Combinatorics of poly-Bernoulli numbers

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Given a set of finite set $\{S_n\}$.

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Given a set of finite set $\{S_n\}$. Determine/bound

$$|S_n|.$$

An example for an extremal question

Question

What is the maximum number of 1's in a 0-1 matrix of size $n \times k$ without the configuration

$$\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}?$$

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The answer

$$n + k - 1.$$

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How many permutation matrices P are there of size $n \times n$ such that P does not contain a submatrix

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The answer

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

the n^{th} Catalan number.

Further examples

Füredi-Hajnal conjecture

Let π be a forbidden configuration where the 1's form a permutation matrix. Then the maximum number of 1's in a matrix of size $n \times n$ without π is

$$\mathcal{O}(n).$$

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Stanley-Wilf conjecture

Let π be any permutation matrix. The number of permutation matrices of size $n \times n$ without the submatrix π is

$$2^{\mathcal{O}(n)}.$$

A connection

Klazar thereom

Füredi-Hajnal conjecture implies Stanley-Wilf conjecture.

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Marcus - Tardos theorem

The Füredi-Hajnal conjecture is true.

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Füredi-Hajnal conjecture implies Stanley-Wilf conjecture.

Marcus - Tardos theorem

The Füredi-Hajnal conjecture is true. Hence the Stanley-Wilf conjecture is true too.

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Observation

The answer should be

$$B_n^{(-k)}, \text{ poly-Bernoulli numbers.}$$

What are the poly-Bernoulli numbers?

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(Kaneko 1997)

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}, \qquad \text{for all } k \in \mathbb{Z}$$

where

$$Li_k(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^k}.$$

Let us see the $B_n^{(k)}$ numbers!

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	$n=0$	1	2	3	4	5	6	7
$k=-5$	1	32	454	4718	41506	329462	2441314	17234438
-4	1	16	146	1066	6902	41506	237686	1315666
-3	1	8	46	230	1066	4718	20266	85310
-2	1	4	14	46	146	454	1394	4246
-1	1	2	4	8	16	32	64	128
0	1	1	1	1	1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0
2	1	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$	$-\frac{38}{2205}$	$-\frac{5}{168}$
3	1	$\frac{1}{8}$	$-\frac{11}{216}$	$-\frac{1}{288}$	$\frac{1243}{54000}$	$-\frac{49}{7200}$	$-\frac{75613}{3704400}$	$\frac{599}{35280}$
4	1	$\frac{1}{16}$	$-\frac{49}{1296}$	$\frac{41}{3456}$	$\frac{26291}{3240000}$	$-\frac{1921}{144000}$	$\frac{845233}{1555848000}$	$\frac{1048349}{59270400}$

What are the poly-Bernoulli numbers of negative upper index?

(Arakawa-Kaneko 1999) $k \in \mathbb{N}$

$$B_n^{(-k)} = \sum_{m=0}^{\min\{n,k\}} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\}.$$

The combinatorial interpretation of Arakawa-Kaneko's formula

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The easy combinatorial definition

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$$B_n^{(k)} := |\mathcal{A}_n^{(k)}|$$

Equivalent combinatorial definitions

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Let $\mathcal{L}_n^{(k)}$ be the set of 0-1 matrices that can be reconstructed from their row and column sums.

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Vesztergombi

Let $\mathcal{V}_n^{(k)}$ be the set of permutations of $[n+k]$ such that

$$-n \leq \pi(i) - i \leq k,$$

for each i.

Equivalent combinatorial definitions II.

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"With Celia Glass and Robert Schumacher, I recently found a combinatorial interpretation of the poly-Bernoulli numbers of negative order ..."

Cameron, Glass, Schumacher

Let $\mathcal{O}_n^{(k)}$ be the set of acyclic orientations of $K_{n,k}$.

Philosophy of enumerative combinatorics

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See

Stanley, Bijective proof problems,
http://www-math.mit.edu/~rstan/bij.pdf

The theorem

Theorem

There is a bijection between the set of 0-1 matrices of size $n \times k$ without the configuration

$$\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}$$

and

$$\mathcal{A}_n^{(k)}.$$

The proof: The first steps

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By knowing this partition of columns we know a lot about our matrix, except elements at the last columns of the ordinary classes.

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In each not all-0 row we define an important 1:

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There is a natural bijection between the classes of the two partitions.

Corollaries

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$$\sum_{i,j \in \mathbb{N}: i+j=N \text{ and } i \text{ even}} B_i^{(-j)} = \sum_{i,j \in \mathbb{N}: i+j=N \text{ and } i \text{ odd}} B_i^{(-j)}.$$

The end

Thank you for your attention