Absence of small solutions and existence of Morse decomposition for a cyclic system of delay differential equations

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Abstract

We consider the unidirectional cyclic system of delay differential equations

$$\dot{x}^i(t) = g^i(x^i(t), x^{i+1}(t - \tau^i), t), \qquad 0 \le i \le N,$$

where the indexes are taken modulo $N + 1$, with $N \in \mathbb{N}_0$, $\tau^i \in [0, \infty)$, $\tau := \sum_{i=0}^{N} \tau^i > 0$, and for all $0 \le i \le N$, the feedback functions $g^i(u, v, t)$ are continuous in $t \in \mathbb{R}$ and C^1 in $(u, v) \in \mathbb{R}^2$, and each of them satisfies either a positive or a negative feedback condition in the delayed term.

We show that all components of a superexponential solution (i.e. nonzero solutions that converge to zero faster than any exponential function) must have infinitely many sign-changes on any interval of length τ. As a corollary we obtain that if a backwards-bounded global pullback attractor exists, then it does not contain any superexponential solutions. In the autonomous case we also prove that the global attractor possesses a Morse decomposition that is based on a discrete Lyapunov function. This generalizes former results by Mallet-Paret [*J. Differential Equations* **72** (1988), 270–315] and Polner [*Nonlinear Anal.* **48** (2002), 377–397] in which the scalar case was studied.

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1. Introduction

Let us consider the unidirectional cyclic system of delay differential equations

$$\dot{w}^i(t) = g^i(w^i(t), w^{i+1}(t - \tau^i), t), \qquad 0 \le i \le N, \tag{1.1}$$

where the indexes are taken modulo $N + 1$, with $N \in \mathbb{N}_0$; $\tau^i \in [0, \infty)$, $\tau := \sum_{i=0}^{N} \tau^i > 0$, and for all $0 \le i \le N$, the feedback functions $g^i(u, v, t)$ are continuous in $t \in \mathbb{R}$ and continuously differentiable in $(u, v) \in \mathbb{R}^2$, moreover, they fulfill the feedback assumptions:

(H_0) there exist $\delta^i := \delta(g^i) \in \{-1, 1\}$, such that $\delta^i v g^i(0, v, t) > 0$ holds for all $v \neq 0$ and $t \in \mathbb{R}$;

(H_1) there exist positive constants $M := M(g)$, $a_1 := a_1(g)$, $b_0 := b_0(g)$ and $b_1 := b_1(g)$ such that

$$|D_1 g^i(u, v, t)| \le a_1 \quad \text{and} \quad b_0 \le \delta^i D_2 g^i(0, v, t) \le b_1$$

hold for all $(u, v, t) \in [-M, M]^2 \times \mathbb{R}$ and $0 \le i \le N$, where D_j denotes differentiation with respect to the j-th variable.

Note that $g^i(0, 0, \cdot) \equiv 0$ follows from continuity of g^i and assumption (H_0), so the origin is an equilibrium.

Such equations were studied in detail in the seminal papers by Mallet-Paret and Sell [29, 30] in a more general setting (with bidirectional connection-topology) – see also the ODE case without delay in

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[16, 17, 31]. These equations have many applications from life sciences, computer science and economy. Without attempting to be exhaustive, equations of the form (1.1) arise in various models of biological regulatory systems [18, 20, 27, 34, 39, 40], physiology [26], population dynamics [7, 13], neural networks [1, 14, 19, 21, 41], as well as in economics – see [32] and some references therein.

A solution w of (1.1) is called *small* if

$$\lim_{t\to\infty} e^{\beta t} w(t) = 0$$

holds for all $\beta \in \mathbb{R}$. A small solution is said to be *nontrivial* if it is not identically 0. Nontrivial small solutions are also called *superexponential*.

The non-existence of superexponential solutions is a central question in the analysis of functional differential equations. Borrowing the words of Mallet-Paret and Sell [30], "Perhaps the most challenging of those [issues which are peculiar to infinite dimensional systems] involves the existence, and non-existence, of so-called superexponential solutions".

Our first main result (Theorem 2.2) establishes that any component w^i of a superexponential solution of (1.1) must change signs infinitely often on any interval of length τ. This was already proved for the autonomous, scalar case (i.e. $N = 0$) with negative feedback by Mallet-Paret [28] and later for the nonautonomous scalar case with either positive or negative feedback by Cao [2]. Further similar results were obtained for the scalar case with state-dependent delay by Cao [3] and Krisztin and Arino [24].

For certain classes of nonlinear cyclic systems, Ducrot [12] proved an analogous result, which is independent of ours: neither implies the other.

For a nonautonomous linear system of equations, Cooke and Verduyn Lunel established rather general sufficient conditions for the non-existence of superexponential solutions [8]. Although solutions of (1.1) can be transformed to solutions of nonautonomous linear equations, their results do not apply for our case. The reason is that their approach requires that the coefficient functions are real analytic, which is not guaranteed in our case. Later, Cooke and Derfel [10, 9] illustrated by counterexamples that the assumption on analyticity is not merely technical. See also Remark 2.10 for more details.

Let us mention two concrete motivational examples for Theorem 2.2:

(1) In [28] and in [37], equation (1.1) was studied in the scalar, autonomous case (with negative, resp. positive feedback), and excluding the existence of superexponential solutions on the global attractor was crucial for the proof of the existence of a Morse decomposition of the global attractor.

(2) Pituk in [36, Corollary 4.4] established an "oscillation by linearization" theorem for a scalar delay differential equation with several delays: he showed that provided all solutions of the linearized equation which tend to zero at infinity are oscillatory, then so is every such solution of the original equation. A key step in the proof is establishing that eventually positive solutions of the original equation cannot be superexponential.

As a consequence of Theorem 2.2 we obtain that provided a backwards-bounded global pullback attractor (or simply the global attractor in the autonomous case) related to (1.1) exists, it does not contain any superexponential solutions (see Theorem 2.9).

Section 3 focuses on the autonomous case under the assumption that the global attractor exists. When studying the long-time behavior of a dynamical system, beyond merely the existence of the global attractor, one is rather interested in its structure. Under some monotonicity assumptions on the feedback functions g^i, Mallet-Paret and Sell could prove a Poincaré–Bendixson-type result in [29]. Although, such a strong result is not expected to hold in the general case, it is possible to construct a Morse decomposition of the attractor, i.e. a finite collection of pairwise disjoint, compact, invariant subsets of the global attractor, which are ordered in the sense, that – roughly speaking – the dynamics on the attractor and outside these sets is gradient-like (see Section 3 for the precise definition). This can be found in Theorem 3.2, which is the other main result of the paper. The decomposition is based on a discrete Lyapunov function, introduced by Mallet-Paret and Sell [30], which essentially counts the number of sign-changes on segments of the solutions. This function has proven to be a very efficient tool in analyzing the structure of the global attractor. Our theorem and its proof are analogs of those from [28, 37] (resp. [15]), for the scalar delay-differential (resp. delay-difference) equation. The argument – both in the scalar and in our case – extensively uses the properties of this Lyapunov function, and also requires detailed information on the spectrum of the linearized equation. These have all been available in the general setting, thanks to [29, 30] (see also Section 3.1). However, there is a particular part in the argument (Proposition 3.9 in our paper), where the absence of superexponential solutions – more precisely, an estimate on the decay of the norm of the solution in terms of the number of sign-changes – seems to be inevitable, since this allows one to describe the asymptotic behavior of solutions on the stable manifold of the origin by the associated linear equation. These results are given in Theorem 2.2 and Proposition 2.8.

Let us also point out, that a recent work of Ivanov and Lani-Wayda [23] provides sufficient conditions for the existence of a periodic solution of a somewhat less general version of (1.1), suggesting that the decomposition given in Theorem 3.2 is nontrivial (see also Remark 3.4).

The paper is organized as follows. Section 2 deals with the general, nonautonomous case. We provide estimates on the maximal rate of decay of solutions in terms of the number of sign-changes of the solutions on certain intervals. The main result of the section (Theorem 2.2) states that components of superexponential solutions must change sign infinitely often on any interval of length τ. As a corollary one obtains the absence of superexponential solutions on backwards-bounded global pullback attractors (or simply on the global attractor in the autonomous case) – see Theorem 2.9. Section 3 is devoted to the construction of a Morse decomposition of the global attractor in the autonomous case, which is given in Theorem 3.2.

2. Superexponential solutions and superhigh-frequency oscillations

2.1. Preliminaries

Since our focus is on superexponential solutions and on the global attractor, we will often assume that the solution we study exists globally in forward time. Following [30] and using notations

$$\sigma^0 = 1, \quad \gamma^0 = 0 \quad \text{and} \quad \sigma^i := \prod_{j=0}^{i-1} \delta(g^j), \quad \gamma^i := \sum_{j=0}^{i-1} \tau^j \quad \text{for all } 1 \le i \le N, \tag{2.1}$$

and

$$f^i(u, v, t) = \sigma^i \tau g^i(\sigma^i u, \sigma^{i+1} v, \tau t - \gamma^i), \tag{2.2}$$

the invertible transformation $x^i(t) := \sigma^i w^i(\tau t - \gamma^i)$ takes system (1.1) into the canonical form

$$\begin{aligned} \dot{x}^i(t) &= f^i(x^i(t), x^{i+1}(t), t), \qquad && 0 \le i \le N-1 \\ \dot{x}^N(t) &= f^N(x^N(t), x^0(t-1), t), && \end{aligned} \tag{2.3}$$

where the nonlinearities $f^i(u, v, t)$ are continuous in $t \in \mathbb{R}$, C^1 in $(u, v) \in \mathbb{R}^2$ and satisfy conditions (H_0) and (H_1) – with some appropriate constants $a_1(f)$, $b_0(f)$, $b_1(f)$, $M(f)$, $\delta(f^i)$ – such that here $\delta(f^N) \in \{-1, 1\}$ and $\delta(f^i) = 1$ for all $0 \le i \le N-1$. Determined by the sign of $\delta(f^N)$ we say that system (2.3) has negative (resp. positive) feedback.

For an interval $I \subseteq \mathbb{R}$, let $\tilde{I}$ denote the interval $\{s + t : s \in [-1, 0],\ t \in I\}$. We say that a continuous function $x\colon \tilde{I} \to \mathbb{R}^{N+1}$ is a *solution of equation* (2.3) *on* I, if it is differentiable on I and satisfies (2.3) there. A differentiable function $x\colon \mathbb{R} \to \mathbb{R}^{N+1}$ is called an entire solution of (2.3) if it fulfills the equation for all $t \in \mathbb{R}$.

Using notation $\mathbb{K} := [-1, 0] \cup \{1, \dots, N\}$, the natural state space for (2.3) – in accordance with [30] – is the Banach space of continuous functions $\mathcal{C}_{\mathbb{K}} := C(\mathbb{K}, \mathbb{R})$ equipped with the supremum norm that we will denote by $\|\cdot\|_{\mathcal{C}_{\mathbb{K}}}$. For arbitrary $t_0 \in \mathbb{R}$ and $\varphi \in \mathcal{C}_{\mathbb{K}}$, there exists a unique maximal forward solution x on (t_0, t^*) for some $t^* \in (t_0, \infty]$, that fulfills the initial condition $x_{t_0} = \varphi$, by which we mean

$$x^0|_{[t_0-1, t_0]} = \varphi|_{[-1,0]} \quad \text{and} \quad x^i(t_0) = \varphi(i) \quad \text{for all} \quad 1 \le i \le N. \tag{2.4}$$

It is worth here noticing that, although we do not have backward uniqueness, the zero solution is unique in backward time (see [29, p. 451]).

It is clear from (2.1) and the transformation $x^i(t) := \sigma^i w^i(\tau t - \gamma^i)$ that a solution w of (1.1) is superexponential if and only if x is a superexponential solution of (2.3).

Considering equation in the form (2.3) has several advantages: one of the most important among those is the fact that, according to [30], there are discrete Lyapunov functions available for (2.3). More precisely, let us introduce the functions

$$V^+ \colon \mathcal{C}_{\mathbb{K}} \setminus \{0\} \to \{0, 2, 4, \dots, \infty\}, \qquad V^- \colon \mathcal{C}_{\mathbb{K}} \setminus \{0\} \to \{1, 3, 5, \dots, \infty\},$$

$$V^+(\varphi) = \begin{cases} \mathrm{sc}(\varphi, \mathbb{K}), & \text{if } \mathrm{sc}(\varphi, \mathbb{K}) \text{ is even or infinite,} \\ \mathrm{sc}(\varphi, \mathbb{K}) + 1, & \text{if } \mathrm{sc}(\varphi, \mathbb{K}) \text{ is odd,} \end{cases}$$

$$V^-(\varphi) = \begin{cases} \mathrm{sc}(\varphi, \mathbb{K}), & \text{if } \mathrm{sc}(\varphi, \mathbb{K}) \text{ is odd or infinite,} \\ \mathrm{sc}(\varphi, \mathbb{K}) + 1, & \text{if } \mathrm{sc}(\varphi, \mathbb{K}) \text{ is even,} \end{cases}$$

where $\mathrm{sc}(\varphi, A)$ denotes the number of sign-changes of a real function φ on the set $A \subseteq \operatorname{dom} \varphi$, that is

$$\begin{aligned} \mathrm{sc}(\varphi, A) = \sup\bigl\{k \geq 0 : & \text{ there exist } \theta_i \in A, \text{ for } 0 \leq i \leq k \\ & \text{with } \theta_{i-1} < \theta_i \text{ and } \varphi(\theta_{i-1})\varphi(\theta_i) < 0 \text{ for } 1 \leq i \leq k\bigr\} \end{aligned} \tag{2.5}$$

Then, as stated in the next proposition, function V^+ (resp. V^-), is nonincreasing along solutions of (2.3) with positive (resp. negative) feedback.

Proposition 2.1 ([29, Theorem 4.1]). *Let V denote either V^+ or V^- (determined by the sign of $\delta(f^N)$), and let $x \colon [t_0 - 1, \infty) \to \mathbb{R}^{N+1}$ be a nontrivial solution of (2.3). Then $V(x_{t_1}) \geq V(x_{t_2})$ holds for all $t_1 > t_2 \geq t_0$.*

Now we will relate solutions of (2.3) to some nonautonomous linear delay equations. Take $M = M(f)$ from hypothesis (H_1), and suppose that $x \colon [t_0 - 1, \infty) \to \mathbb{R}^{N+1}$ is a solution of (2.3) such that $|x^i(t)| \leq M$ for all $0 \leq i \leq N$ and $t \geq t_0 - 1$. Then x is also a solution of equation

$$\begin{aligned} \dot{x}^i(t) &= \alpha^i(t)x^i(t) + \beta^i(t)x^{i+1}(t), \qquad 0 \leq i \leq N-1, \\ \dot{x}^N(t) &= \alpha^N(t)x^N(t) + \beta^N(t)x^0(t-1) \end{aligned} \tag{2.6}$$

with continuous functions

$$\alpha^i(t) = \int_0^1 D_1 f^i(hx^i(t), x^{i+1}(t), t)\, dh \tag{2.7}$$

and

$$\beta^i(t) = \int_0^1 D_2 f^i(0, hx^{i+1}(t), t)\, dh, \tag{2.8}$$

where $x^{N+1}(t) \coloneqq x^0(t-1)$.

Here we have used that $f^i(0, 0, t) = 0$ for all $t \in \mathbb{R}$. Note also that

$$\beta^i(t) = \begin{cases} \dfrac{f^i(0, x^{i+1}(t), t)}{x^{i+1}(t)}, & \text{if } x^{i+1}(t) \neq 0 \\ D_2 f^i(0, 0, t), & \text{otherwise.} \end{cases}$$

Observe that feedback condition (H_1) implies that the functions α^i and β^i satisfy hypothesis (H_1') (on the interval $t \in [t_0, \infty)$), which reads as follows:

(H_1') there exist positive constants α_1, β_0 and β_1, such that $|\alpha^i(t)| \leq \alpha_1$ and $\beta_0 \leq \delta^i \beta^i(t) \leq \beta_1$ hold for all t and for all $0 \leq i \leq N$ with $\delta^N \in \{-1, 1\}$ and $\delta^i = 1$ for all $0 \leq i \leq N-1$.

Furthermore, note that equation (2.6) is a special case of (2.3), and consequently Proposition 2.1 also applies. We will take advantage of this fact, however, for our purposes it will be convenient to apply for a solution $x \colon [t_0 - 1, \infty) \to \mathbb{R}^{N+1}$ of (2.6) a simple transformation

$$z : [-1, \infty) \to \mathbb{R}^{N+1}, \qquad z^i(t) \coloneqq x^i\bigl(t_0 + \tfrac{t+i}{N+1}\bigr), \qquad 0 \leq i \leq N,$$

in order to obtain the following more symmetric system of equations:

$$\dot{z}^i(t) = a^i(t)z^i(t) + b^i(t)z^{i+1}(t-1), \qquad 0 \leq i \leq N, \tag{2.9}$$

with $z^{N+1}(t) \coloneqq z^0(t)$, where for $t \geq 0$,

$$a^i(t) = \frac{\alpha^i\big(t_0 + \frac{t+i}{N+1}\big)}{N+1}, \qquad b^i(t) = \frac{\beta^i\big(t_0 + \frac{t+i}{N+1}\big)}{N+1}. \tag{2.10}$$

Clearly, the functions a^i and b^i are continuous and on the interval $[0, \infty)$ they fulfill the hypothesis

(H_1'') there exist positive constants a_1, b_0 and b_1, such that $|a^i(t)| \leq a_1$ and $b_0 \leq \delta^i b^i(t) \leq b_1$ hold for all t and for all $0 \leq i \leq N$ with $\delta^N \in \{-1, 1\}$ and $\delta^i = 1$ for all $0 \leq i \leq N-1$.

We say that a function z is a *solution of equation* (2.9) *on an interval* I, if $z \colon \tilde{I} \to \mathbb{R}^{N+1}$ is continuous and it is differentiable on I and satisfies (2.9) there. The natural *state-space* for equations of the form (2.9) is the Banach space $\mathcal{C} \coloneqq C([-1,0], \mathbb{R})^{N+1}$ equipped with the supremum norm.

Now, for a solution $z \colon [t_0 - 1, \infty)$ of equation (2.9) that satisfies (H_1''), the change of variables $y^i(t) \coloneqq \exp\big(-\int_{t_0}^t a^i(s)\, ds\big) z^i(t)$ for $t \geq t_0$ we obtain that $y \colon [t_0, \infty) \to \mathbb{R}^{N+1}$ is a solution of the following system of delay differential equations:

$$\dot{y}^i(t) = c^i(t) y^{i+1}(t-1), \qquad 0 \leq i \leq N, \quad t \geq t_0 + 1$$

with

$$c^i(t) = b^i(t) \exp\left(\int_{t_0}^{t-1} a^{i+1}(s)\, ds - \int_{t_0}^{t} a^i(s)\, ds\right) \quad \text{and} \quad a^{N+1}(t) \coloneqq a^0(t).$$

The above properties of functions a^i and b^i imply that functions c^i are continuous and for any bounded interval there exist $0 < c_0 < c_1 < \infty$, such that $c_0 \leq \delta^i c^i(t) \leq c_1$ for all $0 \leq i \leq N$, with $\delta^N \in \{-1, 1\}$ and $\delta^i = 1$ for all $0 \leq i \leq N-1$.

2.2. *Main result*

The main result of this section is the following theorem.

Theorem 2.2. *Assume that conditions* (H_0) *and* (H_1) *are satisfied, and suppose that x is a solution of* (2.3) *on an interval* $[t_0, \infty)$ *fulfilling* $V(x_{t_1}) < \infty$ *for some* $t_1 \in [t_0, \infty)$ *with* $V = V^+$ *(resp.* $V = V^-$*) in case of positive (resp. negative) feedback. Then x is not superexponential.*

Remark 2.3.

(1) Another formulation of Theorem 2.2 is that superexponential solutions of (2.3) must change sign infinitely many times on any subinterval of length 1. In view of the formulas (2.1)–(2.2) and transformation $x^i(t) \coloneqq \sigma^i w^i(\tau t - \gamma^i)$, this yields, that the component w^0 of a superexponential solution of the original equation (1.1) changes sign infinitely often on any subinterval of length τ. Due to symmetry of equation (1.1), this statement holds for any component w^i, $0 \leq i \leq N$.

(2) If (2.3) is autonomous, then – thanks to the C^1-smoothness of the right-hand side – assumption (H_1) reduces to

(H_1^a) $D_2 f^i(0,0) \neq 0$ for all $0 \leq i \leq N$.

The rest of this subsection is devoted to the proof of the above theorem. The proof is carried out in several steps via a number of auxiliary results.

Motivated by the arguments and transformations presented in Section 2.1, let us consider equations of the form

$$\dot{y}^i(t) = c^i(t) y^{i+1}(t-1), \qquad 0 \leq i \leq N, \quad t \in I, \tag{2.11}$$

where I is an interval, and the functions c^i are continuous for all $0 \leq i \leq N$ and fulfill the following hypothesis.

(H_2) There exist $0 < c_0 < c_1 < \infty$, such that $c_0 \leq \delta^i c^i(t) \leq c_1$ for all $0 \leq i \leq N$ and $t \in I$, with $\delta^N \in \{-1, 1\}$ and $\delta^i = 1$ for all $0 \leq i \leq N-1$.

For a solution $z\colon [t_0-1,\infty)\to\mathbb{R}^{N+1}$ of equation (2.9) and for $t\geq t_0$, we denote by z_t the map in $\mathcal{C}$ for which $z_t(s)=z(t+s)$ holds for all $s\in[-1,0]$. We apply the same notation for solutions of (2.11), too. Note that we also use this notation for segments of solutions x of the equation (2.3), where $x_t\in\mathcal{C}_{\mathbb{K}}$ (see (2.4)). This should not cause any confusion, as y_t and z_t will always be in $\mathcal{C}$, whereas x_t is in $\mathcal{C}_{\mathbb{K}}$. Nonetheless, these notations will mostly appear with norms applied to them, which will make it even more unambiguous, since the supremum norms are denoted differently in these spaces (i.e. $\|\cdot\|$, resp. $\|\cdot\|_{\mathcal{C}_{\mathbb{K}}}$).

The below results and their proofs follow the arguments and ideas presented in [25] and [24] for the scalar case and slow oscillations. As is also noted in those works, the main ideas originate from [28, 2] and [3].

The indexes should be understood modulo $N+1$ throughout the proofs.

Lemma 2.4. *Let I be an interval and y be a solution of (2.11) on I with continuous functions c^i satisfying (H_2) on I for all $0\leq i\leq N$. Then for all $\varepsilon\in(0,1)$ and $k\in\mathbb{N}$, there exists $C_k(\varepsilon,c_0)>0$ such that the inequality*

$$\min_{s\in J}|y^i(s)|\leq C_k\|y_t\| \tag{2.12}$$

holds for all $0\leq i\leq N$ and for any closed interval $J\subset[t-k,t]$ of length ε, provided $[t-k+1,t]\subseteq I$.

Proof. First of all, the statement holds obviously for $k=1$ with $C_1:=1$. Similarly, for arbitrary $k\in\mathbb{N}$, if $J\cap[t-1,t]\neq\emptyset$, then choosing $C_k\geq 1$ guarantees that (2.12) holds. Thus by assuming $C_k\geq1$ we may suppose that $J\subseteq[t-k,t-1]$ holds.

We prove the lemma by mathematical induction over k.

If $J:=[s_1-1,s_2-1]\subseteq[t-2,t-1]$ with $t-1\leq s_1<s_1+\varepsilon=s_2\leq t$, then the following estimates hold for all $0\leq i\leq N$:

$$2\|y_t\|\geq\left|y^{i-1}(s_2)-y^{i-1}(s_1)\right|=\left|\int_{s_1}^{s_2}c^{i-1}(u)y^i(u-1)\,du\right|\geq c_0\varepsilon\min_{s\in J}\left|y^i(s)\right|.$$

Hence the statement holds with $C_2(\varepsilon,c_0)=\frac{2}{c_0\varepsilon}$.

Now assume that the statement holds for some $k=m\geq2$ and consider intervals $J\subseteq[t-m-1,t-1]$ of length ε. If the length of the interval $J\cap[t-m,t-1]$ is at least $\frac{\varepsilon}{2}$, then inequality (2.12) holds by letting $C_{m+1}(\varepsilon,c_0):=C_m(\frac{\varepsilon}{2},c_0)$.

There remains the case when the length of the interval $J\cap[t-m-1,t-m]$ is strictly greater than $\frac{\varepsilon}{2}$. Then we can choose $t_1,t_2\in[t-m,t-(m-1)]$ such that $t_2=t_1+\frac{\varepsilon}{2}$ and $[t_1-1,t_2-1]\subset J$. Let us consider two subintervals $[t_1,t_1+\frac{\varepsilon}{6}]$ and $[t_2-\frac{\varepsilon}{6}]$ of $[t_1,t_2]$ and therefore of $[t-m,t-(m-1)]$. Fix an arbitrary index $0\leq i\leq N$. By the inductive hypothesis, there exist $u_1\in[t_1,t_1+\frac{\varepsilon}{6}]$ and $u_2\in[t_2-\frac{\varepsilon}{6},t_2]$ such that the inequalities

$$\left|y^{i-1}(u_j)\right|\leq C_m(\tfrac{\varepsilon}{6},c_0)\|y_t\|\qquad(j=1,2) \tag{2.13}$$

hold. Note that $\frac{\varepsilon}{6}\leq u_2-u_1$.

From the mean value theorem applied to y^{i-1} on the interval $[u_1,u_2]$ and in view of inequality (2.13) we obtain that there exists a $t^*\in[u_1,u_2]\subset[t_1,t_2]$, such that

$$\left|\dot{y}^{i-1}(t^*)\right|=\frac{\left|y^{i-1}(u_2)-y^{i-1}(u_1)\right|}{u_2-u_1}\leq\frac{12C_m(\frac{\varepsilon}{6},c_0)}{\varepsilon}\|y_t\|$$

holds. Noting that $t^*-1\in J$ and using hypothesis (H_2) and the fact that y is a solution of (2.11) we infer

$$\left|y^i(t^*-1)\right|\leq\frac{12C_m(\frac{\varepsilon}{6},c_0)}{\varepsilon c_0}\|y_t\|=:C_{m+1}(\varepsilon,c_0)\|y_t\|.$$

Summarizing the above cases, we obtain that (2.12) holds also for $m+1$ with

$$C_{m+1}(\varepsilon,c_0):=\max\left\{1,C_m(\tfrac{\varepsilon}{2},c_0),\frac{12C_m(\frac{\varepsilon}{6},c_0)}{\varepsilon c_0}\right\}.$$

This completes the proof. ∎

Proposition 2.5. *Fix $n \in \mathbb{N}_0$ and let y be a nontrivial solution of (2.11) on an interval I, where functions c^i are continuous and satisfy (H_2) on I for all $0 \leq i \leq N$. Then there exists a constant $k = k(n, N, c_0, c_1) > 0$, such that for all $t_0 \in I$, for which $t_0 - N - n - 3 \in I$ and y^0 has at most n sign-changes on any subinterval of $[t_0 - N - n - 4, t_0 - n - 1]$ of length 1, the inequality*

$$\|y_{t_0-1}\| \leq k\|y_{t_0}\|$$

holds.

Proof. Set $K := N + n + 1$ and let $\varepsilon \in (0, \frac{1}{2K})$ and $C := C_{K+2}(\varepsilon, c_0)$ from Lemma 2.4. Furthermore, for all $0 \leq m \leq n + N + 1$ define $k_m = (1 + \varepsilon c_1)^{K-m}C$. We claim that the statement holds with $k := k_0$. As always, all upper indexes are to be understood modulo $N + 1$.

Assume to the contrary that there exists $0 \leq i \leq N$ and $t^* \in [t_0 - 2, t_0 - 1]$ such that $|y^i(t^*)| > k_0\|y_{t_0}\|$. For definiteness assume that $y^i(t^*) > 0$ (the case $y^i(t^*) < 0$ is similar).

Applying Lemma 2.4 we obtain that there exists $l_0 \in [t^* - \varepsilon, t^*] \subset [t_0 - 3, t_0 - 1]$ and $r_0 \in [t^*, t^* + \varepsilon] \subset [t_0 - 2, t_0]$, such that $|y^i(l_0)| \leq C\|y_{t_0}\|$ and $|y^i(r_0)| \leq C\|y_{t_0}\|$ hold. The mean value theorem implies that there exist $s_{1,0} \in [l_0, t^*] \subset [t_0 - 3, t_0 - 1]$ and $s_{1,1} \in [t^*, r_0] \subset [t_0 - 2, t_0]$, such that inequalities

$$\dot{y}^i(s_{1,0}) > \frac{(k_0 - C)\|y_{t_0}\|}{\varepsilon} \quad \text{and} \quad \dot{y}^i(s_{1,1}) < -\frac{(k_0 - C)\|y_{t_0}\|}{\varepsilon}$$

hold. Combining this with (2.11) and (H_2) we obtain using the notation $t_{1,j} = s_{1,j} - 1$ $(j = 0, 1)$, that

$$\delta^i y^{i+1}(t_{1,0}) > \frac{(k_0 - C)\|y_{t_0}\|}{\varepsilon c_1} \geq \frac{(k_0 - k_1)\|y_{t_0}\|}{\varepsilon c_1} = k_1\|y_{t_0}\|$$

and

$$-\delta^i y^{i+1}(t_{1,1}) > \frac{(k_0 - C)\|y_{t_0}\|}{\varepsilon c_1} \geq \frac{(k_0 - k_1)\|y_{t_0}\|}{\varepsilon c_1} = k_1\|y_{t_0}\|$$

hold (note that $\delta^N \in \{-1, 1\}$ and $\delta^i = 1$ for all $0 \leq i \leq N - 1$). This, in particular, implies that y^{i+1} has at least one sign-change on the interval $[t^* - 1 - \varepsilon, t^* - 1 + \varepsilon] \subseteq [t_0 - 4, t_0 - 1]$.

We claim that for all $1 \leq m \leq K$ there exists $t_{m,0} < t_{m,1} < \cdots < t_{m,m}$, all from the interval $[t^* - m(1 + \varepsilon), t^* - m(1 - \varepsilon)]$, such that

$$y^{i+m}(t_{m,j}) \cdot (-1)^j \prod_{\ell=i}^{i+m-1} \delta^\ell > 0, \quad \text{for all } 0 \leq j \leq m, \tag{2.14}$$

holds and moreover, inequalities

$$|y^{i+m}(t_{m,0})| > k_m\|y_{t_0}\| \quad \text{and} \quad |y^{i+m}(t_{m,m})| > k_m\|y_{t_0}\| \tag{2.15}$$

are fulfilled. This, in particular, implies that function y^{i+m} changes sign at least m times on the interval $[t^* - m(1 + \varepsilon), t^* - m(1 - \varepsilon)]$.

We will prove this claim using mathematical induction. Note that we have just verified it for $m = 1$.

Now, let us assume that the claim holds for some $1 \leq m \leq K - 1$ and let us verify properties (2.14) and (2.15) for $m + 1$. Note that by our assumptions and the choice of ε,

$$t_0 - (K + 2) \leq t^* - m - (m + 1)\varepsilon \leq t_{m,0} - \varepsilon < t_{m,m} + \varepsilon \leq t^* - m + (m + 1)\varepsilon \leq t_0,$$

hence Lemma 2.4 can be applied to obtain that there exist $l_m \in [t_{m,0} - \varepsilon, t_{m,0}]$ and $r_m \in [t_{m,m}, t_{m,m} + \varepsilon]$ such that $|y^{i+m}(l_m)| \leq C\|y_{t_0}\|$ and $|y^{i+m}(r_m)| \leq C\|y_{t_0}\|$. Bearing this in mind, let us apply the mean value theorem (for the function $y^{i+m}\prod_{\ell=i}^{i+m-1}\delta^\ell$, resp. $y^{i+m}(-1)^m\prod_{\ell=i}^{i+m-1}\delta^\ell$ on the interval $[l_m, t_{m,0}]$, resp. $[t_{m,m}, r_m]$) and use assumption (2.14) to obtain that there exist $s_{m+1,0} \in [l_m, t_{m,0}]$ and $s_{m+1,m+1} \in [t_{m,m}, r_m]$, such that

$$\dot{y}^{i+m}(s_{m+1,0}) \cdot \prod_{\ell=i}^{i+m-1} \delta^\ell > \frac{(k_m - C)\|y_{t_0}\|}{\varepsilon}$$

and

$$\dot{y}^{i+m}(s_{m+1,m+1})\cdot(-1)^{m+1}\prod_{\ell=i}^{i+m-1}\delta^\ell > \frac{(k_m-C)\|y_{t_0}\|}{\varepsilon}.$$

Then (2.11) and (H_2) imply that for $t_{m+1,0} \coloneqq s_{m+1,0}-1$ and $t_{m+1,m+1} \coloneqq s_{m+1,m+1}-1$, the estimates

$$y^{i+m+1}(t_{m+1,0})\cdot\prod_{\ell=i}^{i+m}\delta^\ell > \frac{(k_m-C)\|y_{t_0}\|}{\varepsilon c_1} \geq \frac{(k_m-k_{m+1})\|y_{t_0}\|}{\varepsilon c_1} = k_{m+1}\|y_{t_0}\|$$

and

$$y^{i+m+1}(t_{m+1,m+1})\cdot(-1)^{m+1}\prod_{\ell=i}^{i+m}\delta^\ell > \frac{(k_m-C)\|y_{t_0}\|}{\varepsilon c_1} \geq \frac{(k_m-k_{m+1})\|y_{t_0}\|}{\varepsilon c_1} = k_{m+1}\|y_{t_0}\|$$

hold, that is, we have verified (2.15) for $m+1$, and (2.14) for $t_{m+1,0}$ and $t_{m+1,m+1}$.

As for the rest of (2.14), apply again the mean value theorem (for function $y^{i+m}\cdot\prod_{\ell=i}^{i+m-1}\delta^\ell$ on each interval $[t_{m,j-1},t_{m,j}]$ with $1\leq j\leq m$) to obtain that there exist $s_{m+1,j}\in(t_{m,j-1},t_{m,j})$, such that

$$\dot{y}^{i+m}(s_{m+1,j})\cdot(-1)^j\prod_{\ell=i}^{i+m-1}\delta^\ell > 0, \quad \text{for all } 1\leq j\leq m.$$

Finally, if we define $t_{m+1,j}=s_{m+1,j}-1$ for $0\leq j\leq m$, the application of (2.11) yields that

$$y^{i+m+1}(t_{m+1,j})\cdot(-1)^j\prod_{\ell=i}^{i+m}\delta^\ell > 0,$$

holds for all $1\leq j\leq m$. By definition, $t_{m+1,j}<t_{m+1,\ell}$ clearly holds for all $0\leq j<\ell\leq m$. From $t^*-m(1+\varepsilon)\leq t_{m,0}<t_{m,m}\leq t^*-m(1-\varepsilon)$ one also immediately infers $t^*-(m+1)(1+\varepsilon)\leq t_{m+1,0}<t_{m+1,m+1}\leq t^*-(m+1)(1-\varepsilon)$. This completes the proof of the claim.

Now let

$$m_1 \coloneqq \min\{m\in\mathbb{N} : m>n \text{ and } (i+m) \text{ is a multiple of } (N+1)\}.$$

Obviously, $n+1\leq m_1\leq K$ must hold. Then, according to the above claim and bearing in mind that $2K\varepsilon<1$ and $t^*\in[t_0-2,t_0-1]$, we obtain that y^{i+m_1} (i.e. y^0) changes sign at least m_1 times on the interval $[t^*-m_1(1+\varepsilon),t^*-m_1(1-\varepsilon)]$, which has length at most 1 and is a subinterval of $[t_0-N-n-4,t_0-n-1]$. This contradicts to the assumption that y^0 has at most n sign-changes on any such interval of length 1. ☐

Corollary 2.6. *Fix $n\in\mathbb{N}_0$ and set $\hat{K}\coloneqq N+n+4$. Let z be a nontrivial solution of (2.9) on an interval I, where functions a^i and b^i are continuous and they satisfy (H_1'') on I for all $0\leq i\leq N$. Then there exists a constant $\hat{k}=\hat{k}(n,N,a_1,b_0,b_1)>0$, such that for all $t_0\in I$, for which $t_0-\hat{K}\in I$ and z^0 has at most n sign-changes on any subinterval of $[t_0-\hat{K},t_0-n-1]$ of length 1,*

$$\|z_{t_0-1}\|\leq\hat{k}\|z_{t_0}\|$$

holds.

Proof. Let $s_0=t_0-\hat{K}$ and apply the change of variables $y^i(t)\coloneqq\exp\big(-\int_{s_0}^t a^i(s)\,ds\big)z^i(t)$ for $t\in[s_0,t_0]$. Then $y\colon[s_0,t_0]\to\mathbb{R}^{N+1}$ is a solution of (2.11) on the interval $[s_0+1,t_0]$ with $c^i\colon[s_0+1,t_0]\to\mathbb{R}$, $c^i(t)=b^i(t)\exp\big(\int_{s_0}^{t-1}a^{i+1}(s)\,ds-\int_{s_0}^t a^i(s)\,ds\big)$ for all $0\leq i\leq N$. Moreover, y changes sign exactly where z does on $[s_0,t_0]$, thus, in particular, at most n times on each subinterval of $[t_0-\hat{K},t_0-n-1]$ of length 1.

The functions c^i are obviously continuous and they fulfill hypothesis (H_2) with $c_0\coloneqq b_0\exp(-2\hat{K}a_1)$ and $c_1\coloneqq b_1\exp(2\hat{K}a_1)$.

Therefore, by virtue of Proposition 2.5 and (H_1''), we obtain that there exist $k>0$ such that the estimates

$$\begin{aligned}
\|z_{t_0-1}\| &= \max_{\substack{0\le i\le N\\ s\in[t_0-2,t_0-1]}} \left\{ \left| \exp\left(\int_{s_0}^{s} a^i(u)\,du\right) y^i(s) \right| \right\} \\
&\le e^{a_1(t_0-1-s_0)} \|y_{t_0-1}\| \\
&\le k e^{a_1 \hat{K}} \|y_{t_0}\| \\
&= k e^{a_1 \hat{K}} \max_{\substack{0\le i\le N\\ s\in[t_0-1,t_0]}} \left\{ \left| \exp\left(-\int_{s_0}^{s} a^i(u)\,du\right) z^i(s) \right| \right\} \\
&\le k e^{2a_1 \hat{K}} \|z_{t_0}\|
\end{aligned}$$

hold, which proves our statement. □

It would be desirable to have an estimate on the decay rate similar to the one in Corollary 2.6, but in the $\mathcal{C}_{\mathbb{K}}$ norm and for solutions of (2.6). A particular reason for this is that we will need such an estimate for the proof of Proposition 3.9, which is essential for the proof of our other main result, Theorem 3.2. To achieve this, we need first the following technical lemma that gives a bound on the growth rate in forward time.

Lemma 2.7. *Let x be a solution of equation (2.6) on the interval $[t_0,t_1]$ and assume that there exist positive constants α_1 and β_1, such that $|\alpha^i(t)|\le\alpha_1$ and $|\beta^i(t)|\le\beta_1$ hold for all $t\in[t_0,t_1]$ and for all $0\le i\le N$. Then the inequality*

$$\|x_t\|_{\mathcal{C}_{\mathbb{K}}} \le \exp\left(\alpha_1(t-t_0)+\beta_1(t-t_0)e^{2\alpha_1(t-t_0)}\right)\|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}}$$

holds for all $t\in[t_0,t_1]$.

Proof. From equation (2.6) we obtain

$$\frac{d}{dt}\left(e^{-\int_{t_0}^{t}\alpha^i(s)\,ds}x^i(t)\right) = e^{-\int_{t_0}^{t}\alpha^i(s)\,ds}\beta^i(t)x^{i+1}(t),$$

from which we deduce

$$x^i(t) = e^{\int_{t_0}^{t}\alpha^i(s)\,ds}\left(x^i(t_0)+\int_{t_0}^{t} e^{-\int_{t_0}^{u}\alpha^i(s)\,ds}\beta^i(u)x^{i+1}(u)\,du\right).$$

This combined with the triangle inequality and the assumptions on α^i and β^i implies that the inequalities

$$\begin{aligned}
|x^i(t)| &\le e^{\alpha_1(t-t_0)}|x^i(t_0)| + e^{\alpha_1(t-t_0)}\int_{t_0}^{t} e^{\alpha_1(u-t_0)}\beta_1|x^{i+1}(u)|\,du \\
&\le e^{\alpha_1(t-t_0)}\|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}} + e^{\alpha_1(t-t_0)}\int_{t_0}^{t} e^{\alpha_1(u-t_0)}\beta_1\|x_u\|_{\mathcal{C}_{\mathbb{K}}}\,du
\end{aligned}$$

hold for all $t\in[t_0,t_1]$.

Using the definition of the norm $\|\cdot\|_{\mathcal{C}_{\mathbb{K}}}$, one easily obtains that

$$\|x_t\|_{\mathcal{C}_{\mathbb{K}}} \le e^{\alpha_1(t-t_0)}\|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}} + \int_{t_0}^{t} e^{\alpha_1(t+u-2t_0)}\beta_1\|x_u\|_{\mathcal{C}_{\mathbb{K}}}\,du$$

holds for all $t\in[t_0,t_1]$. Thus by a Grönwall-type inequality (see e.g. [35, Theorem 1.4.2]) we deduce

$$\begin{aligned}
\|x_t\|_{\mathcal{C}_{\mathbb{K}}} &\le e^{\alpha_1(t-t_0)}\|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}}\exp\left(e^{\alpha_1(t-2t_0)}\beta_1\int_{t_0}^{t} e^{\alpha_1 u}\,du\right) \\
&\le \|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}}\exp\left(\alpha_1(t-t_0)+\beta_1(t-t_0)e^{2\alpha_1(t-t_0)}\right).
\end{aligned}$$

□

Proposition 2.8. *Fix $n \in \mathbb{N}_0$ and let V denote either V^+ or V^- (determined by the feedback). Let x be a nontrivial solution of (2.6) on an interval I. Assume that the functions α^i and β^i are continuous and satisfy hypothesis (H_1') on I. Suppose further that $V(x_t) \le n$ for all $t \in I$.*

Then there exists a constant $k_1 = k_1(n, N, \alpha_1, \beta_0, \beta_1) > 0$, such that for all $t_0 \in I$, for which $I \supseteq I_1 := [t_0 - 3 - \frac{n+2}{N+1}, t_0 + 1]$,

$$\|x_{t_0-1}\|_{\mathcal{C}_{\mathbb{K}}} \le k_1 \|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}}$$

holds.

Proof. Let $z^i(t) := x^i\big(t_0 + \frac{t+i}{N+1}\big)$. Then, on the one hand, z is a solution of (2.9)–(2.10) on the interval including $[-3(N+1)-n-2, 0]$. Moreover, the functions a^i and b^i fulfill hypothesis (H_1'') with $a_1 := \frac{\alpha_1}{N+1}$, $b_0 := \frac{\beta_0}{N+1}$ and $b_1 := \frac{\beta_1}{N+1}$.

These show that the inequalities

$$\begin{aligned}
\|x_{t_0-1}\|_{\mathcal{C}_{\mathbb{K}}} &= \max_{\substack{s\in[t_0-2,t_0-1]\\ 1\le i\le N}} \left\{|x^0(s)|, |x^i(t_0-1)|\right\} \\
&= \max_{\substack{s\in[-2(N+1),-(N+1)]\\ 1\le i\le N}} \left\{|z^0(s)|, |z^i(-(N+1+i))|\right\} \\
&\le \max_{0\le i\le N} \|z_{t_1-i}\|,
\end{aligned}$$

hold, where $t_1 := -(N+1)$.

On the other hand, by our assumptions and the definition of V, x^0 has at most n sign-changes on any subinterval of I_1 of length 1. Consequently z^0 has at most n sign-changes on any subinterval of $[-3(N+1)-n-2, 0]$ of length $N+1$. In particular this holds for subintervals of length 1, therefore we can apply Corollary 2.6 to obtain the inequalities $\|z_{t_1-i}\| \le \hat{k}^i \|z_{t_1}\|$ for $0 \le i \le N$. Assuming – without loss of generality – that $\hat{k} \ge 1$ we deduce

$$\|x_{t_0-1}\|_{\mathcal{C}_{\mathbb{K}}} \le \max_{0\le i\le N} \|z_{t_1-i}\| \le \hat{k}^N \ \|z_{t_1}\| \le \hat{k}^{2N+1} \|z_{t_1+N+1}\| = \hat{k}^{2N+1}\|z_0\|.$$

This in turn yields that there exists some $s \in [-1, 0]$ and $0 \le j \le N$, such that

$$\|x_{t_0-1}\|_{\mathcal{C}_{\mathbb{K}}} \le \hat{k}^{2N+1} |z^j(s)| = \hat{k}^{2N+1} \Big| x^j\Big(t_0 + \tfrac{s+j}{N+1}\Big)\Big| \le \hat{k}^{2N+1}\Big(\|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}} + \|x_r\|_{\mathcal{C}_{\mathbb{K}}}\Big),$$

where $r = t_0 + \frac{s+j}{N+1}$. Since $r \in [t_0, t_0+1]$, Lemma 2.7 implies that

$$\|x_r\|_{\mathcal{C}_{\mathbb{K}}} \le \exp\Big(\alpha_1 + \beta_1 e^{2\alpha_1}\Big) \|x_{t_0}\|_{\mathcal{C}_{\mathbb{K}}},$$

which shows that the statement holds with $k_1 = \hat{k}^{2N+1}\big(1 + \exp\big(\alpha_1 + \beta_1 e^{2\alpha_1}\big)\big)$. $\square$

Now we are in position to prove the main theorem of this subsection.

Proof of Theorem 2.2. Assume to the contrary that x is a superexponential solution of (2.3) on an interval $[t_0, \infty)$. In particular, there exists some $t_2 \ge t_1$ such that $x^i(t) \le M$ for all $t \ge t_2 - 1$ and $0 \le i \le N$, where M is from hypothesis (H_1).

As already shown in Section 2.1, x is also a solution of (2.6)–(2.8) on $[t_2, \infty)$, and the functions α^i and β^i satisfy (H_1') on this interval.

Proposition 2.1 yields that $V(x_t) \le V(x_{t_1})$ holds for all $t \in [t_2, \infty)$. Now setting $t_3 := t_2 + 2 + \frac{n+2}{N+1}$, we obtain by Proposition 2.8 that there exists a constant k_1 such that

$$k_1^m \|x_{t_3+m}\|_{\mathcal{C}_{\mathbb{K}}} \ge \|x_{t_3}\|_{\mathcal{C}_{\mathbb{K}}}$$

holds for all $m \in \mathbb{N}$. This clearly contradicts to the assumption that x is superexponential. $\square$

2.3. ***No superexponential solutions on the global attractor***

Recall that a unique forward solution to (2.3) with initial condition $x_{t_0} = \varphi \in \mathcal{C}_{\mathbb{K}}$ exists on a maximal interval (t_0, t^*) and that $t^* < \infty$ can only occur if $\|x_t\|_{\mathcal{C}_{\mathbb{K}}} \to \infty$, as $t \to t^*$.

For readers that are not familiar with nonautonomous dynamical systems, we recall below some basic notions. For a general overview of the topic we refer the interested reader to [4].

Let us assume that $t^* = \infty$ for all maximal solutions – this is the case, if e.g. the functions f^i from (2.3) are bounded. Then (2.3) induces a *process* $S(\cdot,\cdot)$ on $\mathcal{C}_{\mathbb{K}}$, i.e. a family of continuous maps $\{S(t,s) : t \geq s\}$ from $\mathcal{C}_{\mathbb{K}}$ to itself with the properties

- $S(t,t) = \mathrm{id}$ for all $t \in \mathbb{R}$,
- $S(t,s) = S(t,r)S(r,s)$ for all $t \geq r \geq s$,
- $(t,s,\varphi) \mapsto S(t,s)\varphi$ is continuous, $t \geq s$, $\varphi \in \mathcal{C}_{\mathbb{K}}$,
- $S(t,t_0)\varphi = x_t$, where x_t is the unique solution of (2.3) with initial condition $x_{t_0} = \varphi$.

We say that the compact family of sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}} \subset \mathcal{C}_{\mathbb{K}}$ is a *global pullback attractor of* S, if

- $\mathcal{A}(t)$ is compact for all $t \in \mathbb{R}$,
- $\mathcal{A}(\cdot)$ is *invariant under* S in the following sense: $S(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ holds for all $t \geq s$, and
- $\mathcal{A}(\cdot)$ *pullback attracts* all bounded subsets $B \subset \mathcal{C}_{\mathbb{K}}$, i.e.

$$\lim_{s\to-\infty} \mathrm{dist}(S(t,s)B, \mathcal{A}(t)) = 0.$$

Finally, we call a family of bounded sets $\{B(t)\}_{t\in\mathbb{R}}$ *backwards-bounded*, if $\bigcup_{s\leq t} B(s)$ is bounded for each $t \in \mathbb{R}$.

Now we are in position to state the main result of this subsection, which is a corollary of Theorem 2.2.

Theorem 2.9. *If* (2.3) *possesses a backwards-bounded global pullback attractor (resp. a global attractor in the autonomous case), then there are no superexponential solutions on that.*

Proof of Theorem 2.9. Recall that a backwards-bounded global pullback attractor (resp. the global attractor in the autonomous case) is uniquely defined and consists of the segments x_t of all backwards-bounded (resp. bounded) entire solutions x [4, Theorem 1.17], (resp. [38, Lemma 2.18]). Then Corollary 4.6 of [29] yields that $V(x_t) < \infty$ holds for all $t \in \mathbb{R}$, provided x is a nontrivial backwards-bounded entire solution, and consequently this holds if x is a solution from the attractor. Then Theorem 2.2 implies our statement. $\square$

Remark 2.10. We note that in case the functions α^i and β^i in (2.6) are analytic, then the main theorem of Cooke and Lunel [8] (see also (3.9)) would guarantee that there are no superexponential solutions at all (i.e. even the additional assumption on V could be omitted). In general, one cannot expect analyticity of those functions, moreover, Cooke and Derfel [10, 9] also illustrated by means of counterexamples that the assumption on analyticity in [8] is not merely a technicality.

Nevertheless, in the autonomous case, if the functions f^i are analytic, then Nussbaum's result [33, Theorem 1] guarantees that all bounded solutions of (2.3) are analytic. The global attractor – provided it exists – coincides with the set of $\varphi \in \mathcal{C}_{\mathbb{K}}$, such that there is a bounded entire solution through φ [38, Lemma 2.18], thus solutions from the attractor are analytic. Now, if we take an arbitrary solution from the attractor, and consider the conjugated equation (2.6), then the coefficient functions α^i and β^i are necessarily analytic, thus the result from [8] can be applied in this case to conclude that the solution cannot be superexponential.

Example 2.11. It is not hard to give sufficient conditions for the existence of a backwards-bounded global pullback attractor (resp. of the global attractor in the autonomous case).

Assume for example that there exists R, L and ε positive numbers, such that conditions

$$f^i(u,v,t) \leq -\varepsilon \quad \text{if } u \geq R, \tag{2.16}$$

$$f^i(u,v,t) \geq \varepsilon \quad \text{if } u \leq -R, \tag{2.17}$$

and

$$|f^i(u,v,t)| \leq L \quad \text{for all } (u,v,t) \in [-R,R]^2 \times \mathbb{R} \tag{2.18}$$

are fulfilled for all $0 \leq i \leq N$. Then it is easy to see that solutions exist globally in forward time, and that for every bounded $B \in \mathcal{C}_{\mathbb{K}}$ there exist $r = r(B)$, such that $S(t,s)B \subset B_{R,L}$ for all $s \leq t - r$, where

$$B_{R,L} \coloneqq \{\varphi \in \mathcal{C}_{\mathbb{K}} : \|\varphi\| \leq R \text{ and } \varphi|_{[-1,0]} \text{ is } L\text{-Lipschitz}\}.$$

Thus $B_{R,L}$ pullback attracts all bounded sets from $\mathcal{C}_{\mathbb{K}}$. On the other hand, by the Arzelà–Ascoli theorem, this set is compact, and consequently (see e.g. [4, Theorem 2.12]) a uniquely defined global pullback attractor $\mathcal{A}(\cdot)$ exists, moreover, $\mathcal{A}(t) \subseteq B_{R,L}$ for all $t \in \mathbb{R}$, so it is uniformly bounded, thus in particular, backwards-bounded.

We note that the above conditions (2.16)–(2.18) could be relaxed to obtain assumptions similar to that in Example 3.1, but here our goal was to give some criteria from which the existence of a backwards-bounded global pullback attractor can be seen very easily. The interested reader may also find sufficient criteria for existence and (backwards-)boundedness of the global pullback attractor in [4], especially in Section 10.2.

Finally, let us give a more concrete class of feedback functions, for which (H_0) and (H_1), as well as (2.16)–(2.18) hold. For all $0 \leq i \leq N$ let $h_1^i, h_2^i : \mathbb{R} \to \mathbb{R}$ be uniformly bounded, and uniformly positive, continuous functions. Furthermore, assume that for each i, $\tilde{f}^i$ is a bounded C^1-function with positive derivative at 0, and assume that $v\tilde{f}^i(v) > 0$ for all $v \in \mathbb{R} \setminus \{0\}$ and $0 \leq i \leq N$ (take e.g. $\tilde{f}^i(v) = \tanh(v)$).

Then the feedback functions

$$f^i(u,v,t) = -h_1^i(t)u + \delta^i h_2^i(t)\tilde{f}^i(v), \qquad 0 \leq i \leq N$$

clearly fulfill hypotheses (H_0) and (H_1), as well as conditions (2.16)–(2.18), where $\delta^N = \pm 1$ and $\delta^i = 1$ for all $0 \leq i \leq N-1$.

3. Morse decomposition of the global attractor

In this section we shall focus on the autonomous equation

$$\begin{aligned} \dot{x}^i(t) &= f^i(x^i(t), x^{i+1}(t)), \qquad && 0 \leq i \leq N-1 \\ \dot{x}^N(t) &= f^N(x^N(t), x^0(t-1)), && \end{aligned} \tag{3.1}$$

where the functions f^i are C^1-smooth on $\mathbb{R}^2$, and they fulfill the feedback assumptions:

(H_0^a) $\delta^i v f^i(0,v) > 0$ holds for all $v \neq 0$ and $0 \leq i \leq N$ with $\delta^N \in \{-1,1\}$ and $\delta^i = 1$ for all $0 \leq i \leq N-1$;

(H_1^a) $D_2 f^i(0,0) \neq 0$ for all $0 \leq i \leq N$.

As already noted in Remark 2.3, since C^1-smoothness is assumed, (H_0)–(H_1) are equivalent to (H_0^a)–(H_1^a) in the autonomous case, and in particular Theorems 2.2 and 2.9 can be applied.

Furthermore, from now on we shall assume

(H_3) solutions to (3.1) exist globally in forward time, and (3.1) possesses a global attractor.

Example 3.1. Suppose that there exists $R > 0$, such that

$$f^i(u,v) < 0 \quad \text{if } u \geq R \text{ and } |v| \leq u \tag{3.2}$$

and

$$f^i(u,v) > 0 \quad \text{if } u \leq -R \text{ and } |v| \leq |u| \tag{3.3}$$

hold for all $0 \leq i \leq N$. Then one can easily prove – in an analogous way as it was done for the scalar case ($N = 0$) in [37, Corollary 3.2] – that solutions to (3.1) exist globally in forward time and a unique, connected global attractor exists.

The main result of this section establishes that under the above hypotheses, the global attractor of (3.1) possesses a *Morse decomposition*, that is, a finite ordered collection

$$\mathcal{M}_1 \prec \mathcal{M}_2 \prec \cdots \prec \mathcal{M}_m$$

of pairwise disjoint, compact and invariant *Morse sets* $\mathcal{M}_1, \dots, \mathcal{M}_m \subseteq \mathcal{A}$ such that for all $\varphi \in \mathcal{A}$ and any bounded entire solution x with $x_0 = \varphi$, there exist $i \geq j$ with

(m_1) $\alpha(x) \subseteq \mathcal{M}_i$ and $\omega(\varphi) \subseteq \mathcal{M}_j$,

(m_2) $i = j$ implies $\varphi \in \mathcal{M}_i$ (thus, $x_t \in \mathcal{M}_i$ for all $t \in \mathbb{R}$).

Note that – similarly as in [28] and [37] – in the lack of backward uniqueness, we have to deal with α-limit sets of concrete entire solutions (instead of that of an element in the state space). This is reflected in our notation.

In order to state the main result of this section we need to study the linearization of (3.1), namely

$$\begin{aligned} \dot{x}^i(t) &= \mu^i x^i(t) + \gamma^i x^{i+1}(t), \qquad\qquad 0 \leq i \leq N-1 \\ \dot{x}^N(t) &= \mu^N x^N(t) + \gamma^N x^0(t-1)), \end{aligned} \tag{3.4}$$

where $\mu^i = D_1 f^i(0,0)$, and $\gamma^i = D_2 f^i(0,0)$. Then $\delta^N \gamma^N > 0$ and $\gamma^i \in (0,\infty)$ for all $0 \leq i \leq N-1$ hold. The eigenvalues of (3.4) are exactly those numbers $\lambda \in \mathbb{C}$ that solve the characteristic equation

$$\prod_{i=0}^{N} (\lambda - \mu^i) - e^{-\lambda} \prod_{i=0}^{N} \gamma^i = 0. \tag{3.5}$$

Let us denote by M^* the number of eigenvalues (counting multiplicities) of (3.5) with strictly positive real part. Note that, according to the last statement of Proposition 3.6, M^* is always a finite number. Moreover, let

$$N_+^* := \begin{cases} M^* + 1, & \text{if the origin is nonhyperbolic and } M^* \text{ is odd,} \\ M^*, & \text{otherwise.} \end{cases}$$

$$N_-^* := \begin{cases} M^* + 1, & \text{if the origin is nonhyperbolic and } M^* \text{ is even,} \\ M^*, & \text{otherwise.} \end{cases}$$

Now we are in position to state the main result of the section, which is a generalization of the main result of [37] and of [28, Theorem B].

Theorem 3.2. *If* (3.1) *has a global attractor* $\mathcal{A}$ *and the assumptions* (H_0^a) *and* (H_1^a) *hold, then some finite collection of the following sets forms a Morse decomposition of* $\mathcal{A}$*:*

$$\begin{aligned} \mathcal{M}_n = \Big\{ \varphi \in \mathcal{A} \setminus \{0\} : (3.1) &\text{ has a bounded entire solution } x \text{ through } \varphi \\ &\text{with } V(x_t) \equiv n \text{ on } \mathbb{R} \text{ and } 0 \notin \alpha(x) \cup \omega(\varphi) \Big\}, \quad \text{for } n \in \mathbb{N}_0 \setminus \{N^*\}, \end{aligned}$$

$$\mathcal{M}_{N^*} = \Big\{ \varphi \in \mathcal{A} \setminus \{0\} : (3.1) \text{ has a bounded entire solution } x \text{ through } \varphi \text{ with } V(x_t) \equiv N^* \text{ on } \mathbb{R} \Big\} \cup \{0\},$$

where V *and* N^* *denote* $V^{\pm}$ *and* $N^*_{\pm}$*, respectively, and "$\pm$" indicates the sign of* δ^N*.*

Before we begin with the proof, it is appropriate to add some comments to the theorem.

Remark 3.3. Notice that although the definition of N^* formally differs from the ones given in [28] and [37], our definition, under their assumptions, reduces to those in the scalar case. This is because in that case, even more information is known about the spectrum. In particular, under the assumption $|D_1 f(0,0)| < |D_2 f(0,0)|$ one has that M^* is an odd (resp. even) number in case of positive (resp. negative) feedback, therefore the term "and M^* is odd" can be omitted from the definition of N^*_+, and likewise in N^*_-.

Remark 3.4. We stress that Theorem 3.2 itself does not exclude the possibility that the decomposition is trivial, i.e. $M_{N^*} = \mathcal{A}$. This is certainly the case if the trivial solution is globally attractive. In turn, it is an interesting and highly nontrivial question under what conditions the Morse decomposition is nontrivial.

In the scalar case ($N = 0$), much is known: Mallet-Paret already showed in Theorem D of [28] that in case of negative feedback, the obtained Morse decomposition is nontrivial, provided $N^* > 1$. More precisely, he proved the existence of periodic solutions in the Morse sets having an odd index, that is smaller than N^*. Among others, similar results can be found in the monograph of Krisztin, Walther and Wu for the positive feedback case too, under some additional assumptions on the right-hand side.

For the general $N \geq 1$ case, sufficient conditions for the existence of certain types of periodic solutions are obtained in [5, 6, 14, 22, 41] and in the recent work [23]. These results may provide sufficient conditions for a nontrivial Morse decomposition.

The rest of the paper is devoted to the proof of Theorem 3.2, which follows very closely the argument presented in [37] for the scalar case ($N = 0$) and positive feedback. That proof is based on the ideas of [28] but exploiting some results from [29] and [30]. Since the latter results all apply to equation (3.1), essentially the only thing that has been missing for the proof of Theorem 3.2 is the absence of small solutions on the attractor, more precisely, the estimate on the decay from Proposition 2.8.

The proof is carried out in a sequence of lemmas and auxiliary statements, many of which are straightforward modifications of the corresponding ones from [37]. In such cases, we shall leave the proofs for the reader, and only a reference for the analogous result will be given. Nevertheless, some steps may require less obvious modifications in the arguments. These will be discussed in detail.

The proof extensively uses some results from [29] and [30], which we will recall in the following subsection for the reader's convenience.

3.1. ***Preliminaries***

In the proof we will benefit from the properties of the discrete Lyapunov function $V^{\pm}$. We summarize these in Proposition 3.5.

Let

$$\mathcal{C}^1_{\mathbb{K}} \coloneqq \{\varphi \in \mathcal{C}_{\mathbb{K}} : \text{the restriction } \varphi|_{[-1,0]} \text{ is continuously differentiable}\},$$

and observe that for a solution x on $[t_0, \infty)$ one has that $x_t \in \mathcal{C}^1_{\mathbb{K}}$ for all $t \in [t_0, \infty)$. We furnish this space with the norm $\|\cdot\|_{\mathcal{C}^1_{\mathbb{K}}}$, where

$$\|\varphi\|_{\mathcal{C}^1_{\mathbb{K}}} \coloneqq \|\varphi\|_{\mathcal{C}_{\mathbb{K}}} + \sup_{\theta\in[-1,0]} |\dot\varphi(\theta)|.$$

Furthermore, let us define the following subsets of $\mathcal{C}^1_{\mathbb{K}}$:

$$\begin{aligned}
S^0 &\coloneqq \{\varphi \in \mathcal{C}^1_{\mathbb{K}} : \text{if } \varphi(0) = 0, \text{ then } \delta^0\dot\varphi(0)\varphi(1) > 0\},\\
S^{-1} &\coloneqq \{\varphi \in \mathcal{C}^1_{\mathbb{K}} : \text{if } \varphi(-1) = 0, \text{ then } \delta^N\varphi(N)\dot\varphi(-1) < 0\},\\
S^* &\coloneqq \{\varphi \in \mathcal{C}^1_{\mathbb{K}} : \text{if } \varphi(\theta) = 0, \text{ for some } \theta \in [-1,0], \text{ then } \dot\varphi(\theta) \neq 0\},\\
S^i &\coloneqq \{\varphi \in \mathcal{C}^1_{\mathbb{K}} : \text{if } \varphi(i) = 0 \text{ then } \delta^i\varphi(i-1)\varphi(i+1) < 0\}
\end{aligned}$$

for $1 \leq i \leq N$ with $\varphi(N+1) \coloneqq \varphi(-1)$ and set

$$S \coloneqq \left(\bigcap_{i=-1}^{N} S^i\right) \cap S^*.$$

The proof of the proposition below can be found in [29] (see Theorems 4.3 and 4.4).

Proposition 3.5. *The following statements hold for $V = V^{\pm}$.*

(i) The function V is lower semicontinuous, i.e. for every $\varphi \in \mathcal{C}_{\mathbb{K}} \setminus \{0\}$ and for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_{\mathbb{K}} \setminus \{0\}$ with $\varphi_n \to \varphi$ (as $n \to \infty$),

$$V(\varphi) \leq \liminf_{n\to\infty} V(\varphi_n).$$

(ii) *The function V is finite and continuous, hence locally constant, on the open dense subset $S \subset \mathcal{C}^1_{\mathbb{K}} \setminus \{0\}$ in the $\mathcal{C}^1_{\mathbb{K}}$ topology; that is, if $\varphi \in S$, then there exists $\varepsilon > 0$ such that*

$$V(\psi) = V(\varphi) < \infty \quad \text{if} \quad \|\psi - \varphi\|_{\mathcal{C}^1_{\mathbb{K}}} < \varepsilon.$$

(iii) *Suppose that x is a solution of equation (2.3) on an interval I, and that for some $t \in I$ also $t-4 \in I$ and $V(x_t) = V(x_{t-4}) < \infty$ hold. Then $x_t \in S$.*

Note that assertion (iii) is stated for the general equation (2.3), and as such, it applies to solutions of (2.9), (2.11) or (3.1), as well.

It will be crucial to have information on the distribution of the eigenvalues of the linearized equation (3.4) and about the oscillatory properties of the sums of eigensolutions. These are summarized in the next two propositions.

Let

$$P := \{\operatorname{Re}\lambda : \lambda \in \mathbb{C} \text{ is a root of } (3.5)\}.$$

Furthermore, for some $\sigma \in P$, denote by $\mathcal{G}_\sigma \subset \mathcal{C}_{\mathbb{K}}$ the realified generalized eigenspace of the generator of the semigroup given by the linear equation (3.4), that is associated with the spectral set

$$\mathfrak{S}_\sigma := \{\alpha \in \mathbb{C} : \alpha \text{ is an eigenvalue of } (3.4), \text{ such that } \operatorname{Re}\alpha = \sigma\}. \tag{3.6}$$

Proposition 3.6 ([30, Corollary 3.3, Lemma 7.4])**.** *If $\delta^N = -1$, then the real parts $\sigma = \operatorname{Re}\lambda$ of the roots of the characteristic equation (3.5) can be ordered (with multiplicity)*

$$\sigma^0 \geq \sigma^1 > \sigma^2 \geq \sigma^3 > \sigma^4 \geq \cdots, \tag{3.7}$$

and one has

$$V^-(x^i_t) = \begin{cases} i+1, & \text{if } i \text{ is even,} \\ i, & \text{if } i \text{ is odd,} \end{cases}$$

for all $t \in \mathbb{R}$ and for any solution x^i from $\mathcal{G}_{\sigma^i} \setminus \{0\}$.

If $\delta^N = 1$, then one has

$$\sigma^0 > \sigma^1 \geq \sigma^2 > \sigma^3 \geq \sigma^4 > \cdots, \tag{3.8}$$

and

$$V^+(x^i_t) = \begin{cases} i, & \text{if } i \text{ is even,} \\ i+1, & \text{if } i \text{ is odd,} \end{cases}$$

holds for all $t \in \mathbb{R}$ and for any solution x^i from $\mathcal{G}_{\sigma^i} \setminus \{0\}$.

In both cases ($\delta^N = \pm 1$), $\dim \mathcal{G}_\sigma \leq 2$ for all $\sigma \in P$, moreover, there are only finitely many roots of (3.5) to the right of any vertical line in the complex plane.

Motivated by the above statements, we will sometimes take the liberty to omit the subscript t and simply write $V^\pm(x)$ in case $t \mapsto V^\pm(x_t)$ is constant on $\mathbb{R}$.

Proposition 3.7 ([30, Theorem 3.1])**.** *Let V denote either V^+ or V^- (determined by the sign of δ^N) and assume that $\sigma^{i_1} < \sigma^{i_2} < \cdots < \sigma^{i_n}$ belong to P for some $n \in \mathbb{N}$. Suppose further that each $x^{i_j}(\cdot)$ is a solution of (3.4) such that $x^{i_j}_t \in \mathcal{G}_{\sigma^{i_j}}$ for all $t \in \mathbb{R}$, and let*

$$x(t) = \sum_{j=1}^{n} x^{i_j}(t).$$

If neither x^{i_1}, nor x^{i_n} are the zero solution, then

$$\lim_{t \to -\infty} V(x_t) = V(x^{i_1}), \qquad \lim_{t \to \infty} V(x_t) = V(x^{i_n})$$

hold.

3.2. ***Proof of Theorem 3.2***

Throughout the proof we will use the notation V to express that a statement holds both with V^+, in case of positive feedback, and with V^-, for negative feedback.

The following proposition – which is an analog of [37, Lemma 3.8] – suggests that among the level sets of V, the set $V^{-1}(N^*)$ has a distinguished role.

Proposition 3.8.

(i) If x is a nontrivial bounded solution of (3.4) on $(-\infty,0]$, then $V(x_t)\le N^*$ for all $t\le 0$.

(ii) If x is a nontrivial bounded solution of (3.4) on $[0,\infty)$, then $V(x_t)\ge N^*$ for all $t\ge 0$.

Proof. Let us only consider the negative feedback case, i.e. $\delta^N=-1$ (the proof for the positive feedback case is completely analogous).

First note that (3.4) has no superexponential solutions. Indeed, this follows directly from the result of Cooke and Verduyn Lunel [8], which states that the equation

$$\dot{x}(t)=A(t)x(t)+B(t)x(t-1), \tag{3.9}$$

with $A(\cdot)$ and $B(\cdot)$ bounded, real analytic $(N+1)\times(N+1)$-matrix valued functions, does not have any superexponential solutions, provided $|\det B(t)|>0$ for all $t\in\mathbb{R}$. Should x be a superexponential solution of equation (3.4), then z with $z^i(t)=x^i\big(\frac{t+i}{N+1}\big)$ for $0\le i\le N$ would be a superexponential solution of equation

$$\dot{z}^i(t)=\frac{\mu^i}{N+1}z^i(t)+\frac{\gamma^i}{N+1}z^{i+1}(t-1),\qquad 0\le i\le N,$$

which is in the form (3.9), where matrices A and B are constant, therefore trivially bounded and analytic, and $|\det B|=\big|(N+1)^{-N-1}\prod_{i=0}^{N}\gamma^i\big|$, which is nonzero, thanks to assumption (H_1^a) and $\gamma^i=D_2f^i(0,0)$.

Therefore the system of eigenvectors and generalized eigenvectors of the generator of the semigroup given by (3.4) is complete (see Theorem 4.3 in [11]).

Note that in view of the last statement of Proposition 3.6, the definition of M^*, and (3.7), one has

$$\sigma^i>0\quad\text{for } 0\le i\le M^*-1\quad\text{and}\quad\sigma^i\le 0\quad\text{for } i\ge M^*. \tag{3.10}$$

Proof of (i). First consider the hyperbolic case. Hyperbolicity combined with (3.7) and (3.10) imply that $\sigma^i>0$ for all $0\le i\le M^*-1$, and $\sigma^i<0$ for all $i\ge M^*$. Hence, thanks to completeness, any nontrivial solution x of (3.4) that is bounded on $(-\infty,0]$ is of the form

$$x(t)=\sum_{i=0}^{j}x^i(t), \tag{3.11}$$

where $x_t^i\in\mathcal{G}_{\sigma^i}$ for all $t\in(-\infty,0]$ and $0\le i\le j$ with some $j\in\mathbb{N}_0$, $j\le M^*-1$ and $x^j\not\equiv 0$. Then Propositions 3.6 and 3.7 imply that

$$\lim_{t\to-\infty}V(x_t)=V(x^j)\le j+1\le M^*=N^*.$$

Finally, from Proposition 2.1 it follows that $V(x_t)\le N^*$ for all $t\in(-\infty,0]$.

If the origin is nonhyperbolic and M^* is odd, then it follows from (3.7) and (3.10) that $\sigma^i<\sigma^{M^*}=0$ holds for all $i\ge M^*+1$. Thus a nontrivial solution x of (3.4) that is bounded on $(-\infty,0]$ can be written in the form (3.11) with some $j\in\mathbb{N}_0$, $j\le M^*$ such that $x^j\not\equiv 0$. This implies that $\lim_{t\to-\infty}V(x_t)=V(x^j)\le M^*=N^*$ holds, and Proposition 2.1 proves the assertion.

If the origin is nonhyperbolic and M^* is even, then it follows from (3.10) that $\sigma^{M^*}=0$ and $N^*=M^*+1$. We infer from (3.7) that $\sigma^i<\sigma^{M^*+1}\le 0$ holds for all $i\ge M^*+2$. Therefore, in view of completeness, a nontrivial solution x of (3.4) that is bounded on $(-\infty,0]$ can be written in the form (3.11) with some $j\in\mathbb{N}_0$, $j\le M^*+1$ such that $x^j\not\equiv 0$. As M^*+1 is now odd, Propositions 3.6 and 3.7 imply that $\lim_{t\to-\infty}V(x_t)=V(x^j)\le M^*+1=N^*$ holds, and Proposition 2.1 concludes the proof of statement (i).

Proof of (ii). Thanks to completeness and by virtue of (3.10), a nontrivial solution x of (3.4), that is bounded on $[0,\infty)$, can be written in the form

$$x(t)=\sum_{i=j}^{\infty}x^i(t), \tag{3.12}$$

where $x^i_t\in\mathcal{G}_{\sigma^i}$ for all $t\in[0,\infty)$ and $i\geq j$ with some $j\geq M^*$ and $x^j\not\equiv 0$. Since $\dim\mathcal{G}_{\sigma^j}\leq 2$, the set $\mathfrak{S}_{\sigma^j}$ is either a complex pair of eigenvalues $\sigma^j\pm\iota\cdot\omega^j$ ($\omega^j>0$), or solely the real eigenvalue σ^j with a multiplicity of at most 2.

Thus x^j takes either the form

$$(x^j)^\ell(t)=\gamma^\ell e^{\sigma^j t}\cos(\omega^j t+\nu^\ell),\quad \text{for all } 0\leq\ell\leq N \tag{3.13}$$

for some $\gamma^\ell\in\mathbb{R}$, and $\nu^\ell\in[0,2\pi)$ (where $(x^j)^\ell$ denotes the ℓ-th component of the vector function x^j), or

$$(x^j)^\ell(t)=p^\ell(t)e^{\sigma^j t},$$

where $p^\ell(t)$ is a linear polynomial in t.

In the former case, x has the asymptotic expansion (see Theorem 5.4 of [11, Ch. 1])

$$x^\ell(t)=\gamma^\ell e^{\sigma^j t}\big(\cos(\omega^j t+\nu^\ell)+o(1)\big)\quad \text{for all } 0\leq\ell\leq N. \tag{3.14}$$

Recall that by Proposition 3.6, $V(x^j_t)$ is constant on $[0,\infty)$. Now, fix a $t_0>0$ and let $\theta_0<\cdots<\theta_k$ be from $\mathbb{K}$ such that they realize the $\mathrm{sc}(x^j_{t_0},\mathbb{K})$ number of sign-changes, i.e. $x^j_{t_0}(\theta_{i-1})x^j_{t_0}(\theta_i)<0$ for all $1\leq i\leq k$. Obviously, k can only be $V(x^j_{t_0})$ or $V(x^j_{t_0})-1$ and the latter may only happen if k is even.

Using periodicity of the cosine function and expansion (3.14) we obtain that for all $0\leq i\leq k$ and $n\in\mathbb{N}$ large enough,

$$\operatorname{sgn} x_{t_n}(\theta_i)=\operatorname{sgn} x^j_{t_0}(\theta_i),\quad \text{with } t_n=t_0+\frac{2n\pi}{\omega^j},$$

holds, and hence, $\mathrm{sc}(x_{t_n})\geq\mathrm{sc}(x^j_{t_n})$ for large enough $n\in\mathbb{N}$. This readily implies that

$$\limsup_{t\to\infty}V(x_t)\geq V(x^j)\geq\left\{\begin{matrix}M^*+1, & \text{if } M^* \text{ is even,}\\ M^*, & \text{if } M^* \text{ is odd}\end{matrix}\right\}\geq N^*.$$

Thus, by monotonicity of V (Proposition 2.1), we obtain that $V(x_t)\geq N^*$ holds for all $t\geq 0$.

An analogous – and even simpler – argument can be applied to prove the statement if $\sigma^j\in\mathbb{R}$. This completes the proof of (ii). □

The next proposition is one of the key tools for the proof of Theorem 3.2, and this is the point where the findings of Section 2 (in particular, the estimate in Proposition 2.8) are essential.

Proposition 3.9. *There exists an open neighborhood U of 0 in $\mathcal{A}$, such that for all nontrivial entire solutions $x\colon\mathbb{R}\to\mathbb{R}^{N+1}$ of equation (3.1) the following statements hold.*

(i) If $x_t\in\overline{U}$ for all $t\leq 0$, then $V(x_t)\leq N^*$ for all $t\in\mathbb{R}$.

(ii) If $x_t\in\overline{U}$ for all $t\geq 0$, then $V(x_t)\geq N^*$ for all $t\in\mathbb{R}$.

Proof. The argument of the proof relies on Propositions 2.1, 2.8, 3.5 and 3.8, and it is completely analogous to that of [37, Lemma 3.9]. □

Obviously, for the existence of the Morse decomposition, we must prove that only finitely many of the Morse sets are nonempty. The next theorem states somewhat more.

Theorem 3.10. *The Lyapunov function V is bounded on $\mathcal{A}\setminus\{0\}$.*

As in [37], the proof of this requires the following three lemmas. Although the proofs of these are quite lengthy and technical, they are straightforward modifications of the ones from [37], and therefore they are omitted here.

Lemma 3.11. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}\setminus\{0\}$ and let $\varphi \in \mathcal{A}$ such that $\varphi_n \to \varphi$ and $V(\varphi_n) \to \infty$ as $n \to \infty$. Then there exist a bounded entire solution x of equation (3.1) and a number $\sigma \in [-1, 0]$, such that $x_0 = \varphi$ and $x^i(\sigma - k) = 0$ for all $0 \le i \le N$ and all $k \in \mathbb{N}_0$.*

Lemma 3.12. *If x is a bounded entire solution of equation (3.1) such that $x^i(-k) = 0$ for all $k \in \mathbb{N}_0$ and all $0 \le i \le N$, then $x^i(t) = 0$ for all $t \in \mathbb{R}$ and $0 \le i \le N$.*

The following lemma is a straightforward corollary of the previous two.

Lemma 3.13. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}\setminus\{0\}$ and $\varphi \in \mathcal{A}$ such that $\varphi_n \to \varphi$ and $V(\varphi_n) \to \infty$ as $n \to \infty$. Then $\varphi = 0$.*

The proof of Theorem 3.10 is analogous to that of [37, Theorem 3.4], however, for the reader's convenience, we include a detailed proof.

Proof of Theorem 3.10. Arguing by contradiction, assume that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{A}\setminus\{0\}$, such that $V(\varphi_n) \to \infty$, as $n \to \infty$. By compactness, we may assume that $\varphi_n \to \varphi$ for some $\varphi \in \mathcal{A}$, as $n \to \infty$. By virtue of Lemma 3.13, $\varphi = 0$ holds.

Since $V(\varphi_n) \to \infty$ and $\varphi_n \to 0$, there exists some $n_0 \in \mathbb{N}$, such that $\varphi_n \in U$ and $V(\varphi_n) > N^*$ hold for all $n \ge n_0$, where $U \subseteq \mathcal{A}$ is given by Proposition 3.9. Now, for any $k \ge n_0$, there exists a bounded entire solution x^k of (3.1) with $x^k_0 = \varphi_k$ and $x^k_t \in \mathcal{A}$ for all $t \in \mathbb{R}$.

By Proposition 2.1 we obtain that $V(x^k_t) > N^*$ for all $t \le 0$, thus Proposition 3.9 implies that there exists $t \le 0$ such that $x^k_t \notin \overline{U}$. Therefore there must exist $t_k \le 0$ such that $\psi_k := x^k_{t_k} \in \partial U$. As $(\psi_k)_{k\ge n_0}$ is a sequence in the compact set $\mathcal{A}\cap\partial U$, we may assume (by passing to a subsequence) that $\psi_k \to \psi$ for some $\psi \in \mathcal{A}\cap\partial U$.

Using monotonicity of V one has

$$V(\psi_k) = V(x^k_{t_k}) \ge V(x^k_0) = V(\varphi_k) \to \infty \quad \text{as } k \to \infty,$$

which, in view of Lemma 3.13, implies $\psi = 0$, a contradiction to $\psi \in \partial U$ and $0 \notin \partial U$. □

Before we can give the proof of the main result of the section, some further lemmas are necessary.

Lemma 3.14. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ and $\varphi \in \mathcal{A}$ such that $\varphi_n \to \varphi$ as $n \to \infty$. Then $(\varphi_n)_{n\in\mathbb{N}}$ has a subsequence converging to φ in the $\mathcal{C}^1_{\mathbb{K}}$ norm.*

Proof. The proof is completely analogous to [37, Lemma 4.3] and therefore omitted here. □

Lemma 3.15. *Let $\varphi \in \mathcal{A}\setminus\{0\}$ and x denote a bounded entire solution of equation (3.1) for which $x_0 = \varphi$. Then the following statements hold.*

(i) If $\lim_{t\to\infty} V(x_t) = n$ for some $n \in \mathbb{N}_0$, then $V(\psi) = n$ for all $\psi \in \omega(\varphi)\setminus\{0\}$.

(ii) If $\lim_{t\to-\infty} V(x_t) = n$ for some $n \in \mathbb{N}_0$, then $V(\psi) = n$ for all $\psi \in \alpha(x)\setminus\{0\}$.

Proof. The proof is analogous to that of [37, Lemma 4.4] and it is based on Propositions 2.1 and 3.5, Theorem 3.10, and Lemma 3.15. □

Lemma 3.16. *Let $\varphi \in \mathcal{A}\setminus\{0\}$ and x denote a bounded entire solution of equation (3.1) for which $x_0 = \varphi$. Then the following statements hold.*

(i) If $\lim_{t\to\infty} V(x_t) \ne N^*$, then either $\omega(\varphi) = 0$ or $0 \notin \omega(\varphi)$.

(ii) If $\lim_{t\to-\infty} V(x_t) \ne N^*$, then either $\alpha(x) = 0$ or $0 \notin \alpha(x)$.

Proof. The proof is analogous to that of [37, Lemma 4.5] and it is based on Propositions 2.1, 2.8 and 3.9 and Lemmas 2.7 and 3.14. □

The next lemma is an analogue of [37, Lemma 4.6] and it follows readily from the previous two lemmas. As we will see, this auxiliary result will be used to establish the ordering between the Morse sets.

Lemma 3.17. *Let $n \in \mathbb{N}_0 \setminus \{N^*\}$, $\varphi \in \mathcal{A} \setminus \{0\}$ and x denote a bounded entire solution of equation (3.1) for which $x_0 = \varphi$. Then the following statements hold.*

(i) If $\lim_{t\to\infty} V(x_t) = n$, then either $\omega(\varphi) = \{0\}$ or $\omega(\varphi) \subseteq \mathcal{M}_n$.

(ii) If $\lim_{t\to-\infty} V(x_t) = n$, then either $\alpha(x) = \{0\}$ or $\alpha(x) \subseteq \mathcal{M}_n$.

The next lemma is used for establishing that the Morse sets are compact.

Lemma 3.18. *For every $n \in \mathbb{N}_0 \setminus \{N^*\}$, there exists an open neighborhood $\tilde{U}$ in $\mathcal{A}$ of the origin, such that $\mathcal{M}_n \cap \tilde{U} = \emptyset$.*

Proof. The proof relies on Propositions 2.1, 3.5 and 3.9 and Lemma 3.15, and the argument given in [37, Lemma 4.7] applies almost verbatim for our case as well. □

Lemma 3.19. *The set $\mathcal{M}_n$ is compact for each $n \in \mathbb{N}_0$.*

Proof. The proof of [37, Lemma 4.8] applies almost verbatim for our case, in which one has to use Lemma 3.18, Proposition 3.5, and Theorem 3.10. □

After all these preparations, we are now in position to actually prove Theorem 3.2.

Proof of Theorem 3.2. The argument follows the lines of the proof presented in [15, Theorem 4.1] for an analogous discrete time result.

By definition, the sets $\mathcal{M}_n$ are pairwise disjoint and invariant, and, according to Lemma 3.19, they are compact. By virtue of Theorem 3.10, only finitely many Morse sets can be nonempty, hence it is only left to prove that the Morse properties (m_1)–(m_2) hold.

Note that thanks to the backward uniqueness of the zero solution, (m_1)–(m_2) hold trivially for $\varphi = 0$. So consider an arbitrary $\varphi \in \mathcal{A} \setminus \{0\}$, and let x be a bounded entire solution of (3.1) for which $x_0 = \varphi$ holds. Furthermore, define

$$i := \lim_{t\to-\infty} V(x_t) \quad \text{and} \quad j := \lim_{t\to\infty} V(x_t).$$

From Proposition 2.1 one obtains that $i \geq j$.

First, observe that if $j = N^*$, then $\omega(\varphi) \subseteq \mathcal{M}_{N^*}$. To see this, choose an arbitrary $\psi \in \omega(\varphi)$. If $\psi = 0$, then $\psi \in \mathcal{M}_{N^*}$ holds by definition, so we may assume that $\psi \neq 0$. By Lemma 3.15 we infer that $V(\psi) = N^*$. Moreover, by the invariance of $\omega(\varphi) \setminus \{0\}$, there exists an entire solution $\tilde{x}$ in $\omega(\varphi) \setminus \{0\}$, such that $\tilde{x}_0 = \psi$. Hence, by virtue of Lemma 3.15, $V(\tilde{x}_t) = N^*$ holds for all $t \in \mathbb{R}$, and, in particular, $\psi \in \mathcal{M}_{N^*}$. A similar argument can be applied to prove that $i = N^*$ implies that $\alpha(x) \subseteq \mathcal{M}_{N^*}$ holds.

We will distinguish four cases in terms of the values of i and j.

Case 1. If $i = j = N^*$, then $\alpha(x) \cup \omega(\varphi) \subseteq \mathcal{M}_{N^*}$ holds by the above observation. Moreover, from the monotonicity of V it follows that $V(x_t) \equiv N^*$ on $\mathbb{R}$, thus $x_t \in \mathcal{M}_{N^*}$ for all $t \in \mathbb{R}$, and both (m_1) and (m_2) hold.

Case 2. If $i > j = N^*$, then $\omega(\varphi) \subseteq \mathcal{M}_{N^*}$ holds. On the other hand, $\alpha(x) \neq \{0\}$ holds. Otherwise Proposition 3.9 would imply $V(x_t) \leq N^*$ for $t \in \mathbb{R}$, and thus $i \leq N^* = j$, which is a contradiction. Hence Lemma 3.17 yields $\alpha(x) \subseteq \mathcal{M}_i$, and property (m_1) is fulfilled. Note that (m_2) holds automatically, as the two Morse sets in question, i.e. $\mathcal{M}_{N^*}$ and $\mathcal{M}_i$, are different.

Case 3. A similar argument applies in the case when $i = N^* > j$.

Case 4. If $i \neq N^* \neq j$, then Lemma 3.17 yields that either $\omega(\varphi) = \{0\}$ or $\omega(\varphi) \subseteq \mathcal{M}_j$. Similarly, either $\alpha(x) = \{0\}$ or $\alpha(x) \subseteq \mathcal{M}_i$ holds. Note that $\omega(\varphi)$ and $\alpha(x)$ cannot be both $\{0\}$ in this case, because then Proposition 3.9 would imply that $V(x_t) \equiv N^*$ on $\mathbb{R}$, contradicting $i \neq N^* \neq j$.

If none of $\omega(\varphi)$ and $\alpha(x)$ is the origin, then from Lemma 3.17 we obtain that $\omega(\varphi) \subseteq \mathcal{M}_j$ and $\alpha(x) \subseteq \mathcal{M}_i$ hold, so (m_1) is fulfilled. If $i = j$, then the definition of $\mathcal{M}_i$ and $\mathcal{M}_j$ imply that $V(x_t) = i = j$ for all $t \in \mathbb{R}$. On the other hand, Lemma 3.16 ensures that $0 \notin \alpha(x) \cup \omega(\varphi)$, thus $x_t \in \mathcal{M}_i$ holds for all $t \in \mathbb{R}$. This establishes property (m_2).

If $\omega(\varphi) = \{0\} \neq \alpha(x)$, then $\omega(\varphi) \subseteq \mathcal{M}_{N^*}$ holds by definition. Furthermore, Proposition 3.9 implies that $V(x_t) \geq N^*$ holds for all $t \in \mathbb{R}$, and consequently $N^* < j \leq i$. On the other hand, Lemma 3.17 yields that $\alpha(x) \subseteq \mathcal{M}_i$, so (m_1) holds. Property (m_2) is fulfilled automatically.

An analogous argument applies for the case when $\omega(\varphi) \neq \{0\} = \alpha(x)$.

We have listed all possible cases, so the proof is complete. □

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