On the rate of convergence of the St. Petersburg game *

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Abstract

We investigate the repeated and sequential portfolio St. Petersburg games. For the repeated St. Petersburg game, we show an upper bound on the tail distribution, which implies a strong law for a truncation. Moreover, we consider the problem of limit distribution. For the sequential portfolio St. Petersburg game, we obtain tight asymptotic results for the growth rate of the game.

1 Introduction

Consider the simple St. Petersburg game, where the player invests 1$ and a fair coin is tossed until a tail first appears, ending the game. If the first tail

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appears in step k then the payoff X is 2^k and the probability of this event is 2^{-k}:

$$\mathbb{P}\{X = 2^k\} = 2^{-k}. \tag{1}$$

The distribution function of the gain is

$$F(x) = \mathbb{P}\{X \leq x\} = \begin{cases} 0, & \text{if } x < 2\,, \\ 1 - \frac{1}{2^{\lfloor \log_2 x \rfloor}} = 1 - \frac{2^{\{\log_2 x\}}}{x}, & \text{if } x \geq 2\,, \end{cases} \tag{2}$$

where $\lfloor x \rfloor$ is the usual integer part of x, $\{x\}$ stands for the fractional part and $\log_2$ denotes the logarithm with base 2

Since $\mathbb{E}\{X\} = \infty$, this game has delicate properties (cf. Bernoulli [2]). In the literature, usually the repeated St. Petersburg game (called iterated St. Petersburg game, too) means multi-period game such that it is a sequence of simple St. Petersburg games, where in each round the player invests 1$. Let X_n denote the payoff for the n-th simple game. Assume that the sequence $\{X_n\}_{n=1}^{\infty}$ is i.i.d. After n rounds the player's gain in the repeated game is $\sum_{i=1}^{n} X_i$, then Feller [12] proved that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n \log_2 n} = 1$$

in probability.

In Section 2 we revisit the a.s. properties of the repeated St. Petersburg game, in Section 3 we investigate the limit distributions of the truncated sums under different truncation levels. This analysis allow us to understand 'where the important things happen'. In Section 4 we show the consequences for sequential portfolio games with fair St. Petersburg components, while Section 5 contains some further refinements concerning the asymptotics.

2 Almost sure properties

Chow and Robbins [4] and Adler [1] proved that

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n \log_2 n} = 1 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n \log_2 n} = \infty \quad \text{a.s.}$$

For the sum with the largest payoff withheld, one has that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i - \max_{1\le i\le n} X_i}{n\log_2 n} = 1$$

a.s. (cf. Csörgő and Simons [11]).

Theorem 1 *For $c > 2$ let denote $X^{(c)}$ the St. Petersburg random variable cut at c, that is*

$$X^{(c)} = \begin{cases} X, & \text{if } X \le c, \\ c, & \text{if } X > c. \end{cases}$$

Introduce the notation $S_n^{(c)} = \sum_{k=1}^n X_k^{(c)}$ for the sums. For any $\varepsilon > 0$, we have that

$$\mathbb{P}\left\{ \left| \frac{S_n^{(n)} - \mathbb{E}\{S_n^{(n)}\}}{n\log_2 n} \right| > \varepsilon \right\} < 2n^{4-\varepsilon \ln\ln n \log_2 e}.$$

Proof. For any $\lambda > 0$, we apply the Chernoff bounding technique:

$$\begin{aligned} \mathbb{P}\left\{ \frac{\sum_{i=1}^n (X_i^{(n)} - \mathbb{E}\{X_i^{(n)}\})}{n\log_2 n} > \varepsilon \right\} &= \mathbb{P}\left\{ \sum_{i=1}^n (X_i^{(n)} - \mathbb{E}\{X_i^{(n)}\}) > \varepsilon n \log_2 n \right\} \\ &\le \frac{\mathbb{E}\left\{ e^{\lambda \sum_{i=1}^n (X_i^{(n)} - \mathbb{E}\{X_i^{(n)}\})} \right\}}{e^{\lambda\varepsilon n\log_2 n}} \\ &= \frac{\mathbb{E}\left\{ e^{\lambda (X_1^{(n)} - \mathbb{E}\{X_1^{(n)}\})} \right\}^n}{e^{\lambda\varepsilon n\log_2 n}}. \end{aligned}$$

For any $k \ge 2$, we have that

$$\begin{aligned} \mathbb{E}\left\{ (X_1^{(n)})^k \right\} &= \sum_{i=1}^{\lfloor \log_2 n\rfloor} 2^{ki}2^{-i} + 2^{k\log_2 n} \sum_{i=\lfloor \log_2 n\rfloor+1}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{\lfloor \log_2 n\rfloor} 2^{(k-1)i} + 2^{k\log_2 n}2^{-\lfloor \log_2 n\rfloor} \\ &\le 3n^{k-1}, \end{aligned}$$

while for $k = 1$ we have

$$\mathbb{E}\left\{ X_1^{(n)} \right\} \le \log_2 n + 1\,.$$

If I denotes the indicator function then using the inequality

$$\begin{aligned}
\mathbb{E}\left[\left(X_1^{(n)} - \mathbb{E}X_1^{(n)}\right)^k\right] &\leq \mathbb{E}\left[I_{\{X_1^{(n)} > \mathbb{E}X_1^{(n)}\}}\left(X_1^{(n)}\right)^k\right] \\
&\quad + \mathbb{E}\left[I_{\{X_1^{(n)} \leq \mathbb{E}X_1^{(n)}\}}(\mathbb{E}X_1^{(n)})^k\right] \\
&\leq \mathbb{E}\left[\left(X_1^{(n)}\right)^k\right] + (\log_2 n + 1)^k \\
&\leq 4n^{k-1},
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathbb{E}\left\{e^{\lambda(X_1^{(n)} - \mathbb{E}\{X_1^{(n)}\})}\right\} &= \sum_{k=0}^{\infty} \mathbb{E}\left\{\frac{(\lambda(X_1^{(n)} - \mathbb{E}\{X_1^{(n)}\}))^k}{k!}\right\} \\
&= 1 + \sum_{k=2}^{\infty} \mathbb{E}\left\{\frac{(\lambda(X_1^{(n)} - \mathbb{E}\{X_1^{(n)}\}))^k}{k!}\right\} \\
&\leq 1 + \sum_{k=2}^{\infty} \mathbb{E}\left\{\frac{\lambda^k 4n^{k-1}}{k!}\right\},
\end{aligned}$$

and so

$$\mathbb{E}\left\{e^{\lambda(X_1^{(n)} - \mathbb{E}\{X_1^{(n)}\})}\right\} \leq 1 + \frac{4}{n}\sum_{k=0}^{\infty} \mathbb{E}\left\{\frac{\lambda^k n^k}{k!}\right\} = 1 + \frac{4}{n}e^{\lambda n}.$$

Thus, the choice $\lambda = \frac{\ln \ln n}{n}$ implies that

$$\begin{aligned}
\mathbb{P}\left\{\frac{\sum_{i=1}^n (X_i^{(n)} - \mathbb{E}\{X_i^{(n)}\})}{n \log_2 n} > \varepsilon\right\} &\leq \frac{(1 + \frac{4}{n}e^{\lambda n})^n}{e^{\lambda \varepsilon n \log_2 n}} \\
&\leq \frac{e^{4e^{\lambda n}}}{e^{\lambda \varepsilon n \log_2 n}} \\
&\leq \frac{e^{4 \ln n}}{e^{\ln \ln n \varepsilon \log_2 n}} \\
&= n^{4 - \varepsilon \ln \ln n \log_2 e}.
\end{aligned}$$

In the same way we get that

$$\mathbb{P}\left\{\frac{\sum_{i=1}^n (\mathbb{E}\{X_i^{(n)}\} - X_i^{(n)})}{n \log_2 n} > \varepsilon\right\} \leq n^{4 - \varepsilon \ln \ln n \log_2 e},$$

therefore

$$\mathbb{P}\left\{\left|\frac{\sum_{i=1}^{n}(X_i^{(n)}-\mathbb{E}\{X_i^{(n)}\})}{n\log_2 n}\right|>\varepsilon\right\}<2n^{4-\varepsilon\ln\ln n\log_2 e}.$$

■

This theorem and the Borel – Cantelli lemma imply that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}X_i^{(n)}}{n\log_2 n}=1\quad\text{a.s.}$$

We can achieve some asymptotic results for $S_n-S_n^{(n)}$, where $S_n=X_1+\cdots+X_n$ stands for the whole sum. Writing

$$S_n-S_n^{(n)}=\sum_{i=1}^{n}(X_i-\min\{X_i,n\})=\sum_{i=1}^{n}(X_i-n)^+\,,$$

where $x^+=\max\{x,0\}$ stands for the positive part of x, Feller's weak law and Theorem 1 imply that

$$\frac{\sum_{i=1}^{n}(X_i-n)^+}{n\log_2 n}\to 0\quad\text{in probability,}$$

while the almost sure lim inf and lim sup results have the consequences

$$\liminf_{n\to\infty}\frac{\sum_{i=1}^{n}(X_i-n)^+}{n\log_2 n}=0\quad\text{a.s., and}$$
$$\limsup_{n\to\infty}\frac{\sum_{i=1}^{n}(X_i-n)^+}{n\log_2 n}=\infty\quad\text{a.s.}$$

3 Limit distribution properties

As in the previous section, let $X_1,X_2,\ldots$ be independent St. Petersburg random variables and let denote $S_n=X_1+\cdots+X_n$ its partial sums. Since the bounded oscillating function $2^{\{\log_2 x\}}$ in the numerator of (2) is not slowly varying at infinity, by the classical Doeblin – Gnedenko criterion (cf. [13]) the underlying St. Petersburg distribution is *not* in the domain of attraction of any stable law. That is there is no asymptotic distribution for $(S_n-c_n)/a_n$, in the usual sense, whatever the centering and norming constants are. This is where the main difficulty lies for the St. Petersburg games.

However, asymptotic distributions do exist along subsequences of the natural numbers. Martin-Löf [16] 'clarified the St. Petersburg paradox', showing that $S_{2^k}/2^k - k$ converge in distribution, as $k \to \infty$. It turned out in [8] that there are continuum different types of asymptotic distributions of $S_n/n - \log_2 n$ along different subsequences of $\mathbb{N}$. As Csörgő wrote [6] there are continuum many different clarification of the St. Petersburg paradox. In order to state the necessary and sufficient condition for the existence of the limit, we introduce the positional parameter $\gamma_n = n/2^{\lceil \log_2 n \rceil} \in (1/2, 1]$, which shows the position of n between two consecutive powers of 2. We say that a sequence $\{\lambda_n\}_{n=1}^{\infty}$ *circularly converge* to $\lambda \in (1/2, 1]$, $\lambda_n \xrightarrow{\text{cir}} \lambda$, if $\lambda_n \to \lambda \in (1/2, 1]$ in the usual sense, or $\lambda_n \to 1/2$, or $\{\lambda_n\}_{n=1}^{\infty}$ has exactly two limit points: 1 and 1/2. In the latter two cases the circular limit is defined to be 1. In this terminology, the theorem of Csörgő and Dodunekova [8] states that $S_{n_k}/n_k - \log_2 n_k$ converges in distribution to a nondegenerate limit as $k \to \infty$, if and only if $\gamma_{n_k} = n_k/2^{\lceil \log_2 n_k \rceil} \xrightarrow{\text{cir}} \gamma$. In this case the limit random variable W_γ, $\gamma \in (1/2, 1]$ has characteristic function

$$\mathbf{g}_\gamma(t) = \mathbb{E}\left(e^{itW_\gamma}\right) = \int_{-\infty}^{\infty} e^{itx}\, dG_\gamma(x) = e^{y_\gamma(t)}, \quad t \in \mathbb{R},$$

and distribution function $G_\gamma(x) = \mathbb{P}\{W_\gamma \le x\}$, $x \in \mathbb{R}$, where

$$\begin{aligned} y_\gamma(t) &= its_\gamma + \sum_{l=0}^{-\infty} \left(\exp\left\{ \frac{it2^l}{\gamma} \right\} - 1 - \frac{it2^l}{\gamma} \right) \frac{\gamma}{2^l} + \sum_{l=1}^{\infty} \left(\exp\left\{ \frac{it2^l}{\gamma} \right\} - 1 \right) \frac{\gamma}{2^l} \\ &= \exp\left\{ it\left[s_\gamma + u_\gamma\right] + \int_0^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR_\gamma(x) \right\}, \end{aligned}$$

with finite constants $s_\gamma = -\log_2 \gamma$ and

$$u_\gamma = \sum_{l=1}^{\infty} \frac{\gamma^2}{\gamma^2 + 4^l} - \sum_{l=0}^{\infty} \frac{1}{1 + \gamma^2 4^l}$$

and right-hand-side Lévy function

$$R_\gamma(x) = -\frac{\gamma}{2^{\lfloor \log_2(\gamma x) \rfloor}} = -\frac{\gamma}{2^{\lfloor \log_2(\gamma x) \rfloor}} = -\frac{2^{\{\log_2(\gamma x)\}}}{x}, \quad x > 0.$$

From this form, it is clear that W_γ is a semistable random variable with characteristic exponent 1.

Moreover, it turned out that a so-called merging theorem holds: Csörgő [5] showed that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n}{n} - \log_2 n \le x \right\} - G_{\gamma_n}(x) \right| \to 0. \tag{3}$$

We note that this behavior holds in a more general setup when the underlying random variables are from the domain of geometric partial attraction of a semistable law, see Csörgő, Megyesi [10]. For more precise asymptotics for the St. Petersburg sums we refer to Csörgő [5], [7] and Csörgő, Kevei [9].

Now we turn to the asymptotic behavior of the sums of the truncated variables, under different truncation levels. The following two theorems say that in some sense the exact truncation is at level n, since in this case both the truncated variables, both the truncations shows the same limit behavior as the whole sums.

First we consider the truncations. Also notice the analogue of this theorem and the lim inf and lim sup results at the end of the previous section.

Theorem 2 *The normalized sum*

$$\frac{\sum_{i=1}^{n}(X_i - n)^+}{n}$$

converges in distribution along a subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ *to a nondegenerate limit, if and only if* $\gamma_{n_k} \xrightarrow{\text{cir}} \gamma$ *as* $k \to \infty$, *for some* $\gamma \in (1/2, 1]$ *and in this case the limit variable* Y_γ *has characteristic function*

$$\mathbb{E}(e^{itY_\gamma}) = \exp\left\{ \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dQ_\gamma^1(x) \right\} ,$$

where

$$Q_\gamma^1(x) = -\frac{2^{\{\log_2[\gamma(x+1)]\}}}{x+1} .$$

Proof. According to (2) the distribution function of one summand is

$$F_n(x) = \mathbb{P}\left\{ \frac{(X_i - n)^+}{n} \le x \right\} = \mathbb{P}\{X_i \le n(x+1)\} = 1 - \frac{1}{2^{\lfloor \log_2(n(x+1)) \rfloor}} ,$$

for $x \ge 0$, and 0 elsewhere. By Theorem 25.1 in [13], $\sum_{i=1}^{n}(X_i - n)^+/n - A_n$ converges in distribution along the subsequence $\{n_k\}$, for some appropriate centering sequence A_n, if and only if

$$\begin{aligned} n_k\left[1 - F_{n_k}(x)\right] &= n_k \frac{2^{\{\log_2(n_k(x+1))\}}}{2^{\log_2(n_k(x+1))}} \\ &= \frac{1}{x+1} 2^{\{\log_2(n_k(x+1))\}} \to -R(x) \end{aligned} \tag{4}$$

in every continuity point of R, and

$$\lim_{\varepsilon\to 0}\limsup_{k\to\infty} n_k \int_{|x|<\varepsilon} x^2\, dF_{n_k}(x) = 0\,.$$

Here the right-sided Lévy function R is a non-decreasing right-continuous function, for which $\lim_{x\to\infty} R(x) = 0$ and $\int_0^\varepsilon x^2 dR(x) < \infty$ for every $\varepsilon > 0$. In this case the centering constant can be chosen as $A_n = n\int_{|x|<\tau} x dF_n(x)$, $\tau > 0$.

The latter condition is trivially hold along the whole sequence of the natural numbers, since

$$n\int_{|x|<\varepsilon} x^2\, dF_n(x) \le \varepsilon^2 n \int_{0<x<\varepsilon} dF_n(x) \le \varepsilon^2 n[1 - F_n(0)] \le 2\,\varepsilon^2.$$

Similar calculation shows that we may choose $A_n \equiv 0$.

It is easy to check that (4) holds if and only if $\gamma_{n_k} \xrightarrow{\text{cir}} \gamma$ for some $\gamma \in (1/2, 1]$, and in this case Q_γ^1 has the desired form. ∎

Exactly the same way we can prove a limit theorem for the sum $S_n^{(n)}$ of the truncated variables. We omit the proof.

Theorem 3 *The centered and weighted sum*

$$\frac{S_{n_k}^{(n_k)}}{n_k} - \log_2 n_k$$

has a nondegenerate limit, if and only if $\gamma_{n_k} \xrightarrow{\text{cir}} \gamma$ *as* $k \to \infty$, *for some* $\gamma \in (1/2, 1]$, *and in this case the characteristic function of the limit variable* Z_γ *is*

$$\mathbb{E}(e^{itZ_\gamma}) = \exp\left\{\int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dQ_\gamma^2(x)\right\}\,,$$

where

$$Q_\gamma^2(x) = \begin{cases} -\frac{2^{\{\log_2(\gamma x)\}}}{x}, & \text{if } x < 1\,, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that if we define $Q_\gamma^1(x) = 0$ on $(-1, 0]$, then $Q_\gamma^1(x-1) + Q_\gamma^2(x) = R_\gamma(x)$, $x > 0$.

Next we investigate the asymptotic normality of the truncated variables under general truncation. Theorem 3 says that truncation at n is not enough for the existence of a usual limit. According to the following theorem a slightly stronger truncation implies asymptotic normality. In a very informal way these three theorems together show that the interesting things happen around n.

Theorem 4 *The asymptotic normality*

$$\frac{S_n^{(c_n)} - \mathbb{E}S_n^{(c_n)}}{\sqrt{\operatorname{Var} S_n^{(c_n)}}} \xrightarrow{\mathcal{D}} N(0,1)$$

holds, if and only if $c_n/n \to 0$.

Proof. We use the Lindeberg – Lévy central limit theorem, which says that the distributional convergence

$$\frac{S_n^{(c_n)} - \mathbb{E}S_n^{(c_n)}}{\sqrt{\operatorname{Var} S_n^{(c_n)}}} \xrightarrow{\mathcal{D}} N(0,1)$$

holds, if and only if $L_n(\varepsilon) \to 0$ for each $\varepsilon > 0$, where

$$L_n(\varepsilon) = \frac{n}{s_n^2} \int_{\left\{|X^{(c_n)} - \mathbb{E}X^{(c_n)}| > \varepsilon s_n\right\}} (X^{(c_n)} - \mathbb{E}X^{(c_n)})^2 \, d\mathbb{P} \,,$$

and $s_n^2 = \operatorname{Var} S_n^{(c_n)}$.

To exclude trivial cases, we assume that $c_n \to \infty$.

First we show the sufficiency part, that is we assume that $c_n/n \to 0$. For the second moment of a truncated variable we have

$$\begin{aligned} \mathbb{E}\left\{\left[X^{(c)}\right]^2\right\} &= \sum_{k=1}^{\lfloor \log_2 c \rfloor} \frac{1}{2^k} 2^{2k} + \frac{1}{2^{\lfloor \log_2 c \rfloor}} c^2 = 2\left(2^{\lfloor \log_2 c \rfloor} - 1\right) + c 2^{\{\log_2 c\}} \\ &= \left(2^{1-\{\log_2 c\}} + 2^{\{\log_2 c\}}\right) c - 2. \qquad (5) \end{aligned}$$

So for the variance we obtain

$$s_n^2 = \operatorname{Var} S_n^{(c_n)} = n \operatorname{Var} X^{(c_n)} = n c_n \left(2^{1-\{\log_2 c_n\}} + 2^{\{\log_2 c_n\}} + o(1)\right) ,$$

where $o(1) \to 0$ as $n \to \infty$. Therefore, using that $2^x + 2^{1-x} \in [2\sqrt{2}, 3]$ for $x \in [0,1]$, and the assumption $c_n/n \to 0$, we obtain that $c_n/s_n \to 0$. Since we cut at c_n, in this case the domain of integration in the definition of $L_n(\varepsilon)$ will be empty for n large enough, so clearly $L_n(\varepsilon) \to 0$, for every $\varepsilon > 0$.

For the converse we indirectly assume that $c_n/n \not\to 0$. This means that we can choose an $\varepsilon_0 > 0$ and a subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, for which $c_{n_k}/n_k > \varepsilon_0$. The same asymptotic as in the previous part of the proof shows that $c_{n_k}/s_{n_k} > \varepsilon_0/2 = \varepsilon$, for k large enough.

The asymptotic order of the expectation $\log c_n$ is negligible to the variance c_n, therefore easy computation shows that instead of the integral

$$\int_{\left\{|X^{(c_n)}-\mathbb{E}X^{(c_n)}|>\varepsilon s_n\right\}} (X^{(c_n)} - \mathbb{E}X^{(c_n)})^2 \, d\mathbb{P}$$

we may investigate

$$\int_{\left\{X^{(c_n)}>\varepsilon s_n\right\}} (X^{(c_n)})^2 \, d\mathbb{P} \, .$$

Writing n_k instead of n, the latter can be computed as

$$\begin{aligned}
&= \sum_{j=\lfloor \log_2(\varepsilon s_{n_k})\rfloor+1}^{\lfloor \log_2 c_{n_k}\rfloor} 2^j + c_{n_k}^2 \, \mathbb{P}\{X > c_{n_k}\} \\
&= c_{n_k}\left(2^{1-\{\log_2 c_{n_k}\}} + 2^{\{\log_2 c_{n_k}\}}\right) - 2 - (\varepsilon s_{n_k} 2^{1-\{\log_2(\varepsilon s_{n_k})\}} - 2) \\
&= c_{n_k}\left(2^{1-\{\log_2 c_{n_k}\}} + 2^{\{\log_2 c_{n_k}\}}\right) - \varepsilon s_{n_k} 2^{1-\{\log_2(\varepsilon s_{n_k})\}} \, .
\end{aligned}$$

Multiplying by $n_k/s_{n_k}^2$, it is clear that the first term converges to 1, while the second cannot converge to 1 for all $\varepsilon > 0$. So $L_n(\varepsilon)$ does not converge to 0, that is Lindeberg's condition fails, and the theorem is completely proved now. ∎

Finally, as a counterpart of the previous theorem we note that if the truncation level c_n is asymptotically greater than n, that is $c_n/n \to \infty$, then for the sum of the truncations we have

$$\frac{\sum_{i=1}^n (X_i - c_n)^+}{n} \to 0$$

in probability.

4 Growth rate of sequential St. Petersburg portfolio games

According to the previous results $\sum_{i=1}^n X_i \approx n \log_2 n$. Györfi and Kevei [14] introduced sequential St. Petersburg game and sequential St. Petersburg portfolio game, having exponential growth. The sequential St. Petersburg game means that the player starts with initial capital $C_0 = 1\$$, and there is a sequence of simple St. Petersburg games, and for each simple game the player reinvests his capital. If C_{n-1} is the capital after the $(n-1)$-th simple game then the invested capital is $C_{n-1}/4$, while $3C_{n-1}/4$ is the proportional cost of the simple game with commission factor $c = 3/4$. It means that after the n-th round the capital is

$$C_n = C_{n-1}X_n/4 = \prod_{i=1}^{n}(X_i/4).$$

Because of its multiplicative definition, S_n has exponential trend:

$$C_n = 2^{nW_n} \approx 2^{nW},$$

with average growth rate $W_n := \frac{1}{n}\log_2 C_n$ and with asymptotic average growth rate $W := \lim_{n\to\infty}\frac{1}{n}\log_2 C_n$. The strong law of large numbers implies that

$$W = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log_2 X_i/4 = \mathbb{E}\{\log_2 X_1/4\} = 0,$$

i.e., the growth rate of this sequential game is 0. In the sequel, the sequential game with payoff $X_1/4$ is called fair.

Using the model of constantly rebalanced portfolio (CRP) one can achieve positive growth rate out of financial instruments of zero growth rate such that the sequential portfolio game is operating on several simple games and cash, and in each round of the games the player rebalance his wealth according to a portfolio. The aim is to achieve the best possible growth rate of the wealth. In the model of log-optimal portfolio theory, one can access d fair St. Petersburg component and cash and a portfolio vector is denoted by $\mathbf{b} = (b^{(1)}, \dots, b^{(d+1)})$. The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in financial instrument j ($j \le d$), while $b^{(d+1)}$ denotes the weight of the cash. We assume that the portfolio vector $\mathbf{b}$ has nonnegative components sum up to 1. The set of portfolio vectors is denoted by

$$\Delta_{d+1} = \left\{\mathbf{b} = (b^{(1)}, \dots, b^{(d+1)});\, b^{(j)} \ge 0,\, \sum_{j=1}^{d+1} b^{(j)} = 1\right\}.$$

The behavior of the market is given by the sequence of return vectors $\{\mathbf{x}_n\}$ as follows:

$$\mathbf{x}_n = (x_n^{(1)}, \dots, x_n^{(d)}, 1)$$

such that the j-th component $x_n^{(j)}$ of the return vector $\mathbf{x}_n$ denotes the amount obtained after investing a unit capital in the j-th financial instrument on the n-th round ($1 \leq j \leq d$).

In case of CRP we fix a portfolio vector $\mathbf{b} \in \Delta_{d+1}$. At the beginning of the first round the initial capital $C_0 = 1$ such that $b^{(j)}$ is invested into financial instrument j, and it results in return $b^{(j)} x_1^{(j)}$, therefore at the end of the first round the investor's wealth becomes

$$C_1 = \sum_{j=1}^{d+1} b^{(j)} x_1^{(j)} = \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle \, ,$$

where $\langle \cdot \, , \, \cdot \rangle$ denotes inner product. For the second round, C_1 is the new initial capital

$$C_2 = C_1 \cdot \langle \mathbf{b} \, , \, \mathbf{x}_2 \rangle = \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b} \, , \, \mathbf{x}_2 \rangle \, .$$

By induction, for the round n the initial capital is C_{n-1}, therefore

$$C_n = C_{n-1} \langle \mathbf{b} \, , \, \mathbf{x}_n \rangle = \prod_{i=1}^{n} \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle \, .$$

The average growth rate of this portfolio selection is

$$\frac{1}{n} \log_2 C_n = \frac{1}{n} \sum_{i=1}^{n} \log_2 \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle \, .$$

If the market process $\{\mathbf{X}_i\}$ is memoryless, i.e., it is a sequence of independent and identically distributed (i.i.d.) random return vectors then the best constantly rebalanced portfolio (BCRP) is the log-optimal portfolio:

$$\mathbf{b}^* := \arg\max_{\mathbf{b} \in \Delta_{d+1}} \mathbb{E}\{\log_2 \langle \mathbf{b} \, , \, \mathbf{X}_1 \rangle\}.$$

This optimality means that if $C_n^* = C_n(\mathbf{b}^*)$ denotes the capital after round n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$, then for any portfolio strategy $\mathbf{b}$ with finite $\mathbb{E}\{\log_2 \langle \mathbf{b} \, , \, \mathbf{X}_1 \rangle\}$ and with capital $C_n = C_n(\mathbf{b})$ and for any memoryless market process $\{\mathbf{X}_n\}_{n=1}^{\infty}$,

$$\lim_{n \to \infty} \frac{1}{n} \log_2 C_n \leq \lim_{n \to \infty} \frac{1}{n} \log_2 C_n^* \quad \text{almost surely}$$

and maximal asymptotic average growth rate is

$$\lim_{n \to \infty} \frac{1}{n} \log_2 C_n^* = W^* := \mathbb{E}\{\log_2 \langle \mathbf{b}^* \, , \, \mathbf{X}_1 \rangle\} \quad \text{a.s.}$$

(cf. Breiman [3] and Kelly [15]).

Consider the portfolio game, where a fraction of the capital is invested in simple fair St. Petersburg games and the rest is kept in cash, i.e., it is a CRP problem with the return vector

$$\mathbf{X} = (X^{(1)}, \dots, X^{(d)}, X^{(d+1)}) = (X_1/4, \dots, X_d/4, 1)$$

($d \geq 1$) such that the first d i.i.d. components of the return vector $\mathbf{X}$ are fair St. Petersburg payoffs, while the last component is the cash. The main aim is to calculate the largest growth rate W_d^*.

Györfi and Kevei [14] proved that, for $d = 1$, $\mathbf{b}^* = (0.385, 0.615)$ and, for $d = 2$, $\mathbf{b}^* = (0.364, 0.364, 0.272)$. For $d \geq 3$, the best portfolio is the uniform portfolio such that the cash has zero weight:

$$\mathbf{b}^* = (1/d, \dots, 1/d, 0)$$

and the asymptotic average growth rate is

$$W_d^* = \mathbb{E}\left\{\log_2\left(\frac{1}{4d}\sum_{i=1}^{d} X_i\right)\right\}.$$

Here are the first few values:

d	1	2	3	4	5	6	7	8
W_d^*	0.149	0.289	0.421	0.526	0.606	0.669	0.721	0.765

Table 1: Numerical results

Györfi and Kevei [14] proved that

$$\log_2 \frac{S_n}{n \log_2 n} \to 0$$

in L_1, which implies that

$$\begin{aligned} W_d^* &= \mathbb{E}\left\{\log_2\left(\frac{1}{4d}\sum_{i=1}^{d} X_i\right)\right\} \\ &= \mathbb{E}\left\{\log_2 \frac{S_d}{d \log_2 d}\right\} + \log_2 \log_2 d - 2 \approx \log_2 \log_2 d - 2 \end{aligned}$$

such that

$$-\frac{0.8}{\ln 2}\frac{1}{\log_2 d} \leq W_d^* - \log_2 \log_2 d + 2 \leq \frac{\log_2 \log_2 d + 4}{\ln 2 \log_2 d}.$$

Next we slightly extend this asymptotics which implies that

$$W_d^* \approx \log_2 \log_2 d - 2 + \frac{\log_2 \log_2 d}{\ln 2 \log_2 d}.$$

Theorem 5 *We have*

$$\frac{0.16 + o(1)}{\log_2 n} \leq \mathbb{E}\left\{\log_2 \frac{S_n}{n \log_2 n}\right\} - \frac{\log_2 \log_2 n}{\ln 2 \log_2 n} \leq \frac{2.52 + o(1)}{\log_2 n}.$$

From this theorem we get that

$$0.16 \leq \log_2 n \left(\mathbb{E}\left\{\log_2 \frac{S_n}{n \log_2 n}\right\} - \frac{\log_2 \log_2 n}{\ln 2 \log_2 n}\right) \leq 2.52.$$

Introduce the notation

$$\begin{aligned} \Delta_n &:= \log_2 n \left(\mathbb{E}\left\{\log_2 \frac{S_n}{n \log_2 n}\right\} - \frac{\log_2 \log_2 n}{\ln 2 \log_2 n}\right) \\ &= \log_2 n \left(\mathbb{E}\{\log_2 S_n\} - \log_2 n - \log_2 \log_2 n - \frac{\log_2 \log_2 n}{\ln 2 \log_2 n}\right). \end{aligned}$$

We conjecture that the limit

$$c := \lim_{n \to \infty} \Delta_n$$

exists. In order to prove or disprove it, we performed some simulations. Table 2 contains some values of $\mathbb{E}\{\log_2 S_n\}$, $\mathrm{Var}\,(\log_2 S_n)$ (which goes to 0, according to Remark 2), Δ_n and W_n^* such that, for each n, there were 20000

n	8	16	32	64
$\log_2 2n$	4	5	6	7
$\mathbb{E}\{\log_2 S_n\}$	5.76	6.97	8.17	9.35
$\mathrm{Var}\,(\log_2 S_n)$	1.56	1.40	1.20	1.10
Δ_n	1.24	1.02	0.92	0.85
$W_n^* = \mathbb{E}\{\log_2 S_n\} - \log_2 n - 2$	0.76	0.97	1.17	1.35
$\mathbb{E}\{\log_2 S_n^{(c_n)}\}$	5.35	6.69	7.92	9.13
$\mathrm{Var}\,(\log_2 S_n^{(c_n)})$	0.31	0.36	0.35	0.32

Table 2: Simulation results

runs. Since $S_n \geq 2n$, we include $\log_2 2n$, too. Notice that the exact value of W_8^* in Table 1 and the simulated value in Table 2 are close to each other. In addition, we included in Table 2 values of $\mathbb{E}\{\log_2 S_n^{(c_n)}\}$, $\mathrm{Var}\,(\log_2 S_n^{(c_n)})$ with $c_n = n\log_2 n$, too.

In order to prove Theorem 5, apply the decomposition

$$\log_2 \frac{S_n}{n\log_2 n} = \log_2 \frac{S_n}{S_n^{(c_n)}} + \log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\,,$$

where $S_n^{(c_n)}$ is defined in Theorem 1 with $c_n = n\log_2 n$. Next we formulate bounds for the two terms on the right hand side.

Concerning the first term one has that

$$0 \leq \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}}\right\} \leq \frac{2n}{c_n \ln 2}$$

(cf. Györfi and Kevei [14]).

Proposition 1 *For $c_n = n\log_2 n$ we have*

$$\frac{1+o(1)}{\log_2 n} \leq \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}}\right\} \leq \frac{2+o(1)}{\log_2 n}.$$

Proof. If $X_1 \leq c_n, \ldots, X_n \leq c_n$ then $S_n = S_n^{(c_n)}$, therefore

$$\mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}}\right\} = \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}} I_{\{X_1>c_n,\ldots,X_k>c_n\}} I_{\{X_{k+1}\leq c_n,\ldots,X_n\leq c_n\}}\right\}.$$

For $k=1$

$$\begin{aligned}
& n\mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}} I_{\{X_1>c_n\}} I_{\{X_2\leq c_n,\ldots,X_n\leq c_n\}}\right\} \\
&= n\mathbb{E}\left\{\log_2 \frac{X_1+\sum_{i=2}^n X_i}{c_n+\sum_{i=2}^n X_i} I_{\{X_1>c_n\}} I_{\{X_2\leq c_n,\ldots,X_n\leq c_n\}}\right\} \\
&= n\int_0^\infty \mathbb{P}\left\{\log_2 \frac{X_1+\sum_{i=2}^n X_i}{c_n+\sum_{i=2}^n X_i} I_{\{X_1>c_n\}} I_{\{X_2\leq c_n,\ldots,X_n\leq c_n\}} \geq x\right\} dx \\
&= n\int_0^\infty \mathbb{P}\left\{\frac{X_1+\sum_{i=2}^n X_i}{c_n+\sum_{i=2}^n X_i} \geq 2^x, X_1 > c_n, X_2 \leq c_n, \ldots, X_n \leq c_n\right\} dx \\
&= n\int_0^\infty \mathbb{P}\left\{X_1 \geq 2^x c_n + (2^x-1)\sum_{i=2}^n X_i, X_2 \leq c_n, \ldots, X_n \leq c_n\right\} dx. \quad (6)
\end{aligned}$$

Concerning the lower bound of the proposition, (2) implies that

$$\begin{aligned}
& \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}}\right\} \\
\geq\ & n\mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}} I_{\{X_1>c_n\}} I_{\{X_2\leq c_n,\ldots,X_n\leq c_n\}}\right\} \\
=\ & n\int_0^\infty \mathbb{E}\left\{\mathbb{P}\left\{X_1 \geq 2^x c_n + (2^x-1)\sum_{i=2}^n X_i \Big| X_2 \leq c_n,\ldots,X_n\leq c_n\right\}\right\} dx \\
& \times \mathbb{P}\left\{X_2 \leq c_n,\ldots,X_n \leq c_n\right\} \\
\geq\ & n\int_0^\infty \mathbb{E}\left\{\frac{1}{2^x c_n + (2^x-1)\sum_{i=2}^n X_i^{(c_n)}}\Big| X_2 \leq c_n,\ldots,X_n\leq c_n\right\} dx \\
& \times (1-2/c_n)^{n-1},
\end{aligned}$$

therefore Jensen's inequality implies that

$$\begin{aligned}
& \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}}\right\} \\
\geq\ & n\int_0^\infty \frac{1}{2^x c_n + (2^x-1)\sum_{i=2}^n \mathbb{E}\left\{X_i^{(c_n)} \mid X_2 \leq c_n,\ldots,X_n\leq c_n\right\}} dx \\
& \times (1+o(1)) \\
\geq\ & n\int_0^\infty \frac{1}{2^x c_n + (2^x-1)n\log_2 c_n} dx(1+o(1)) \\
\geq\ & \frac{n}{c_n}\int_0^\infty \frac{1}{2\cdot 2^x - 1} dx(1+o(1)) \\
=\ & \frac{1+o(1)}{\log_2 n},
\end{aligned}$$

where the last equality follows from

$$\int_0^\infty \frac{1}{2\cdot 2^x-1} dx = 1.$$

Concerning the upper bound, (2) and (6) imply that

$$\begin{aligned}
& n\mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}} I_{\{X_1>c_n\}} I_{\{X_2\leq c_n,\ldots,X_n\leq c_n\}}\right\} \\
\leq\ & n\int_0^\infty \mathbb{E}\left\{\frac{2}{2^x c_n + (2^x-1)\sum_{i=2}^n X_i^{(c_n)}}\right\} dx.
\end{aligned}$$

Using Fubini's theorem we have

$$\begin{aligned}&\int_0^\infty \mathbb{E}\left\{\frac{2}{2^x c_n + (2^x-1)\sum_{i=2}^n X_i^{(c_n)}}\right\} dx \\ = \ & \mathbb{E}\left\{\int_0^\infty \frac{2}{2^x c_n + (2^x-1)\sum_{i=2}^n X_i^{(c_n)}} dx\right\}.\end{aligned}$$

For computing the inner integral notice that $\int (e^y-1)^{-1}dy = \ln(e^y-1)-y$, and so straightforward calculation implies that

$$\int_0^\infty \frac{dx}{c2^x - d} = \frac{1}{d}\log_2 \frac{c}{c-d}.$$

Therefore we obtain

$$\begin{aligned}\int_0^\infty \mathbb{E}\left\{\frac{2}{2^x c_n + (2^x-1)\sum_{i=2}^n X_i^{(c_n)}}\right\} dx &= \mathbb{E}\left\{\frac{2}{S_{n-1}^{(c_n)}}\log_2\left(1+\frac{S_{n-1}^{(c_n)}}{c_n}\right)\right\} \\ &= \frac{2}{\ln 2}\mathbb{E}\left\{\frac{1}{S_{n-1}^{(c_n)}}\ln\left(1+\frac{S_{n-1}^{(c_n)}}{c_n}\right)\right\}\end{aligned}$$

By Feller's weak law

$$\mathbb{P}\left\{\left|\frac{S_{n-1}^{(c_n)}}{c_n}-1\right| > \varepsilon\right\} \to 0.$$

Let $A_n(\varepsilon) = \{|S_{n-1}^{(c_n)}/c_n - 1| \le \varepsilon\}$. Using that $[\ln(1+x)]/x \le 1$, $x>0$, and integrating on $A_n(\varepsilon)$ and on $A_n(\varepsilon)^c$ we obtain

$$\frac{1}{c_n}\frac{\ln(2-\varepsilon)+o(1)}{1+\varepsilon} \le \mathbb{E}\left\{\frac{1}{S_{n-1}^{(c_n)}}\ln\left(1+\frac{S_{n-1}^{(c_n)}}{c_n}\right)\right\} \le \frac{1}{c_n}\frac{\ln(2+\varepsilon)+o(1)}{1-\varepsilon}.$$

Since this holds for every $\varepsilon > 0$ we have that

$$n\int_0^\infty \mathbb{E}\left\{\frac{2}{2^x c_n + (2^x-1)\sum_{i=2}^n X_i^{(c_n)}}\right\} dx = \frac{2+o(1)}{\log_2 n}.$$

Thus,

$$
\begin{aligned}
& \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}}\right\} \\
= & \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}} I_{\{X_1 > c_n, \ldots, X_k > c_n\}} I_{\{X_{k+1} \le c_n, \ldots, X_n \le c_n\}}\right\} \\
\le & \frac{2+o(1)}{\log_2 n} + \sum_{k=2}^{n} \binom{n}{k} \mathbb{E}\left\{\log_2 \frac{\sum_{i=1}^{k} X_i + \sum_{i=k+1}^{n} X_i}{k c_n + \sum_{i=k+1}^{n} X_i} I_{\{X_1 > c_n, \ldots, X_k > c_n\}}\right\} \\
\le & \frac{2+o(1)}{\log_2 n} + \sum_{k=2}^{n} \binom{n}{k} \mathbb{E}\left\{\log_2 \frac{\sum_{i=1}^{k} X_i}{k c_n} I_{\{X_1 > c_n, \ldots, X_k > c_n\}}\right\} \\
= & \frac{2+o(1)}{\log_2 n} + \sum_{k=2}^{n} \binom{n}{k} \mathbb{E}\left\{\log_2 \frac{\sum_{i=1}^{k} X_i}{k c_n} \Big| X_1 > c_n, \ldots, X_k > c_n\right\} \\
& \times \mathbb{P}\left\{X_1 > c_n, \ldots, X_k > c_n\right\}.
\end{aligned}
$$

From the union bound and from (2) we get that

$$
\begin{aligned}
& \mathbb{E}\left\{\log_2 \frac{\sum_{i=1}^{k} X_i}{k c_n} \Big| X_1 > c_n, \ldots, X_k > c_n\right\} \\
= & \int_0^\infty \mathbb{P}\left\{\log_2 \frac{\sum_{i=1}^{k} X_i}{k c_n} \ge x \Big| X_1 > c_n, \ldots, X_k > c_n\right\} dx \\
\le & \int_0^\infty \mathbb{P}\left\{\cup_{i=1}^{k} \{X_i \ge 2^x c_n\} \Big| X_1 > c_n, \ldots, X_k > c_n\right\} dx \\
\le & k \int_0^\infty \mathbb{P}\left\{X_1 \ge 2^x c_n \mid X_1 > c_n\right\} dx \\
\le & k \int_0^\infty \frac{2/(2^x c_n)}{1/c_n} dx = k \frac{2}{\ln 2},
\end{aligned}
$$

therefore

$$
\begin{aligned}
\sum_{k=2}^{n} \binom{n}{k} \mathbb{E}\left\{\log_2 \frac{\sum_{i=1}^{k} X_i}{k c_n} I_{\{X_1 > c_n, \ldots, X_k > c_n\}}\right\} &\le \sum_{k=2}^{n} \binom{n}{k} k \frac{2}{\ln 2} \left(\frac{2}{c_n}\right)^k \\
&\le \sum_{k=2}^{n} \left(\frac{2}{\log_2 n}\right)^k \frac{2}{\ln 2} = o(1/\log_2 n),
\end{aligned}
$$

and the upper bound in the proposition is proved. ∎

The following remark is a kind of probabilistic essence of the St.Petersburg distribution, and also gives another proof for the upper bound in the previous proposition.

Remark 1 *For each $c > 0$ we have that*

$$\mathbb{E}\left\{\log_2 \frac{S_k}{c}\Big| X_1 > c, \dots, X_k > c\right\} \le \mathbb{E}\left\{\log_2 S_k\right\}.$$

Proof. The idea behind the simple proof is that a St. Petersburg random variable X can be represented as 2^Y, where Y is the memoryless discrete random variable, that is a geometric random variable with parameter $1/2$, $\mathbb{P}\{Y = k\} = 2^{-k}$, $k = 1, 2, \dots$. We have

$$\begin{aligned}
&\mathbb{E}\left\{\log_2 \frac{S_k}{c}\Big| X_1 > c, \dots, X_k > c\right\} \\
&\qquad = \sum_{j_i \ge \lfloor \log_2 c\rfloor + 1} \log_2 \frac{\sum_{i=1}^k 2^{j_i}}{c} \frac{1}{\prod_{i=1}^k 2^{j_i}} 2^{k\lfloor \log_2 c\rfloor} \\
&\qquad = \sum_{j_i \ge \lfloor \log_2 c\rfloor + 1} \log_2 \left(\sum_{i=1}^k 2^{j_i - \log_2 c}\right) \frac{1}{\prod_{i=1}^k 2^{j_i - \lfloor \log_2 c\rfloor}} \\
&\qquad \le \sum_{j_i \ge 1} \log_2 \left(\sum_{i=1}^k 2^{j_i}\right) \frac{1}{\prod_{i=1}^k 2^{j_i}} = \mathbb{E}\left\{\log_2 S_k\right\},
\end{aligned}$$

which proves our statement. ∎

Proposition 2 *We have*

$$\frac{-0.84 + o(1)}{\log_2 n} \le \mathbb{E}\left\{\log_2 \frac{S_n^{(c_n)}}{n \log_2 n}\right\} - \frac{\log_2 \log_2 n}{\ln 2 \log_2 n} \le \frac{0.52 + o(1)}{\log_2 n}.$$

Proof. Concerning the upper bound, we apply the inequality

$$\ln z \le (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3},\ z > 0,$$

which implies that

$$
\begin{aligned}
\mathbb{E}\left\{\log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right\} &= \frac{1}{\ln 2}\mathbb{E}\left\{\ln \frac{S_n^{(c_n)}}{n\log_2 n}\right\} \\
&\leq \frac{1}{\ln 2}\left\{\mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n}-1\right)-\frac{1}{2}\mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n}-1\right)^2\right. \\
&\qquad\left.+\frac{1}{3}\mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n}-1\right)^3\right\}. \qquad (7)
\end{aligned}
$$

Easy computation shows that

$$
\begin{aligned}
\mathbb{E}\{X^{(c)}\} &= \sum_{k=1}^{\lfloor\log_2 c\rfloor} 1 + c\,\mathbb{P}\{X>c\} = \lfloor\log_2 c\rfloor + \frac{c}{2^{\lfloor\log_2 c\rfloor}} \\
&= \log_2 c + 2^{\{\log_2 c\}} - \{\log_2 c\},
\end{aligned}
$$

therefore for the first order term in the inequality (7) we get the bound

$$
\mathbb{E}\left\{\frac{S_n^{(c_n)}}{n\log_2 n}-1\right\} = \mathbb{E}\left\{\frac{X_1^{(c_n)}}{\log_2 n}-1\right\} \leq \frac{\log_2\frac{c_n}{n}+1}{\log_2 n}.
$$

For the second order term in the inequality (7), (5) implies the bound

$$
\begin{aligned}
\mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n}-1\right)^2 &= \frac{\operatorname{Var} S_n^{(c_n)} + \left(\mathbb{E}S_n^{(c_n)} - n\log_2 n\right)^2}{n^2\log_2^2 n} \\
&\geq \frac{\operatorname{Var} S_n^{(c_n)}}{n^2\log_2^2 n} = \frac{n\operatorname{Var} X_1^{(c_n)}}{n^2\log_2^2 n} \\
&= \frac{\mathbb{E}\left\{\left[X^{(c_n)}\right]^2\right\} - \left\{\mathbb{E}\left[X^{(c_n)}\right]\right\}^2}{n\log_2^2 n} \\
&\geq \frac{c_n\left(2^{1-\{\log_2 c_n\}} + 2^{\{\log_2 c_n\}}\right) - 2 - (\log_2 c_n + 1)^2}{n\log_2^2 n} \\
&\geq \frac{c_n 2\sqrt{2} - 2 - (\log_2 c_n + 1)^2}{n\log_2^2 n}.
\end{aligned}
$$

Concerning the third order term in the inequality (7), we get that

$$\begin{aligned}
\mathbb{E}\left\{\left[X^{(c)}-\mathbb{E}\left\{X^{(c)}\right\}\right]^3\right\} &\le \mathbb{E}\left\{\left[X^{(c)}\right]^3\right\}\\
&= \sum_{k=1}^{\lfloor\log_2 c\rfloor}\frac{1}{2^k}2^{3k}+\frac{1}{2^{\lfloor\log_2 c\rfloor}}c^3\\
&= 4\frac{4^{\lfloor\log_2 c\rfloor}-1}{3}+c^2 2^{\{\log_2 c\}}\\
&= \left(\frac{4^{1-\{\log_2 c\}}}{3}+2^{\{\log_2 c\}}\right)c^2-\frac{4}{3}\\
&\le \frac{7c^2}{3}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left\{\left[S_n^{(c_n)}-n\log_2 n\right]^3\right\}\\
= &\ \mathbb{E}\left\{\left[S_n^{(c_n)}-\mathbb{E}S_n^{(c_n)}+\mathbb{E}S_n^{(c_n)}-n\log_2 n\right]^3\right\}\\
= &\ \mathbb{E}\left[S_n^{(c_n)}-\mathbb{E}S_n^{(c_n)}\right]^3+3\mathbb{E}\left[S_n^{(c_n)}-\mathbb{E}S_n^{(c_n)}\right]^2\left(\mathbb{E}S_n^{(c_n)}-n\log_2 n\right)\\
&+\left(\mathbb{E}S_n^{(c_n)}-n\log_2 n\right)^3\\
= &\ n\mathbb{E}\left[X^{(c_n)}-\mathbb{E}X^{(c_n)}\right]^3+n^3\left(\log_2\frac{c_n}{n}+2^{\{\log_2 c_n\}}-\{\log_2 c_n\}\right)^3\\
&+3n^2\,\mathbb{E}\left[X^{(c_n)}-\mathbb{E}X^{(c_n)}\right]^2\left(\log_2\frac{c_n}{n}+2^{\{\log_2 c_n\}}-\{\log_2 c_n\}\right),
\end{aligned}$$

which imply that

$$\mathbb{E}\left[\frac{S_n^{(c_n)}}{n\log_2 n}-1\right]^3 \le \frac{7c_n^2}{3n^2\log_2^3 n}+\frac{\left(\log_2\frac{c_n}{n}+1\right)^3}{\log_2^3 n}+9\frac{c_n\left(\log_2\frac{c_n}{n}+1\right)}{n\log_2^3 n},$$

where at the estimation of the last term we used (5). Summarizing these inequalities we get the upper bound

$$\begin{aligned}
\mathbb{E}\left\{\log_2\frac{S_n^{(c_n)}}{n\log_2 n}\right\} \le \frac{1}{\ln 2}\Bigg(&\frac{\log_2\frac{c_n}{n}+1}{\log_2 n}-\frac{c_n 2\sqrt{2}-2-(\log_2 c_n+1)^2}{2n\log_2^2 n}\\
&+\frac{7c_n^2}{9n^2\log_2^3 n}+\frac{\left(\log_2\frac{c_n}{n}+1\right)^3}{3\log_2^3 n}+3\frac{c_n\left(\log_2\frac{c_n}{n}+1\right)}{n\log_2^3 n}\Bigg).
\end{aligned}$$

Choose $c_n = n\log_2 n$, then we have that

$$\mathbb{E}\left\{\log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right\} \quad \leq \quad \frac{1}{\ln 2}\frac{\log_2\log_2 n}{\log_2 n} + \frac{1-\sqrt{2}+7/9}{\ln 2\log_2 n} + o(1/\log_2 n).$$

Concerning the lower bound, for fixed $0 < \varepsilon < 1$, let b_ε be the solution of the equation

$$\ln(1-\varepsilon) = -\varepsilon - b_\varepsilon \varepsilon^2,$$

i.e.,

$$b_\varepsilon = \frac{-\ln(1-\varepsilon)-\varepsilon}{\varepsilon^2},$$

then

$$\lim_{\varepsilon\to 0} b_\varepsilon = 1/2.$$

The definition of b_ε implies that, for all $1-\varepsilon \leq z$, we have

$$\ln z \geq (z-1) - b_\varepsilon(z-1)^2.$$

Moreover, for $z \geq 2/\log_2 n$,

$$\ln z \geq -a_n,$$

where $a_n = \ln\log_2 n$. These inequalities imply that, for $z \geq 2/\log_2 n$,

$$\ln z \geq (z-1) - b_\varepsilon(z-1)^2 - a_n I_{\{z<1-\varepsilon\}}. \tag{8}$$

From inequality (8) we get the lower bound

$$\begin{aligned}\mathbb{E}\left\{\log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right\} \quad \geq \quad & \frac{1}{\ln 2}\left(\frac{\mathbb{E}\{S_n^{(c_n)}\}}{n\log_2 n} - 1 - b_\varepsilon \mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n} - 1\right)^2\right. \\ & \left. -a_n \mathbb{P}\left\{\frac{S_n^{(c_n)}}{n\log_2 n} < 1-\varepsilon\right\}\right). \end{aligned} \tag{9}$$

For the first term in the inequality (9) we obtain

$$\frac{\mathbb{E}\{S_n^{(c_n)}\}}{n\log_2 n} - 1 = \frac{\mathbb{E}X^{(c_n)} - \log_2 n}{\log_2 n} \geq \frac{\log_2 \frac{c_n}{n} + c^*}{\log_2 n},$$

where

$$c^* = \min_c \left(2^{\{\log_2 c\}} - \{\log_2 c\}\right) = 0.914.$$

Using (5), for the second term in the inequality (9) we have

$$\begin{aligned}
\mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n}-1\right)^2 &= \frac{\operatorname{Var} S_n^{(c_n)}+\left(\mathbb{E}S_n^{(c_n)}-n\log_2 n\right)^2}{n^2\log_2^2 n}\\
&\le \frac{3nc_n+n^2\left(\log_2 c_n-\log_2 n+1\right)^2}{n^2\log_2^2 n}\\
&= \frac{3c_n}{n\log_2^2 n}+\frac{\left(\log_2\frac{c_n}{n}+1\right)^2}{\log_2^2 n}.
\end{aligned}$$

For the third term in the inequality (9) and for $c_n > n$, Theorem 1 implies that

$$\begin{aligned}
\mathbb{P}\left\{\frac{S_n^{(c_n)}}{n\log_2 n}<1-\varepsilon\right\} &\le \mathbb{P}\left\{\frac{S_n^{(n)}}{n\log_2 n}<1-\varepsilon\right\}\\
&= \mathbb{P}\left\{\frac{S_n^{(n)}-\mathbb{E}\{S_n^{(n)}\}}{n\log_2 n}<1-\varepsilon-\frac{\mathbb{E}\{S_n^{(n)}\}}{n\log_2 n}\right\}\\
&= \mathbb{P}\left\{\frac{S_n^{(n)}-\mathbb{E}\{S_n^{(n)}\}}{n\log_2 n}<1-\varepsilon-\frac{n\mathbb{E}\{X_1^{(n)}\}}{n\log_2 n}\right\}\\
&\le \mathbb{P}\left\{\frac{S_n^{(n)}-\mathbb{E}\{S_n^{(n)}\}}{n\log_2 n}<-\varepsilon\right\}\\
&\le n^{4-\varepsilon\ln\ln n\log_2 e}.
\end{aligned}$$

Summarizing these inequalities we get that

$$\begin{aligned}
&\mathbb{E}\left\{\log_2\frac{S_n^{(c_n)}}{n\log_2 n}\right\}\\
&\ge \frac{1}{\ln 2}\left(\frac{\mathbb{E}\{S_n^{(c_n)}\}}{n\log_2 n}-1-b_\varepsilon\mathbb{E}\left(\frac{S_n^{(c_n)}}{n\log_2 n}-1\right)^2-a_n\mathbb{P}\left\{\frac{S_n^{(c_n)}}{n\log_2 n}<1-\varepsilon\right\}\right)\\
&\ge \frac{1}{\ln 2}\left(\frac{\log_2\frac{c_n}{n}+c^*}{\log_2 n}-b_\varepsilon\left(\frac{3c_n}{n\log_2^2 n}+\frac{\left(\log_2\frac{c_n}{n}+1\right)^2}{\log_2^2 n}\right)-a_n n^{4-\varepsilon\ln\ln n\log_2 e}\right).
\end{aligned}$$

Choose $c_n = n\log_2 n$, then

$$\begin{aligned}
&\mathbb{E}\left\{\log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right\}\\
&\geq \frac{1}{\ln 2}\left(\frac{\log_2\log_2 n + c^*}{\log_2 n} - \frac{3b_\varepsilon}{\log_2 n} - \frac{b_\varepsilon\left(\log_2\log_2 n + 1\right)^2}{\log_2^2 n} - a_n n^{4-\varepsilon\ln\ln n\log_2 e}\right)\\
&= \frac{\log_2\log_2 n}{\ln 2\ \log_2 n} + \frac{c^* - 3b_\varepsilon}{\ln 2\ \log_2 n} + o(1/\log_2 n),
\end{aligned}$$

which implies the lower bound, since for small ε, $b_\varepsilon \approx 1/2$. ∎

5 Further refinements

Next we show that

$$\log_2 \frac{S_n}{n\log_2 n} \to 0$$

in L_2:

Theorem 6 *We have that*

$$\mathbb{E}\left\{\left(\log_2 \frac{S_n}{n\log_2 n}\right)^2\right\} = O(1/\ln n).$$

Proof. Apply the notations of the previous section, then

$$\begin{aligned}
\mathbb{E}\left\{\left(\log_2 \frac{S_n}{n\log_2 n}\right)^2\right\} &= \mathbb{E}\left\{\left(\log_2 \frac{S_n}{S_n^{(c_n)}} + \log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right)^2\right\}\\
&\leq 2\,\mathbb{E}\left\{\left(\log_2 \frac{S_n}{S_n^{(c_n)}}\right)^2\right\} + 2\,\mathbb{E}\left\{\left(\log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right)^2\right\},
\end{aligned}$$

with $c_n = n \log_2 n$. For the first term, (2) implies that

$$\begin{aligned}
\mathbb{E}\left\{\left(\log_2 \frac{S_n}{S_n^{(c_n)}}\right)^2\right\} &= \int_0^\infty 2x\mathbb{P}\left\{\log_2 \frac{S_n}{S_n^{(c_n)}} > x\right\} dx \\
&\leq \int_0^\infty 2x\mathbb{P}\left\{\cup_{i=1}^n \{X_i > 2^x X_i^{(c_n)}\}\right\} dx \\
&\leq \int_0^\infty 2xn\mathbb{P}\left\{X_1 > 2^x X_1^{(c_n)}\right\} dx \\
&= \int_0^\infty 2xn\mathbb{P}\left\{X_1 > 2^x c_n\right\} dx \\
&\leq \int_0^\infty 2x\frac{2n}{2^x c_n} dx = O(1/\ln n).
\end{aligned}$$

Concerning the second term, for $x > 1$ we have that

$$\ln x \leq x - 1,$$

while for $2/\log_2 n \leq x \leq 1$ and $\varepsilon = 1/2$, (8) means that

$$\ln x \geq (x-1) - b_{1/2}(x-1)^2 - a_n I_{\{x<1/2\}},$$

therefore

$$|\ln x| \leq |x-1| + (x-1)^2 + a_n I_{\{x<1/2\}} \leq 2|x-1| + a_n I_{\{x<1/2\}}.$$

Thus,

$$(\ln x)^2 \leq (2|x-1| + a_n I_{\{x<1/2\}})^2 \leq 8(x-1)^2 + 2a_n^2 I_{\{x<1/2\}}.$$

These two bounds imply that, for $x \geq 2/\log_2 n$, one has

$$(\ln x)^2 \leq 8(x-1)^2 + 2a_n^2 I_{\{x<1/2\}}.$$

Let's apply the inequalities in the proof of Proposition 2, then

$$\begin{aligned}
&\mathbb{E}\left\{\left(\log_2 \frac{S_n^{(c_n)}}{n\log_2 n}\right)^2\right\} \\
\leq\ & \frac{8}{(\ln 2)^2}\mathbb{E}\left\{\left(\frac{S_n^{(c_n)}}{n\log_2 n} - 1\right)^2\right\} + \frac{2a_n^2}{(\ln 2)^2}\mathbb{P}\left\{\frac{S_n^{(c_n)}}{n\log_2 n} < 1/2\right\} \\
=\ & O(1/\ln n).
\end{aligned}$$

■

Remark 2 *Theorem 6 implies the surprising limit:*

$$\text{Var }(\log_2 S_n) = \text{Var }\left(\log_2 \frac{S_n}{n\log_2 n}\right) \le \mathbb{E}\left\{\left(\log_2 \frac{S_n}{n\log_2 n}\right)^2\right\} \to 0.$$

Figures 1, 2, 3 and 4 show histograms for $\log_2 S_n$ and for $\log_2 S_n^{(c_n)}$ with $c_n = n\log_2 n$. The main advantage of the log-scale is that from these figures we can observe that these histograms are approximately the mixtures of two distributions. One component is nearly normal, and the other one has small weight with fluctuation of period 1. This second component of the mixture is not convergent.

It is interesting to note that asymptotic normality still does not hold for $\log_2 S_n$. We have

$$\begin{aligned}\mathbb{P}\left\{\frac{\log_2 S_n - \mathbb{E}\log_2 S_n}{\sqrt{\text{Var }\log_2 S_n}} \le x\right\} &= \mathbb{P}\left\{S_n \le 2^{x\sqrt{\text{Var }\log_2 S_n}+\mathbb{E}\log_2 S_n}\right\}\\ &= \mathbb{P}\left\{\frac{S_n}{n} - \log_2 n \le \log_2 n\left(2^{h_n(x)}-1\right)\right\},\end{aligned}$$

where

$$h_n(x) = x\sqrt{\text{Var }\log_2 S_n} + \mathbb{E}\log_2 S_n - \log_2 n - \log_2\log_2 n,$$

and since $\text{Var }\log_2 S_n = O(1/\ln n)$, $h_n(x) \to 0$ for each fix x. Therefore

$$2^{h_n(x)} - 1 \sim \ln 2\, h_n(x)$$

and we may continue writing

$$\begin{aligned}&= \mathbb{P}\left\{\frac{S_n}{n} - \log_2 n \le \log_2 n\ \ln 2\, h_n(x) + o(1)\right\}\\ &= G_{\gamma_n}(\log_2 n\ \ln 2\, h_n(x)) + o(1),\end{aligned}$$

where at the last equality we used (3). Since

$$h_n(x) = x\sqrt{\text{Var }\log_2 S_n} + O(1/\log_2 n) + \frac{\log_2\log_2 n}{\ln 2\ \log_2 n},$$

for every $x \ge 0$ we have that $h_n(x)\log_2 n \to \infty$, which means that the right side goes to 1. That is for every $x \ge 0$

$$\mathbb{P}\left\{\frac{\log_2 S_n - \mathbb{E}\log_2 S_n}{\sqrt{\text{Var }\log_2 S_n}} \le x\right\} \to 1,$$

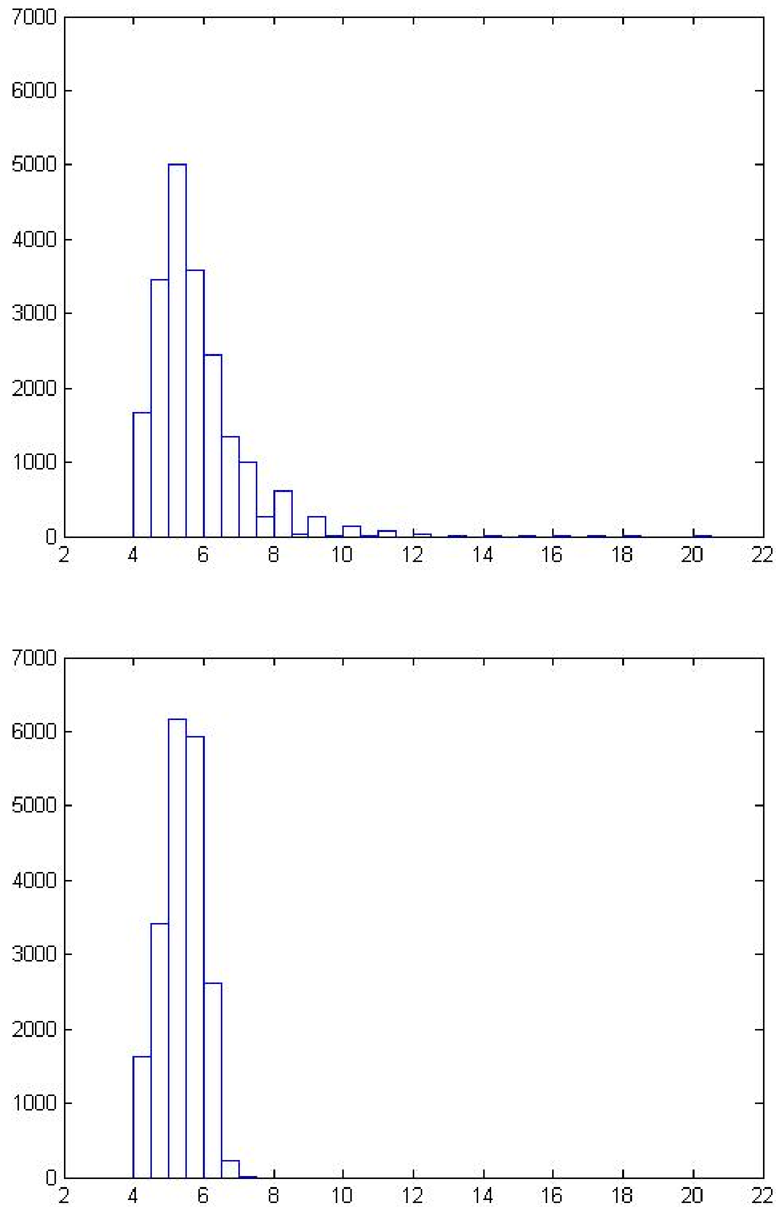

Figure 1: The histogram for $\log_2 S_8$ and for $\log_2 S_8^{(24)}$

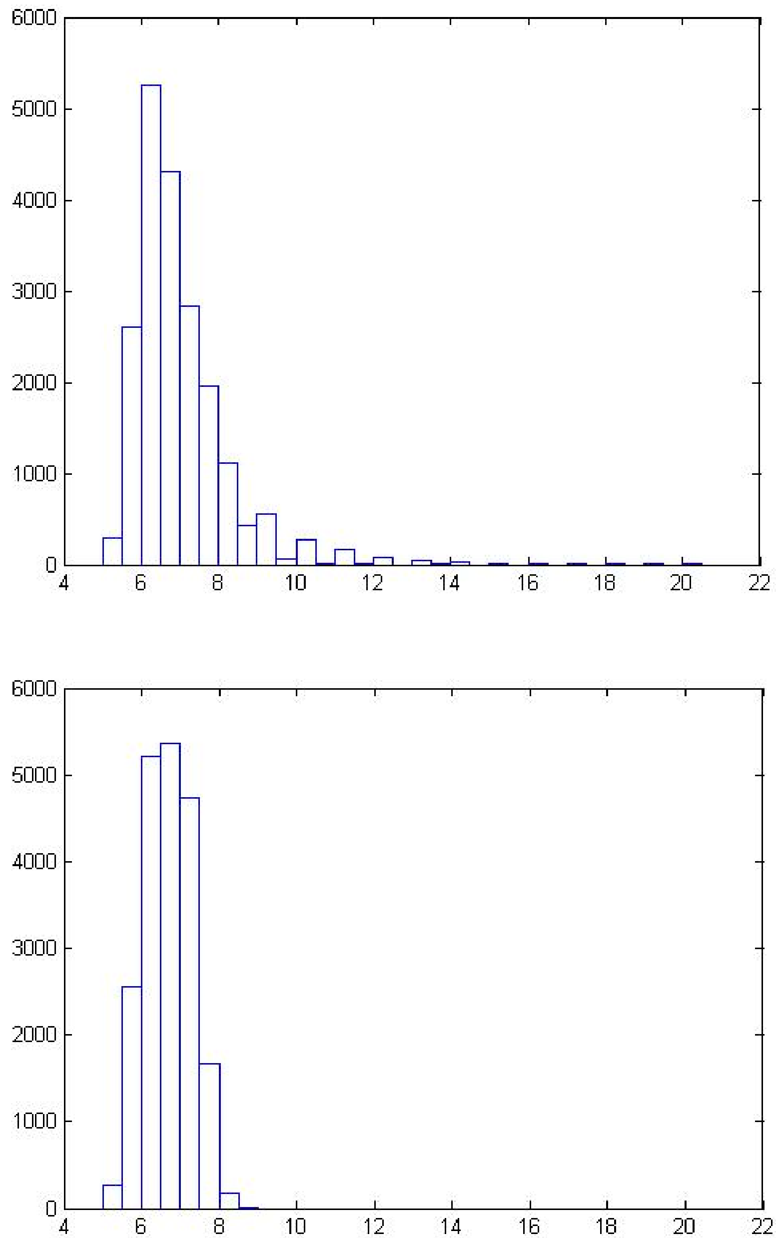

Figure 2: The histogram for $\log_2 S_{16}$ and for $\log_2 S_{16}^{(64)}$

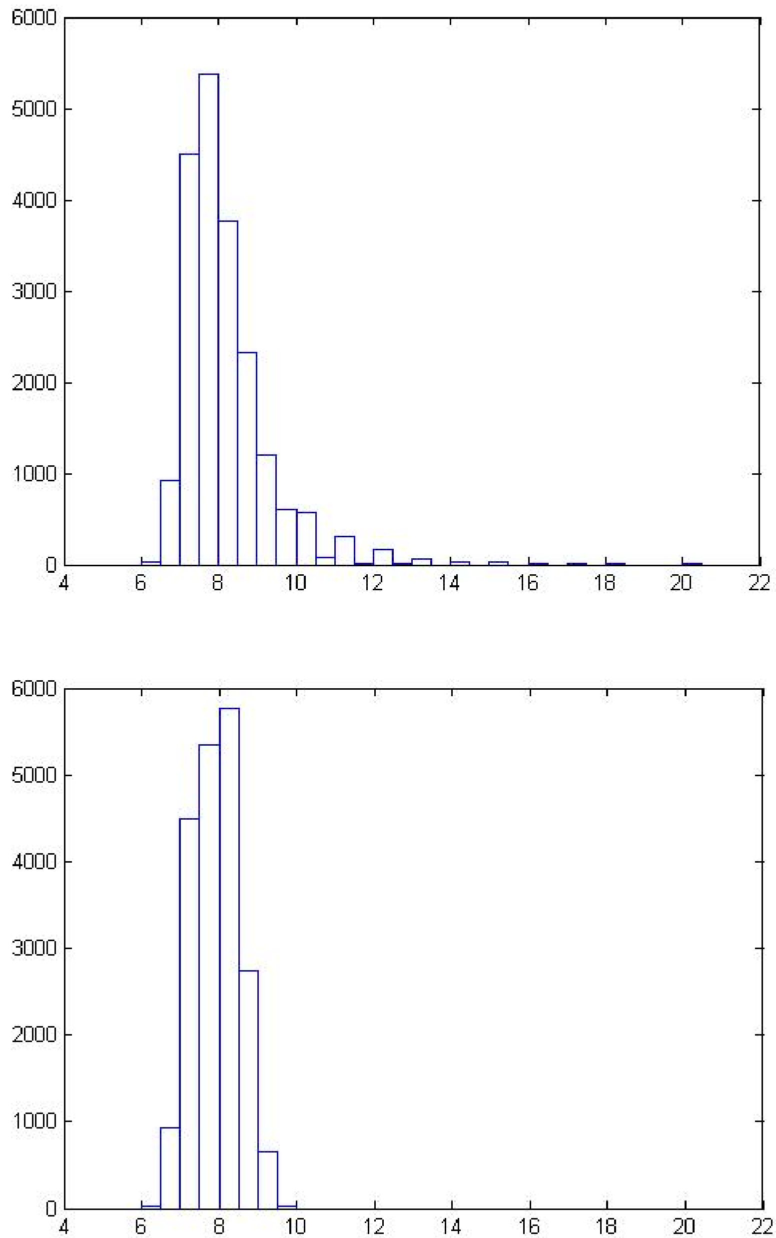

Figure 3: The histogram for $\log_2 S_{32}$ and for $\log_2 S_{32}^{(160)}$

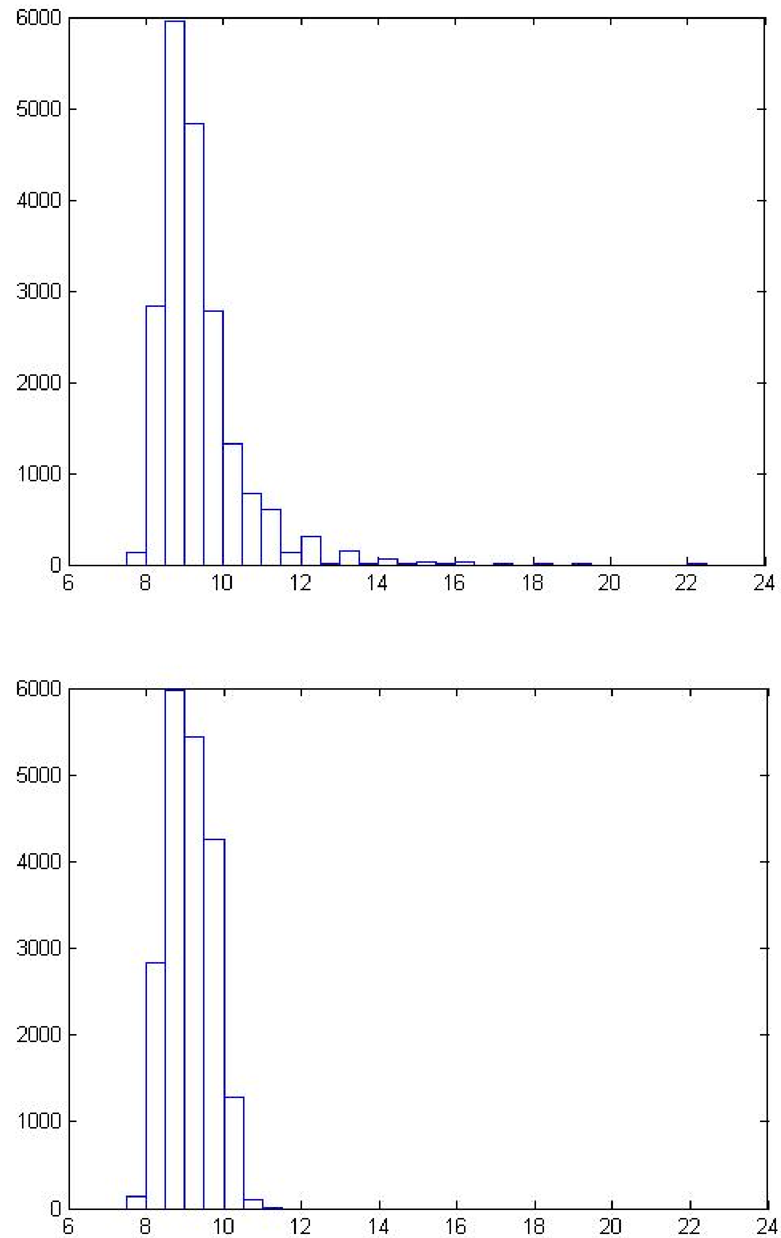

Figure 4: The histogram for $\log_2 S_{64}$ and for $\log_2 S_{64}^{(384)}$

so asymptotic normality does not hold.

Finally, we note that the same method shows that there is no limit distribution for $(\log_2 S_n - c_n)/a_n$ whatever the centering and norming constants are.

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