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UNIQUE PERIODIC ORBITS OF A DELAY DIFFERENTIAL EQUATION WITH PIECEWISE LINEAR FEEDBACK FUNCTION

Ábel Garab

Bolyai Institute, University of Szeged
Aradi vértanúk tere 1.
Szeged, H-6720, Hungary

Abstract. In this paper we study the scalar delay differential equation $\dot{x}(t) = -ax(t) + bf(x(t-\tau))$ with feedback function $f(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|)$ and with real parameters $a > 0$, $\tau > 0$ and $b \neq 0$, which can model a single neuron or a group of synchronized neurons. We give necessary and sufficient conditions for existence and uniqueness of periodic orbits with prescribed oscillation frequencies. We also investigate the period of the slowly oscillating periodic solution as a function of the delay. Based on the obtained results we state an analogous theorem concerning existence and uniqueness of periodic orbits of a certain type of system of delay differential equations. The proofs are based among others on theory of monotone systems and discrete Lyapunov functionals.

1. **Introduction.**

Consider the following delay differential equation

$$\dot{x}(t) = -ax(t) + bf(x(t-\tau)), \tag{1}$$

where $a > 0,\ \tau > 0,\ b \neq 0$ and the feedback function is defined by

$$f(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|) = \begin{cases} -1 & \text{if } \xi \leq -1, \\ \xi & \text{if } -1 < \xi < 1, \\ 1 & \text{if } 1 \leq \xi. \end{cases}$$

These hypotheses are assumed throughout the paper. When $b > 0$ ($b < 0$), the feedback is said to be positive (negative). We also investigate the unidirectional ring of delay differential equations of the following form

$$\begin{aligned} \dot{x}^0(t) &= -ax^0(t) + bf(x^1(t)), \\ &\vdots \\ \dot{x}^{n-1}(t) &= -ax^{n-1}(t) + bf(x^n(t)), \\ \dot{x}^n(t) &= -ax^n(t) + \delta bf(x^0(t-1)), \end{aligned} \tag{2}$$

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where $a > 0$, $b > 0$, $n \geq 1$, $\delta \in \{-1, 1\}$ and f is the same as in equation (1). The feedback is positive / negative if $\delta = 1/-1$, respectively.

These equations are often applied in models of neural networks, especially in cellular neural networks (CNN). In (1) x represents the electric potential of a self-excited neuron or the average potential of a group of synchronized neurons and time delay appears due to finite conduction velocities or synaptic transmission and due to finite switching speed of amplifiers in CNNs. System (2) is a model of a unidirectional ring of interconnecting neurons. In neural systems, periodic solutions are of great importance. For a general overview of neural networks we refer the reader to [16]. From the papers of Cao, Krisztin and Walther [1, 8, 9, 11] and from the monograph of Krisztin, Walther and Wu [12] we get a very detailed and clear picture of the periodic orbits of equation (1) in the case when f is a smooth feedback function satisfying some convexity properties. The results in [1, 8, 9, 11] and [12] use smoothness of f and the assumption that $f'(\xi) \neq 0$ for all $\xi \in \mathbb{R}$.

The purpose of this present paper is to give necessary and sufficient conditions for existence and uniqueness of periodic orbits of (1) and (2) with prescribed oscillation frequencies when the feedback function is the above defined piecewise linear function. The global attractor of equation (1) is described by Krisztin, Walther and Wu when the feedback function is from a special class of sigmoid functions. In this case, existence and uniqueness of periodic orbits in certain oscillation frequency regions is also known. It seems reasonable to approximate our piecewise linear feedback function with functions from the aforementioned class, but the problem is that the global attractor is only upper semicontinuous, hence this approach cannot provide uniqueness results on periodic orbits of (1). Another technical difficulty is that our feedback function f is neither smooth, nor strictly monotone, therefore the solution operator is neither differentiable everywhere nor injective. For this reason the Poincaré-Bendixson-type theorem of Mallet-Paret and Sell [14] cannot be applied directly in this case.

Győri and Hartung [6] showed that if $0 < b < a$ and $\tau = 1$, then the trivial solution is globally attractive and they conjectured that every solution of (1) is convergent for any choice of parameters $a > 0$ and $b > 0$. Vas [15] disproved their conjecture by showing that there exists $b_0 = b_0(a) > a$ such that if $b \geq b_0$, then there exists a slowly oscillating periodic solution of the equation. Let us note here that their equations were somewhat more general than (1) and now we only pay attention to what they said about our case. However, it remained an open problem whether periodic solutions exist if $a \leq b < b_0$. As a special case of Proposition 3.7 it turns out that the answer is no. We have to emphasize here that this does not prove the Győri-Hartung conjecture for (1) with parameters satisfying $b \in [a, b_0)$, since we do not have a Poincaré-Bendixson-type theorem. Therefore it remains still a challenging problem to prove (or disprove) a Poincaré-Bendixson-type theorem for equation (1) or the conjecture of Győri and Hartung in this parameter region.

The positive and the negative feedback cases can be treated similarly both in equation (1) and in system (2), thus we are going to focus on the positive feedback case. The difference only occurs in concrete values of the bifurcation points and of the discrete Lyapunov functional which is going to be defined in the next section. In our main theorems (Theorem 4.3 and Theorem 5.1) we summarize the results for both negative and positive feedback.

2. **Preliminaries.** Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of reals and positive integers, respectively. $C_\tau := C([-\tau, 0], \mathbb{R})$ denotes the set of real-valued functions with domain $[-\tau, 0]$. For a simple closed curve $c : [s_0, s_1] \to \mathbb{R}^2$, $\operatorname{int}(c), \operatorname{ext}(c)$ and $|c|$ denote the interior, exterior and the trace of c, respectively.

For our purposes, consideration of the following linear delay differential equation is also essential

$$\dot{x}(t) = -ax(t) + bx(t - \tau), \tag{3}$$

where the same assumptions are made about a, b and τ.

The natural phase space for (1) and (3) is C_τ equipped with the maximum norm. We say that a continuous function $x : [t_0 - \tau, \infty) \to \mathbb{R}$ is a solution of equation (1) or (3) if it is differentiable on (t_0, ∞) and it satisfies (1) or (3), respectively. Similarly, $x : \mathbb{R} \to \mathbb{R}$ is a solution of (1) or (3) if it is differentiable and satisfies everywhere (1) or (3), respectively. If $x(\cdot)$ is a real function on some interval I with $t,\ t - \tau \in I$, then we let $x_t \in C_\tau$ be defined by

$$x_t(\theta) = x(t + \theta) \quad \text{for all} \quad \theta \in [-\tau, 0].$$

From the method of steps, it is clear that every $\varphi \in C_\tau$ uniquely determines a solution $x^\varphi : [-\tau, \infty) \to \mathbb{R}$ of equation (1) and a solution $y^\varphi : [-\tau, \infty) \to \mathbb{R}$ of equation (3) such that $x_0^\varphi = y_0^\varphi = \varphi$. Now we are going to define a discrete Lyapunov functional developed by Mallet-Paret and Sell [13].

$$V_\tau^+ : C_\tau \setminus \{0\} \to \{0, 2, 4, \dots, \infty\}, \quad V_\tau^- : C_\tau \setminus \{0\} \to \{1, 3, 5, \dots, \infty\},$$

$$V_\tau^+(\varphi) = \begin{cases} \operatorname{sc}(\varphi) & \text{if} \quad \operatorname{sc}(\varphi) \quad \text{is even or infinite,} \\ \operatorname{sc}(\varphi) + 1 & \text{if} \quad \operatorname{sc}(\varphi) \quad \text{is odd,} \end{cases}$$

$$V_\tau^-(\varphi) = \begin{cases} \operatorname{sc}(\varphi) & \text{if} \quad \operatorname{sc}(\varphi) \quad \text{is odd or infinite,} \\ \operatorname{sc}(\varphi) + 1 & \text{if} \quad \operatorname{sc}(\varphi) \quad \text{is even,} \end{cases}$$

where the sign change functional $\operatorname{sc} : C_\tau \setminus \{0\} \to \mathbb{N} \cup \{0\}$ is defined as follows:

$$\begin{aligned} \operatorname{sc}(\varphi) = \quad & \sup\{k \in \mathbb{N} : \text{ There is a strictly increasing finite sequence } \{s^i\}_0^k \subset [-\tau, 0] \\ & \text{with } \varphi(s^{i-1})\varphi(s^i) < 0 \text{ for all } i \in \{1, 2, \dots, k\}\} \leq \infty. \end{aligned}$$

By definition, let $\sup \emptyset = 0$. According to [13] the following proposition holds.

Proposition 2.1. *Let $x : [t_0 - \tau, \infty) \to \mathbb{R}$ be a solution of equation (1) or (3). Let V denote V_τ^+ in case of positive feedback and V_τ^- in the negative feedback case. Then the following statement hold.*

(i) $V(x_t)$ is a nonincreasing function of $t > t_0$ for as long as $x_t \in C_\tau$ is not the zero function.

(ii) If $t_1 \geq t_0 + 3\tau$ is such that $x_{t_1} \in C_\tau$ is not the zero element and $x(t_1) = x(t_1 - \tau) = 0$ then $V(x_{t_1}) < V(x_{t_0})$ or else $V(x_{t_1}) = \infty$.

(iii) If $x : \mathbb{R} \to \mathbb{R}$ is a periodic solution of equation (1) or (3), then $t \mapsto V(x_t)$ is constant and finite for all $t \in \mathbb{R}$.

According to the last statement, if x is a periodic solution of equation (1) or (3), then we may omit subscript t and write e.g. $V_\tau^+(x) = 2$ instead of $V_\tau^+(x_t) = 2$ for all $t \in \mathbb{R}$. From the method of steps it is obvious that if $x : [t_0 - \tau, \infty) \to \mathbb{R}$ is a solution of equation (1) or (3) and $x_{t_0} \neq 0 \in C_\tau$, then $x_t \neq 0$ for all $t > t_0$. Let

$$\tau_k^+ = \frac{2k\pi - \arccos\frac{a}{b}}{\sqrt{b^2 - a^2}} \quad \text{and} \quad \tau_k^- = \frac{(2k-1)\pi - \arccos\frac{a}{|b|}}{\sqrt{b^2 - a^2}}.$$

The following proposition can be obtained by consideration of the characteristic equation and from classical results on linear functional differential equations [7].

Proposition 2.2.

(i) *If $b > 0$, then x is a nonconstant periodic solution of (3) if and only if $\tau = \tau_k^+$ for some $k \in \mathbb{N}$ and there exists $\zeta_k \in ((2k-1)\pi/\tau, 2k\pi/\tau)$ and $A > 0,\ d > 0$ such that $x(t) = A\cos(\zeta_k t + d)$. In this case $V_\tau^+(x) = 2k$ follows.*

(ii) *If $b < 0$, then x is a nonconstant periodic solution of (3) if and only if $\tau = \tau_k^-$ for some $k \in \mathbb{N}$ and there exists $\zeta_k \in ((2k-2)\pi/\tau, (2k-1)\pi/\tau)$ and $A > 0,\ d > 0$ such that $x(t) = A\cos(\zeta_k t + d)$. In this case $V_\tau^-(x) = 2k-1$ follows.*

3. **Periodic orbits, nonexistence and uniqueness results.** First of all we note that if $|b| < a$, then according to Gopalsamy and He [5], the trivial solution 0 is globally asymptotically stable regardless of the value of τ. This remains true if $0 < b = a$, according to [15], hence there cannot exist any nonconstant periodic solution of equation (1) in these cases. It is easy to see that there exists no periodic solution of (1) when $0 > b = -a$, either. Thus from this point we shall restrict our attention only to the case when $a < |b|$ holds.

Proposition 3.1. *If $x : [t_0 - \tau, \infty) \to \mathbb{R}$ is a solution of (1) for which there exists $t_1 \in [t_0, \infty)$ such that $x(t_1) = 0$ then $x(t) \in (-\frac{|b|}{a}, \frac{|b|}{a})$ for all $t \geq t_1$.*

Proof. First note that $-ax(t) - |b| \leq \dot{x}(t) = -ax(t) + bf(x(t-\tau)) \leq -ax(t) + |b|$. Now, let $u_\pm(t)$ be the unique solutions of the following initial value problems, respectively

$$\begin{aligned} \dot{y}(t) &= -ay(t) \pm |b|, \\ y(t_1) &= 0. \end{aligned}$$

This yields $u_-(t) \leq x(t) \leq u_+(t)$ for all $t \geq t_1$. By simple calculation we obtain

$$u_-(t) = \frac{|b|}{a} e^{a(t_1 - t)} - \frac{|b|}{a} > -\frac{|b|}{a}$$

and

$$u_+(t) = -\frac{|b|}{a} e^{a(t_1 - t)} + \frac{|b|}{a} < \frac{|b|}{a}$$

for all $t > t_1$, which proves our claim. □

Proposition 3.2. *If $x : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of (1), then for all $t_0 \in \mathbb{R}$, x has a sign change on the half-line (t_0, ∞).*

Proof. We consider the positive and negative feedback cases separately.

1. If the feedback is negative, then, by way of contradiction, let us assume that x is a nonconstant periodic solution of (1) and there exists $t_0 \in \mathbb{R}$ such that $x(t_0) > 0$ and $x(t) \geq 0$ for all $t \geq t_0$ (the other case is similar). Now, $\dot{x}(t) \leq -ax(t)$ holds for all $t \geq t_0 + \tau$ and $0 \leq x(t) \leq x(t_0)e^{-a(t-t_0)}$ follows for all $t \geq t_0 + \tau$ which is in contradiction with the periodicity of x.

2. If the feedback is positive, then let us assume that x is a periodic solution of equation (1) with minimal period $T_x > 0$ and that t_0 is such that $x(t) \geq 0$ for $t \geq t_0$. Then from periodicity of x we get that $x(t) \geq 0$ for all $t \in \mathbb{R}$. The continuity of x guarantees that there exists $t_{min} \in [0, T_x]$ such that

$$x(t_{min}) = \min_{t \in [0, T_x]} x(t).$$

If $x(t_{min}) \geq 1$, then (1) can be written in the form

$$\dot{x}(t) = -ax(t) + b \quad \text{for all } t \in \mathbb{R},$$

but this equation has no periodic solution, so $x(t_{min}) \geq 1$ cannot occur. Equation (1) and the definition of t_{min} yield

$$0 = \dot{x}(t_{min}) = -ax(t_{min}) + bf(x(t_{min} - \tau)). \tag{4}$$

Now, if $x(t_{min}) < 1$, then from $a < b$ we get $x(t_{min}) < \frac{b}{a}$, from which we obtain $b > bf(x(t_{min} - \tau))$. Thus $bf(x(t_{min} - \tau)) = bx(t_{min} - \tau)$ and from equation (4) one obtains

$$x(t_{min} - \tau) = \frac{a}{b} x(t_{min}).$$

This can only happen when $x(t_{min} - \tau) = x(t_{min}) = 0$, which contradicts to Proposition 2.1. $\square$

Definition 3.3. For a periodic C^1 function $x : [0, T_x) \to \mathbb{R}$ with period T_x, let X denote the *D-trajectory of* x defined by $X : [0, T_x) \to \mathbb{R}^2, t \mapsto (x(t), \dot{x}(t))$.

Proposition 3.4. *Let $x : \mathbb{R} \to \mathbb{R}$ be a periodic solution of equation (1) or (3) with minimal period $T_x > 0$, and let X denote the D-trajectory of the solution. Then the following statements hold.*

(i) X is a simple closed curve and $0 \in \operatorname{int}(X)$.
(ii) Monotonicity: There exist $t_0 \in \mathbb{R}$ and $t_1 \in (t_0, t_0 + T_x)$ such that $0 < \dot{x}(t)$ for all $t \in (t_0, t_1)$, $x(\mathbb{R}) = [x(t_0), x(t_1)]$ and $\dot{x}(t) < 0$ for all $t \in (t_1, t_0 + T_x)$.
(iii) Special symmetry: $x\left(t + \frac{T_x}{2}\right) = -x(t)$ for all $t \in \mathbb{R}$.
(iv) If the feedback is positive, then there exists $k \in \mathbb{N}$ such that $V_\tau^+(x) = 2k$.
(v) If the feedback is negative, then there exists $k \in \mathbb{N}$ such that $V_\tau^-(x) = 2k - 1$.

Proof. For nonconstant periodic solutions of (3), these statements are straightforward consequences of Proposition 2.2.

In the case when x is a nonconstant periodic solution of (1), statement (iii) is proved in [10]. They also proved a slightly weaker statement than (ii), namely that there exist $t_0 \in \mathbb{R}$ and $t_1 \in (t_0, t_0 + T_x)$ such that $0 \leq \dot{x}(t)$ for all $t \in (t_0, t_1)$, $x(\mathbb{R}) = [x(t_0), x(t_1)]$ and $\dot{x}(t) \leq 0$ for all $t \in (t_1, t_0 + T_x)$. Assertions (iv) and (v) are consequences of Proposition 2.1 and 3.2. It is now clear that in order to prove statement (i) and the strict monotonicity part of (ii), it is sufficient to prove the following proposition. $\square$

Proposition 3.5. *Let $x : \mathbb{R} \to \mathbb{R}$ be a periodic solution of (1) with minimal period $T_x > 0$. If $t_0 \in \mathbb{R}$ is such that $\dot{x}(t_0) = 0$ then $\ddot{x}(t_0)$ exists and $\ddot{x}(t_0) \neq 0$.*

Proof. Since $0 = \dot{x}(t_0) = -ax(t_0) + bf(x(t_0 - \tau))$, Proposition 3.1 yields

$$\frac{|b|}{a} > x(t_0) = \frac{b}{a} f(x(t_0 - \tau)) > -\frac{|b|}{a}$$

from which we obtain $|x(t_0 - \tau)| < 1$ and thus by using the equation above we obtain $x(t_0 - \tau) = \frac{a}{b} x(t_0)$. From continuity of x it also follows that there exists $\varepsilon > 0$ such that for any t with $|t - t_0| < \varepsilon$ equation

$$\dot{x}(t) = -ax(t) + bx(t - \tau)$$

holds. Hence $\ddot{x}(t_0)$ exists and

$$\ddot{x}(t_0) = -a\dot{x}(t_0) + b\dot{x}(t_0 - \tau) = b\dot{x}(t_0 - \tau).$$

Now, assume to the contrary that $\ddot{x}(t_0) = 0$. From the equations above we obtain that $\dot{x}(t_0 - \tau) = 0$ and $x(t_0 - \tau) = \frac{a}{b}x(t_0)$. Now, recall that part (ii) of Proposition 3.4 is proved for the case when we do not ask for *strict* monotonicity. Using this, statement (iii) of Proposition 3.4, $x(t_0 - \tau) = \frac{a}{b}x(t_0)$ and $|b| > a$ we get that x cannot have an extremum point in $t_0 - \tau$. Now, from $|x(t_0 - \tau)| < 1$ it follows that $\ddot{x}(t_0 - \tau)$ exists and is equal to 0.

It follows by induction that for all $n \in \mathbb{N}$, equations $x(t_0 - n\tau) = (\frac{a}{b})^n x(t_0)$ and $\dot{x}(t_0 - n\tau) = 0$ hold. By periodicity of x it follows that $\forall n \in \mathbb{N} : \exists t_n \in [0, T_x]$ such that $x(t_n) = x(t_0 - n\tau)$. Let $\{n_j,\ j \in \mathbb{N}\}$ such that for some $t^* \in [0, T_x]$: $\lim_{j\to\infty} t_{n_j} = t^*$. By continuity of x it follows that $x(t^*) = \lim_{j\to\infty} x(t_{n_j}) = \lim_{j\to\infty} (\frac{a}{b})^{n_j} x(t_0) = 0$ and $\dot{x}(t^*) = \lim_{j\to\infty} \dot{x}(t_{n_j}) = \lim_{j\to\infty} \dot{x}(t_0 - n_j\tau) = 0$, too, which is a contradiction to Proposition 2.1. This contradiction completes our proof. □

Remark 3.6. Győri and Hartung [6] proved that in case of $0 < b < a$ and $\tau = 1$, all solutions of equation (1) tend to an equilibrium point. Six years later, Vas [15] showed that if $b_0 = b_0(a)$ is defined by the equation $\sqrt{{b_0}^2 - a^2} + \arccos\frac{a}{b_0} = 2\pi$ and $b \geq b_0$, then there exists a periodic solution of (1) in $(V_1^+)^{-1}(2)$. It remained an open problem whether there exists a periodic function in the case of $a < b < b_0$. As a corollary of the following proposition we obtain that there exists no periodic solution in this case. We note that the following techniques apply to the case of negative feedback, too. Similarly, in Proposition 3.8 and 3.9, we restrict our attention only to the case of positive feedback, but in 4.3 we also summarize the results for both cases.

Proposition 3.7. *Suppose that $a, b, \tau > 0$ and $k \in \mathbb{N}$ are fixed in a way that $\tau\sqrt{b^2 - a^2} + \arccos\frac{a}{b} < 2k\pi$. Then there exists no periodic solution x of (1) for which $V_\tau^+(x) = 2k$.*

Proof. The proof is based on the Cao-Krisztin-Walther technique [1, 11]. As in [11], the proof is divided into two parts. The first part almost coincides with the corresponding part of the aforementioned proof, but for the sake of completeness, the argument is repeated here. The difficulties induced by non-strict monotonicity and not everywhere differentiability of f appear mainly in the second part of the proof.

Assume to the contrary that $x : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (1) satisfying $V_\tau^+(x) = 2k$ and let X denote the D-trajectory of x. Furthermore let

$$\alpha = \alpha(a, b) = \frac{2k\pi - \arccos\frac{a}{b}}{\sqrt{b^2 - a^2}} > \tau. \tag{5}$$

By Proposition 2.2 we obtain that there exists a periodic function $y : \mathbb{R} \to \mathbb{R}$ which satisfies the following delayed differential equation:

$$\dot{y}(t) = -ay(t) + by(t - \alpha). \tag{6}$$

Moreover, if $y(\cdot)$ is any fixed nonconstant periodic solution of (6), then $V_\alpha^+(y) = 2k$. Since equation (6) is linear, for any $\gamma > 0$, $\gamma y(\cdot)$ is also a nonconstant periodic solution of (6) and $V_\alpha^+(\gamma y) = 2k$. There exists a unique $\gamma > 0$ so that, denoting the D-trajectory of $z = \gamma y$ by Z, we have $|Z| \subset |X| \cup \mathrm{ext}(X)$ and $|Z| \cap |X| \neq \emptyset$. Let T_z be the minimal period of z. Using $|Z| \cap |X| \neq \emptyset$, we may assume $X(0) = Z(0)$ without loss of generality, i.e.,

$$x(0) = z(0) \quad \text{and} \quad \dot{x}(0) = \dot{z}(0).$$

Using the continuity of $x, z, \dot{x}$ and $\dot{z}$ we may assume that $x(0)$ is maximal in the following sense:

$$x(0) = \max\{x(t) : \quad t \in \mathbb{R},\ X(t) \in |Z| \text{ and } \dot{x}(t) \geq 0\}. \tag{7}$$

Proposition 3.4 implies that all zeros of x and z are simple and the distances between two consecutive zeros of x and z are $T_x/2$ and $T_z/2$, respectively. This fact combined with $V_\tau^+(x) = V_\alpha^+(z) = 2k$ yields

$$\left(k - \frac{1}{2}\right) T_x \leq \tau \leq kT_x \quad \text{and} \quad \left(k - \frac{1}{2}\right) T_z \leq \alpha \leq kT_z.$$

Using statement (ii) of Proposition 2.1 we infer

$$\left(k - \frac{1}{2}\right) T_x < \tau < kT_x \quad \text{and} \quad \left(k - \frac{1}{2}\right) T_z < \alpha < kT_z. \tag{8}$$

Now, we distinguish two cases.

Case 1: $\dot{x}(0) = \dot{z}(0) = 0$.

We may assume that $x(0) = z(0) = d > 0$. Since properties (ii) and (iii) of Proposition 3.4 hold for x and z, we obtain that

$$d = \max_{t\in\mathbb{R}} x(t) = \max_{t\in\mathbb{R}} z(t), \quad -d = \min_{t\in\mathbb{R}} x(t) = \min_{t\in\mathbb{R}} z(t),$$

$$\dot{x}(t) > 0 \quad \text{for } -\frac{T_x}{2} < t < 0, \quad \dot{z}(t) > 0 \quad \text{for } -\frac{T_z}{2} < t < 0,$$

$$x(-T_x/2) = -d, \quad \dot{x}(-T_x/2) = 0, \quad z(-T_z/2) = -d, \quad \dot{z}(-T_z/2) = 0.$$

Let $T^* = \min\{T_x, T_z\}$.

Claim: $z(s) \leq x(s)$ for $-T^*/2 \leq s \leq 0$. Proof of the claim: Let x^{-1} and z^{-1} denote the inverses of the functions

$$[-T_x/2, 0] \ni t \mapsto x(t) \in \mathbb{R} \quad \text{and} \quad [-T_z/2, 0] \ni t \mapsto z(t) \in \mathbb{R},$$

respectively. Then the domain of x^{-1} and z^{-1} is $[-d, d]$. Define the following two functions:

$$\phi_x : [-d, d] \ni u \mapsto \dot{x}(x^{-1}(u)) \in \mathbb{R} \quad \text{and} \quad \phi_z : [-d, d] \ni u \mapsto \dot{z}(z^{-1}(u)) \in \mathbb{R}.$$

The arcs

$$\Omega_x = \{X(t) : t \in [-T_x/2, 0]\} \quad \text{and} \quad \Omega_z = \{Z(t) : t \in [-T_z/2, 0]\},$$

coincide with the graphs

$$\{(u, \phi_x(u)) : u \in [-d, d]\} \quad \text{and} \quad \{(u, \phi_z(u)) : u \in [-d, d]\},$$

respectively. From the special symmetry of x and z we obtain

$$|X| = \Omega_x \cup (-\Omega_x) \quad \text{and} \quad |Z| = \Omega_z \cup (-\Omega_z).$$

Hence

$$\mathrm{int}(X) = \{(u, v) : u \in (-d, d),\ -\phi_x(-u) < v < \phi_x(u)\}.$$

From $|Z| \subset |X| \cup \mathrm{ext}(X)$ we conclude

$$\phi_x(u) \leq \phi_z(u) \quad \text{for } -d \leq u \leq d.$$

The functions x and z satisfy

$$\dot{x}(t) = \phi_x(x(t)) \quad \text{for all } t \in [-T_x/2, 0]$$

and

$$\dot{z}(t) = \phi_z(z(t)) \quad \text{for all } t \in [-T_z/2, 0].$$

For $-T_z/2 < s_1 < s_2 < 0$ the last equation and the inequality $\dot{z}(t) > 0$ for $-T_z/2 < t < 0$ combined yield

$$\int_{z(s_1)}^{z(s_2)} \frac{\mathrm{d}u}{\phi_z(u)} = \int_{s_1}^{s_2} \frac{\dot{z}(t)}{\phi_z(z(t))} \mathrm{d}t = s_2 - s_1.$$

Also,

$$\int_{x(s_1)}^{x(s_2)} \frac{\mathrm{d}u}{\phi_z(u)} = s_2 - s_1 \quad \text{for } -\frac{T_x}{2} < s_1 < s_2 < 0.$$

The continuity of z and x at 0 yields

$$\int_{z(s)}^{d} \frac{\mathrm{d}u}{\phi_z(u)} = -s \quad \text{for } -\frac{T_z}{2} < s \le 0$$

and

$$\int_{x(s)}^{d} \frac{\mathrm{d}u}{\phi_x(u)} = -s \quad \text{for } -\frac{T_x}{2} < s \le 0.$$

We obtain immediately that for $-T^*/2 < s \le 0$

$$\int_{z(s)}^{d} \frac{\mathrm{d}u}{\phi_z(u)} = \int_{x(s)}^{d} \frac{\mathrm{d}u}{\phi_x(u)}$$

holds and using $0 < \phi_x(u) \le \phi_z(u)$ on $(-d, d)$ we infer

$$z(s) \le x(s) \quad \text{for } -\frac{T^*}{2} < s \le 0.$$

Using continuity we complete the proof of the claim.

If $T_z > T_x$ then from the claim above and from $x(-T_x/2) = -d$ we obtain $z(-T_x/2) \le x(-T_x/2) = -d$. This is impossible since $-T_z/2 < -T_x/2 < 0$, $z(-T_z/2) = -d$ and $\dot{z}(t) > 0$ for $-T_z/2 < t < 0$. So

$$T_z \le T_x.$$

Combining this with $\alpha > \tau$ and (8) we get

$$\left(k - \frac{1}{2}\right) T_z \le \left(k - \frac{1}{2}\right) T_x < \tau < \alpha < kT_z \le kT_x \tag{9}$$

and hence

$$-kT_z < -\alpha < -\tau < -\left(k - \frac{1}{2}\right) T_z. \tag{10}$$

Using $\dot{z}(t) > 0$ for $-T_z/2 < t < 0$, the periodicity and the special symmetry of z we conclude that

$$\dot{z}(t) < 0 \quad \text{for } -kT_z < t < -\left(k - \frac{1}{2}\right) T_z.$$

This inequality combined with (10) implies

$$z(-\alpha) > z(-\tau).$$

Again, from the periodicity and the special symmetry of z we find

$$z(-\tau) = z(-\tau + kT_z) = -z\left(-\tau + \left(k - \frac{1}{2}\right) T_z\right).$$

From (10) we obtain

$$-\frac{T_z}{2} < -\tau + \left(k - \frac{1}{2}\right) T_z < 0.$$

These inequalities, our claim above and $T^* = T_z$ combined yield

$$z\left(-\tau + \left(k - \frac{1}{2}\right) T_z\right) \leq x\left(-\tau + \left(k - \frac{1}{2}\right) T_z\right).$$

Using that x is increasing on $[-T_x/2, 0]$ and the consequence

$$-\frac{T_x}{2} \leq -\frac{T_z}{2} < -\tau + \left(k - \frac{1}{2}\right) T_z \leq -\tau + \left(k - \frac{1}{2}\right) T_x < 0$$

of the inequalities $T_z \leq T_x$ and (9), we infer

$$x\left(-\tau + \left(k - \frac{1}{2}\right) T_z\right) \leq x\left(-\tau + \left(k - \frac{1}{2}\right) T_x\right).$$

By periodicity and the special symmetry of x,

$$x\left(-\tau + \left(k - \frac{1}{2}\right) T_x\right) = -x(-\tau + kT_x) = -x(-\tau)$$

holds. Consequently,

$$\begin{aligned} z(-\alpha) > z(-\tau) = &- z\left(-\tau + \left(k - \frac{1}{2}\right) T_z\right) \geq -x\left(-\tau + \left(k - \frac{1}{2}\right) T_z\right) \\ &\geq - x\left(-\tau + \left(k - \frac{1}{2}\right) T_x\right) = x(-\tau). \end{aligned} \tag{11}$$

Using equations (1) and (6) and $\dot{x}(0) = \dot{z}(0) = 0$, $x(0) = z(0) = d > 0$ we obtain

$$z(-\alpha) > 0, \quad x(-\tau) > 0$$

and

$$bz(-\alpha) = bf(x(-\tau)) \leq bx(-\tau).$$

Hence we obtain $z(-\alpha) \leq x(-\tau)$, a contradiction to (11).

Case 2: $\dot{z}(0) = \dot{x}(0) \neq 0$.

Let $\beta = x(0) = z(0)$. Then there exists $\varepsilon > 0$ such that $\dot{x}(t) > 0$ and $\dot{z}(t) > 0$ for all $t \in (-\varepsilon, \varepsilon)$. Then there exists a $\delta > 0$ so that there are inverses

$$x^{-1} : (\beta - \delta, \beta + \delta) \to \mathbb{R}, \quad z^{-1} : (\beta - \delta, \beta + \delta) \to \mathbb{R}$$

of restrictions of x and z to open intervals in $(-\varepsilon, \varepsilon)$, respectively. Define the following maps

$$\begin{aligned} \eta_x &: (\beta - \delta, \beta + \delta) \ni u \mapsto \dot{x}(x^{-1}(u)) \in \mathbb{R}, \\ \eta_z &: (\beta - \delta, \beta + \delta) \ni u \mapsto \dot{z}(z^{-1}(u)) \in \mathbb{R}. \end{aligned}$$

Since z is a solution of (6) it follows that z is C^2-smooth and we have

$$\eta_z{}'(u) = \ddot{z}(z^{-1}(u)) \frac{\mathrm{d}}{\mathrm{d}u} z^{-1}(u) = \frac{\ddot{z}(z^{-1}(u))}{\dot{z}(z^{-1}(u))} \quad \text{for all } u \in (\beta - \delta, \beta + \delta).$$

In particular,

$$\eta_z{}'(\beta) = \frac{\ddot{z}(0)}{\dot{z}(0)}. \tag{12}$$

Now, according to the value of $x(-\tau)$ we distinguish two further cases:

Case 2.1: $|x(-\tau)| \neq 1$.

If $x(-\tau) > 1$ or $x(-\tau) < -1$ then by the continuity of x we may choose ε and δ so that $x(t-\tau) > 1$ or $x(t-\tau) < -1$ for all $t \in (-\varepsilon, \varepsilon)$, respectively. Similarly if $|x(-\tau)| < 1$ then let ε and δ be such that $|x(t-\tau)| < 1$ for all $t \in (-\varepsilon, \varepsilon)$.

Therefore from (1) it follows that in both cases $\ddot{x}(x^{-1}(u))$ exists and

$$\eta_x'(u) = \frac{\ddot{x}(x^{-1}(u))}{\dot{x}(x^{-1}(u))} \quad \text{for all } u \in (\beta - \delta, \beta + \delta).$$

In particular we have

$$\eta_x'(\beta) = \frac{\ddot{x}(0)}{\dot{x}(0)}. \tag{13}$$

The sets

$$\{(u, \eta_x(u)) : u \in (\beta - \delta, \beta + \delta)\}, \quad \{(u, \eta_z(u)) : u \in (\beta - \delta, \beta + \delta)\}$$

are graph representations of pieces of $|X|$ and $|Z|$, respectively. Since $|Z| \subset |X| \cup \operatorname{ext}(X)$ and $(\beta, \eta_x(\beta)) = (\beta, \eta_z(\beta))$, we infer

$$\eta_x'(\beta) = \eta_z'(\beta).$$

This fact combined with $\dot{x}(0) = \dot{z}(0)$ and with equations (12) and (13) implies that

$$\ddot{x}(0) = \ddot{z}(0).$$

Differentiation of equations (1) and (6) at $t \in (-\varepsilon, \varepsilon)$ yields

$$\begin{aligned} \ddot{x}(t) &= -a\dot{x}(t) + bf'(x(t-\tau))\dot{x}(t-\tau), \\ \ddot{z}(t) &= -a\dot{z}(t) + b\dot{z}(t-\alpha). \end{aligned} \tag{14}$$

From $\ddot{x}(0) = \ddot{z}(0)$ and $\dot{x}(0) = \dot{z}(0)$ we obtain

$$\dot{z}(-\alpha) = f'(x(-\tau))\dot{x}(-\tau). \tag{15}$$

Using equations (1) and (6) and that $X(0) = Z(0)$ we obtain

$$f(x(-\tau)) = z(-\alpha). \tag{16}$$

Assume $|x(-\tau)| > 1$. From equations (15) and (16) we have $\dot{z}(-\alpha) = 0$ and $|z(-\alpha)| = 1$. Using the strict monotone property of z and the special symmetry of x and z we infer that

$$\max_{t \in (0, T_z]} z(t) = |z(-\alpha)| = 1 < |x(-\tau)| \leq \max_{t \in (0, T_x]} x(t),$$

which contradicts to $|Z| \subset |X| \cup \operatorname{ext}(X)$.

Thus we may assume that $|x(-\tau)| < 1$. Equation (15) yields $\dot{z}(-\alpha) = \dot{x}(-\tau)$ and equation (16) yields $z(-\alpha) = x(-\tau)$, hence $X(-\tau) = Z(-\alpha)$. From (8) we obtain that

$$X(-\tau) \in \{(x(t), \dot{x}(t)) : t \in (0, T_x/2)\}.$$

The choice of the point $X(0) = Z(0)$ and the strict monotone property of x guarantee that $\dot{x}(-\tau) = \dot{z}(-\alpha) < 0$ holds.

We claim that from $X(0) = Z(0)$, $x(-\tau) = z(-\alpha)$, $\dot{x}(-\tau) = \dot{z}(-\alpha) < 0$ and $|Z| \subset |X| \cup \operatorname{ext}(X)$, it follows that there exists $t^* \in (0, \varepsilon)$ such that

$$x(t^*) < z(t^*), \quad x(t^* - \tau) \geq z(t^* - \alpha) \quad \text{and} \quad \dot{x}(t^*) < \dot{z}(t^*).$$

Proof of the claim: for an arbitrary fixed $\delta_0 \in (0, \delta)$ let $t_x(\delta_0) = x^{-1}(\beta + \delta_0)$ and $t_z(\delta_0) = z^{-1}(\beta + \delta_0)$, where x^{-1} and z^{-1} are restricted to the interval $(\beta - \delta, \beta + \delta)$ (recall that δ has been chosen earlier in the proof). A similar argument as in Case 1 shows that

$$\int_\beta^{\beta + \delta_0} \frac{\mathrm{d}u}{\eta_x(u)} = t_x(\delta_0) \quad \text{and} \quad \int_\beta^{\beta + \delta_0} \frac{\mathrm{d}u}{\eta_z(u)} = t_z(\delta_0).$$

The choice of the intersection point $X(0) = Z(0)$ implies

$$\eta_x(u) < \eta_z(u) \quad \text{for all } u \in (\beta, \beta + \delta_0).$$

Combining this with the above equations we obtain $t_x(\delta_0) > t_z(\delta_0)$ and hence the strict monotone property of x yields $x(t_z(\delta_0)) < x(t_x(\delta_0)) = \delta_0 = z(t_z(\delta_0))$. Let $t_0 = t_z(\delta/2)$. Since δ_0 was arbitrary and t_z is continuous, hence $x(t) < z(t)$ holds for all $t \in (0, t_0)$. Now, define $u, v : \mathbb{R} \to \mathbb{R}$ by $u(t) = -x(t-\tau)$ and $v(t) = -z(t-\alpha)$, respectively. Using that f is odd we obtain that u and v are periodic solutions of (1) and (6), respectively and $u(0) = v(0)$, $\dot{u}(0) = \dot{v}(0) > 0$. According to the argument above there exists $t_1 < t_0$ such that $u(t) \leq v(t)$ for all $t \in (0, t_1)$ or equivalently $x(t-\tau) \geq z(t-\alpha)$. Note that here we cannot guarantee strict inequality, because $u(0) = v(0)$ may not possess the maximal property analogous to (7). To complete the proof of our claim let us assume to the contrary that for all $s \in (0, t_1)$: $\dot{x}(s) \geq \dot{z}(s)$. This implies that for all $t \in (0, t_1)$ we have $x(t) = x(0) + \int_0^t \dot{x}(s)\mathrm{d}s \geq z(0) + \int_0^t \dot{z}(s)\mathrm{d}s = z(t)$ which is a contradiction. So the claim is proved, thus we may choose $t^* \in (0, \varepsilon)$ so that

$$x(t^*) < z(t^*), \quad x(t^* - \tau) \geq z(t^* - \alpha) \quad \text{and} \quad \dot{x}(t^*) < \dot{z}(t^*).$$

On the other hand, considering equations (1) and (6) at time t^* yields

$$\begin{aligned}
\dot{x}(t^*) &= -ax(t^*) + bf(x(t^* - \tau)) \\
&= -ax(t^*) + bx(t^* - \tau) \\
&> -az(t^*) + bz(t^* - \alpha) \\
&= \dot{z}(t^*),
\end{aligned}$$

which contradicts to $\dot{x}(t^*) < \dot{z}(t^*)$, therefore Case 2.1 cannot occur.

Case 2.2: $|x(-\tau)| = 1$.

First note that from equations (1) and (6) and from assumption $X(0) = Z(0)$ we obtain

$$x(-\tau) = z(-\alpha) = 1 \quad \text{or} \quad x(-\tau) = z(-\alpha) = -1. \tag{17}$$

Now, recall that from (5) and Proposition 2.2 it follows that equation

$$\dot{y}(t) = -ay(t) + by(t - \tau) \tag{18}$$

has no periodic solution $u(\cdot)$ for which $V_\tau^+(u) = 2k$. This implies that if x is a periodic solution of equation (1) with $V_\tau^+(x) = 2k$ then

$$\max_{t \in \mathbb{R}} x(t) > 1 \quad \text{and} \quad \min_{t \in \mathbb{R}} x(t) < -1, \tag{19}$$

since otherwise x would be a periodic solution of (18) satisfying $V_\tau^+(x) = 2k$ which is impossible. Now, (19) and the monotone property of x imply that ε and δ (defined earlier in the proof) can be chosen in a way that

$$\dot{x}(t - \tau) > 0 \quad \text{for all } t \in (-\varepsilon, \varepsilon) \quad \text{or} \quad \dot{x}(t - \tau) < 0 \quad \text{for all } t \in (-\varepsilon, \varepsilon). \tag{20}$$

Thus it follows that $\eta_x'(u) = \frac{\ddot{x}(x^{-1}(u))}{\dot{x}(x^{-1}(u))}$ exists for all $u \in (\beta - \delta, \beta + \delta) \setminus \{0\}$. This combined with (14) shows that

$$\lim_{u \nearrow \beta} \frac{\ddot{x}(x^{-1}(u))}{\dot{x}(x^{-1}(u))} \quad \text{and} \quad \lim_{u \searrow \beta} \frac{\ddot{x}(x^{-1}(u))}{\dot{x}(x^{-1}(u))}$$

exist, moreover

$$\lim_{u \nearrow \beta} \frac{\ddot{x}(x^{-1}(u))}{\dot{x}(x^{-1}(u))} = \lim_{t \nearrow 0} \frac{\ddot{x}(t)}{\dot{x}(t)} = \frac{\lim_{t \nearrow 0} \ddot{x}(t)}{\dot{x}(0)}$$

and

$$\lim_{u\searrow\beta}\frac{\ddot{x}(x^{-1}(u))}{\dot{x}(x^{-1}(u))}=\lim_{t\searrow 0}\frac{\ddot{x}(t)}{\dot{x}(t)}=\frac{\lim_{t\searrow 0}\ddot{x}(t)}{\dot{x}(0)}.$$

Let

$$\eta'_x(\beta_-)=\frac{\lim_{t\nearrow 0}\ddot{x}(t)}{\dot{x}(0)}\quad\text{and}\quad\eta'_x(\beta_+)=\frac{\lim_{t\searrow 0}\ddot{x}(t)}{\dot{x}(0)}.$$

From $|Z|\in|X|\cup\operatorname{ext}(X)$ we obtain that

$$\eta'_x(\beta_-)\geq\eta'_z(\beta)\geq\eta'_x(\beta_+),$$

which is equivalent to

$$\lim_{t\nearrow 0}\ddot{x}(t)\geq\ddot{z}(0)\geq\lim_{t\searrow 0}\ddot{x}(t).$$

Using (14) it can be written in the following form

$$\begin{aligned}\lim_{t\nearrow 0}[-a\dot{x}(t)+bf'(x(t-\tau))\dot{x}(t-\tau)]&\geq -a\dot{z}(0)+b\dot{z}(-\alpha)\\&\geq\lim_{t\searrow 0}[-a\dot{x}(t)+bf'(x(t-\tau))\dot{x}(t-\tau)].\end{aligned}$$

Since $\dot{x}$ is continuous, thus it is equivalent to the following

$$\dot{x}(-\tau)\cdot\lim_{t\nearrow 0}f'(x(t-\tau))\geq\dot{z}(-\alpha)\geq\dot{x}(-\tau)\cdot\lim_{t\searrow 0}f'(x(t-\tau)).$$

Now (20) guarantees that the inequality above can be written in one of the following forms

$$0\geq\dot{z}(-\alpha)\geq\dot{x}(-\tau)\quad\text{or else}\quad 0\leq\dot{z}(-\alpha)\leq\dot{x}(-\tau).$$

This combined with (17) and $|Z|\subset|X|\cup\operatorname{ext}(X)$ infers

$$X(-\tau)=Z(-\alpha).$$

The choice of $X(0)$ infers that $\dot{x}(-\tau)$ cannot be positive or zero. If $\dot{x}(-\tau)=\dot{z}(-\alpha)<0$ then the same argument as in Case 2.1 shows that there must exist $t^*\in(0,\varepsilon)$ such that

$$x(t^*)<z(t^*),\quad x(t^*-\tau)\geq z(t^*-\alpha)\quad\text{and}\quad\dot{x}(t^*)<\dot{z}(t^*)$$

hold. Note that equation (20) combined with $\dot{x}(-\tau)<0$, $|x(-\tau)|=1$ and $t^*\in(0,\varepsilon)$ yields $x(t^*-\tau)<1$ from which we particularly get $bx(t^*-\tau)\leq bf(x(t^*-\tau))$. This combined with the above inequalities yields

$$\begin{aligned}\dot{x}(t^*)&=-ax(t^*)+bf(x(t^*-\tau))\\&\geq -ax(t^*)+bx(t^*-\tau)\\&>-az(t^*)+bz(t^*-\alpha)\\&=\dot{z}(t^*),\end{aligned}$$

a contradiction to $\dot{x}(t^*)<\dot{z}(t^*)$ which means that $|x(-\tau)|$ cannot be 1, either and thus our proof is now complete! □

Proposition 3.8. *Let $a,b,\tau>0$ be fixed in a way that $\tau\sqrt{b^2-a^2}+\arccos\frac{a}{b}\neq 2l\pi$ for any $l\in\mathbb{N}$. Then for every $k\in\mathbb{N}$, equation (1) has at most one periodic orbit in $(V_\tau^+)^{-1}(2k)$.*

Proof. Suppose to the contrary that x_1 and x_2 are both nonconstant periodic solutions of (1) with minimal periods T_1 and T_2, respectively such that $\{x_{1t} : t \in [0, T_1]\} \neq \{x_{2t} : t \in [0, T_2]\}$ and $V_\tau^+(x_1) = V_\tau^+(x_2) = 2k$ hold. We claim that for the corresponding D-trajectories $|X_1| \neq |X_2|$ holds. If this is not the case, then as in the first part of the previous proof, we get $T_1 \leq T_2$ and $T_1 \geq T_2$, also, moreover $x_1(s) = x_2(s)$ holds for all $s \in [-T_1/2, 0] = [-T_2/2, 0]$. Now, from the special symmetry of periodic solutions one gets a contradiction, which proves the claim. Using statement (i) of Proposition 3.4, we may assume that $|X_2| \cap \text{int}(X_1) \neq \emptyset$, furthermore there exists a unique constant $\gamma > 1$ for which $\gamma|X_2| \subset \text{ext}(X_1) \cup |X_1|$ and $\gamma|X_2| \cap |X_1| \neq \emptyset$. Now let $x := x_1$ and $y := \gamma x_2$. By simple calculation we infer that y is a periodic solution of the following delay differential equation:

$$\dot{y}(t) = -ay(t) + b\gamma f(y(t-\tau)/\gamma), \tag{21}$$

with minimal period $T_y = T_2$. It is clear that x is a periodic solution of (1) with minimal period $T_x = T_1$ and $V_\tau^+(x) = V_\tau^+(y) = 2k$. From this point, the process of the proof is rather similar to the proof of Proposition 3.7 and hence some parts of it are omitted.

Without loss of generality we may assume that $X(0) = Y(0)$ and

$$x(0) = \max\{x(t) : \quad t \in \mathbb{R},\ X(t) \in |Y| \text{ and } \dot{x}(t) > 0\}, \tag{22}$$

where X and Y are the corresponding D-trajectories. We have again two cases.

Case 1: $\dot{x}(0) = \dot{y}(0) = 0$.

The same argument as in the proof of Proposition 3.7 shows that $T_y \leq T_x$ and $y(s) \leq x(s)$ for all $s \in [-T_y/2, 0]$. In addition, now we claim that the first inequality is strict, that is

$$T_y < T_x.$$

To prove this, let us assume that $T_y \geq T_x$, which implies $T_y = T_x$ and $y(s) \leq x(s)$ for all $s \in [-T_x/2, 0]$ also. Using the observations in the proof of Proposition 3.7 and the same notations we obtain

$$s - T_x/2 = \int_{-d}^{x(s)} \frac{1}{\phi_x(u)} \mathrm{d}u \geq \int_{-d}^{x(s)} \frac{1}{\phi_y(u)} \mathrm{d}u \geq \int_{-d}^{y(s)} \frac{1}{\phi_y(u)} \mathrm{d}u = s - T_y/2 = s - T_x/2,$$

for all $s \in [-T_x/2, 0]$, which shows that both the above inequalities are equations and in particular we get

$$x(s) = y(s) \text{ for all } s \in \mathbb{R}.$$

This also implies $\dot{x}(s) = \dot{y}(s)$ for all $s \in \mathbb{R}$. Note that our assumption that $\nexists l \in \mathbb{N} : \tau\sqrt{b^2 - a^2} + \arccos\frac{a}{b} = 2l\pi$ together with Proposition 2.2 yields that x is *not* a solution of the linearized equation

$$\dot{z}(t) = -az(t) + bz(t-\tau).$$

In other words, there exists $s_0 \in \mathbb{R}$ such that $1 < x(s_0)$. By continuity we may choose s_0 in a way that $1 < x(s_0) < \gamma$. Now, we have

$$\dot{x}(s_0 + \tau) = -ax(s_0 + \tau) + bf(x(s_0)) = -ax(s_0 + \tau) + b$$

and

$$\begin{aligned} \dot{y}(s_0 + \tau) &= -ay(s_0 + \tau) + \gamma bf(y(s_0)/\gamma) = -ax(s_0 + \tau) + \gamma bf(x(s_0)/\gamma) \\ &= -ax(s_0 + \tau) + bx(s_0) > -ax(s_0 + \tau) + b, \end{aligned}$$

which contradicts to $\dot{x}(s_0 + \tau) = \dot{y}(s_0 + \tau)$. This contradiction shows that $T_y < T_x$.

From Proposition 3.1 and equation (1) for $t = 0$ we infer $|f(x(-\tau))| < 1$ which implies $|x(-\tau)| < 1$. Now, from $\dot{x}(0) = \dot{y}(0)$ and equations (1) and (21) we get

$$\gamma b f(y(-\tau)/\gamma) = bf(x(-\tau)) = bx(-\tau)$$

or equivalently:

$$f(y(-\tau)/\gamma) = x(-\tau)/\gamma < 1,$$

which leads to $x(-\tau) = y(-\tau)$. On the other hand the same argument as in the proof of Proposition 3.7 shows that

$$-kT_x < -kT_y < -\tau < -\left(k - \frac{1}{2}\right)T_x < -\left(k - \frac{1}{2}\right)T_y,$$

thus by using the monotone property and the special symmetry of x and y we infer

$$y(-\tau) = -y\left(-\tau + T_y/2\right) \geq -x\left(-\tau + T_y/2\right) > -x\left(-\tau + T_x/2\right) = x(-\tau),$$

which is a contradiction. This proves that Case 1 cannot occur.

Case 2: $\dot{x}(0) = \dot{y}(0) \neq 0$.

Let $\beta = x(0) = y(0)$. As in the proof of the previous proposition we may assume that $\varepsilon > 0$ and $\delta > 0$ are such that $\dot{x}(t), \dot{y}(t) > 0$ for all $t \in (-\varepsilon, \varepsilon)$ and

$$x^{-1} : (\beta - \delta, \beta + \delta) \to \mathbb{R}, \quad y^{-1} : (\beta - \delta, \beta + \delta) \to \mathbb{R}$$

are inverses of restrictions of x and y to open intervals in $(-\varepsilon, \varepsilon)$, respectively. Just as before, we define the following two maps:

$$\eta_x : (\beta - \delta, \beta + \delta) \ni u \mapsto \dot{x}(x^{-1}(u)) \in \mathbb{R},$$

$$\eta_y : (\beta - \delta, \beta + \delta) \ni u \mapsto \dot{y}(y^{-1}(u)) \in \mathbb{R}.$$

Note that from $X(0) = Y(0)$ and equations (1) and (21)

$$f(x(-\tau)) = \gamma f(y(-\tau)/\gamma)$$

follows. Hence $|f(y(-\tau)/\gamma)| \leq 1/\gamma < 1$ and by continuity of y and f we obtain that ε may be chosen in a way that $|f(y(t-\tau)/\gamma)| < 1$ for all $t \in (-\varepsilon, \varepsilon)$. Thus on this interval, equation (21) can be written in the form

$$\dot{y}(t) = -ay(t) + by(t - \tau). \tag{23}$$

Hence y is C^2-smooth on the interval $(-\varepsilon, \varepsilon)$ and just as in the preceding proof we find that

$$\eta_y'(\beta) = \frac{\ddot{y}(0)}{\dot{y}(0)}.$$

The rest of Case 2 can be dealt with repetition of the proof of Proposition 3.7 with substituting τ in place of α and y in place of z, everywhere. □

Proposition 3.9. *Let $a, b, \tau > 0$ and $k \in \mathbb{N}$ be such that $\tau\sqrt{b^2 - a^2} + \arccos\frac{a}{b} = 2k\pi$. Then there exists no periodic solution x of equation (1) for which $\max_{t\in\mathbb{R}} x(t) > 1$ and $V_\tau^+(x) = 2k$.*

Proof. Let us assume that the claim is not true and x is a periodic solution of equation (1) satisfying $\max_{t\in\mathbb{R}} x(t) > 1$ and $V_\tau^+(x) = 2k$. As in the proof of Proposition 3.7, y can be chosen to be a periodic solution of the linearized equation (23), such that $V_\tau^+(y) = 2k$ and for the corresponding D-trajectories $|Y| \subset |X| \cup \operatorname{ext}(X)$ and $|Y| \cap |X| \neq \emptyset$ hold. The two cases are treated separately precisely as in the former propositions. In the first case, using the same notations, the same argument yields $T_y \leq T_x$. As assumed earlier, there exists $s_0 \in \mathbb{R}$ such that

$x(s_0) > 1$. Thus a rather similar argument to the one in the proof of Proposition 3.8 can be applied to obtain $T_y < T_x$ and then we get the contradiction similarly.

In the second case of the proof of Proposition 3.7, condition $\alpha > \tau$ was used nowhere, hence the contradiction can be obtained in this case, too. □

4. **Periodic orbits, existence results and the period function.** Let a, b be fixed and $b > a > 0$ hold and let

$$\tau^* = \tau^*(a,b) = \frac{2\pi - \arccos \frac{a}{b}}{\sqrt{b^2 - a^2}}.$$

We know from [15] and from Proposition 3.8 that if $\tau \geq \tau^*$, then equation (1) has a nonconstant periodic solution x, for which $V_\tau^+(x) = 2$ holds and that this solution is unique up to translation of time (and up to scalar multiplication of the solution when $\tau = \tau^*$). We denote the minimal period of that unique periodic solution by $T(\tau)$. Function $T : [\tau^*, \infty) \to \mathbb{R}$, $\tau \mapsto T(\tau)$ is regarded as the period function of equation (1). The following proposition roughly says that "the bigger the delay, the bigger the a D-trajectory is". This proposition as well as Theorems 4.2 and 5.1 are already proved for the case when the feedback is of smooth, sigmoid-type [4]. The proofs are based on the uniqueness of periodic solutions which is guaranteed here by Proposition 3.8.

Proposition 4.1. *Let $a > 0$, $b \neq 0$ and $\tau^* \leq \tau_1 < \tau_2$ be fixed and let x_i be a periodic solution of equation (1) with delay $\tau = \tau_i$, $i \in \{1, 2\}$. In addition, let ε denote the sign of b and X_i denote the D-trajectory of x_i. If $V_{\tau_1}^\varepsilon(x_1) = V_{\tau_2}^\varepsilon(x_2)$, then*

$$|X_2| \subset \operatorname{ext}(X_1) \cup |X_1| \quad \textit{and} \quad |X_2| \cap \operatorname{ext}(X_1) \neq \emptyset.$$

Proof. Again, we only treat the case when the feedback is positive. By way of contradiction, let us suppose that x_1 and x_2 satisfy the assumptions of the proposition but the claim is not true. Then, as in the beginning of the proof of Proposition 3.8, it follows that $|X_1| \neq |X_2|$, hence we may assume that $|X_2| \cap \operatorname{int}(X_1) \neq \emptyset$. Then there exists a unique $\gamma > 1$ such that

$$\gamma|X_2| \subset \operatorname{ext}(X_1) \cup |X_1| \quad \text{and} \quad \gamma|X_2| \cap |X_1| \neq \emptyset$$

hold. Now, if we define $y(t) = \gamma x_2(t)$ for all $t \in \mathbb{R}$, then y is a nonconstant periodic solution of (21) and $V_{\tau_1}^+(x_1) = V_{\tau_2}^+(y) > 0$ holds. Now, the rest of the argument is the same as it was in the proof of Proposition 3.8. The only difference is that we cannot assume that x is not a solution of the linearized equation, but since $\tau_2 > \tau_1 \geq \tau^*$, thus we may assume that y is not a solution of it and contradiction is obtained in the same manner. □

Theorem 4.2. *Let T denote the period function of (1). Let $\tau_1, \tau_2 \in [\tau^*, \infty)$, $\tau_1 < \tau_2$. Then*

$$0 \leq T(\tau_2) - T(\tau_1) < 2(\tau_2 - \tau_1).$$

Proof. The proof presented in [4] works in this case, also and is based on the previous proposition. □

Note that in particular it follows that the period function is continuous. The following theorem is the main result of the section. Here, we summarize Propositions 3.7-3.9 and their analogues for negative feedback. We also state and prove the remaining existence results. The theorem gives a complete picture concerning the number and the oscillation-frequency of the periodic solutions.

Theorem 4.3. *Let $k \in \mathbb{N}$, $a, \tau > 0$ and $b \neq 0$ be fixed and let $\nu = \nu(a, b, \tau) = \tau\sqrt{b^2 - a^2} + \arccos \frac{a}{|b|}$. The following statements hold.*

(i) If the feedback is positive, i.e. $b > 0$, then equation (1) has a periodic solution in $(V_\tau^+)^{-1}(2k)$ if and only if $\nu \geq 2k\pi$ holds. If $\nu > 2k\pi$, then this periodic solution is unique, up to translation of time.

(ii) If $\nu = 2k\pi$ and x is an arbitrary periodic solution of equation (1) in $(V_\tau^+)^{-1}(2k)$, then $\max_{t\in\mathbb{R}} x(t) \leq 1$. If $y \in (V_\tau^+)^{-1}(2k)$ is also a periodic solution of (1), then there exist $A > 0$ and $d > 0$ such that $y(t) = Ax(t + d)$ for all $t \in \mathbb{R}$.

(iii) If the feedback is negative, i.e. $b < 0$, then equation (1) has a periodic solution in $(V_\tau^-)^{-1}(2k-1)$ if and only if $\nu \geq (2k-1)\pi$ holds. If $\nu > (2k-1)\pi$, then this periodic solution is unique, up to translation of time.

(iv) If $\nu = (2k-1)\pi$ and x is an arbitrary periodic solution of equation (1) in $(V_\tau^-)^{-1}(2k-1)$, then $\max_{t\in\mathbb{R}} x(t) \leq 1$. If $y \in (V_\tau^-)^{-1}(2k-1)$ is also a periodic solution of (1), then there exist $A > 0$ and $d > 0$ such that $y(t) = Ax(t + d)$ for all $t \in \mathbb{R}$.

(v) Nonoscillatory periodic solutions do not exist.

Proof. Only the existence parts are left to prove. There are several ways to prove the existence of such periodic solutions, see, e.g. Diekmann et al. [3] and the references therein. We are going to present here a rather elementary proof. Actually, as we have already mentioned earlier, existence was also proved in $(V_\tau^+)^{-1}(2)$ by Vas [15]. We shall prove the remaining existence parts using this fact and Theorem 4.2.

1. Let us suppose that $b > 0$, $2 \leq k \in \mathbb{N}$ and $\nu(a, b, \tau) \geq 2k\pi$. We need to prove that there exists a periodic solution x of equation (1), for which $V_\tau^+(x) = 2k$. Let us consider the following delay differential equation

$$\dot{x}(t) = -ax(t) + bf(x(t - \gamma\tau)). \tag{24}$$

Let $\omega(\gamma) = T(\gamma\tau)$, where $\gamma \in [\gamma^*, \infty)$ and γ^* is defined by the following formula

$$\gamma^*\tau\sqrt{b^2 - a^2} + \arccos \frac{a}{b} = 2\pi. \tag{25}$$

Now, let us assume that there exists $\gamma_0 \in [\gamma^*, \infty)$, such that

$$\tau = \gamma_0\tau + (k-1)T(\gamma_0\tau)$$

holds and that y is the only periodic solution of (24) with $\gamma = \gamma_0$ for which $V_{\gamma_0\tau}^+(y) = 2$. Then we obtain that the following equations hold

$$\begin{aligned}\dot{y}(t) &= -ay(t) + bf(y(t - \gamma_0\tau)) = -ay(t) + bf(y(t - (\gamma_0\tau + (k-1)T(\gamma_0\tau))))\\ &= -ay(t) + bf(y(t - \tau)).\end{aligned}$$

From the special symmetry of the solutions we infer $V_\tau^+(y) = 2k$, thus it is sufficient to prove the existence of such a γ_0. From Theorem 4.2 it follows that ω is continuous on its domain, so it is sufficient to find such $\gamma_1, \gamma_2 \in [\gamma^*, \infty)$ values that one of the the following two expression is nonpositive and one is nonnegative

$$\tau - \gamma_1\tau - (k-1)T(\gamma_1\tau) \quad \text{and} \quad \tau - \gamma_2\tau - (k-1)T(\gamma_2\tau).$$

If $\gamma_1 \geq 1$, then the first expression is clearly negative. Let us consider the case when $\gamma_2 = \gamma^*$. It is easy to see that

$$T(\gamma^*\tau) = \frac{2\pi}{\sqrt{b^2 - a^2}}.$$

Now, from this and from formula (25) we infer that the inequality

$$\tau - \gamma^*\tau - (k-1)T(\gamma^*\tau) \geq 0$$

is equivalent to the following

$$\tau \geq \frac{2\pi - \arccos\frac{a}{b}}{\sqrt{b^2-a^2}} + (k-1)\frac{2\pi}{\sqrt{b^2-a^2}}.$$

This is equivalent to $\nu(a,b,\tau) \geq 2k\pi$, so our claim is proved.

2. If the feedback is negative, i.e. $b < 0$, then suppose that $\nu(a,b,\tau) \geq (2k-1)\pi$. Now, there exists $\gamma^* > 0$ (but not necessarily greater than 1), such that for all $\gamma \in [\gamma^*, \infty)$, equation

$$\dot{x}(t) = -ax(t) - bf(x(t-\gamma\tau)) \tag{26}$$

has a unique periodic solution in $(V^+_{\gamma\tau})^{-1}(2)$. If y is such a solution, we infer the following

$$\begin{aligned}\dot{y}(t) &= -ay(t) - bf(y(t-\gamma\tau)) = -ay(t) - bf(y(t-\gamma\tau-(k-1)T(\gamma\tau)))\\ &= -ay(t) + bf(y(t-\gamma\tau-(k-3/2)T(\gamma\tau))).\end{aligned}$$

We used the special symmetry of y and the oddness of f. Again, from the special form of the solutions, it is clear that if y is the unique solution of (26) in $(V^+_{\gamma_0\tau})^{-1}(2)$, where

$$\tau = \gamma_0\tau - (k-3/2)T(\gamma_0\tau)$$

holds, then y is also a periodic solution of equation (1) with negative feedback and $V^-_\tau(y) = 2k-1$. If we choose γ_1 large enough and $\gamma_2 = \gamma^*$, then a similar argument shows the existence of such a γ_0. This completes our proof. □

5. **A generalization.** Let us consider the following system of delay differential equations:

$$\begin{aligned}\dot{x}^0(t) &= -ax^0(t) + bf(x^1(t)),\\ &\vdots\\ \dot{x}^{n-1}(t) &= -ax^{n-1}(t) + bf(x^n(t)),\\ \dot{x}^n(t) &= -ax^n(t) + \delta bf(x^0(t-1)),\end{aligned} \tag{27}$$

where $a, b > 0$, $n \in \mathbb{N}$, $\delta \in \{-1, 1\}$ and f is the already defined feedback function. This unidirectional ring-type system was studied in papers [2, 4, 13, 14] and [17] in the case when f is of smooth, sigmoid-type. We state that the existence and uniqueness results on periodic solutions proved in [4] and [17] remain true for this type of feedback, also. The argument presented in [4] is valid in this case, as well, and the proof is based on Theorem 4.2.

For a system of the form (27) the natural phase space is the Banach space $C(\mathbb{K},\mathbb{R})$, which is the space of continuous functions from $\mathbb{K}$ to $\mathbb{R}$, where $\mathbb{K} = [-1,0] \cup \{1,2,\dots,n\}$. We shall use the shorter form $C(\mathbb{K})$. If x is a solution of (27) on some interval, then we let $x_{t,\mathbb{K}} \in C(\mathbb{K})$ be defined by

$$x_{t,\mathbb{K}}(\theta) = \begin{cases} x^0(t+\theta) & \text{for} \quad \theta \in [-1,0],\\ x^\theta(t) & \text{for} \quad \theta \in \{1,2,\dots,n\}.\end{cases}$$

In order to state our theorem, we first need to define the Lyapunov functionals for this case, also introduced in [13].

$$V_{\mathbb{K}}^+ : C(\mathbb{K}) \setminus \{0\} \to \{0, 2, 4, \ldots, \infty\}, \quad V_{\mathbb{K}}^- : C(\mathbb{K}) \setminus \{0\} \to \{1, 3, 5, \ldots, \infty\},$$

$$V_{\mathbb{K}}^+(\varphi) = \begin{cases} \mathrm{sc}(\varphi, \mathbb{K}) & \text{if} \quad \mathrm{sc}(\varphi, \mathbb{K}) \quad \text{is even or infinite,} \\ \mathrm{sc}(\varphi, \mathbb{K}) + 1 & \text{if} \quad \mathrm{sc}(\varphi, \mathbb{K}) \quad \text{is odd,} \end{cases}$$

$$V_{\mathbb{K}}^-(\varphi) = \begin{cases} \mathrm{sc}(\varphi, \mathbb{K}) & \text{if} \quad \mathrm{sc}(\varphi, \mathbb{K}) \quad \text{is odd or infinite,} \\ \mathrm{sc}(\varphi, \mathbb{K}) + 1 & \text{if} \quad \mathrm{sc}(\varphi, \mathbb{K}) \quad \text{is even,} \end{cases}$$

where $\mathrm{sc}(\varphi, \mathbb{K})$ denotes the number of sign changes of φ on set $\mathbb{K}$.

Theorem 5.1. *Let $\nu_n = \nu_n(a, b) = \sqrt{b^2 - a^2} + (n + 1) \arccos \frac{a}{b}$ and $k \in \mathbb{N}$. The following statements hold.*

(i) If $\delta = 1$ and $k \geq \frac{n+1}{2}$, then system (27) has a periodic solution in $(V_{\mathbb{K}}^+)^{-1}(2k)$ if and only if $\nu_n \geq 2k\pi$. This solution is unique up to translation of time when $\nu_n > 2k\pi$. If $\nu_n = 2k\pi$ and $x, y \in (V_{\mathbb{K}}^+)^{-1}(2k)$ are both periodic solutions of system (27), then $\max_{t \in \mathbb{R}} x(t) \leq 1$ holds and there exist $A > 0$ and $d > 0$ such that $y(t) = Ax(t + d)$ for all $t \in \mathbb{R}$.

(ii) If $\delta = 1$, $k < \frac{n+1}{2}$ and $\nu_n \geq 2k\pi$, then system (27) has a periodic solution in $(V_{\mathbb{K}}^+)^{-1}(2k)$.

(iii) If $\delta = -1$ and $k \geq \frac{n+2}{2}$ then system (27) has a periodic solution in $(V_{\mathbb{K}}^-)^{-1}(2k - 1)$ if and only if $\nu_n \geq (2k - 1)\pi$. This solution is unique up to translation of time when $\nu_n > (2k - 1)\pi$. If $\nu_n = (2k - 1)\pi$ and $x, y \in (V_{\mathbb{K}}^-)^{-1}(2k - 1)$ are both periodic solutions of system (27), then $\max_{t \in \mathbb{R}} x(t) \leq 1$ holds and there exist $A > 0$ and $d > 0$ such that $y(t) = Ax(t + d)$ for all $t \in \mathbb{R}$.

(iv) If $\delta = -1$, $k < \frac{n+2}{2}$ and $\nu_n \geq (2k - 1)\pi$, then system (27) has a periodic solution in $(V_{\mathbb{K}}^-)^{-1}(2k - 1)$.

Sketch of proof. The proof is based on Theorem 4.2 and on the fact that if $\delta \in \{-1, 1\}$ is fixed and $x : \mathbb{R} \to \mathbb{R}^{n+1}$ is a nonconstant periodic solution of system (27), then there exists $r_\delta \in \mathbb{R}$ such that $(x^1(t), \ldots, x^n(t), \delta x^0(t - 1)) = (x^0(t + r_\delta), \ldots, x^n(t + r_\delta))$ holds for all $t \in \mathbb{R}$. For details, see [17]. The existence of such an r_δ depends on the fact that if x and y are two nonconstant periodic solutions of system (27) with distinct orbits, then for all $i \in \{0, \ldots, n\}$ the trace of $t \mapsto (x^i(t), x^{i+1}(t))$ and $t \mapsto (y^i(t), y^{i+1}(t))$ are disjoint simple closed curves on the plane, where by $x^{n+1}(t)$ and $y^{n+1}(t)$ we denote $x^0(t-1)$ and $y^0(t-1)$, respectively. This is proved for $n = 0$ in Proposition 2.4 of [10], but their argument also holds when $n \geq 1$. For more details we refer the reader to [4]. $\square$

We conjecture that the restrictions for k in statements (i) and (iii) are not necessary. Note that if $n = 0$, then $k \geq (n + 1)/2$ holds for all $k \in \mathbb{N}$, so in this case Theorem 5.1 gives back Theorem 4.3. To prove the conjecture, according to [4], it is sufficient to prove that the period function T of equation (1) is such that the function defined by $[\tau^*, \infty) \ni \tau \mapsto T(\tau)/\tau$ is monotone nonincreasing. Numerical simulations suggest that this is true, moreover, with Gabriella Vas, we showed analytically that there exists $\tau^{**} \geq \tau^*$ such that $T(\tau)/\tau$ is monotone nonincreasing on $[\tau^{**}, \infty)$.

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E-mail address: garab@math.u-szeged.hu