Delayed differential and difference equations modeling biological phenomena

Outline of PhD thesis

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1 Introduction

The thesis investigates delayed differential and difference equations modelling neural networks and a discrete time population dynamical model. We give necessary and sufficient conditions in terms of the parameters on the existence and uniqueness of periodic solutions and global stability. The obtained theorems contribute to the better understanding of the global behaviour of the models.

The dissertation is based on four papers of the author [1–4]. In this outline we use the same numbering and notations as in the thesis.

2 Neural models

In this chapter, we consider the neural network modelled by the following system of delayed differential equations:

$$\begin{aligned} \dot{x}^0(t) &= -\alpha x^0(t) + f_\beta(x^1(t)), \\ &\vdots \\ \dot{x}^{n-1}(t) &= -\alpha x^{n-1}(t) + f_\beta(x^n(t)), \\ \dot{x}^n(t) &= -\alpha x^n(t) + \delta f_\beta(x^0(t-\tau)), \end{aligned} \tag{2.4}$$

where x^j represents the electric potential of the jth neuron, $\alpha > 0$ and $\tau > 0$ are parameters, and $\delta \in \{-1, 1\}$. According to the sign of δ we distinguish the positive and negative feedback case. The delay τ is present due to the finite propagation velocity of the electric signal. The feedback function $f_\beta \colon \mathbb{R} \to \mathbb{R}$ is either defined by $f_\beta(x) = \beta f_0(x)$, where $f_0(x) = (|x+1| - |x-1|)/2$ or $f_\beta \in S$, that is f_β is a continuous, strictly increasing, odd function having $f'_\beta(0) = \beta$, and possessing the property that the map $\xi \mapsto \xi f'(\xi)/f(\xi)$ is strictly decreasing on $(0, \infty)$. We note that equation

$$\dot{x}^j(t) = -\alpha x^i(t) \pm f_\beta(x^{i+1}(t-\tau_i)), \quad i = \{0, 1, \dots, n\},$$

can be easily transformed into the form of (2.4), where the indices are modulo $(n+1)$.

The former function is the most common feedback function in the theory of certain artificial neural networks, the so-called cellular neural networks, which play an important role in the research of artificial intelligence (e.g. solving image processing and optimization problems. In these models, neurons (cells) are put on a d-dimensional grid and neighbouring cells are connected (with some conditions on the boundary). The above model corresponds to a 1-dimensional cellular neural network with periodic boundary condition. We shall refer the case of this type of feedback function as the “piecewise linear case”.

The latter, so-called sigmoid type of feedback functions are widely used in models of (real) neural networks. The most common examples of sigmoid functions are the tangent hyperbolic and the inverse tangent functions.

Throughout the chapter, non-constant periodic solutions of the above system of delay differential equations are investigated. Periodic solutions are of great importance in neural networks. A technical difficulty in the piecewise linear case is that the function f_0 is neither smooth, nor strictly monotonic, therefore the solution operator is neither differentiable everywhere nor injective and for this reason the Poincaré–Bendixson type theorem of Mallet-Paret and Sell [13] and several other related theorems cannot be applied directly.

Preliminaries

The natural phase space for equation (2.4) is the Banach space of continuous real functions $C(\mathbb{K}_\tau) = C(\mathbb{K}_\tau, \mathbb{R})$ equipped with the supremum-norm, where $\mathbb{K}_\tau = \mathbb{K}_{\tau,n} = [-\tau, 0] \cup \{1, 2, \ldots, n\}$. We shall use the notation $\mathbb{K}$ for $\mathbb{K}_1$.

Definition 2.7. Let $t_0 \in \mathbb{R}$ be fixed. A function $x = (x^0, \ldots, x^n)$ is a solution of equation (2.4) on the interval (t_0, ∞), if $x^0 \in C([t_0 - \tau, \infty), \mathbb{R})$, $x^i \in C([t_0, \infty), \mathbb{R})$ and x^i is continuously differentiable on interval (t_0, ∞) for all $i \in \{0, 1, \ldots, n\}$ and x satisfies equation (2.4) for all $t > t_0$. We say that $x\colon \mathbb{R} \to \mathbb{R}^{n+1}$ is a solution of equation (2.4) on $\mathbb{R}$, if it is a solution on interval (t_0, ∞) for all $t_0 \in \mathbb{R}$.

Assume that x is a solution of equation (2.4) on (t_0, ∞). Let $x_t \in C(\mathbb{K}_\tau)$ be defined for all $t \geq t_0$ as follows:

$$x_t(\theta) = \begin{cases} x^0(t+\theta) & \text{if } \theta \in [-\tau, 0], \\ x^\theta(t) & \text{if } \theta \in \{1, \ldots, n\}. \end{cases}$$

One can readily show by the method of steps that for all $\varphi \in C(\mathbb{K}_\tau)$ there exists a unique solution x of equation (2.4) on the interval $(0, \infty)$ such that $x_0(\theta) = \varphi(\theta)$ for all $\theta \in \mathbb{K}_\tau$.

According to Mallet-Paret and Sell [14] we introduce to following discrete Lyapunov functionals.

$$V^+_{\mathbb{K}_\tau}\colon C(\mathbb{K}_\tau) \setminus \{0\} \to \{0, 2, 4, \ldots, \infty\}, \quad V^-_{\mathbb{K}_\tau}\colon C(\mathbb{K}_\tau) \setminus \{0\} \to \{1, 3, 5, \ldots, \infty\},$$

$$V^+_{\mathbb{K}_\tau}(\varphi) = \begin{cases} \mathrm{sc}(\varphi, \mathbb{K}_\tau) & \text{if } \mathrm{sc}(\varphi, \mathbb{K}_\tau) \text{ is even of infinite}, \\ \mathrm{sc}(\varphi, \mathbb{K}_\tau) + 1 & \text{if } \mathrm{sc}(\varphi, \mathbb{K}_\tau) \text{ is odd}, \end{cases}$$

$$V^-_{\mathbb{K}_\tau}(\varphi) = \begin{cases} \mathrm{sc}(\varphi, \mathbb{K}_\tau) & \text{if } \mathrm{sc}(\varphi, \mathbb{K}_\tau) \text{ is odd or infinite}, \\ \mathrm{sc}(\varphi, \mathbb{K}_\tau) + 1 & \text{if } \mathrm{sc}(\varphi, \mathbb{K}_\tau) \text{ is even}, \end{cases}$$

where $\mathrm{sc}(\varphi, H)$ denotes the number of sign changes φ has on subset H of its domain. For brevity, we shall use notation $V^\pm_\tau$ in the case when $n = 0$.

Let V denote $V^\pm_{\mathbb{K}_\tau}$ determined by the sign of δ. Then according to the results of Mallet-Paret and Sell [14], if $x : \mathbb{R} \to \mathbb{R}^{n+1}$ is a periodic solution of equation (2.4), then $V(x_t)$ is finite and constant for all $t \in \mathbb{R}$. Thus in case of periodic solutions we shall usually omit the lower index and write $V(x)$.

The theorem of Gopalsamy and He [7] implies that for $\beta < \alpha$, all solutions of (2.4) converge to (the unique) equilibrium point. By their argument one can easily prove that there are no non-constant periodic solutions of equation (2.4) when $\beta = \alpha$, which proves the lemma below.

Lemma 2.9. *If $0 < \beta \leq \alpha$, then there exist no non-constant periodic solutions of equation* (2.4)*.*

According to the above lemma, for the investigation of non-constant periodic solutions, it is sufficient to concentrate on the case when $\beta > \alpha$.

Categorization and number of periodic orbits for one equation

In Section 2.4 we investigate the non-constant periodic solutions of equation (2.4) in the case when $n = 0$, that is we consider the following equation:

$$\dot{x}(t) = -\alpha x(t) \pm f_\beta(x(t-\tau)). \tag{2.15}$$

In the sigmoid case, a very detailed picture of the global attractor is available due to the monograph of Krisztin, Walther and Wu [11] and to a sequence of papers [5,8–10]. In particular, necessary and sufficient conditions in terms of the parameters α, β and τ are known for the existence and uniqueness of periodic solutions. The section is devoted to prove analogous theorems in the piecewise linear case. It seems reasonable to approximate our piecewise linear feedback function with functions from the sigmoid class, but the problem is that the global attractor is only upper semi-continuous, hence this approach cannot provide uniqueness and non-existence results.

Theorem 2.27 summarizes the results of the section. The proof consists of several steps: first we prove some technical lemmas, which we do not detail here, then we prove the non-existence and uniqueness parts of the theorem. The definition below and the following theorem has an important role throughout the hole section.

Definition 2.19. Let $x : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable, periodic function with minimal period T_x. We call the curve $X\colon [0, T_x] \ni t \mapsto (x(t), \dot{x}(t)) \in \mathbb{R}^2$ the D-trajectory of function x.

It can be shown that – in both feedback function cases – the D-trajectory X of a non-constant periodic solution x is a simple closed curve on the plane containing the origin in its interior. Let $|X|$ denote the trace of X, which separates the plane into one bounded and one unbounded connected set, which are denoted by $\mathrm{int}(X)$ and $\mathrm{ext}(X)$, respectively. The following theorem states basically that bigger delays imply bigger D-trajectories.

Theorem 2.25 (Garab, Krisztin [4]; Garab [3])**.** *Let α, β, f_β, and τ_1, τ_2 be fixed and $0 < \tau_1 < \tau_2$. Furthermore, let x_1 and x_2 be non-constant periodic solutions of equation* (2.15) *with delays $\tau = \tau_i$, $i \in \{1, 2\}$ and D-trajectories X_i, respectively. Let ε denote* *the sign* $+$*, resp.* $-$*, in the case of positive, resp. negative feedback. If* $V^{\varepsilon}_{\tau_1}(x_1) = V^{\varepsilon}_{\tau_2}(x_2)$*, then*

$$|X_2| \subset \operatorname{ext}(X_1) \cup |X_1| \quad \textit{and} \quad |X_2| \cap \operatorname{ext}(X_1) \neq \emptyset.$$

The results of Krisztin and Walther [10] imply in the sigmoid case, that if a so-called slowly oscillating periodic solution exists ($V^+_\tau(x) = 2$ type), then for fixed parameters it is uniquely defined (up to translation of time). The uniqueness parts of Theorem 2.27 – proved in the dissertation – guarantee this property also in the piecewise linear case. Therefore we can define the period of that solution as a function of the delay. The following theorem on the period function plays an essential role in the proof of the existence of periodic solutions of equation (2.15) and in the investigation of the periodic solutions of system (2.4).

Theorem 2.26 (Garab, Krisztin [4]; Garab [3]). *Let T denote the period function of equation* (2.15) *with positive feedback and* $\alpha > 0$*,* $\beta > 0$ *fixed. Assume that* τ_1 *are* τ_2 *are from the domain of T and that* $\tau_1 < \tau_2$ *holds. Then*

$$0 \leq T(\tau_2) - T(\tau_1) < 2(\tau_2 - \tau_1).$$

The next theorem is one of the main results of the chapter and it summarizes the statements on existence, non-existence and uniqueness of periodic orbits of equation (2.15) for the piecewise linear case $f_\beta = \beta f_0$.

Theorem 2.27 (Garab [3]). *Let* $\alpha, \beta, \tau > 0$*,* $f_\beta = \beta f_0$ *and* $k \geq 1$ *be fixed, and*

$$\nu = \nu(\alpha, \beta, \tau) = \tau\sqrt{\beta^2 - \alpha^2} + \arccos\frac{\alpha}{\beta}.$$

Then the following statements hold.

(i) *In case of positive feedback, equation* (2.15) *has a periodic solution x such that* $V^+_\tau(x) = 2k$ *if and only if* $\beta > \alpha$ *and* $\nu \geq 2k\pi$ *hold. There exist no non-constant periodic solutions of* $V^+_\tau = 0$ *type.*

  (a) *If* $\nu > 2k\pi$*, then the solution of this type is unique up to translation of time.*

  (b) *If* $\nu = 2k\pi$*, then the non-constants periodic solutions of equation* (2.15) *are the functions* $x(t) = A\cos(t\sqrt{\beta^2 - \alpha^2} + \Delta)$*, where constants* $\Delta \in \mathbb{R}$ *and* $A \in (0, 1]$ *can be arbitrarily chosen. In this case* $V^+_\tau(x) = 2k$ *necessarily holds.*

(ii) *In case of negative feedback, equation* (2.15) *has a periodic solution x such that* $V^-_\tau(x) = 2k - 1$ *if and only if* $\beta > \alpha$ *and* $\nu \geq (2k-1)\pi$ *hold.*

  (a) *If* $\nu > (2k-1)\pi$*, then the solution of this type is unique up to translation of time.*

  (b) *If* $\nu = (2k-1)\pi$*, then the non-constants periodic solutions of equation* (2.15) *are the functions* $x(t) = A\cos(t\sqrt{\beta^2 - \alpha^2} + \Delta)$*, where constants* $\Delta \in \mathbb{R}$ *and* $A \in (0, 1]$ *can be arbitrarily chosen. In this case* $V^-_\tau(x) = 2k - 1$ *necessarily holds.*

The proof consists of several parts. The sufficient condition for the existence of slowly oscillating periodic orbits was given by Vas [20]. Combining this with Theorem 2.26 we are able to prove the statement of the theorem on existence in the negative feedback case and for faster oscillations as well. The proof of the statements on uniqueness and non-existence are carried out by applying and suitably modifying the Cao–Krisztin–Walther technique and is based on the investigation of the D-trajectories.

The periodic solutions of a ring-like system of equations

In Section 2.5, we consider the non-constant periodic solutions of system (2.4) for $n \geq 1$ and with both feedback functions. We note that $\tau = 1$ may be assumed without loss of generality. The main result of the section is that we give necessary and sufficient conditions on the existence and uniqueness of relatively quickly oscillating periodic solutions and we give sufficient conditions on the existence and non-existence for periodic solutions that oscillate slower. The results are summarized in Theorem 2.34 and Theorem 2.35.

Following and, at some parts, modifying the argument of Yi, Chen and Wu [21], one can derive the theorem below, which shows the strong connection between the periodic solutions of system (2.4) and the slowly oscillating periodic solutions of equation (2.15).

Theorem 2.32 (Garab, Krisztin [4]; Garab [3]). *Let T denote the period function of equation* (2.15) *with positive feedback. Then in case of positive feedback there is a one-to-one correspondence between the periodic solutions of system* (2.4) *of $V^{+}_{\mathbb{K}_\tau} = 2k \geq 2$ type and the intersection points of the following two curves:*

$$\mathrm{dom} T \ni \gamma \mapsto (\gamma, T(\gamma)) \quad \textit{and} \quad \mathbb{R} \ni \zeta \mapsto \left(\frac{(n-k+1)\zeta + 1}{n+1}, \zeta \right).$$

Analogously, for negative feedback, there is a one-to-one correspondence between the non-constant periodic solutions of system (2.4) *of $V^{-}_{\mathbb{K}_\tau} = 2k-1$ type and the intersection points of the following two curves:*

$$\mathrm{dom} T \ni \gamma \mapsto (\gamma, T(\gamma)) \quad \textit{and} \quad \mathbb{R} \ni \zeta \mapsto \left(\frac{(n-k+3/2)\zeta + 1}{n+1}, \zeta \right).$$

Combining this result with Theorem 2.26, we obtain Theorem 2.34 and Theorem 2.35, which are one of the main results of the section. The following theorem is a generalization of the results of Yi, Chen and Wu [6,21].

Theorem 2.34 (Garab, Krisztin [4]; Garab [3]). *Let $\tau = 1$ and use the following notation:*

$$\nu_n(\alpha, \beta) = \sqrt{\beta^2 - \alpha^2} + (n+1) \arccos \frac{\alpha}{\beta}.$$

Assume that $f_\beta \in S$ and $\delta = 1$. Then the following statements hold.

(i) If $\mathbb{N} \ni k \geq \frac{n+1}{2}$, then equation (2.4) *has a periodic solution x for which $V_{\mathbb{K}}^{+}(x) = 2k$ holds if and only if $\nu_n(\alpha,\beta) > 2k\pi$. This solution is uniquely defined (up to translation of time).*

(ii) If $\mathbb{N} \ni k < \frac{n+1}{2}$, then $\nu_n(\alpha,\beta) > 2k\pi$ implies that equation (2.4) *has a non-constant periodic solution x for which $V_{\mathbb{K}}^{+}(x) = 2k$ holds.*

(iii) There exist no non-constant periodic solutions of $V_{\mathbb{K}}^{+} = 0$ type.

Analogously, for negative feedback, if $f_\beta \in S$ and $\delta = -1$, then the following statements hold.

(iv) If $\mathbb{N} \ni k \geq \frac{n+2}{2}$, then equation (2.4) *has a periodic solution x for which $V_{\mathbb{K}}^{-}(x) = 2k-1$ holds if and only if $\nu_n(\alpha,\beta) > (2k-1)\pi$. This solution is uniquely defined (up to translation of time).*

(v) If $\mathbb{N} \ni k < \frac{n+2}{2}$, then $\nu_n(\alpha,\beta) > (2k-1)\pi$ implies that equation (2.4) *has a non-constant periodic solution x for which $V_{\mathbb{K}}^{-}(x) = 2k-1$ holds.*

In the case when $f_\beta = \beta f_0$, the relations "$>$" should be changed to "$\geq$" everywhere. When equation holds, then the periodic solutions are unique up to translation of time and to a constant multiple.

Note that if $n = 0$, then Theorem 2.34 gives the statements of Theorem 2.27 (and the analogues for the sigmoid case). Moreover, if the feedback is positive and $n = 1$, then the above theorem gives necessary and sufficient conditions on existence and uniqueness of all types of periodic solutions. The following theorem is a simple consequence of Theorem 2.26 and the results of Nussbaum [17].

Theorem 2.35. *Let $\tau = 1$ and use the following notation:*

$$\tau^* = \frac{2\pi - \arccos\frac{\alpha}{\beta}}{\sqrt{\beta^2 - \alpha^2}}.$$

Then the following statements hold.

(i) Assume that $\frac{n+1}{4} < k < \frac{n+1}{2}$, $\delta = 1$ and $\tau^*(4k - n - 1) \geq 3$ hold. Then equation (2.4) *has no periodic solution of $V_{\mathbb{K}}^{+} = 2k$ type.*

(ii) Assume that $\frac{n+3}{4} < k < \frac{n+2}{2}$, $\delta = -1$, and $\tau^*(4k - n - 3) \geq 3$ hold. Then equation (2.4) *has no periodic solution of $V_{\mathbb{K}}^{-} = 2k - 1$ type.*

Yi, Chen and Wu [21] formulated a conjecture – for the sigmoid, positive feedback case which they considered – that statement (i) of Theorem (2.34) holds for all $k \geq 1$. Using Theorem 2.32, it can be easily shown that in order to prove the consejture – for the negative and positive feedback case and for both type of feedback functions – it is sufficient to prove our conjecture below. The conjecture can be strengthened by computer simulations.

Conjecture 2.36. (Garab, Krisztin [4]; Garab [3]). *Let T denote the period function of equation* (2.15) *with positive feedback. Then the map* $\mathrm{dom}T \ni \tau \mapsto T(\tau)/\tau$ *is monotonically non-increasing.*

3 Global stability analysis of second order difference equations

In this chapter we give necessary and sufficient conditions on the global stability of the equilibrium of two second order, parametric difference equations. In Section 3.1 we consider the following difference equation:

$$x_{n+1} = mx_n - \alpha \tanh(x_{n-1}),$$

where $(\alpha, m) \in \mathbb{R}^2$. This equation can be regarded as a discrete-time single neuron model, but one can also easily transform it to get a Clark type population dynamical model, as well.

In Section 3.2, we investigate the following delayed Ricker type population dynamical model

$$x_{n+1} = x_n e^{\alpha - x_{n-d}},$$

where x_n denotes the size of the population living in a certain area at time instant n, α is a positive parameter, and $d > 0$ is the delay in the self-regulatory system. The model was first formulated by Ricker [18] in 1954 (at that time without delay) to model the dynamics of baleen whale populations. Since then it became one of the most widespread population dynamical models.

The proof of the global asymptotic stability is carried out similarly for the two equations. We combine different validated computer aided methods with analytical tools. Essentially, the proof consists of the following steps. First we construct a uniform neighbourhood of the equilibrium that is independent of the parameters and belongs to the basin of attraction of the fixed point for all parameters for which local asymptotic stability holds. For parameters near the critical values, this is done in both models by analysing the (resonant) normal form of the Neimark–Sacker bifurcation. To the best of our knowledge, such application of the normal form of the Neimark–Sacker bifurcation is new in the literature. Thereafter, we show by validated computer aided methods that every trajectory enters this small neighbourhood, which proves that local asymptotic stability of the equilibrium implies its global asymptotic stability. The latter part of the proof is done by Ferenc Bartha using graph representations and interval arithmetical tools.

The term "computer aided" means here that some of our calculations and estimates are done by using a computer program which gives validated results, that is, all possible numerical errors are controlled. This allows us to prove mathematical theorems from the obtained outputs.

A discrete-time neural model

Section 3.1 is devoted to the analysis of the global stability of the delayed difference equation

$$x_{n+1} = mx_n - \alpha\varphi(x_{n-1}), \tag{3.1}$$

where $(\alpha, m) \in \mathbb{R}^2$ and φ is a bounded, continuous, real function satisfying the following Yorke type condition:

$$\min\{0, x\} < \varphi(x) < \max\{0, x\} \quad \text{for all} \quad x \neq 0. \tag{3.2}$$

Instead of equation (3.1), we shall consider the following equivalent two-dimensional map:

$$F_{\alpha,m} \colon \mathbb{R}^2 \to \mathbb{R}^2, \; F_{\alpha,m} \colon (x, y) \mapsto (y, my - \alpha\varphi(x)). \tag{3.3}$$

The main result of the section is that we give necessary and sufficient conditions on the global asymptotic stability of the $(0, 0)$ fixed point of the map (3.3) for the case when $\varphi(x) \equiv \tanh(x)$, which is one of the most common feedback functions in the field of neural networks.

Computer aided graph representational methods may be applied only on a bounded domain of the phase space. By consideration of the global dynamics by elementary techniques we obtain – among others – the following corollary.

Corollary 3.8 (Bartha, Garab [1]). *Suppose that* $(\alpha, m) \in [0, 1]^2$ *and* M *is a strict upper bound of function* φ. *Then the bounded set*

$$\left[-\tfrac{2M}{\max\{m,1-m\}}, \tfrac{2M}{\max\{m,1-m\}}\right]^2 \subset \mathbb{R}^2$$

containes a compact subset, which is positive invariant and globally attractive with respect to the map (3.3).

This allows us to restrict the phase space to the above bounded domain in the analysis of the long-time behaviour.

We concentrate on the case $\varphi(x) \equiv \tanh(x)$ in the sequel, thus we consider the map

$$F \colon \mathbb{R}^2 \to \mathbb{R}^2, \; F(x, y) = F_{\alpha,m}(x, y) = (y, my - \alpha \tanh(x)). \tag{3.5}$$

We shall use notation F^k for the kth iterate of the map F. It can be easily seen that global asymptotic stability of the fixed point may only hold if

$$(\alpha, m) \in \mathcal{R}(m) = [|m| - 1, 1] \times [-1, 1] \setminus \{(0, -1), (0, 1)\}.$$

We show that this condition is also sufficient. As the tanh function is odd, one may assume that $m \geq 0$. General results of Nenya and Nenya et al. [15, 16] guarantee that the trivial fixed point of the map (3.5) is globally asymptotically stable if $\alpha \leq \frac{m^2+1}{|m|+1}$ and $(\alpha, m) \in \mathcal{R}(m)$. To our knowledge, that is the best known result so far for the global stability of the map (3.5). The following theorem – which is one of the main results of the chapter – gives necessary and sufficient conditions on the global asymptotic stability of the trivial equilibrium of the map (3.5).

Theorem 3.11 (Bartha, Garab [1]). *The* $(0, 0)$ *fixed point of* (3.5) *is globally asymptotically stable if and only if* $(\alpha, m) \in \mathcal{R}(m)$.

In order to prove the theorem, first we have to construct a neighbourhood belonging to the basin of attraction of the fixed point.

Theorem 3.12 (Bartha, Garab [1]). *Let* $\alpha \in \left[\frac{1}{2}, 1\right)$*,* $m \in [0,1]$*, and*

$$\varepsilon(\alpha) = \sqrt[4]{\tfrac{27}{800}}\sqrt{1-\sqrt{\alpha}}.$$

If $(x,y) \in \big(-\varepsilon(\alpha), \varepsilon(\alpha)\big)^2$*, then* $\lim_{k\to\infty} F^k(x,y) = (0,0)$ *holds.*

The above theorem can be proved by analysing the linearised map. As α tends to the critical value $\alpha = 1$, the size of the neighbourhood given in the theorem converges to zero, hence if α is near to 1, then one cannot show by validated interval arithmetical tools that every trajectory enters the obtained neighbourhood. Therefore we need a different approach. The next theorem gives a neighbourhood which is in the basin of attraction of the fixed point and whose size is independent of the parameters.

Theorem 3.13 (Bartha, Garab [1]). *Let* $\alpha \in [0.98, 1]$*,* $m \in [0,1]$*, and*

$$\varepsilon(\alpha) = \frac{1}{6},$$

If $(x,y) \in \big(-\varepsilon(\alpha), \varepsilon(\alpha)\big)^2$*, then* $\lim_{k\to\infty} F^k(x,y) = (0,0)$ *holds.*

At $(\alpha, m) = (1,0)$ a strong 1:4 resonance occurs, hence the proof of the above theorem is carried out by analysing the resonant normal form of the Neimark–Sacker bifurcation. Throughout the proof, one has to give uniform estimates on the coefficients of the normal form and on the higher order error terms, too.

Now, let $\varepsilon(\alpha)$ be defined as in Theorem 3.12 if $\alpha \in \left[\frac{1}{2}, 0.98\right)$ and let it be $\frac{1}{6}$ (as in Theorem 3.13) for $\alpha \in [0.98, 1]$. It can be proved by using validated computer aided tools that if $(\alpha, m) \in \left[\frac{1}{2}, 1\right] \times [-1, 1]$ and $(x,y) \in [-4,4]^2$, then there exists an integer $k \geq 0$, such that $F^k(x,y) \in (-\varepsilon(\alpha), \varepsilon(\alpha))^2$ holds. Combining this with Corollary 3.8, Theorems 3.12 and 3.13 and with the results of Nenya et al. completes to proof of Theorem 3.11.

A Ricker type population dynamical model

In 1976, Levin and May formulated a conjecture on a class of delayed difference equations involving $x_{n+1} = x_n e^{\alpha - x_{n-d}}$, that the local asymptotic stability of the positive equilibrium implies its global attractivity [12]. We are interested in the case of $d = 1$. Accordingly, let us consider the following two-dimensional map:

$$F_\alpha \colon \mathbb{R}^2 \ni (x,y) \mapsto (y, ye^{\alpha - x}) \in \mathbb{R}^2,$$

where $\alpha > 0$ is a parameter. We use notation F_α^k for the kth iterate of F_α. The map F_α has two fixed points: $(0,0)$ and (α,α). According to the conjecture, the (α,α) fixed point is globally asymptotically stable (in the sense that $\mathbb{R}^2_+$ is in its basin of attraction) for all $\alpha \in (0,1]$. The best result so far in the topic follows from the general theorem of Tkachenko and Trofimchuk [19] and states that if $\alpha \in (0, 0.875)$, then the positive equilibrium is globally asymptotically stable. The following theorem is one of the main results of the thesis, in which we prove the conjecture in the case of $d = 1$.

Theorem 3.15 (Bartha, Garab, Krisztin [2]). *If $0 < \alpha \leq 1$, then the (α, α) fixed point of the map F_α is locally asymptotically stable and $F_\alpha^n(x, y) \to (\alpha, \alpha)$ for all $(x, y) \in \mathbb{R}_+^2$, as $n \to \infty$.*

The proof is similar to what we have seen in the previous part. First we construct for all parameter values $\alpha \in [0.875, 1]$ a compact, positive invariant $S(\alpha)$ set, such that for all $(x, y) \in \mathbb{R}_+^2$ there exists $k \in \mathbb{N}$, so that $F^k(x, y) \in S(\alpha)$ holds. The next step is to give a neighbourhood of the fixed point using one of the following theorems (according to the value of α), which belongs to the basin of attraction of the equilibrium point.

Theorem 3.18 (Bartha, Garab, Krisztin [2]). *The set*

$$\left\{(x, y) \in \mathbb{R}^2 : |x - \alpha| < \frac{1}{37}, |y - \alpha| < \frac{1}{37}\right\}$$

belongs to the basin of attraction of the (α, α) fixed point of the map F_α for all $\alpha \in [0.875, 1]$.

Theorem 3.19 (Bartha, Garab, Krisztin [2]). *The set*

$$\left\{(x, y) \in \mathbb{R}^2 : |x - \alpha| < \frac{1}{22}, |y - \alpha| < \frac{1}{22}\right\}$$

belongs to the basin of attraction of the (α, α) fixed point of the map F_α for all $\alpha \in [0.999, 1]$.

As there is no resonance in this case, the above theorems are proved by estimating the non-resonant normal form of the Neimark–Sacker bifurcation. It is natural to ask why it is not enough to prove Theorem 3.18. That is because at and near the critical value $\alpha = 1$ the convergence is so slow that if we intend to prove that every trajectory enters into the neighbourhood $(\alpha - 1/37, \alpha + 1/37)^2$, then the 128 GB memory capacity of the computer cluster would not be enough to store the necessary graph representation of the map.

We complete the proof of Theorem 3.15 by showing by validated computer aided tools that for all $\alpha \in [0.875, 0.999]$, resp. $\alpha \in [0.999, 1]$, and $(x, y) \in S(\alpha)$ there exists an integer $k \geq 0$, such that $F_\alpha^k(x, y)$ is in the neighbourhood defined in Theorem 3.18, resp. Theorem 3.19.

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[1] F. Bartha and Á. Garab. Necessary and sufficient condition for the global stability of a delayed discrete-time single neuron model. *J. Comput. Dyn.*, accepted.

[2] F. A. Bartha, Á. Garab, and T. Krisztin. Local stability implies global stability for the 2-dimensional ricker map. *J. Difference Equ. Appl.*, 19(12), 2043–2078, 2013.

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