alpha, p) = \frac{1}{6} + \frac{1}{3} \frac{1}{1 - L_{\alpha,p}} \quad \text{and} \quad C_9 = C_9(\alpha, p) = \frac{1}{12} + \frac{1}{4} \frac{1}{1 - L_{\alpha,p}}.$$

Using these bounds, the second statement of Lemma 1 and (24), we get

$$\begin{aligned}(25)\qquad |R_{n,5}^{\alpha,p}(t)| \leq \widetilde{C}_2 |t|^3 \sum_{k=1}^{n} p_{k,n}^{3/\alpha} + R_{n,8}^{\alpha,p}(t) + C_9 \sum_{k=1}^{n} |y_{k,n}^{\alpha,p}(t)|^3 + \frac{C_2^2 |t|^4}{2} \sum_{k=1}^{n} p_{k,n}^{4/\alpha} \\ + C_2 t^2 \sum_{k=1}^{n} p_{k,n}^{2/\alpha} v_{\alpha,p}(t p_{k,n}^{1/\alpha}) + C_2 |s_1^{\alpha,p}| |t|^3 \sum_{k=1}^{n} p_{k,n}^{3/\alpha},\end{aligned}$$

for all $t \in [-T_n^{\alpha,p}, T_n^{\alpha,p}]$, where $R_{n,8}^{\alpha,p}(t)$ is an upper bound on $|R_{n,2}^{\alpha,p}(t)|$, given by

$$R_{n,8}^{\alpha,p}(t) = \sum_{l=2}^{\infty} \frac{1}{l!} \left[C_8 \sum_{k=1}^{n} |y_{k,n}^{\alpha,p}(t)|^2 + C_2 |t|^2 \sum_{k=1}^{n} p_{k,n}^{2/\alpha} \right]^l.$$

For simplicity we now separate the three main cases: $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. In the followings we will use the simple identity $\int_0^\infty t^\eta \mathrm{e}^{-ct^\alpha} \mathrm{d}t = \frac{\Gamma((\eta+1)/\alpha)}{\alpha c^{(\eta+1)/\alpha}}$, for $\eta > -1$, $c > 0$ and the inequality $\sum_{k=1}^{n} p_{k,n}^{\beta} \leq \overline{p}_n^{\beta-1}$ for $\beta > 1$.

Consider first the case $\alpha \in (0,1)$. Since $|y_{k,n}^{\alpha,p}(t)| = |p_{k,n} x_{k,n}^{\alpha,p}(t)| \leq p_{k,n} C_5 |t|^\alpha$ and $v_{\alpha,p}(|t|) = C_7 |t|^\alpha$, we have by (25),

$$\begin{aligned}\Delta_{\boldsymbol{p}_n,1}^{\alpha,p} \leq \overline{p}_n^{\frac{3}{\alpha}-1} 2\widetilde{C}_2 \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} + \overline{p}_n^2 2 C_5^3 C_9 \frac{\Gamma(3)}{\alpha C_1^3} + \overline{p}_n^{\frac{4}{\alpha}-1} C_2^2 \frac{\Gamma(4/\alpha)}{\alpha C_1^{4/\alpha}} \\ + \overline{p}_n^{\frac{2}{\alpha}} 2 C_2 C_7 \frac{\Gamma((2+\alpha)/\alpha)}{\alpha C_1^{(2+\alpha)/\alpha}} + \overline{p}_n^{\frac{3}{\alpha}-1} 2 C_2 |s_1^{\alpha,p}| \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} \\ + 2 \int_0^{T_n^{\alpha,p}} \frac{1}{t} \mathrm{e}^{-C_1 t^\alpha} |R_{n,8}^{\alpha,p}(t)| \, \mathrm{d}t.\end{aligned}$$

Substituting the bounds into $R_{n,8}^{\alpha,p}(t)$, we obtain

$$
\begin{aligned}
|R_{n,8}^{\alpha,p}(t)| \leq & \sum_{l=2}^{\infty} \frac{1}{l!} \left[C_8 C_5^2 |t|^{2\alpha} \overline{p}_n + C_2 |t|^2 \overline{p}_n^{(2-\alpha)/\alpha} \right]^l \\
\leq & \sum_{l=2}^{\infty} \frac{1}{l!} \left[\left(C_8 C_5^2 + C_2 |t|^{2-2\alpha} \overline{p}_n^{(2-2\alpha)/\alpha} \right) |t|^{2\alpha} \overline{p}_n \right]^l \\
\leq & \sum_{l=2}^{\infty} \frac{1}{l!} \left[\left(C_8 C_5^2 + C_2 (2K^{1/\alpha})^{2-2\alpha} \right) |t|^{2\alpha} \overline{p}_n \right]^l ,
\end{aligned}
$$

where we used that $|t|^{2-2\alpha} \overline{p}_n^{2/\alpha - 2} \leq (T_n^{\alpha,p})^{2-2\alpha} \overline{p}_n^{2/\alpha-2} = (2K^{1/\alpha})^{2-2\alpha}$. Then the same calculation as in [3], page 325, yields

$$
2 \int_0^{T_n^{\alpha,p}} \frac{\mathrm{e}^{-C_1 t^\alpha}}{t} |R_{n,8}^{\alpha,p}(t)| \, \mathrm{d}t \leq \overline{p}_n^2 \frac{2}{\alpha 4^\alpha C_1^2 K^2} \frac{3R^2 - 2R^3}{(1-R)^2},
$$

provided that $K = K_{\alpha,p}$ is small enough to make

$$
R = R(\alpha, p) = 2^\alpha K \frac{C_8 C_5^2 + C_2 (2K^{1/\alpha})^{2-2\alpha}}{C_1} < 1.
$$

After an easy check on the powers of $\overline{p}_n$ the proof is ready in this case.

Now consider the case $\alpha = 1$. Elementary analysis shows that for each $\delta \in (0,1)$ the function $f(t) = t^\delta \left(r + \frac{p}{q} \log_r \frac{2}{\overline{p}_n t} \right)$ is monotone increasing on $(0, T_n^{1,p})$ if $K < \mathrm{e}^{-1/\delta}$. Recall that $T_n^{1,p} = 2K/\overline{p}_n$. The monotonicity of f easily implies that

$$
(p_{k,n} t)^\delta \left(r + \frac{p}{q} \log_r \frac{2}{p_{k,n} t} \right) \leq (2K)^\delta \left(r + \frac{p}{q} \log_r \frac{1}{K} \right)
$$

for $t \in (0, T_n^{1,p})$, $k = 1, 2, \ldots, n$. Applying this for $\delta = 1/3$ we get

$$
\frac{y_{k,n}^{1,p}(t)^3}{t} = p_{k,n}^3 t^2 \left(r + \frac{p}{q} \log_r \frac{2}{t p_{k,n}} \right)^3 \leq p_{k,n}^2 t \left[\left(r + \frac{p}{q} \log_r \frac{1}{K} \right) (2K)^{1/3} \right]^3 ,
$$

if $K < \mathrm{e}^{-3}$. Using also the inequality $t p_{k,n} \log_r \frac{1}{t p_{k,n}} \leq 2K \log_r \frac{1}{2K}$ and integrating the bounds in (25), we obtain

$$
\begin{aligned}
\Delta_{\boldsymbol{p}_n,1}^{1,p} \leq & \overline{p}_n^2 2 \widetilde{C}_2 \frac{\Gamma(3)}{C_1^3} + \overline{p}_n 2 C_9 \left[\left(r + \frac{p}{q} \log_r \frac{1}{K} \right) (2K)^{1/3} \right]^3 \frac{\Gamma(2)}{C_1^2} + \overline{p}_n^3 C_2^2 \frac{\Gamma(4)}{C_1^4} \\
& + \overline{p}_n^2 2 C_2 C_7 \frac{\Gamma(3)}{C_1^3} + \overline{p}_n \frac{4 C_2 p \Gamma(2)}{q C_1^2} 2K \log_r \frac{1}{2K} + \Delta_{\boldsymbol{p}_n,3}^{1,p},
\end{aligned}
$$

where $\Delta^{1,p}_{\boldsymbol{p}_n,3} = 2\int_0^{T^{1,p}_n} \mathrm{e}^{-C_1 t}\, t^{-1}\, R^{1,p}_{n,8}(t)\,\mathrm{d}t$. The monotonicity of f also implies the inequality $p_{k,n}(r + \frac{p}{q}\log_r \frac{2}{tp_{k,n}})^2 \le \overline{p}_n(r + \frac{p}{q}\log_r \frac{2}{t\overline{p}_n})^2$, $k = 1, 2, \dots, n$, if $K < \mathrm{e}^{-2}$. Hence we obtain

$$\begin{aligned} R^{1,p}_{n,8}(t) &\le \sum_{l=2}^{\infty} \frac{1}{l!}\left[C_2 t^2 \overline{p}_n + C_8 t^2 \sum_{k=1}^{n} p^2_{k,n}\left(r + \frac{p}{q}\log_r \frac{2}{tp_{k,n}}\right)^2\right]^l \\ &\le \sum_{l=2}^{\infty} \frac{1}{l!}\left[C_2 t^2 \overline{p}_n + C_8 t^2 \overline{p}_n\left(r + \frac{p}{q}\log_r \frac{2}{t\overline{p}_n}\right)^2\right]^l, \end{aligned}$$

and since $1 - \frac{2}{l} + \frac{2\delta}{l} \ge \delta$ for every $l \ge 2$, the inequality

$$\left[C_2 + C_8\left(r + \frac{p}{q}\log_r \frac{2}{\overline{p}_n t}\right)^2\right] t^{1-\frac{2}{l}+\frac{2\delta}{l}} \le \left[C_2 + C_8\left(r + \frac{p}{q}\log_r \frac{1}{K}\right)^2\right](T^{1,p}_n)^{1-\frac{2}{l}+\frac{2\delta}{l}}$$

holds on $(0, T^{1,p}_n)$, if K is so small that $K < \mathrm{e}^{-2/\delta}$. Substituting these bounds into $\Delta^{1,p}_{\boldsymbol{p}_n,3}$ and using that $C_1(1,p) = 2/\pi$, we get

$$\begin{aligned} \Delta^{1,p}_{\boldsymbol{p}_n,3} &= \sum_{l=2}^{\infty} \frac{2\overline{p}_n^{\,l}}{l!}\int_0^{\frac{2K}{\overline{p}_n}} \left[C_2 t^2 + C_8 t^2\left(r + \frac{p}{q}\log_r \frac{2}{t\overline{p}_n}\right)^2\right]^l \frac{\mathrm{e}^{-\frac{2}{\pi}t}}{t}\,\mathrm{d}t \\ &= \sum_{l=2}^{\infty} \frac{2\overline{p}_n^{\,l}}{l!}\int_0^{\frac{2K}{\overline{p}_n}} \left[\left\{C_2 + C_8\left(r + \frac{p}{q}\log_r \frac{2}{t\overline{p}_n}\right)^2\right\} t^{1-\frac{2}{l}+\frac{2\delta}{l}}\right]^l t^{l+1-2\delta}\,\mathrm{e}^{-\frac{2}{\pi}t}\,\mathrm{d}t \\ &\le \frac{2\overline{p}_n^{\,2-2\delta}}{(2K)^{2-2\delta}} \sum_{l=2}^{\infty} \frac{(2K)^l}{l!}\left[C_2 + C_8\left(r + \frac{p}{q}\log_r \frac{1}{K}\right)^2\right]^l \left(\frac{\pi}{2}\right)^{l+2-2\delta} \Gamma(l + 2 - 2\delta) \\ &\le \frac{\overline{p}_n^{\,2-2\delta}\pi^{2-2\delta}}{2^{3-4\delta}\,K^{2-2\delta}} \sum_{l=2}^{\infty}(l+1)\left[\pi C_2 K + \pi C_8\left(r + \frac{p}{q}\log_r \frac{1}{K}\right)^2 K\right]^l \\ &= \frac{\overline{p}_n^{\,2-2\delta}\pi^{2-2\delta}}{2^{3-4\delta}\,K^{2-2\delta}} \sum_{l=2}^{\infty}(l+1)R^l = \overline{p}_n^{\,2-2\delta}\,\frac{\pi^{2-2\delta}(3R^2 - 2R^3)}{2^{3-4\delta}\,K^{2-2\delta}(1-R)^2}, \end{aligned}$$

provided that K is small enough to make

$$R = R_{1,p} = \pi C_8(1,p)\left(\frac{1}{q} + \frac{p}{q}\log_r \frac{1}{K_{1,p}}\right)^2 K_{1,p} + \pi C_2(1,p) K_{1,p} < 1.$$

For simplicity here we used the inequality $\Gamma(l + 2 - 2\delta) < \Gamma(l + 2) = (l + 1)!$ for all $l = 2, 3, \dots$. Choosing now $\delta < 1/2$ and collecting all terms, we see that the order is indeed $O(\overline{p}_n)$ as claimed.

In the final case $\alpha > 1$, we have $|y_{k,n}^{\alpha,p}(t)| = |p_{k,n}x_{k,n}^{\alpha,p}(t)| \le (C_6+1)|t|p_{k,n}^{1/\alpha}$ and $v_{\alpha,p}(|t|) = C_7|t|^\alpha$. Substituting into $\Delta_{\boldsymbol{p}_n,1}^{\alpha,p}$, by (25) we obtain

$$\begin{aligned}\Delta_{\boldsymbol{p}_n,1}^{\alpha,p} \le{}& \overline{p}_n^{\frac{3}{\alpha}-1} 2\widetilde{C}_2 \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} + \overline{p}_n^{\frac{3}{\alpha}-1} 2C_9(C_6+1)^3 \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} + \overline{p}_n^{\frac{4}{\alpha}-1} C_2^2 \frac{\Gamma(4/\alpha)}{\alpha C_1^{4/\alpha}} \\ &+ \overline{p}_n^{\frac{2}{\alpha}} 2C_2C_7 \frac{\Gamma((2+\alpha)/\alpha)}{\alpha C_1^{(2+\alpha)/\alpha}} + \overline{p}_n^{\frac{3}{\alpha}-1} 2C_2 \big|s_1^{\alpha,p}\big| \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} \\ &+ 2\int_0^{T_n^{\alpha,p}} \frac{\mathrm{e}^{-C_1t^\alpha}}{t} \big|R_{n,8}^{\alpha,p}(t)\big| \,\mathrm{d}t.\end{aligned}$$

Using the inequality

$$|R_{n,8}^{\alpha,p}(t)| \le \sum_{l=2}^{\infty} \frac{1}{l!} \Big[C_8(C_6+1)^2|t|^2 \overline{p}_n^{(2-\alpha)/\alpha} + C_2|t|^2\overline{p}_n^{(2-\alpha)/\alpha} \Big]^l,$$

and referring again to [3], page 329, we get

$$\int_0^{T_n^{\alpha,p}} \frac{\mathrm{e}^{-C_1t^\alpha}}{t} |R_{n,8}^{\alpha,p}(t)| \,\mathrm{d}t \le \begin{cases} O\big(\overline{p}_n\big), & \text{if } 1 < \alpha < 4/3, \\ O\big(\overline{p}_n^{2(2-\alpha)/\alpha}\big), & \text{if } 4/3 \le \alpha < 2. \end{cases}$$

Collecting all the terms and taking into account that $1/\alpha < (4-2\alpha)/\alpha$ if and only if $\alpha < 3/2$, the statement in the final case also follows. ■

Proof of Corollary 1. For simplicity we show for $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ that

$$\sup_{x\in\mathbb{R}} \big|\boldsymbol{P}\big\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\big\} - G_{\alpha,p,\boldsymbol{p}_n}(x)\big| \le (1+\varepsilon)\frac{C_7^2}{2\pi\alpha C_1^2}\,\overline{p}_n,$$

$$\sup_{x\in\mathbb{R}} \big|\boldsymbol{P}\big\{S_{\boldsymbol{p}_n}^{1,p} \le x\big\} - G_{1,p,\boldsymbol{p}_n}(x)\big| \le (1+\varepsilon)\frac{p^2}{2q^2\pi C_1^2}\,\overline{p}_n \log_r^2 \frac{1}{\overline{p}_n},$$

and

$$\sup_{x\in\mathbb{R}} \big|\boldsymbol{P}\big\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\big\} - G_{\alpha,p,\boldsymbol{p}_n}(x)\big| \le (1+\varepsilon)\frac{\Gamma(2/\alpha)\left([s_1^{\alpha,p}]^2 + p/(q-q^{2/\alpha})\right)}{2\pi\alpha C_1^{2/\alpha}}\,\overline{p}_n^{(2-\alpha)/\alpha},$$

respectively, for all n large enough, where the strategy $\boldsymbol{p}_n$, with $\overline{p}_n \to 0$, corresponds to the given strategy $\boldsymbol{q}_n$ as described before Theorem 2. Then Corollary 1 follows by these statements exactly as Theorem 2 follows from Theorem 1.

First, if $\alpha < 1$, then $\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\sum_{k=1}^n p_{k,n}^2[y_{\gamma_{k,n}}^{\alpha,p}(t)]^2/2$ is the leading remainder term in $\boldsymbol{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$. We can estimate its inverse Fourier – Stieltjes transform $M_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(\cdot)$, which is not $G_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(\cdot)$, by the extended Gil-Pelaez – Rosén formula in Section 3:

$$M_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(x) = -\frac{1}{\pi}\int_0^\infty \frac{\Im\mathfrak{m}\left\{\mathrm{e}^{-\mathrm{i}tx}\boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\frac{1}{2}\sum_{k=1}^n p_{k,n}^2[y_{\gamma_{k,n}}^{\alpha,p}(t)]^2\right\}}{t}\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

Whence by (16) and Lemma 2,

$$\begin{aligned}\big|M^{(0,2)}_{\alpha,p,\boldsymbol{p}_n}(x)\big| &\le \frac{1}{2\pi}\int_0^\infty \frac{1}{t}\,\mathrm{e}^{\sum_{k=1}^n p_{k,n}\,\mathfrak{Re}\, y^{\alpha,p}_{\gamma_{k,n}}(t)}\sum_{k=1}^n p_{k,n}^2 |y^{\alpha,p}_{\gamma_{k,n}}(t)|^2 \mathrm{d}t \\ &\le \frac{C_7^2}{2\pi}\sum_{k=1}^n p_{k,n}^2 \int_0^\infty \mathrm{e}^{-C_1 t^\alpha}\, t^{2\alpha-1}\,\mathrm{d}t \le \frac{C_7^2}{2\pi\alpha C_1^2}\,\overline{p}_n\end{aligned}$$

for every $x\in\mathbb{R}$, finishing the first case.

Next, if $\alpha=1$, then $\boldsymbol{g}_{1,p,\boldsymbol{p}_n}(t)\,\frac{p^2t^2}{2q^2}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}$ is the leading remainder term in $\boldsymbol{g}^{1,p}_{\boldsymbol{p}_n}(t)$. For its inverse Fourier – Stieltjes transform $M^{(2,0)}_{1,p,\boldsymbol{p}_n}(\cdot)$, which differs from $G^{(2,0)}_{1,p,\boldsymbol{p}_n}(\cdot)$ only in a constant factor, we obtain

$$M^{(2,0)}_{1,p,\boldsymbol{p}_n}(x) = -\frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}tx}\boldsymbol{g}_{1,p,\boldsymbol{p}_n}(t)\frac{t^2p^2}{2q^2}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}\right\}}{t}\,\mathrm{d}t,\quad x\in\mathbb{R},$$

by the extended Gil-Pelaez – Rosén formula. Thus, again by (16) and Lemma 2,

$$\begin{aligned}\big|M^{(2,0)}_{1,p,\boldsymbol{p}_n}(x)\big| &\le \frac{p^2}{2q^2\pi}\int_0^\infty t\,\mathrm{e}^{\sum_{k=1}^n p_{k,n}\,\mathfrak{Re}\, y^{1,p}_{\gamma_{k,n}}(t)}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}\,\mathrm{d}t \\ &\le \frac{p^2}{2q^2\pi}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}\int_0^\infty \mathrm{e}^{-C_1 t}\,t\,\mathrm{d}t \le \frac{p^2}{2\pi q^2 C_1^2}\,\overline{p}_n\log_r^2\frac{1}{\overline{p}_n}\end{aligned}$$

for every $x\in\mathbb{R}$, where the last inequality comes from the fact that the function $x\mapsto x\log_r^2 x$ is monotone increasing near 0.

Finally, if $\alpha>1$, then the leading remainder term in $\boldsymbol{g}^{\alpha,p}_{\boldsymbol{p}_n}(t)$ is

$$\boldsymbol{m}^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(t) = \boldsymbol{g}_{\alpha,p,\boldsymbol{p}_n}(t)\frac{t^2}{2}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\sum_{k=1}^n p_{k,n}^{2/\alpha}.$$

For its inverse Fourier – Stieltjes transform $M^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(\cdot)$, differing again from $G^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(\cdot)$ in a constant factor, by a final application of the extended Gil-Pelaez – Rosén formula we have

$$M^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(x) = -\frac{1}{\pi}\int_0^\infty\frac{\mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}tx}\boldsymbol{m}^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(t)\right\}}{t}\,\mathrm{d}t,\quad x\in\mathbb{R}.$$

Therefore, using (16) and Lemma 2 for the last time, for all $x\in\mathbb{R}$ we obtain

$$\begin{aligned}\big|M^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(x)\big| &\le \frac{1}{2\pi}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\int_0^\infty t\,\mathrm{e}^{\sum_{k=1}^n p_{k,n}\,\mathfrak{Re}\, y^{\alpha,p}_{\gamma_{k,n}}(t)}\sum_{k=1}^n p_{k,n}^{2/\alpha}\,\mathrm{d}t \\ &\le \frac{1}{2\pi}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\sum_{k=1}^n p_{k,n}^{2/\alpha}\int_0^\infty \mathrm{e}^{-C_1t^\alpha}\,t\,\mathrm{d}t \\ &\le \frac{\Gamma(2/\alpha)}{2\pi C_1^{2/\alpha}}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\overline{p}_n^{(2-\alpha)/\alpha},\end{aligned}$$

completing the proof. ■

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