for every $k\in N$. Then, for arbitrary $\sigma_0, \ldots, \sigma_{n-1}\in M^N$ it holds

$$h^i(g^0(\sigma_0, \ldots, \sigma_{n-1}), \ldots, g^{m-1}(\sigma_0, \ldots, \sigma_{n-1}))(k) =$$
$$= g^{k \bmod m}(\sigma_0, \ldots, \sigma_{n-1})(n(k \operatorname{div} m)+i) =$$
$$= \sigma_{(n(k \operatorname{div} m)+i) \bmod k}(m((n(k \operatorname{div} m)+i) \operatorname{div} n)+k \bmod m) = \sigma_i(k).$$

The identities in the second line of (6) can be verified in the same way. Thus, $S\in\mathscr{A}_{m,n}$, as required.

For a variety, to contain free algebras which have m-element and also n-element free generating sets $(m, n\in N;\ m\neq n)$ is a strong Mal'cev property ([7], p. 400), characterized by the identities (6). Hence we can conclude that the fulfilment of a Mal'cev condition does not exclude ubiquity. Using selective algebras, it is easy to establish that several other syntactical properties of varieties, e.g. equational completeness, definability by regular identities, and definability by linear identities are independent from ubiquity as well.

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SELECTIVE ALGEBRAS AND COMPATIBLE VARIETIES

BÉLA CSÁKÁNY

1. Introduction

In this paper, the notion of a selective algebra is introduced and applied to characterize equational theories which have models over every variety. Another characterization was already proposed by Isbell [9]; the use of selective algebras makes it possible to prove or refute this property for several concrete equational theories.

We shall use the standard terminology of universal algebra [7]. A *non-trivial* set or algebra always has at least two elements. The set consisting of the first k non-negative integers will be denoted by $\mathbf{k}$.

Let P and $M_p (p\in P)$ be arbitrary non-empty sets and k a natural number. We define a k-ary operation f on $S:=\prod_{p\in P} M_p$ in the following way. We consider two mappings $f_1\colon P\to\mathbf{k}$ and $f_2\colon P\to P$, such that, for all $p\in P$, $M_{f_2(p)}\subseteq M_p$ and $M_{f_2(p)}$ is non-trivial if M_p is non-trivial. Let $\sigma_0, \ldots, \sigma_{k-1}\in S$. Put

$$f(\sigma_0, \ldots, \sigma_{k-1})(p) = \sigma_{f_1(p)}(f_2(p)), \tag{1}$$

for every $p\in P$. In words, in order to get the p-component of the result, first we select the $f_1(p)^{\text{th}}$ operand, and then the $f_2(p)$-component of it. Operations f obtained in this way will be called *selective operations*. The mappings f_1 and f_2 will be referred to as the *first* and *second selectors* of f. We say that $\langle S; F\rangle$ is a *selective algebra* if each $f\in F$ is a selective operation on S. If $M_p=M$ for every $p\in P$ (i.e. $S=M^P$), we call $\langle S; F\rangle$ a *regular selective algebra*.

Special kinds of selective algebras have been in use for a long time. A selective algebra $\langle S; F\rangle$ with $P=\mathbf{k}$, f k-ary, and $f_1(p)=f_2(p)=p$ for each $p\in P$ is a *k-dimensional diagonal algebra* (Płonka [13]) which often appears in the study of free spectra of varieties (see, e.g. [10]). Diagonal algebras of a given dimension form a variety in which regularity in the above sense means freeness. *Rectangular bands, left* and *right zero semigroups* are examples of diagonal algebras, hence also of selective algebras. A further example is the *k-dimensional die*, introduced by Fajtlowicz [4]; such an object is a free k-dimensional diagonal algebra whose structure is enriched by a further unary selective operation c with $c_2(i)\equiv i-1 \pmod k$ for every $i\in\mathbf{k}$. Regular selective groupoids with two-element P and non-trivial cyclic selectors were

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characterized by Evans [3] by means of identities; a more general result for regular selective groupoids was obtained by Saade [15]. Regular selective algebras with k-element P and with all possible selective operations appear at Taylor [16] as members of the k^{th} *power-variety of sets*.

Regular selective algebras are a special case of the *wreath algebras* introduced and applied to the study of completeness properties of finite algebras by Rosenberg [14]. Take a selective operation f on M^P and a mapping π of P into the symmetric group over M. Define the operation $w_{f,\pi}$ on M^P by $w_{f,\pi}(\sigma_0, \ldots, \sigma_{k-1})(p) = \pi(p)(f(\sigma_0, \ldots, \sigma_{k-1})(p))$. The operations arising in this way are the wreath operations; and wreath algebras are the ones with wreath basic operations.

Now we make some observations we will need in the sequel.

Polynomials of selective algebras are selective operations.

Indeed, each projection on a product set $S = \prod_{p\in P} M_p$ is a selective operation having the first selector constant and the second selector the identical map. Further, if f and $g^0, \ldots, g^{k-1}$ are n-ary, resp. k-ary, selective operations on S, then, for any $\sigma_0, \ldots, \sigma_{k-1} \in S$ and $p \in P$

$$(2) \quad f\big(g^0(\sigma_0, \ldots, \sigma_{k-1}), \ldots, g^{n-1}(\sigma_0, \ldots, \sigma_{k-1})\big)(p) = \sigma_{g_1^{f_1(p)}(f_2(p))}\big(g_2^{f_1(p)}(f(p))\big),$$

as, in view of (1), both sides are equal to $g^{f_1(p)}(\sigma_0, \ldots, \sigma_{k-1})\big(f_2(p)\big)$. Note also $M_{g_2^{f_1(p)}(f_2(p))} \subseteq M_p$; thus we see that $f(g^0, \ldots, g^{n-1}) = h$ is a selective operation on S with selectors $h_1\colon p \mapsto g_1^{f_1(p)}\big(f_2(p)\big)$ and $h_2\colon p \mapsto g_2^{f_1(p)}\big(f_2(p)\big)$.

This consideration also shows that we can attribute a well-determined pair of selectors to every polynomial symbol h of a selective algebra $\mathbf{S}$, which are also selectors of the polynomial induced by h in $\mathbf{S}$.

For a product set $S = \prod_{p\in P} M_p$, the *support* of S is the set $Q = \{p \in P\colon |M_p| > 1\}$.

An n-ary selective operation f on $\mathbf{S}$ *depends essentially on its i^{th} variable* ($i \in \mathbf{n}$) *if and only if the image of the support of S under f_1 contains i.* This follows directly from the definition.

LEMMA. *Two selective operations f and g of the same arity on S are equal iff their first selectors as well as their second selectors coincide on the support of S.*

The easy proof may be omitted. We note only that $f(\sigma_0, \ldots, \sigma_{n-1})(p) = a$ for all $p \in P$ such that $M_{f_2(p)} = \{a\}$, and also that for $p \in P \setminus Q$ we have $M_{f_2(p)} = M_p$ (because $M_{f_2(p)} \subseteq M_p$). Thus, without loss of generality we may assume that $f_2|(P \setminus Q) = \mathrm{id}_{P\setminus Q}$.

2. Compatibility of varieties

An *n-ary operation over an algebra* $\mathbf{A}$ is a homomorphism $h\colon \mathbf{A}^n \to \mathbf{A}$. For the algebra $\mathbf{A} = \langle A; \emptyset \rangle$ (i.e., a set) this is the common notion of the operation. Expressing it in other way, f is an operation over $\mathbf{A}$ iff f commutes with all operations of $\mathbf{A}$, i.e. belongs to the centralizer of $\mathbf{A}$ ([1], p. 127; cf. [12], [11]).

$\mathbf{B} = \langle A; H \rangle$ is an *algebra over* $\mathbf{A}$ if every $h \in H$ is an operation over $\mathbf{A}$. We can thus speak of algebras of a given type over $\mathbf{A}$, and of algebras over $\mathbf{A}$ which are models of a given equational theory, i.e. belong to a given variety.

Following Isbell [9], for two varieties $\mathscr{V}$ and $\mathscr{W}$, we say that $\mathscr{V}$ is compatible with $\mathscr{W}$ if there exists an algebra $A \in \mathscr{V}$ over a nontrivial $B \in \mathscr{W}$. For operations f, g the relation f commutes with g is symmetric, hence compatibility of varieties is symmetric, too. We say that a variety $\mathscr{V}$ is *ubiquitous* if $\mathscr{V}$ is compatible with every variety. Isbell proved ([9], Theorem 1.1) that every variety compatible with the variety of Boolean algebras is ubiquitous. The next proposition slightly extends this result, and throws light on the relationship of ubiquity and selective algebras.

PROPOSITION. *For a variety $\mathscr{V}$ the following are equivalent:*
(I) *$\mathscr{V}$ is compatible with a variety generated by a primal algebra.*
(II) *$\mathscr{V}$ contains a nontrivial regular selective algebra.*
(III) *$\mathscr{V}$ contains a nontrivial selective algebra.*
(IV) *$\mathscr{V}$ is ubiquitous.*

PROOF. Our proposition is implied by the following four claims:

CLAIM 1. *Let a variety $\mathscr{W}$ be generated by a primal algebra M. If* $\mathbf{B}$ *is a nontrivial algebra over an algebra* $\mathbf{A} \in \mathscr{W}$, *then* $\mathbf{B}$ *is a dense subalgebra of a regular selective algebra on a power of M.* ($A \subseteq M^P$ *is dense* if $A|P' = M^{P'}$ *for every finite* $P' \subseteq P$).

CLAIM 2. *If some dense subalgebra of a regular selective algebra* $\mathbf{S}$ *belongs to the variety $\mathscr{V}$, then* $\mathbf{S}$ *belongs to $\mathscr{V}$.*

CLAIM 3. *If a variety $\mathscr{V}$ contains a nontrivial selective algebra then for an arbitrary nontrivial set M, the variety $\mathscr{V}$ contains a regular selective algebra on some power of M.*

CLAIM 4. *For an arbitrary algebra* $\mathbf{K}$, *every selective operation on a power K^P commutes with every operation of K^P.*

Indeed, (II)→(III) and (IV)→(I) are obvious; (I)→(II) follows from Claims 1 and 2; and (III)→((II)→)(IV) follows from Claims 3 and 4. Hence it remains to prove the Claims.

1. Let $\mathbf{B} = \langle A; F \rangle$ be an algebra over $\mathbf{A} \in \mathscr{W}$. As $\mathbf{M}$ is primal, $\mathbf{A}$ is isomorphic to a subdirect power of $\mathbf{M}$. (Concerning primal algebras, consult [7], pp. 177—180, 401—403.) Hence the maximal congruences of $\mathbf{A}$ are exactly those having $|M|$ distinct congruence classes. We can represent $\mathbf{A}$ as a subdirect product of all factoralgebras modulo maximal congruences, which is the same as a subdirect product of copies of $\mathbf{M}$ indexed by the set P of all maximal congruences of $\mathbf{A}$. Thus, $\mathbf{A}$ is, up to isomorphism, a subalgebra of $\mathbf{M}^P$, and the primality of $\mathbf{M}$ implies that A is dense.

Consider an n-ary operation $f \in F$, i.e. a homomorphism $f\colon \mathbf{A}^n \to \mathbf{A}$. Let $\pi \in P$ and, for $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle, \langle \alpha'_0, \ldots, \alpha'_{n-1} \rangle \in \mathbf{A}^n$, put $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \sim \langle \alpha'_0, \ldots, \alpha'_{n-1} \rangle$ if the π-components of $f(\alpha_0, \ldots, \alpha_{n-1})$ and $f(\alpha'_0, \ldots, \alpha'_{n-1})$ coincide. Then $\sim$ is a maximal congruence of A^n. As the algebras in $\mathscr{W}$ may be considered as lattices with additional operations, the congruences of A^n are factorizable [5]. Thus, $\sim = \iota_A \times \ldots \times \pi' \times \iota_A \times \ldots$, where $\pi' \in P$ and π' is the k_π^{th} factor. This shows that the π-component of $f(\alpha_0, \ldots, \alpha_{n-1})$ is a bijective function of the π'-component of $\alpha_{k_\pi - 1}$. As f is a homomorphism, this function is an automorphism of $\mathbf{M}$, hence identical, because $\mathbf{M}$ is primal. We obtained that f is the restriction to $\mathbf{A}$ of a selective operation f' on M^P with $f'_1(\pi) = k_{\pi-1}$ and $f_2(\pi) = \pi'$ for every $\pi \in P$. Hence $\mathbf{A}$ is a dense subalgebra of a regular selective algebra on M^P, as asserted. (Note that this consideration may also be formulated using the Stone-Hu duality for primal algebra theory [8]).

2. Let $\mathbf{S}=\langle M^P; F\rangle$ be a regular selective algebra, and $\mathbf{D}$ a dense subalgebra of $\mathbf{S}$. We have to prove that the identities of $\mathbf{D}$ are satisfied in $\mathbf{S}$, too. This means that distinct (n-ary) polynomials h, h' of $\mathbf{S}$ can be distinguished by suitable $\delta_0, \dots, \delta_{n-1}\in\mathbf{D}$. We can suppose that $\mathbf{S}$ is non-trivial, hence the support of $\mathbf{S}$ is P. Now, by the Lemma, $h\neq h'$ on $\mathbf{S}$ means that at least one of $h_1\neq h_1'$ and $h_2\neq h_2'$ is valid. First suppose that $h_1\neq h_1'$ and let $p\in P$ be such that $h_1(p)\neq h_1'(p)$. Take distinct elements m_1, m_2 from M. As $\mathbf{D}$ is dense, there exist $\delta, \delta'\in D$ with $\delta(p)=m_1$, $\delta'(p)=m_2$. Let $\delta_{h_1(p)}=\delta$, $\delta_{h_1'(p)}=\delta'$ and choose all the remaining $\delta_i\in D$ $(i\in\mathbf{n};\ i\neq h_1(p), h_1'(p))$ arbitrarily. Then

$$h(\delta_0, \dots, \delta_{n-1})(p) = m_1 \neq m_2 = h'(\delta_0, \dots, \delta_{n-1})(p). \tag{3}$$

Assume $h_1=h_1'$; then there is a $p\in P$ with $h_2(p)\neq h_2'(p)$. As $\mathbf{D}$ is dense, there exists $\delta\in D$ with $\delta(h_2(p))=m_1\neq m_2=\delta(h_2'(p))$. Let $\delta_{h_1(p)}=\delta$ and choose the other δ_i's arbitrarily. Under these assumptions again (3) holds. Thus, h is distinct from h' on D, as stated.

3. Let $\mathbf{S}=\langle S; F\rangle$ be a non-trivial selective algebra. For an arbitrary non-trivial set M we present a regular selective algebra on some power of M which is the same type as $\mathbf{S}$ and satisfies all the identities of $\mathbf{S}$.

S has the form $\prod_{p\in P} M_p$ with non-empty support $Q\subseteq P$. Take an operation f of $\mathbf{S}$. Restrict f_1, f_2 to Q, thus obtaining f_1', f_2'. Let f' be the selective operation on M^Q determined by selectors f_1', f_2'. Now, $\mathbf{S}'=\langle M^Q; f'\colon f\in F\}\rangle$ is the regular selective algebra in question. Indeed, if g and h are polynomial symbols of $\mathbf{S}$, and $\mathbf{S}$ satisfies $g=h$, then, by the Lemma, $\mathbf{S}'$ satisfies $g'=h'$, where g', h' are the corresponding polynomial symbols of $\mathbf{S}'$.

4. Let $\mathbf{S}=\langle K^P; F\rangle$ be a selective algebra and take an n-ary $f\in F$. We have to show that f is a homomorphism of $(\mathbf{K}^P)^n$ into $\mathbf{K}^P$. Let g be an m-ary operation of $\mathbf{K}^P$. Choose m elements from $(\mathbf{K}^P)^n$ arbitrarily: $\langle\mu_0^i, \dots, \mu_{n-1}^i\rangle$ $(i=0, \dots, m-1)$. Then

$$\begin{aligned} f(\langle g(\mu_0^0, \dots, \mu_0^{m-1}), \dots, g(\mu_{n-1}^0, \dots, \mu_{n-1}^{m-1})\rangle)(p) &= g(\mu_{f_1(p)}^0, \dots, \mu_{f_1(p)}^{m-1})(f_2(p)) = \\ &= g(\mu_{f_1(p)}^0(f_2(p)), \dots, \mu_{f_1(p)}^{m-1}(f_2(p))) = \\ &= g(f\langle\mu_0^0, \dots, \mu_{n-1}^0\rangle(p), \dots, f\langle\mu_0^{m-1}, \dots, \mu_{n-1}^{m-1}\rangle(p)) = \\ &= g(f\langle\mu_0^0, \dots, \mu_{n-1}^0\rangle, \dots, f\langle\mu_0^{m-1}, \dots, \mu_{n-1}^{m-1}\rangle)(p) \end{aligned}$$

holds for each $p\in P$, i.e. f commutes with g, as required, and the Proposition is proved.

3. Applications

The fact that ubiquitous varieties can be characterized by the presence of algebras with a quite transparent structure allows us to decide on several varieties whether they are ubiquitous.

No congruence modular variety is ubiquitous.

We prove this by showing that there is no non-trivial regular selective algebra in a congruence modular variety. Let $\mathscr{V}$ be congruence modular. By the Mal'cev type theorem of Day [2], there exist quaternary polynomial symbols $d^0, \dots, d^n$ $(n\geq 1)$ such that for $i=0, \dots, n-1$ the following identities hold in $\mathscr{V}$:

$$d^i(x, y, y, x) = x, \tag{4}$$

$$\begin{cases} d^0(x, y, z, u) = x, & \\ d^i(x, y, y, u) = d^{i+1}(x, y, y, u) & \text{for } i \text{ odd} \\ d^i(x, x, u, u) = d^{i+1}(x, x, u, u) & \text{for } i \text{ even} \\ d^n(x, y, z, u) = u. & \end{cases} \tag{5}$$

Assume that there exists a regular selective algebra $\mathbf{S}=\langle M^P; F\rangle$ in $\mathscr{V}$. Set $e^i(x, y)=d^i(x, y, y, x)$ $(i=0, \dots, n-1)$. Then for arbitrary $\sigma_0, \sigma_1\in\mathbf{S}$ and for every $p\in P$

$$e^i(\sigma_0, \sigma_1)(p) = \sigma_0(p).$$

Applying (1), it follows $\sigma_0(p)=\sigma_{e_1^i(p)}(e_2^i(p))$, and the right side equals $\sigma_0(e_2^i(p))$ if $d_1^i(p)\in\{0, 3\}$ while it equals $\sigma_1(e_2^i(p))$ if $d_1^i(p)\in\{1, 2\}$. As we can choose σ_0 and σ_1 with $\sigma_0(p)\neq\sigma_1(e_2^i(p))$, the second case cannot occur, i.e., $d_1^i(p)\in\{0, 3\}$ for each i and p. This means that no d^i depends essentially upon its second and third variables. Hence, by (5), $\mathbf{S}$ satisfies $x=u$, thus $\mathbf{S}$ is trivial, a contradiction.

As a consequence, no varieties of quasigroups, groups, rings, or lattices are ubiquitous. As for semigroups, an easy argument shows that a variety of semigroups is ubiquitous if and only if it contains a non-trivial rectangular band.

Varieties $\mathscr{A}_{m,n}$ (with natural numbers m and n) having n-ary operations $g^0, \dots, g^{m-1}$ and m-ary operations $h^0, \dots, h^{n-1}$ which satisfy for each meaningful i

$$\begin{aligned} h^i(g^0(x_0, \dots, x_{n-1}), \dots, g^{m-1}(x_0, \dots, x_{n-1})) &= x_i, \\ g^i(h^0(x_0, \dots, x_{m-1}), \dots, h^{n-1}(x_0, \dots, x_{m-1})) &= x_i \end{aligned} \tag{6}$$

were first studied by Goetz and Ryll-Nardzewski [6]. They have the notable property that a free algebra in $\mathscr{A}_{m,n}$ with an m-element free generating set has also an n-element free generating set. Hence, for $m\neq n$, these varieties do not contain non-trivial finite algebras. Here we prove:

The varieties $\mathscr{A}_{m,n}$ are ubiquitous.

By the Proposition, we have to produce a selective algebra $\mathbf{S}$ with operations g^i $(i=0, \dots, m-1)$, h^i $(i=0, \dots, n-1)$ satisfying (6). Take a non-trivial set M. We shall define $\mathbf{S}$ on the set $M^{\mathbf{N}}$ where $\mathbf{N}=\{1, 2, \dots\}$. Write $i \operatorname{div} j$ for the quotient of the Euclidean division of i by j, and $i \bmod j$ for the remainder of that. Define g^i and h^j by their selectors as follows:

$$g_1^i(k) = k \bmod n, \quad g_2^i(k) = m(k \operatorname{div} n)+i,$$

$$h_1^j(k) = k \bmod m, \quad h_2^j(k) = n(k \operatorname{div} m)+j,$$