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Algebras whose subalgebras and reducts are trivial

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Abstract. Algebras having only trivial subalgebras and reducts are determined up to equivalence. They are simple, and their clones are simple, too; thus, in a sense, they are the smallest algebras.

In this note we determine all nontrivial algebras having the properties in the title. Nontrivial algebras with only trivial reducts are exactly those having a minimal clone of term operations (cf. [1], [2]). Hence they are of form $\mathbf{A} = (A; f)$, where f is idempotent or unary. We consider algebras up to equivalence; thus we can write $(A; \mathcal{C})$ instead of $(A; f)$ with $\mathcal{C}$ the clone of term operations of $(A; f)$.

Suppose that all proper subalgebras (if any) of $\mathbf{A} = (A; f)$ are trivial, and let f be idempotent. Then $\mathbf{A}$ is simple. The simple algebras with no nontrivial subalgebras are called *plain* (or, strictly simple; cf. [1], [2]). Finite idempotent plain algebras with minimal clones on an at least three-element universe were determined by Á. Szendrei in [3], Lemma 3.3. In fact, we can restrict ourselves to the finite case, by virtue of the following

Lemma. *An idempotent plain algebra with a minimal clone is finite.*

Proof. Let $\mathbf{A} = (A; \mathcal{C})$ be an idempotent plain algebra with $\mathcal{C}$ a minimal clone. Assume $|A| > 2$. Then Theorem 2.1 and the proof of Lemma 3.3 in [3] show that one of the following conditions holds:

(i) $\mathbf{A}$ is locally quasi-primal,

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(ii) there is an element $0 \in A$, and a local binary operation $*$ of $\mathbf{A}$ such that $x * y = 0$ if $x = 0$ or $y = 0$, and $x * y = x$ otherwise,

(iii) $\mathbf{A}$ is locally term equivalent to the full idempotent reduct of a module with universe A.

Suppose (i). Let $a, b \in A$ be distinct. The discriminator on A is a local operation of $\mathbf{A}$, hence there is a ternary $t \in \mathcal{C}$ which interpolates the discriminator on $\{a, b\}$. As t is not a projection and $\mathcal{C}$ is minimal, t generates $\mathcal{C}$. Since $\{a, b\}$ is closed under t, the set $\{a, b\}$ is a subuniverse of $\mathbf{A}$. But $\mathbf{A}$ is plain, so $A = \{a, b\}$, in contrary to the assumption $|A| > 2$.

Suppose (ii). Let $a \in A$, $a \neq 0$. There is a binary operation $s \in \mathcal{C}$ which interpolates $*$ on $\{0, a\}$. Again, s is not a projection, and we get a contradiction as in the case (i).

We have (iii). Then $\mathbf{A}$ is locally affine, hence every operation $r \in \mathcal{C}$ commutes with the local operation $x - y + z$ of $\mathbf{A}$. Thus, $r(x_1, \dots, x_n)$ is an idempotent R-module operation $r_1x_1 + \cdots + r_nx_n$ for an appropriate ring R, which turns out to be one-generated and hence commutative by the following argument. We may assume $r_1 \neq 0$. If R is not the two-element field, then $r_1 \neq 1$ under an appropriate choice of r. In this case, the operation $r_1x + (1 - r_1)y$ $(= r(x, y, \dots, y) \in \mathcal{C})$ is not a projection. So it generates $\mathcal{C}$, whence r_1 generates the ring R.

Now, $\mathbf{A}$ is a reduct of an R-module $\mathbf{A}'$ with universe A, and $\mathbf{A}'$ is still plain, hence irreducible. It follows that $\mathbf{A}'$ is a one-dimensional vector space over a field which is one-generated as a ring. Hence A is finite, as stated. ∎

Theorem. *All nontrivial algebras with no nontrivial subalgebras or reducts are on the following list:*

(1) *for every prime p, the p-element affine space,*
(2) *the two-element semilattice,*
(3) *the two-element ternary majority algebra,*
(4) *for every prime p, the p-element set with a full cycle,*
(5) *the two-element pointed set (or, equivalently, the two-element zero semigroup).*

Proof. Consider an algebra $\mathbf{A} = (A; f)$ with the required properties. If f is idempotent, then $\mathbf{A}$ is plain. By the Lemma, $\mathbf{A}$ is finite. If $|A| \geq 3$, then Lemma 3.3 in [3] asserts that $\mathbf{A}$ is a p-element affine space for some p, i. e., it is (1) in the above list. For $|A| = 2$, by Post's description of all clones on the two-element set, the idempotent algebras with minimal clone of term operations are exactly (2) and (3). Now, let f be unary. Then either $|A| = p$ and f is a cycle of length p or $|A| = 2$ and f is a constant; i. e., we have the case (4) or (5), completing the proof. ∎

If one regards algebras with only projection operations as nontrivial, one must expand the above list by

(6) *the two-element left-zero (or, equivalently, right-zero) semigroup.*

For algebras $\mathbf{A}$ of form (4), the proper subvarieties of $\mathcal{V}(\mathbf{A})$ are the trivial variety and a variety equivalent to the variety of sets. In the remaining cases, our algebras generate minimal varieties. Thus all of them have not only minimal, but also simple clones of term operations (cf. [5]). Therefore, in a sense, (1)–(5) are the "smallest" nontrivial algebras.

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